# Multi-yield elastoplastic continuum - modeling and computations.

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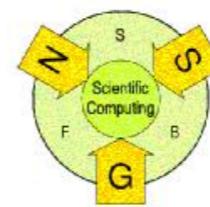
Outline

- Why multi-yield plasticity?

- Mathematical analysis

- Discretized minimization problem

- Algorithm

- Numerical experiments

- Outlook

Why Multi-yield (Two-yield) model?

- More realistic hysteresis stress-strain relation in materials!
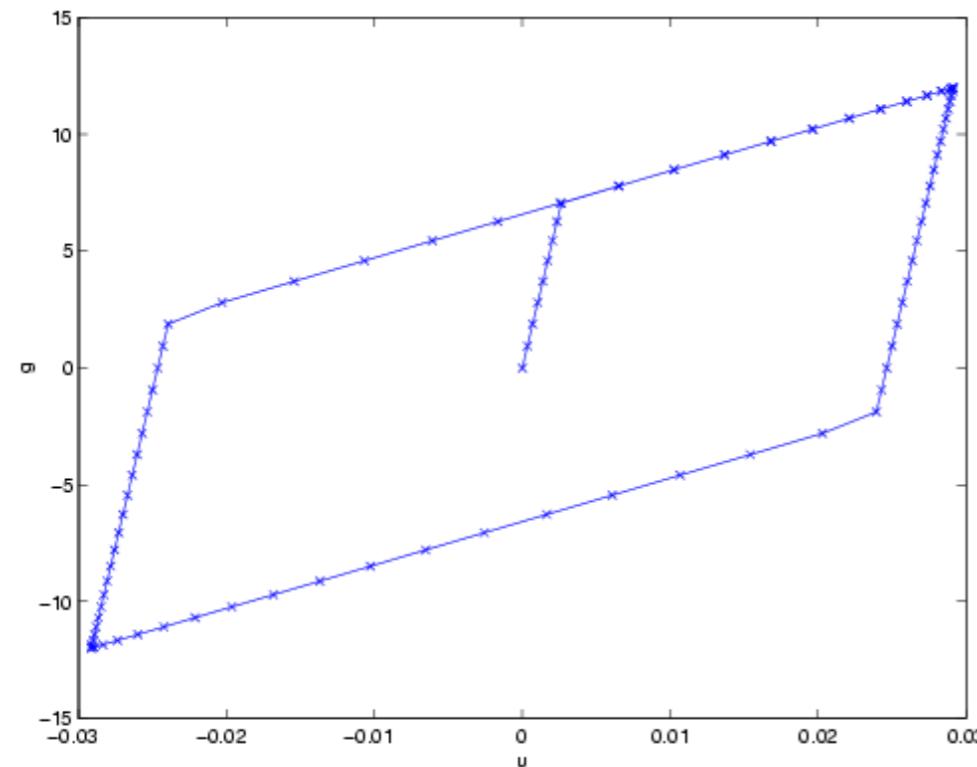
Kinematic hardening model.
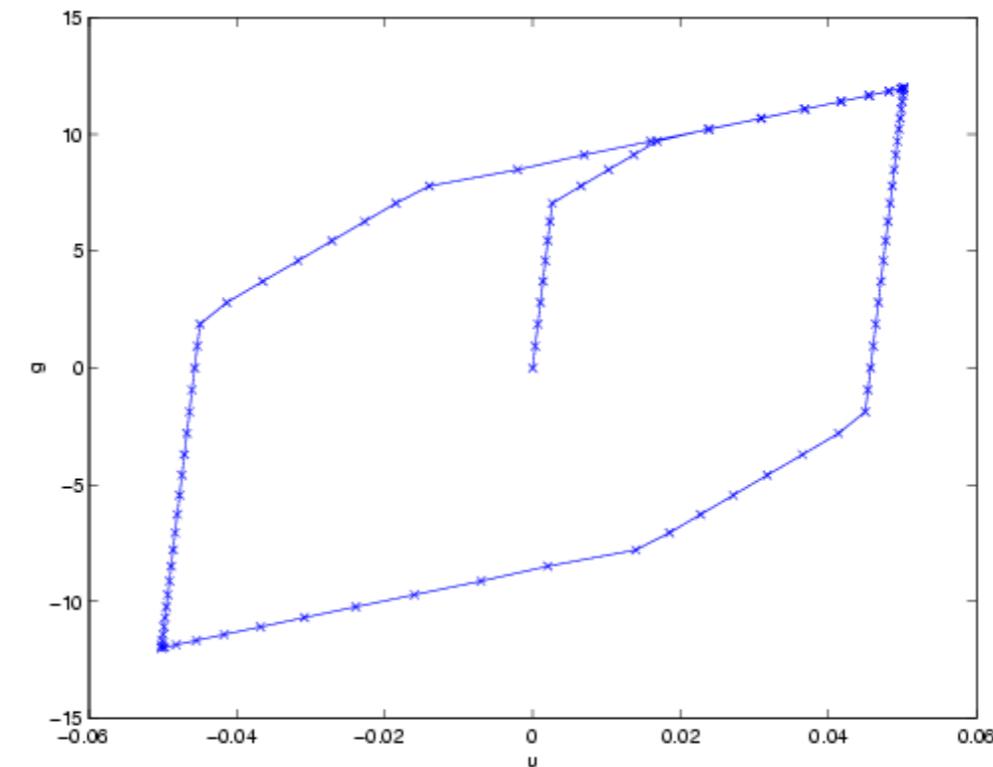
Two-yield hardening model.

Mathematical model of M-yield elastoplasticity

Problem (Prandtl-Ishlinskii): For $l \in H^1(0, T; \mathcal{H}^*), l(0) = 0$, find $w = (u, p^1, \ldots, p^M) : [0, T] \rightarrow \mathcal{H}, w(0) = 0$ s. t.

$$\langle l(t), z - \dot{w}(t) \rangle \leq a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t))$$

$\forall z = (v, \tau^1, \ldots, \tau^M) \in \mathcal{H}.$

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$\forall z = (v, \tau^1, \ldots, \tau^M) \in \mathcal{H}$.

