On a Robust Multigrid-Preconditioned Solver for Incremental Plasticity Problems

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1 Introduction

Plasticity models have a long history in the engineering and mathematical community. The rigorous mathematical and numerical analysis of different elasto-plastic models has been a topic of mathematical research during the last two decades, see e.g. [2], [6], [7], [8] and the literature cited there. The method presented in this paper is based on the approach proposed by C. Carstensen [1]. In contrast to [1], we introduce some regularization of the local minimization problems making the nonsmooth cost functional differentiable. We develop an adjusted multigrid preconditioned conjugate gradient (PCG) method for the Schur-complement problems arising at each incremental step. Moreover, we prove that the elastic multigrid preconditioner is sufficient for an effective and robust solving of large scale elasto-plastic problems.

2 Elasto-plasticity

The stress field σ of a deformed body Ω in $\mathbb{R}^n$ $(n = 2, 3)$ with a Lipschitz-continuous boundary has to fulfill the symmetry and equilibrium equations

$$\begin{aligned} \sigma &= \sigma^T \quad \text{in } \Omega, && (1)\\ -\text{div}\,\sigma &= b \quad \text{in } \Omega, && (2) \end{aligned}$$

with b being the vector field of given body forces. The linearized strain tensor ε is appropriate in the case of small deformations,

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \quad \text{a. e. in } \Omega, \tag{3}$$

where u denotes the displacement vector. The strain is split additively into elastic part $\mathbb{C}^{-1}\sigma$ and plastic part p with the local elasticity tensor $\mathbb{C}$:

$$\varepsilon(u) = \mathbb{C}^{-1}\sigma + p \quad \text{a. e. in } \Omega. \tag{4}$$

Purely elastic material behavior is characterized by $p \equiv 0$. In order to describe the time development $\dot{p} = \frac{\partial p}{\partial t}$, the generalized stress variables (σ, α) are introduced, their values are restricted by a convex dissipation functional φ:

$$\varphi(\sigma, \alpha) < \infty \quad \text{a. e. in } \Omega. \tag{5}$$

Then, the Prandtl-Reuß normality law states

$$\dot{p} : (\tau - \sigma) - \dot{\alpha} : (\beta - \alpha) \leq \varphi(\tau, \beta) - \varphi(\sigma, \alpha) \quad \forall(\tau, \beta) \quad \text{a. e. in } \Omega. \tag{6}$$

The hardening parameter α describes the memory effect of the deformed body. Its structure and dimension depend on the hardening law [1]. The scalar product of matrices is defined by $A : M = \sum_{i,j=1}^{n} A_{ij} M_{ij}$ for all $A, M \in \mathbb{R}^{n\times n}$.

We now are in the position to define the initial value problem: *Find the displacement field u, the plastic strains p, the stress σ and the hardening parameter α such that (1) - (6) together with appropriate initial and boundary conditions are satisfied.*

Using functional-analytic techniques, (1) - (6) can be combined in one time dependent variational inequality. Then, an implicit Euler scheme in time is applied and results in an equivalent minimization problem for each time step. In the case of isotropic hardening the minimization problem in one time step reads:

$$
\begin{aligned}
f(u,p) := {} & \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u) - p] : (\varepsilon(u) - p)dx + \frac{1}{2}\int_\Omega (\alpha_0 + \sigma_y H|p - p_0|)^2 dx \\
& + \int_\Omega \sigma_y |p - p_0| dx - \int_\Omega b\, u\, dx \quad \longrightarrow \quad \min
\end{aligned}
\tag{7}
$$

under the constraint tr $(p - p_0) = 0$. Here, the yield stress $\sigma_y > 0$ and the modulus of hardening $H > 0$ are material parameters. The deviator is defined by $\operatorname{dev} A := A - \frac{1}{n}\operatorname{tr}(A) I$, where $\operatorname{tr}(A) := \sum_{i=1}^{n} A_{ii}$ is the trace of a matrix and I is the identity matrix.

