Fast solver for elastoplasticity

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Outline

- Modelling
- Details of Solution Algorithm
- Numerical examples
- Conclusions and outlook

Modeling

Find $u \in W^{1,2}(0,T;\ H_0^1(\Omega)^n)$, $p \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{n\times n}))$,
$\sigma \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{n\times n}))$, $\alpha \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^m))$ such that

$$\begin{aligned}
-\operatorname{div}\sigma &= b \\
\sigma &= \sigma^T \\
\sigma &= \mathbb{C}(\varepsilon(u) - p) \\
\varepsilon(u) &= \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) \\
\varphi(\sigma,\alpha) &< \infty \\
\dot{p} : (\tau - \sigma) - \dot{\alpha} : (\beta - \alpha) &\leq \varphi(\tau,\beta) - \varphi(\sigma,\alpha)
\end{aligned}$$

are satisfied in the variational sense with $(u, p, \sigma, \alpha)(0) = 0$ for all (τ, β).
b and $\mathbb{C}^{-1}$ are given, $b(0) = 0$.

Numeric-analytic steps

- Time discretization: $t_1 = t_0 + \Delta t$
- Reformulation of the problem using functional-analytic arguments (switching arguments in variational inequalities using a dual functional)
- Equivalent minimization problem:

  Find the minimizer $(u, p, \alpha) \in H \times L^{n\times n}_{sym} \times L^m$ of

$$f(u,p,\alpha) := \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u) - p] : (\varepsilon(u) - p)dx + \frac{1}{2}\int_\Omega |\alpha|^2 dx + \Delta t \int_\Omega \varphi^*(\frac{p - p_0}{\Delta t}, \frac{\alpha_0 - \alpha}{\Delta t})dx - \int_\Omega b\, u\, dx \to \text{min}$$

with φ describing the hardening law.

Minimization problem for isotropic hardening

The minimization problem is under the constraint $\mathrm{tr}(p - p_0) = 0$.

New variable: $\tilde{p} = p - p_0$
A differentiable approximation of $|\tilde{p}|$:

$$|p|_\epsilon := \begin{cases} |p| & \text{if } |p| \geq \epsilon \\ \frac{1}{2\epsilon}|p|^2 + \frac{\epsilon}{2} & \text{if } |p| < \epsilon \end{cases}$$

convex smooth problem

$$f(u, \tilde{p}) := \frac{1}{2}\int_\Omega \mathbb{C}(\varepsilon(u) - \tilde{p} - p_0) : (\varepsilon(u) - \tilde{p} - p_0)\,\mathrm{d}x + \frac{1}{2}\int_\Omega \alpha_0^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \sigma_y^2 H^2 |\tilde{p}|^2\,\mathrm{d}x$$
$$+ \int_\Omega \sigma_y(1 + \alpha_0 H)|\tilde{p}|_\epsilon\,\mathrm{d}x - \int_\Omega bu\,\mathrm{d}x \to \min$$

Computer storage

Two cases: full 3D and 2D plain strain!!! Symmetric stresses and strains are stored as vectors:

$$\sigma = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix}, \qquad p = \begin{pmatrix} p_{11} \\ p_{22} \\ p_{33} \\ p_{12} \\ p_{13} \\ p_{23} \end{pmatrix}$$

or

$$\sigma = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{pmatrix}, \qquad p = \begin{pmatrix} p_{11} \\ p_{22} \\ p_{12} \end{pmatrix}$$

Elastic matrices: $\sigma = \mathbb{C}(\varepsilon - p)$

$$\mathbb{C} := \frac{E}{(1+\nu)(1-2\nu)}[\begin{pmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{pmatrix} \oplus (1-2\nu)]$$

or

$$\mathbb{C} := \frac{E}{(1+\nu)(1-2\nu)}[\begin{pmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{pmatrix} \oplus \begin{pmatrix} 1-2\nu & 0 & 0 \\ 0 & 1-2\nu & 0 \\ 0 & 0 & 1-2\nu \end{pmatrix}]$$

Additional nonzero components

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) + 2\mu(p_{11} + p_{22})$$

$$p_{33} = -p_{11} - p_{22}$$

for the plain-strain model.