Notation: $\mathcal{H} = H_D^1(\Omega) \times \underbrace{L^2(\Omega)_{sym}^{d \times d} \times \cdots \times L^2(\Omega)_{sym}^{d \times d}}_{M \text{ times}}$,

$$a(w, z) = \int_\Omega \left(\mathbb{C}(\varepsilon(u) - \sum_{r=1}^{M} p^r) \right) : \left(\varepsilon(v) - \sum_{r=1}^{M} \tau^r \right) \mathrm{d}x + \sum_{r=1}^{M} \int_\Omega \mathbb{H}^r p^r : \tau^r \, \mathrm{d}x,$$

$$\langle l(t), z \rangle = \int_\Omega f(t) \cdot v \, \mathrm{d}x + \int_{\Gamma_N} g(t) \cdot v \, \mathrm{d}x,$$

$$j(z) = \int_\Omega \sum_{r=1}^{M} D^r(\tau^r) \, \mathrm{d}x, \qquad D^r(x) = \left\{ \begin{array}{ll} \sigma_y^r ||x|| & \text{if } \operatorname{tr} x = 0, \\ +\infty & \text{otherwise.} \end{array} \right. \qquad \text{(von Mises criterion)}$$

Existence and uniqueness

Assumption: positive definite elastic and hardening operators:

$$\mathbb{C}\xi : \xi \geq c||\xi||^2 \ \forall \xi \in \mathbb{R}^d$$

$$\mathbb{H}^r \xi : \xi \geq h^r ||\xi||^2 \ \forall \xi \in \mathbb{R}^d, r = 1, \ldots, M$$

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Results: 1) $a(\cdot, \cdot)$ - *bounded, elliptic bilinear form in* $\mathcal{H}$,

$$|a(w, z)| \leq \Big((M + 1)||\mathbb{C}|| + \max_{r=1,\ldots,M} ||\mathbb{H}^r||\Big)||w||_{\mathcal{H}}||z||_{\mathcal{H}},$$

$$a(w, w) \geq \Big(k \min_{r=1,\ldots,M}\{c, h^r\} \min\{1, K\}\Big)||w||_{\mathcal{H}}^2,$$

where $K > 0$ *(Korn's first inequality) and* $k = k(M) = 1 + \frac{M}{2} - \frac{1}{2}\sqrt{M(M+4)}$.

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2) $j(\cdot)$ - nonnegative, positive homogeneous and Lipschitz continuous functional in $\mathcal{H}$

$$|j(z_1) - j(z_2)| \leq \left(\max_{r=1,\ldots,M}\{\sigma_y^r\}meas(\Omega)^{\frac{1}{2}}M^{\frac{1}{2}}\right)||z_1 - z_2||_{\mathcal{H}}.$$

Existence and uniqueness

Theorem: Let $l \in H^1(0, T; \mathcal{H}')$ with $l(0) = 0$. $\exists! \ w = (u, p^1, \ldots, p^M)(t) \in H^1(0, T; \mathcal{H})$ of **Problem** (Prandtl-Ishlinskii).

Proof: *[Han, Reddy '99.]*

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Generalization to Measure case:
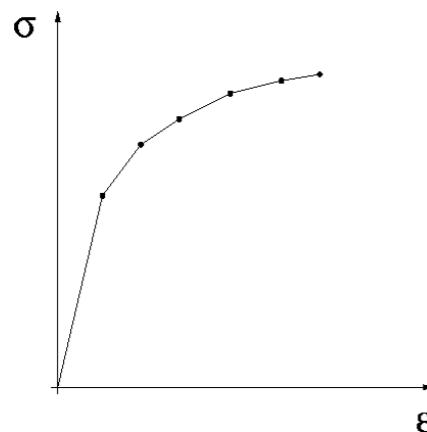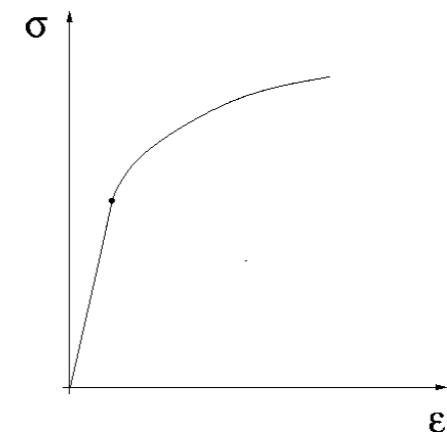
$$p = \sum_{r=1}^{M} p^r \quad \Rightarrow \quad p = \int_{r \in I} p(r) d\mu(r)$$

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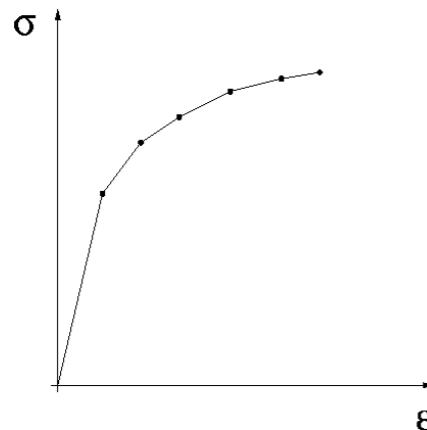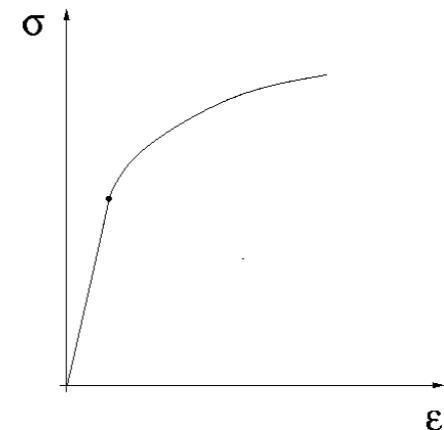
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- *The same existence and uniqueness result with modified constants.*