The structures of the optimization problems are similar for all hardening models. Thus the basic ingredients of the algorithm are presented for the isotropic hardening case as model case.

For given variables (with index 0) of an initial time step t_0, the upgrades of the variables at the time step $t_1 = t + \Delta t$ have to be determined. In addition to the time discretization the spatial discretization is performed by the finite element method.

3 Algorithm

The basic idea for solving the problem (7) at each time step presented by [1] is iterating until the minimizers (u, p) are determined with sufficient accuracy. Additionally, the norm function $|p|$ is smoothed as follows:

$$
|p|_\epsilon := \begin{cases} |p| & \text{if } |p| \geq \epsilon, \\ \frac{1}{2\epsilon}|p|^2 + \frac{\epsilon}{2} & \text{if } |p| < \epsilon, \end{cases}
$$

with a small regularization parameter $\epsilon > 0$. Another simplification is defining the incremental variable $\tilde{p} = p - p_0$, and using it as an argument of the objective (7) instead of p. The spatial discretization is carried out using triangular or tetrahedral finite elements. The minimization problem (7) now reads:

$$
\frac{1}{2}(Bu - \tilde{p})^T \mathbb{C}(Bu - \tilde{p}) + \frac{1}{2}\tilde{p}^T \mathbb{H}(|\tilde{p}|_\epsilon)\tilde{p} + (-B^T\mathbb{C}p_0 - b)^T u \quad \longrightarrow \quad \min \tag{8}
$$

under the local constraint tr $(\tilde{p}|_T) = 0$ on every element T of a triangulation τ. Here, Bu denotes the discretized strain $\varepsilon(u)$. $\mathbb{H}$ depends on $|\tilde{p}|_\epsilon$ and is computed locally as $\mathbb{H} = (\sigma_y^2 H^2 + 2\sigma_y(1 + \alpha_0 H)/|\tilde{p}|_\epsilon)\, \mathbb{Q}$. Since the local constraint tr $(\tilde{p}|_T) = 0$ is linear, the problem (8) is projected onto a hyperplane, where the constraint $\tilde{p} = P\bar{p}$ is satisfied exactly:

$$
\frac{1}{2}\begin{pmatrix} u \\ \bar{p} \end{pmatrix}^T \begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C}P \\ -P^T\mathbb{C}B & P^T(\mathbb{C} + \mathbb{H})P \end{pmatrix} \begin{pmatrix} u \\ \bar{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ P^T\mathbb{C}p_0 \end{pmatrix}^T \begin{pmatrix} u \\ \bar{p} \end{pmatrix} \longrightarrow \min\,. \tag{9}
$$

$\mathbb{H}$ is computed in every iteration step using the current $\tilde{p}$. Apart from this dependence on $\tilde{p}$, (9) is quadratic with a positive definite matrix. Thus the minimizer $(u, P\bar{p})$ has to fulfill the sufficient condition of first kind:

$$\begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C}P \\ -P^T\mathbb{C}B & P^T(\mathbb{C}+\mathbb{H})P \end{pmatrix} \begin{pmatrix} u \\ \bar{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ P^T\mathbb{C}p_0 \end{pmatrix} = 0. \tag{10}$$

The Schur-Complement system in u

$$B^T(\mathbb{C} - \mathbb{C}P(P^T(\mathbb{C}+\mathbb{H})P)^{-1}P^T\mathbb{C})Bu = b + B^T(\mathbb{C} - \mathbb{C}P(P^T(\mathbb{C}+\mathbb{H})P)^{-1}P^T\mathbb{C})\,p_0 \tag{11}$$

is solved by a multigrid preconditioned conjugate gradient method. The minimization in $\tilde{p}$ is done locally for each element, since no connections over several elements (e.g. derivatives) occur. Furthermore, the nested iteration approach is applied. For various 2D and 3D testing geometries the algorithm behaves linearly with respect to the number of unknowns, i.e., it has linear complexity, see [4].