Trace free condition treatment: Substitution $\tilde{p} = P\bar{p}$

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finite element discretization

$$\varepsilon(u) = Bu,$$

where u represent local displacement vector

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{or} \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and B is a discretization of the ε differential operator in terms of a given finite element basis. Regualized Frobenius norm of a symmetric trace free plastic strain and its derivatives:

$$\begin{aligned} |\tilde{p}|_\epsilon &= (\tilde{p}^T b \tilde{p})_\epsilon^{1/2}, \\ \frac{\mathrm{d}|\tilde{p}|_\epsilon}{\mathrm{d}\tilde{p}} &= \begin{cases} \frac{b\tilde{p}}{|\tilde{p}|} & \text{if } |\tilde{p}| \geq \epsilon, \\ \frac{b\tilde{p}}{\epsilon} & \text{if } |\tilde{p}| < \epsilon, \end{cases} \\ \frac{\mathrm{d}^2|\tilde{p}|_\epsilon}{\mathrm{d}\tilde{p}} &= \begin{cases} \frac{b}{|\tilde{p}|} - \frac{b\tilde{p}\tilde{p}^T b}{|\tilde{p}|^3} & \text{if } |\tilde{p}| \geq \epsilon, \\ \frac{b}{\epsilon} & \text{if } |\tilde{p}| < \epsilon, \end{cases} \end{aligned} \tag{1}$$

with a matrix

$$b = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \oplus \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{or} \quad b = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus 2.$$

Then the local problem reads

$$f(u,\bar{p}) = \frac{1}{2}\begin{pmatrix} u \\ \bar{p} \end{pmatrix}^T \begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C}P \\ -P\mathbb{C}B & P^T(\mathbb{C}+\mathbb{H})P \end{pmatrix} \begin{pmatrix} u \\ \bar{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ P^T\mathbb{C}p_0 \end{pmatrix}^T \begin{pmatrix} u \\ \bar{p} \end{pmatrix}$$
$$+\frac{1}{2}\mathbb{C}\tilde{p}_0 : \tilde{p}_0 + \frac{1}{2}\alpha_0^2 \to \text{min},$$

where $\mathbb{H}(\tilde{p}) = \sigma_y^2 H^2 b + 2\sigma_y(1+\alpha_0 H)\frac{b}{|p|_\epsilon}$.

Minimization in $\tilde{p}$

The objective in each integration point writes as

$$F(\tilde{p}) = \frac{1}{2}\tilde{p}^T\mathbb{C}\tilde{p} + p_0^T\mathbb{C}\tilde{p} - \tilde{p}^T\mathbb{C}\varepsilon(u) + \frac{1}{2}\sigma_y^2 H^2|\tilde{p}|^2 + \sigma_y(1+\alpha_0 H)|\tilde{p}|_\epsilon$$

Without regularization, $\epsilon = 0$ a unique solution

$$\tilde{p} = \frac{(||\operatorname{dev} A|| - a)_+}{2\mu + \sigma_y^2 H^2}\frac{\operatorname{dev} A}{||\operatorname{dev} A||}, \qquad (2)$$

where

$$A = \mathbb{C}[\varepsilon(u) - p_0], \quad a = \sigma_y(1+\alpha_0 H).$$

With regularization: Newton method

$$P^T F''(\tilde{p}) P \Delta\bar{p} = -P^T F'(\tilde{p}). \qquad (3)$$

Minimization in u

Simplification: $\mathbb{H} = \mathbb{H}(\tilde{p})$ dependence frozen $\Rightarrow f(u, \tilde{p})$ is perfectly quadratic functional.
A nececesar condition of the minima of (2) is $f'(u, \tilde{p}) = 0$, i.e.,

$$\begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C}P \\ -P\mathbb{C}B & P^T(\mathbb{C}+\mathbb{H})P \end{pmatrix} \begin{pmatrix} u \\ \bar{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ P^T\mathbb{C}p_0 \end{pmatrix} = 0. \tag{4}$$

By eliminating $\tilde{p}$ from (4), we get a linear system for u only

$$S\,u = b + B^T(\mathbb{C} - \mathbb{C}P(P^T(\mathbb{C}+\mathbb{H})P)^{-1}P^T\mathbb{C})\,p_0, \tag{5}$$

where $S := B^T(\mathbb{C} - \mathbb{C}P(P^T(\mathbb{C}+\mathbb{H})P)^{-1}P^T\mathbb{C})B$ represents the Schur-complement.