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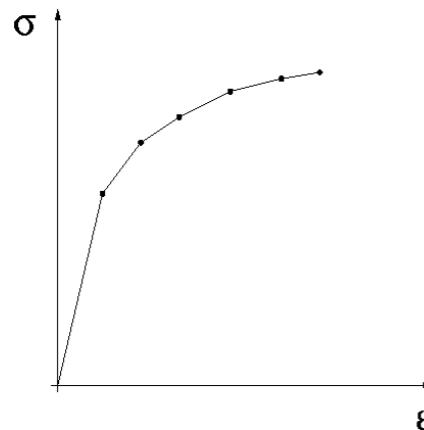
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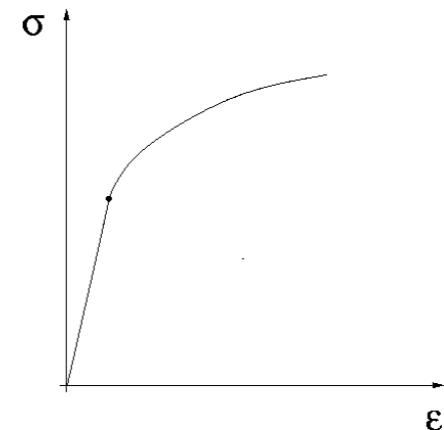
- *The same existence and uniqueness result with modified constants.*

Paper: Brokate, Carstensen, Valdman: Mathematical Modeling of the multi-yield elastoplastic material: Part 1 - Analysis

Discretization

- *in time:* net $\quad 0 = t_0 < t_1 < \cdots < t_N = T$, *and implicit Euler scheme*

$$\dot{X}(t_j) = \frac{X(t_j) - X(t_{j-1})}{t_j - t_{j-1}} \quad \text{for } j = 1, \ldots, N.$$

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- *in space: regular triangulation $\mathcal{T}$ of Ω*

  - *displacement u: $H_D^1(\Omega)$ approximated by* $\mathcal{S}_D^1(\mathcal{T}) := \{v \in H_D^1(\Omega) : \forall T \in \mathcal{T}, v|_T \in \mathbb{P}_1(T)^d\}$

  - *plastic strains $p_1, \ldots, p_M$: $L^2(\Omega)$ approximated by* $\mathcal{S}^0(\mathcal{T}) := \{a \in L^2(\Omega) : \forall T \in \mathcal{T}, a|_T \in \mathbb{R}\}$

Minimization problem in $u, p^1, \ldots, p^M$

$$f(u, p^1, \ldots, p^M) := \frac{1}{2} \int_\Omega \mathbb{C}(\varepsilon(u) - \sum_{r=1}^M p^r) : (\varepsilon(u) - \sum_{r=1}^M p^r) \, dx + \frac{1}{2} \int_\Omega \sum_{r=1}^M |\alpha_0^r|^2 \, dx + \frac{1}{2} \int_\Omega \sum_{r=1}^M |p^r - p_0^r|^2 \, dx$$

$$+ \int_\Omega \sum_{r=1}^M \alpha_0^r : (p^r - p_0^r) \, dx + \int_\Omega \sum_{r=1}^M \sigma_y^r |p^r - p_0^r| \, dx - \int_\Omega bu \, dx \to min.$$

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New variables: $\tilde{p}^1 = p^1 - p_0^1, \ldots.$

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Regulatization of $|\cdot|$: $\qquad |\cdot|_\epsilon := \left\{ \begin{array}{ll} |\cdot| & \text{if } |\cdot| \geq \epsilon \\ \frac{1}{2\epsilon}|\cdot|^2 + \frac{\epsilon}{2} & \text{if } |\cdot| < \epsilon \end{array} \right.$

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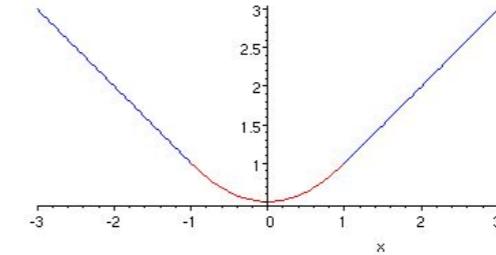
Minimization strategy in each time step:

$$u^{k+1} = \operatorname{argmin}_v \min_q \bar{f}(v, q) = \operatorname{argmin}_v \tilde{f}(v, q_{opt}(v))$$

Then $p^1 = p_0^1 + \tilde{p}^1, \ldots$

Direct minimization problem in u: Two-yield model

Matrix form:

$$f(u, \tilde{p}^1, \tilde{p}^2) = \frac{1}{2} \begin{pmatrix} u \\ \tilde{p}^1 \\ \tilde{p}^2 \end{pmatrix}^T \begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C} & -B^T\mathbb{C} \\ -\mathbb{C}B & \mathbb{C} + \mathcal{D}^1 & \mathbb{C} \\ -\mathbb{C}B & \mathbb{C} & \mathbb{C} + \mathcal{D}^2 \end{pmatrix} \begin{pmatrix} u \\ \tilde{p}^1 \\ \tilde{p}^2 \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}(p_0^1 + p_0^2) \\ \mathbb{C}(p_0^1 + p_0^2) + \mathbb{Q}\alpha_0^1 \\ \mathbb{C}(p_0^1 + p_0^2) + \mathbb{Q}\alpha_0^2 \end{pmatrix}^T \begin{pmatrix} u \\ \tilde{p}^1 \\ \tilde{p}^2 \end{pmatrix}$$

$$+ \frac{1}{2}\mathbb{C}p_0^1 : p_0^1 + \frac{1}{2}\mathbb{C}p_0^2 : p_0^2 + \frac{1}{2}|\alpha_0^1|^2 + \frac{1}{2}|\alpha_0^2|^2 \to min,$$

where $\mathcal{D}^1 = \mathbb{Q}(1 + \frac{2\sigma_y^1}{|\tilde{p}^1|_\epsilon}), \mathcal{D}^2 = \mathbb{Q}(1 + \frac{2\sigma_y^2}{|\tilde{p}^2|_\epsilon}).$