4 Robustness Analysis

Numerical tests indicate that it is not necessary to use the multigrid preconditioner (see [3]) arising from the elasto-plasticity problem (11). The preconditioner for the related elasticity problem with the stiffness matrix $K = B^T\mathbb{C}B$ is sufficient and much faster.

Indeed, the spectral equivalence constants c_1 and c_2 in the spectral inequalities

$$c_1\langle Ku, u\rangle \le \langle Su, u\rangle \le c_2\langle Ku, u\rangle \quad \forall u \in \mathbb{R}^n, \tag{12}$$

with the Schur-Complement matrix S of (11) can obviously be determined as the smallest and largest eigenvalues of the generalized eigenvalue problem

$$\mathbb{S}u = \lambda\mathbb{C}u. \tag{13}$$

$\mathbb{S} = \mathbb{C} - \mathbb{C}P(P^T(\mathbb{C}+\mathbb{H})P)^{-1}P^T\mathbb{C}$ and $\mathbb{C}$ are block diagonal matrices with blocks corresponding to local systems. Consequently, it is sufficient to calculate the generalized eigenvalue problems for the single blocks in (13) only.

In the two-dimensional case, each block in (13) has three eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$ of the form

$$\frac{1}{1+\beta} < \frac{1}{1+\frac{\beta}{4}} < 1, \tag{14}$$

with β depending on the hardening law. In the three-dimensional case, we obtain the same eigenvalues with geometric multiplicity: $\lambda_{1,2} < \lambda_{3,4,5} < \lambda_6 = 1$. The upper bound in (12) is therefore $c_2 = \lambda_{max} = 1$, and the lower bound $c_1 = \lambda_{min}$ can be further analyzed with respect to its behavior in the regularization parameter ϵ. We computed the solutions of the generalized eigenvalue problem for different kinds of hardening laws (isotropic hardening, perfect plasticity, kinematic hardening, and the two-yield kinematic hardening case [5]) using symbolic computation codes.

Table 1 shows the spectral equivalence of the elasto-plastic and elastic matrices for all hardening laws. Surprisingly, the spectral equivalence even holds in the perfect plasticity case (which corresponds to the isotropic hardening case with $H = 0$). Only in the limit case $\epsilon \to 0$ in perfect plasticity the preconditioner fails.

Table 1: β for various hardening laws and their limit case

Hardening law	β	$\beta(\epsilon \to 0)$
isotropic, $H > 0$	$\frac{E\|p\|_\epsilon}{(1+\nu)\sigma_y(2(1+\alpha_0 H) + H^2\sigma_y\|p\|_\epsilon)}$	$\frac{E}{(1+\nu)H^2\sigma_y^2}$
perfect plasticity	$\frac{E\|p\|_\epsilon}{2\sigma_y(1+\nu)}$	∞
kinematic	$\frac{E\|p\|_\epsilon}{(1+\nu)(\|p\|_\epsilon + 2\sigma_y)}$	$\frac{E}{1+\nu}$
2-yield, kinematic	$\frac{2E(\|p_2\|_\epsilon\sigma_1^y + \|p_1\|_\epsilon(\|p_2\|_\epsilon + \sigma_2^y))}{(1+\nu)(\|p_1\|_\epsilon + \sigma_1^y)(\|p_2\|_\epsilon + \sigma_2^y)}$	$\frac{2E}{1+\nu}$

5 Conclusions

In this paper the theory of elasto-plasticity is combined with the nested iteration approach and a multigrid preconditioned conjugate gradient solver. A solution algorithm is designed and analyzed. The elastic multigrid preconditioner is robust for all considered hardening laws except for the limit case of the regularization in perfect plasticity.

Acknowledgment: This work has been supported by the Austrian Science Fund 'Fonds zur Förderung der wissenschaftlichen Forschung (FWF)' under grant SFB F013/F1306.

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