Algorithm

Algorithm 1. *(One time step iteration) Given initial* u*.*

1. *Calculate local* A, a *from (2) and local* $\tilde{p}$ *using Newton method (3).*

2. *Substitute* $\tilde{p}$ *to* $\mathbb{H}$ *in local Schur-complement (5) and assemle the global Schur-complement.*

3. *Solve new* u *from the global stifness matrix using CG-multigrid preconditioned method.*

4. *Repeat steps (1)-(3) until the convergence is reached.*

5. *Upgrade* $p = \tilde{p} + p_0$ *and output* u *and* p*.*

Multi-grid PCG for solving $K_l u_l = f_l, l = 0, \dots, M$

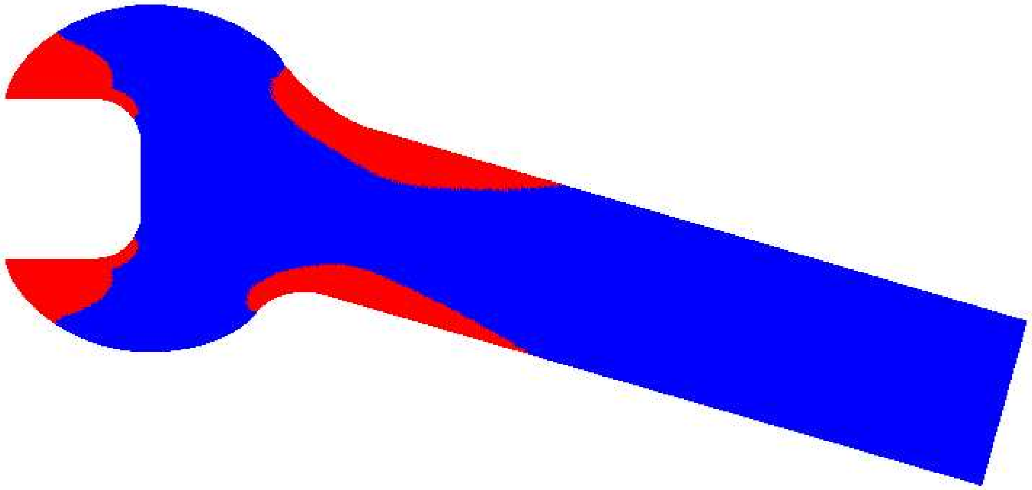
1. Initialization: Let u_l^0 be an initial aproximation of the solution u_l.

$$\begin{aligned} d_l^0 &= f_l - K_l u_l^0 & \text{(defect calculation)} \\ w_l^0 &= B_{l,k}\, d_l^0 & \text{(multigrid preconditioner } B_l(I_l - (M_l)^k)^{-1}) \\ s_l^0 &= w_l^0 \end{aligned}$$

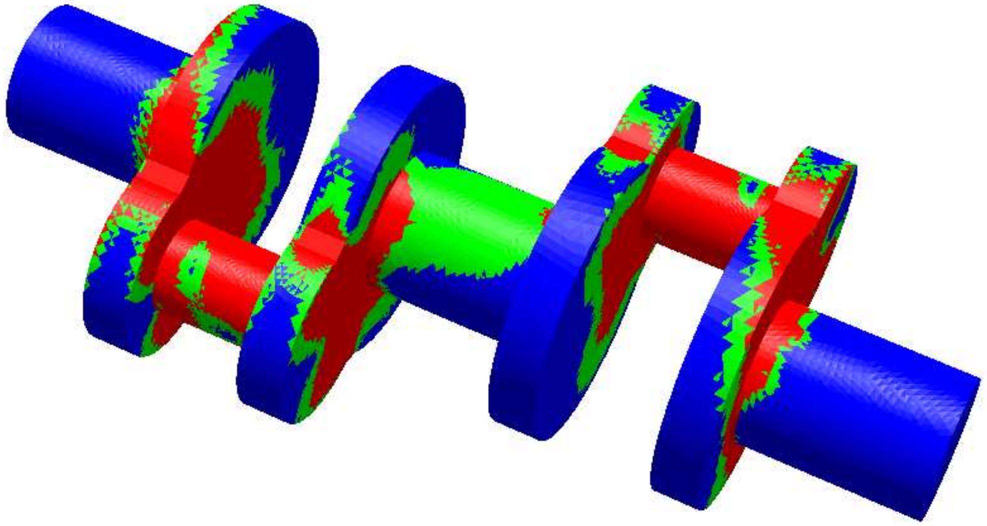
2. Iteration: for $j = 1, \dots, i$

$$\begin{aligned} \alpha_{j+1} &= (w_l, j, d_l^j)/(K_l s_l^j, s_l^j) & \text{(stepsize calculation)} \\ u_l^{j+1} &= u_l^j + \alpha_{j+1} s_l^j & \text{(solution } u \text{ upgrade)} \\ d_l^{j+1} &= d_l^j - \alpha_{j+1} K_l s_l^j & \text{(defect } d \text{ upgrade)} \\ w_l^{j+1} &= B_{l,k}\, d_l^{j+1} & \text{(multigrid preconditioner } B_l(I_l - (M_l)^k)^{-1}) \\ \beta_{j+1} &= (w_l^{j+1}, d_l^{j+1})/(w_l^j, d_l^j) \\ s_l^{j+1} &= w_l^{j+1} + \beta_{j+1} s_l^j \end{aligned}$$