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The Schur-Complement system in u

$$B^T(\mathbb{C} - \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}^T \begin{pmatrix} \mathbb{C}+\mathcal{D}^1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C}+\mathcal{D}^2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix})Bu = b + B^T\mathbb{C}(p_0 - \begin{pmatrix} I \\ I \end{pmatrix}^T \begin{pmatrix} \mathbb{C}+\mathcal{D}^1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C}+\mathcal{D}^2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{C}p_0 + \mathbb{Q}\alpha_0^1 \\ \mathbb{C}p_0 + \mathbb{Q}\alpha_0^2 \end{pmatrix}),$$

where $p_0 = p_0^1 + p_0^2.$ $\Rightarrow$ Multigrid-PCG

Direct minimization problem in $\tilde{p}$: analytical approach

Kinematic hardening model $(M = 1)$:

$$f(q) = \frac{1}{2}(\mathbb{C} + \mathbb{H})q : q - q : A + \sigma_y ||q|| \rightarrow min$$

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Two-yield hardening model $(M = 2)$:

$$f\begin{pmatrix} q^1 \\ q^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbb{C} + \mathbb{H}^1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C} + \mathbb{H}^2 \end{pmatrix}\begin{pmatrix} q^1 \\ q^2 \end{pmatrix} : \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} - \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} : \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} + \sigma_y^1||q^1|| + \sigma_y^2||q^2|| \to min$$

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$$\text{minimizer } (\tilde{p}^1, \tilde{p}^2) = ?$$

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$\tilde{p}^2 \neq 0 \Rightarrow \|\tilde{p}^2\|$ *is a root of a 6-th degree polynomial.*
Symbolic methods (Gröbner basis?)

Direct minimization problem in $\tilde{p}$: Two-yield model - iterative approach

Algorithm (*): *Given* $tolerance \geq 0$.
(a) Choose $(p_1^0, p_2^0) \in \mathrm{dev}\, \mathrm{I\!R}_{sym}^{d\times d} \times \mathrm{dev}\, \mathrm{I\!R}_{sym}^{d\times \longrightarrow \min! d}$, *set*
$i := 0$.
(b) Find $p_2^{i+1} \in \mathrm{dev}\, \mathrm{I\!R}_{sym}^{d\times d}$ *s. t.*

$$f(p_1^i, p_2^{i+1}) = \min_{Q_2 \in \, dev\, \mathrm{I\!R}_{sym}^{d\times d}} f(p_1^i, q_2).$$

(c) Find $p_1^{i+1} \in \mathrm{dev}\, \mathrm{I\!R}_{sym}^{d\times d}$ *s. t.*

$$f(p_1^{i+1}, p_2^{i+1}) = \min_{q_1 \in \, dev\, \mathrm{I\!R}_{sym}^{d\times d}} f(q_1, p_2^{i+1}).$$

(d) If $\frac{||p_1^{i+1} - p_1^i|| + ||p_2^{i+1} - p_2^i||}{||p_1^{i+1}|| + ||p_1^i|| + ||p_2^{i+1}|| + ||p_2^i||} > tolerance$ *set* $i := i + 1$
and goto (b), otherwise output (p_1^{i+1}, p_2^{i+1}).

Direct minimization problem in $\tilde{p}$: Two-yield model - iterative approach

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(d) If $\frac{||p_1^{i+1} - p_1^i|| + ||p_2^{i+1} - p_2^i||}{||p_1^{i+1}|| + ||p_1^i|| + ||p_2^{i+1}|| + ||p_2^i||} > tolerance$ set $i := i + 1$ and goto (b), otherwise output (p_1^{i+1}, p_2^{i+1}).
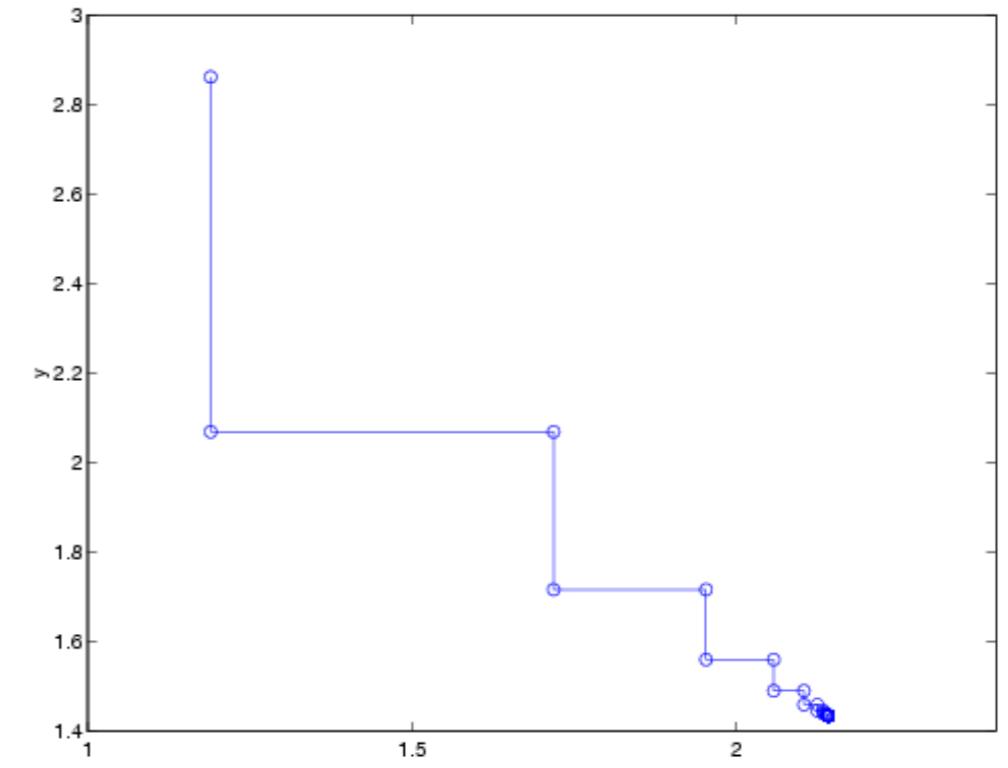
The approximations $p_1^i = (x^i, 0; 0, -x^i), p_2^i = (y^i, 0; 0, -y^i), i = 0, 1, \ldots$ of Algorithm (*) in the $x - y$ coordinate system.

• *global convergence with the rate $1/2$: $||p_1^i - p_1||^2 + ||p_2^i - p_2||^2 \leq C_0 \cdot q^i$*

Matlab calculations in 2D

Elastoplastic domains (black - elastic, brown - first plastic, yellow - second plastic)
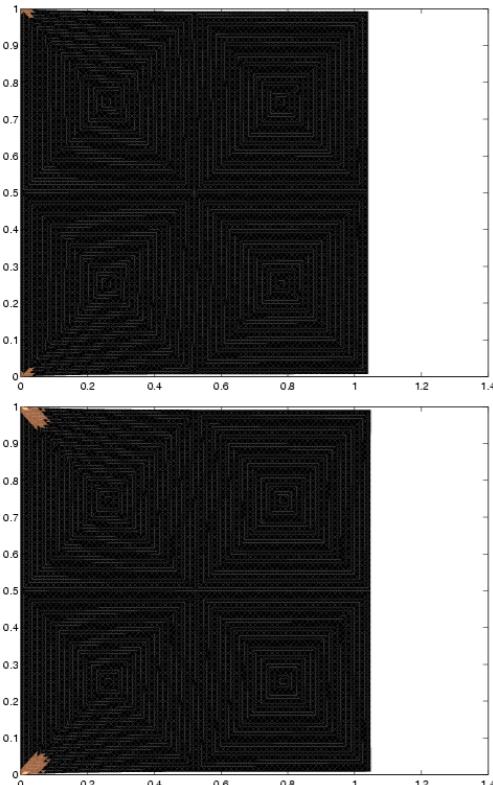
Matlab calculations in 2D

Elastoplastic domains (black - elastic, brown - first plastic, yellow - second plastic)
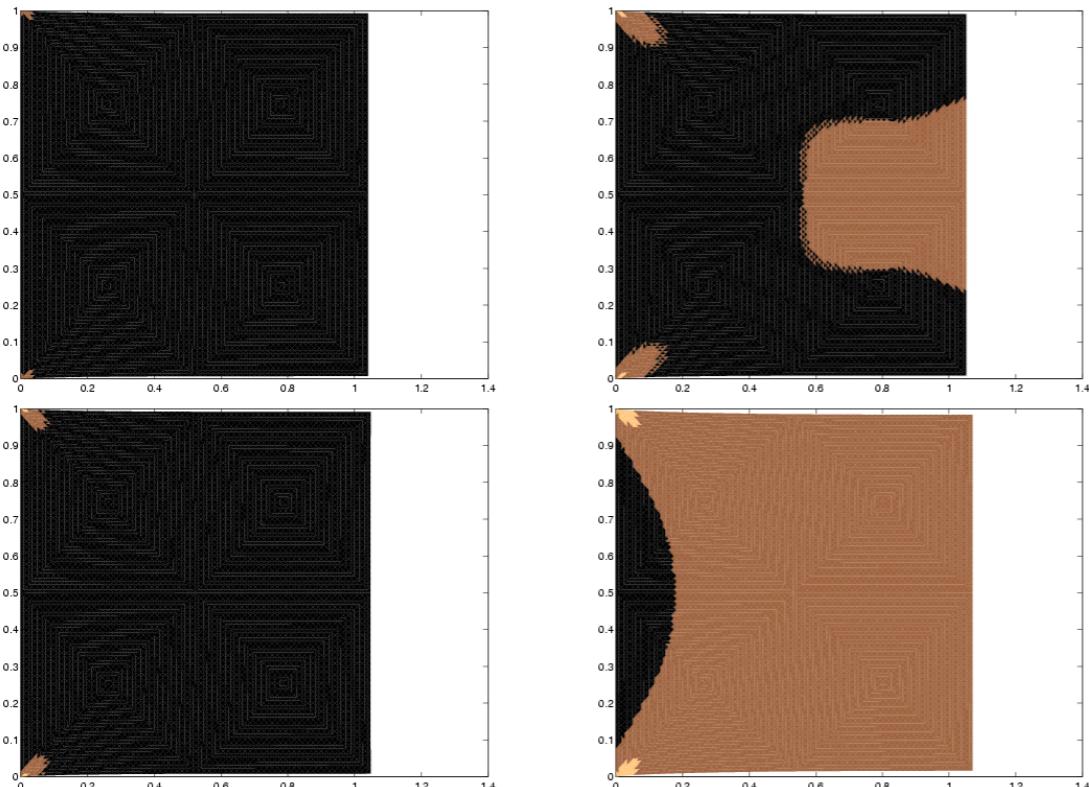
Matlab calculations in 2D