Numerical experiments: elastoplastic zones



2D Screwwrench



3D Crankshaft

Conclusions

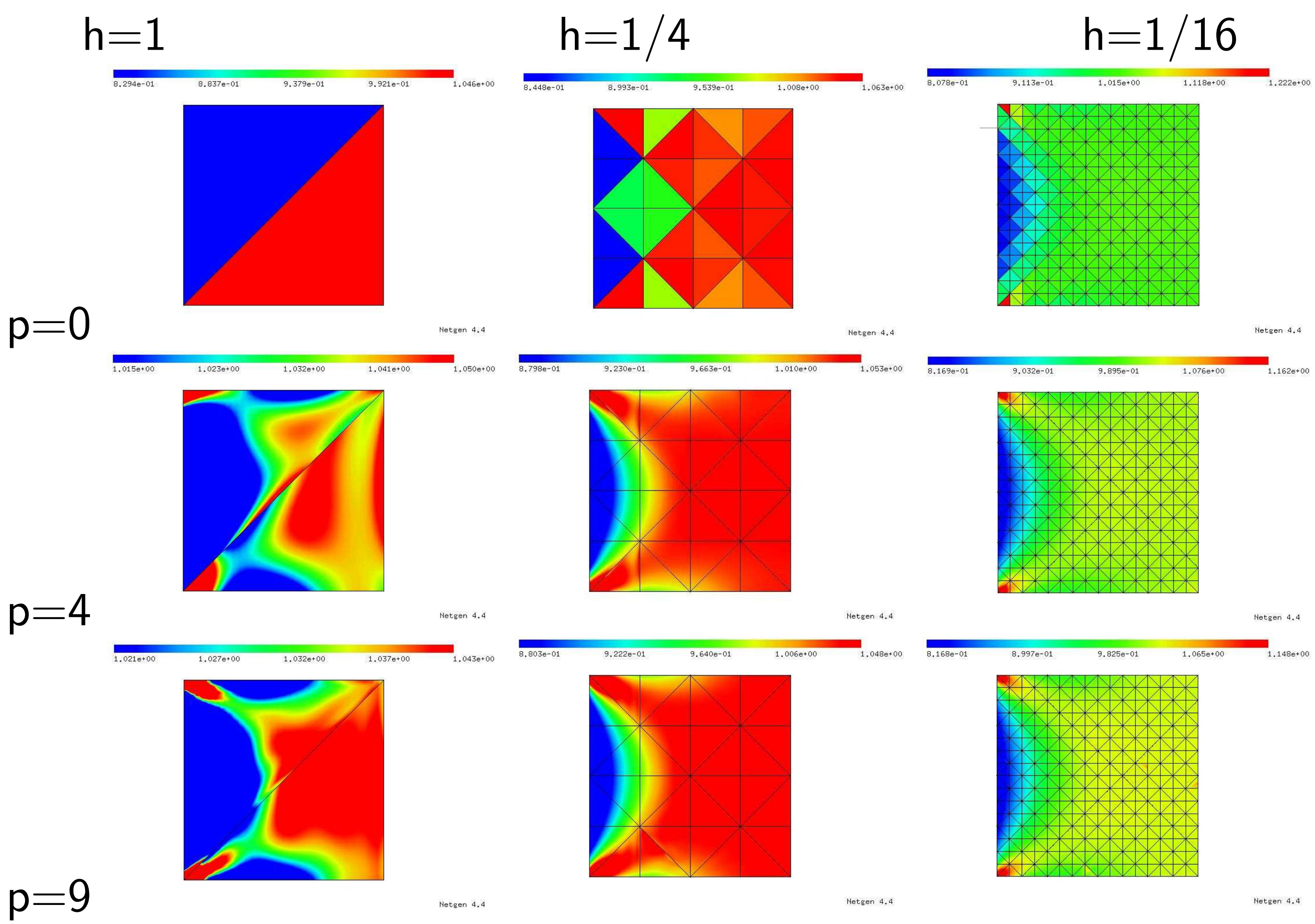
We have considered:

- Modeling
- 2D/3D one time-step algorithm
- Numerical experiments

Outlook

- Combined hpr methods
  - h, r: Singularities
  - p: Smooth solutions
- Level sets use for elastoplastic interface identification
- Application to shells

p and h method in 2D: von Mises stress



p and h method in 3D: von Mises stress

For visualization reasons the stresses are projected onto a H^1 function