Elastoplastic domains (black - elastic, brown - first plastic, yellow - second plastic)
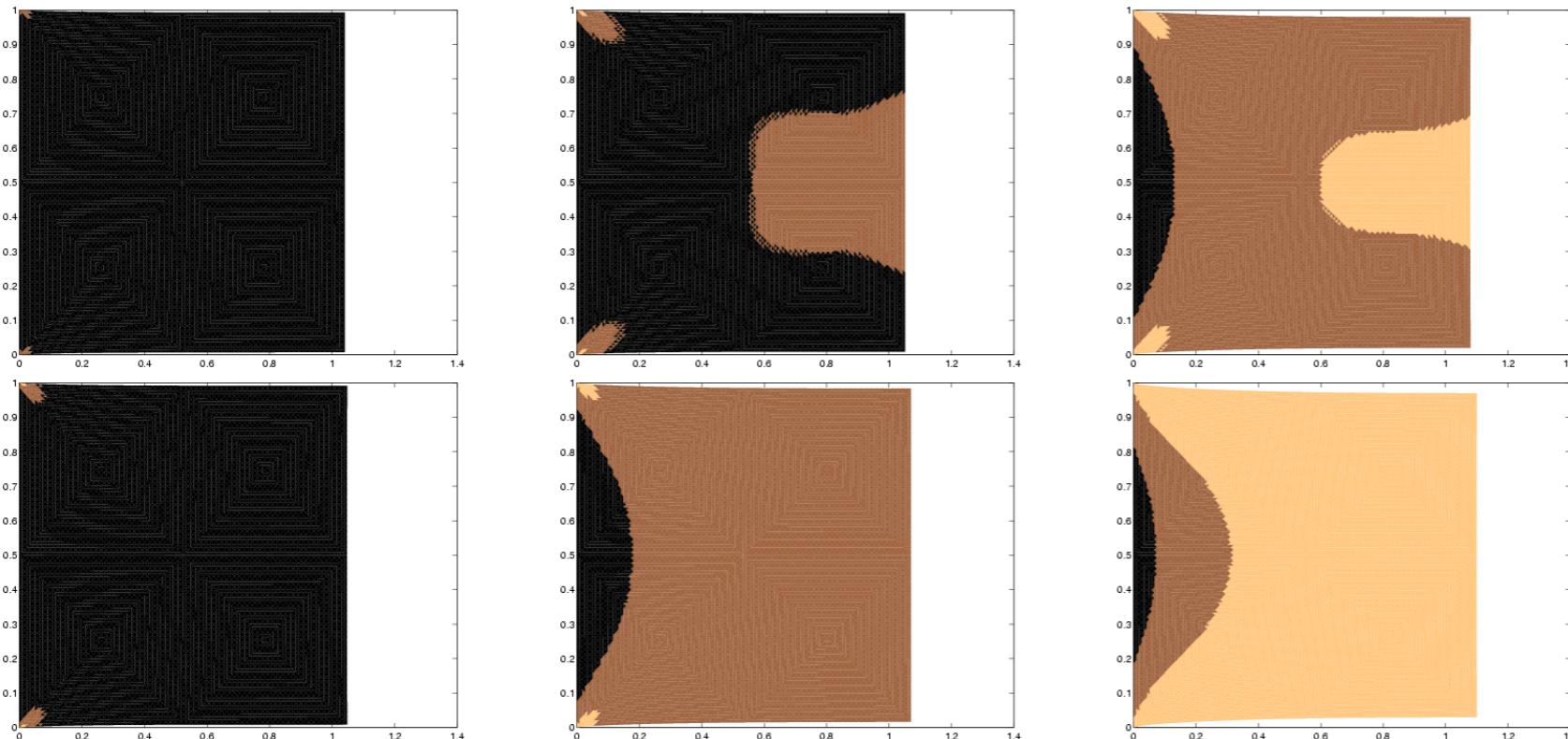
Matlab calculations in 2D

Elastoplastic domains (black - elastic, brown - first plastic, yellow - second plastic)
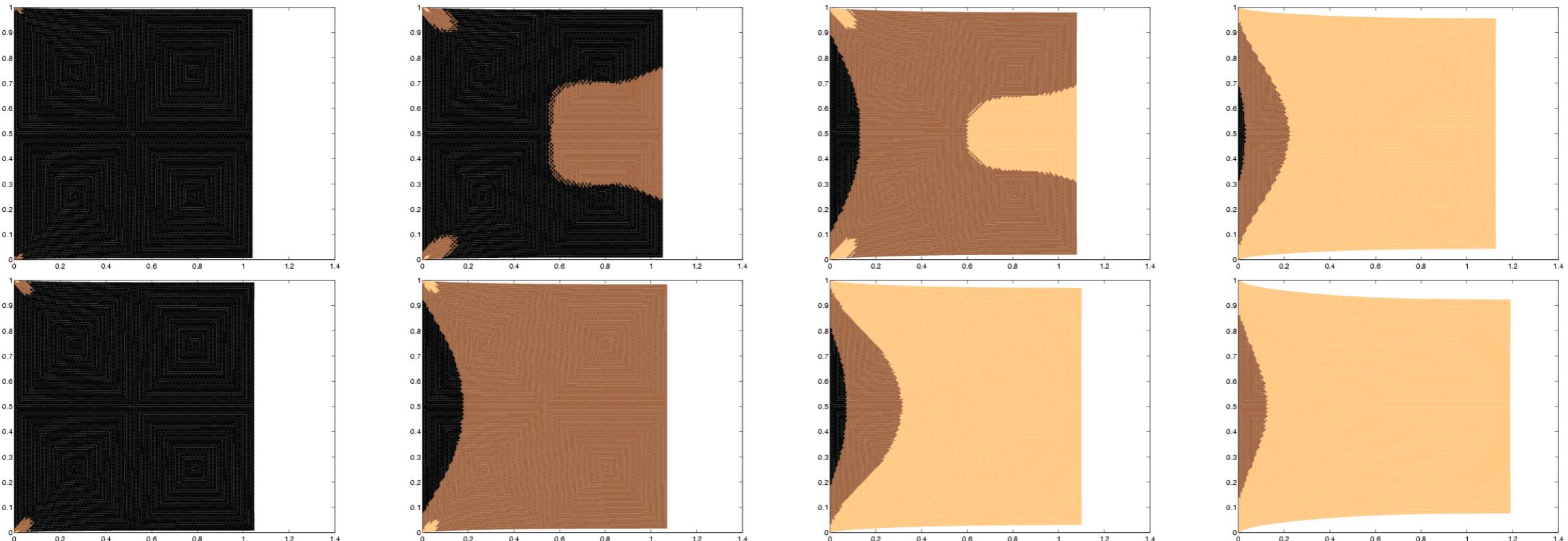
Evolution of elastoplastic zones at 8 graduate discrete times of the two-yield beam. Calculated for 16334 elements, CPU time $= 25.17$ hours.

Matlab calculations in 2D

Elastoplastic domains (black - elastic, brown - first plastic, yellow - second plastic)
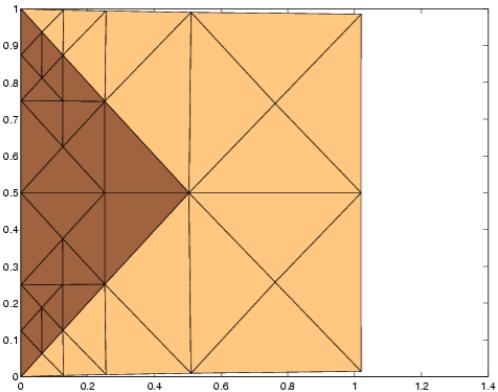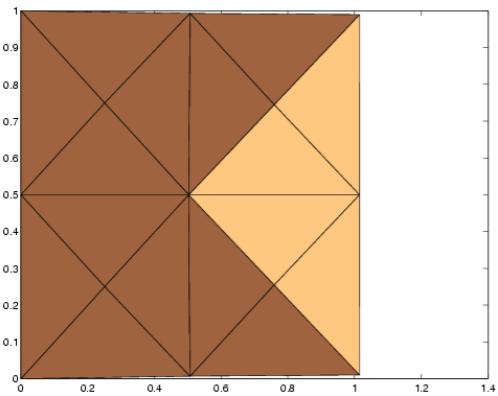
Matlab calculations in 2D

Elastoplastic domains (black - elastic, brown - first plastic, yellow - second plastic)
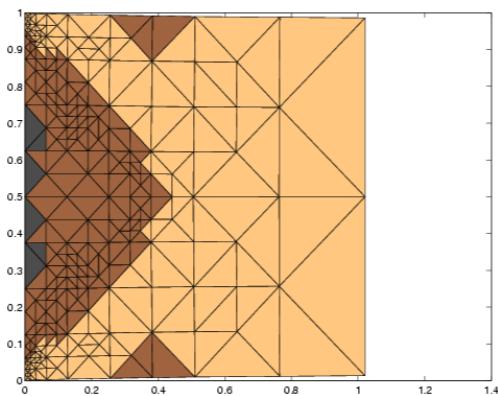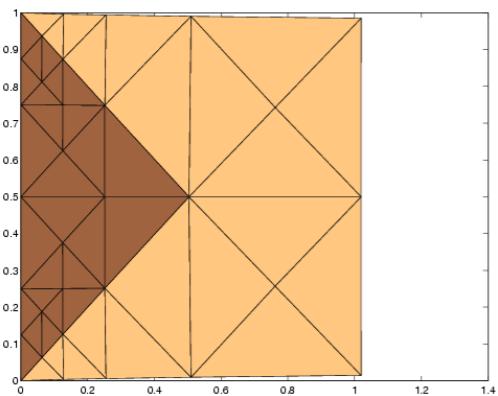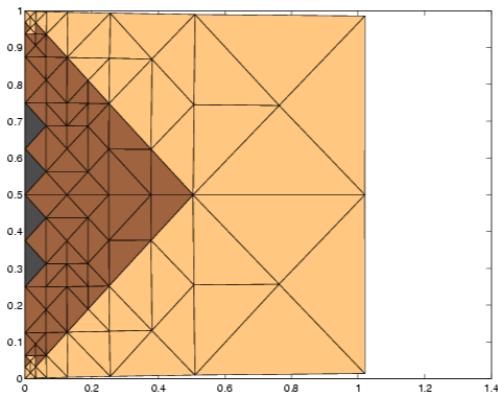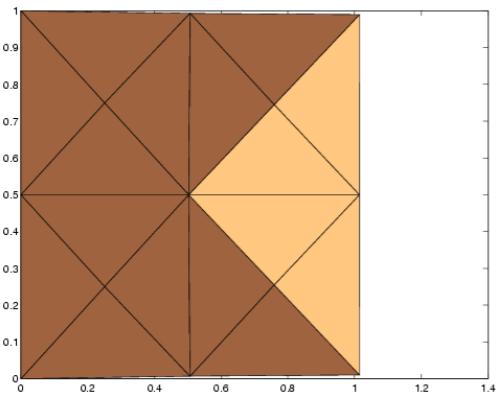
Matlab calculations in 2D

Elastoplastic domains (black - elastic, brown - first plastic, yellow - second plastic)
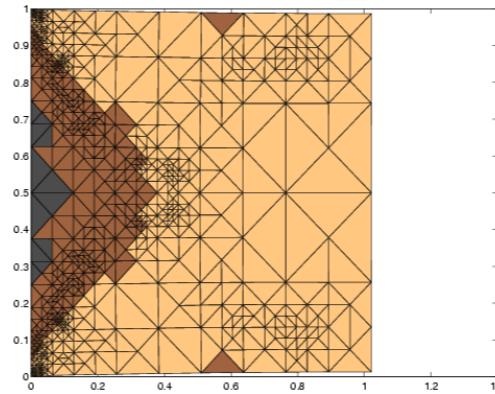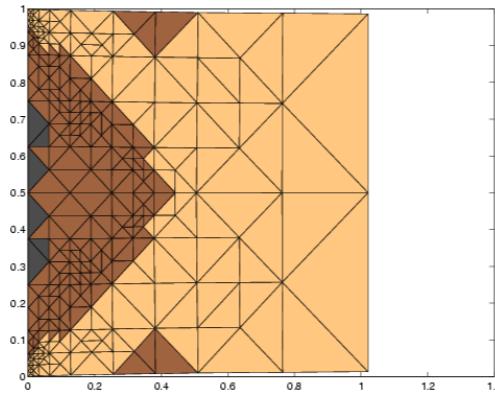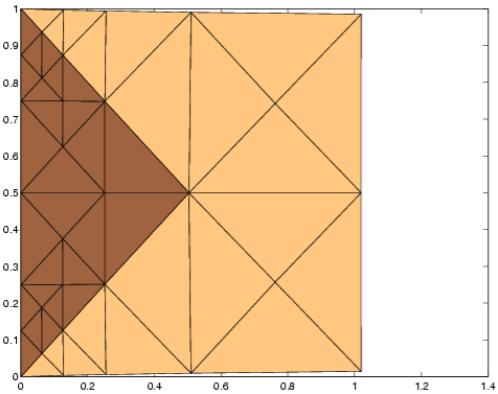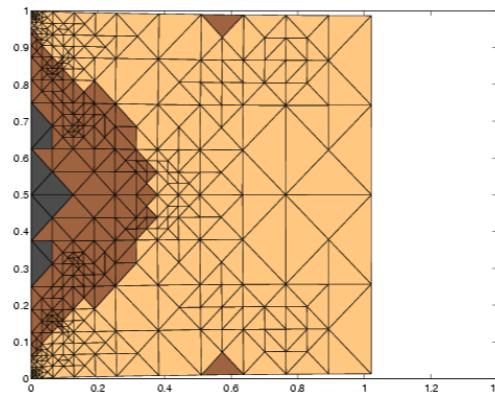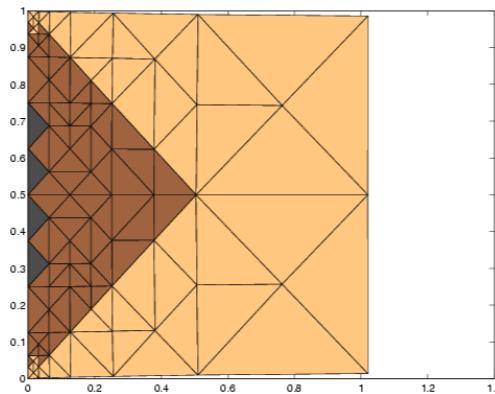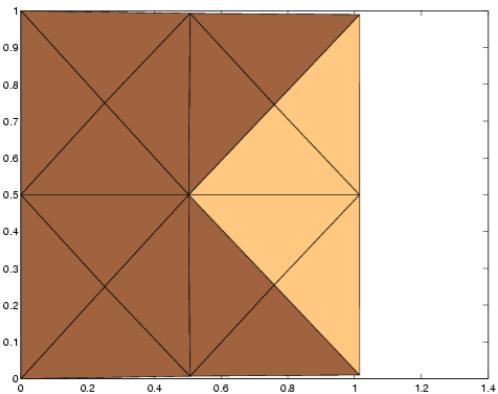
Matlab calculations in 2D

Elastoplastic domains (black - elastic, brown - first plastic, yellow - second plastic)
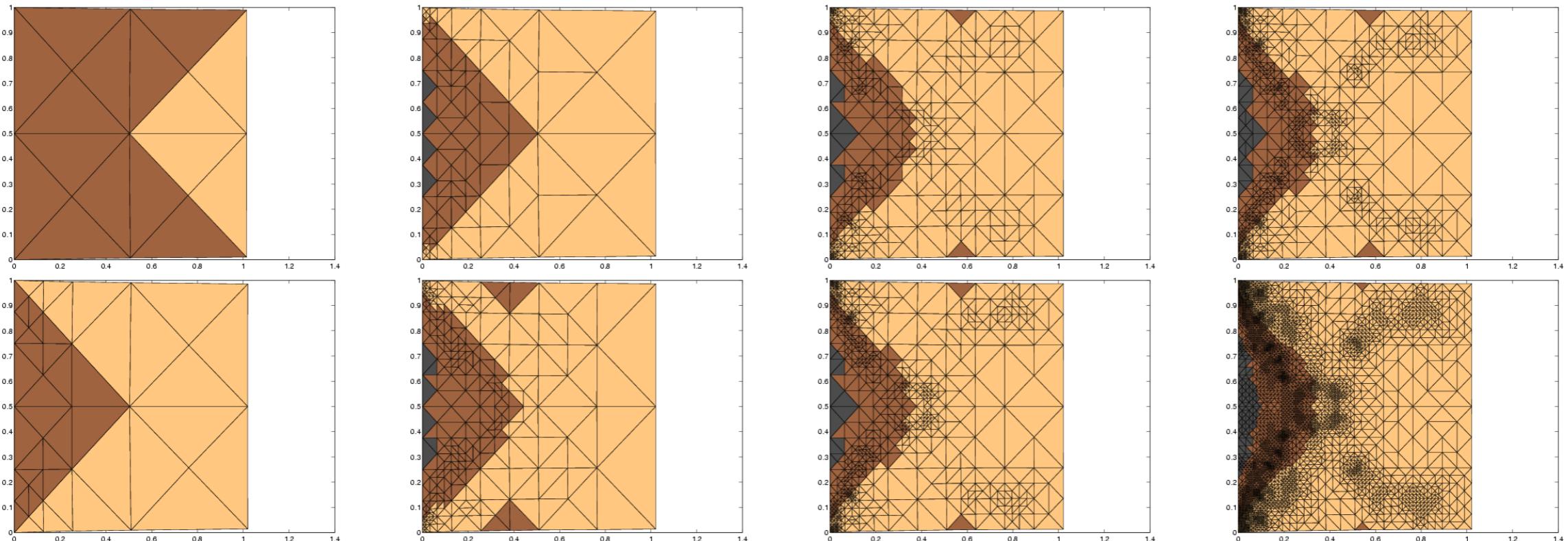

ZZ- adaptively refined meshes and elastoplastic zones, one time-step problem, two-yield beam.

NGSOLVE calculations in 2D

*Elastoplastic domains (blue - elastic, green - first plastic, red - second plastic)*
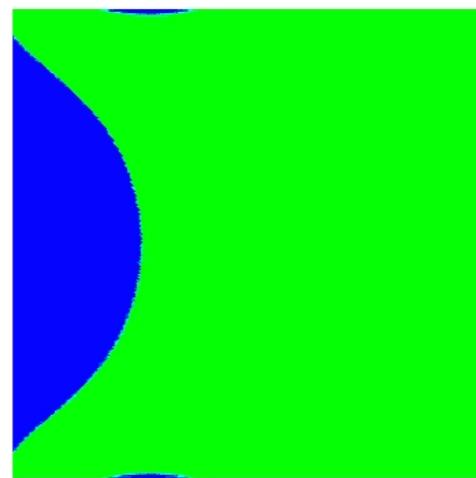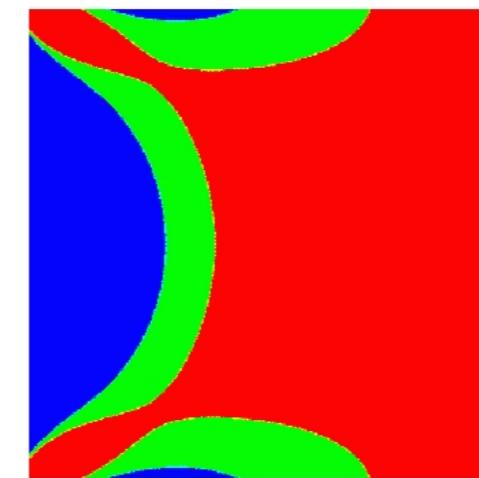
Kinematic hardening model.   *Two-yield hardening model.*

NGSOLVE calculations in 3D

Elastoplastic domains (blue - elastic, green - first plastic, red - second plastic)
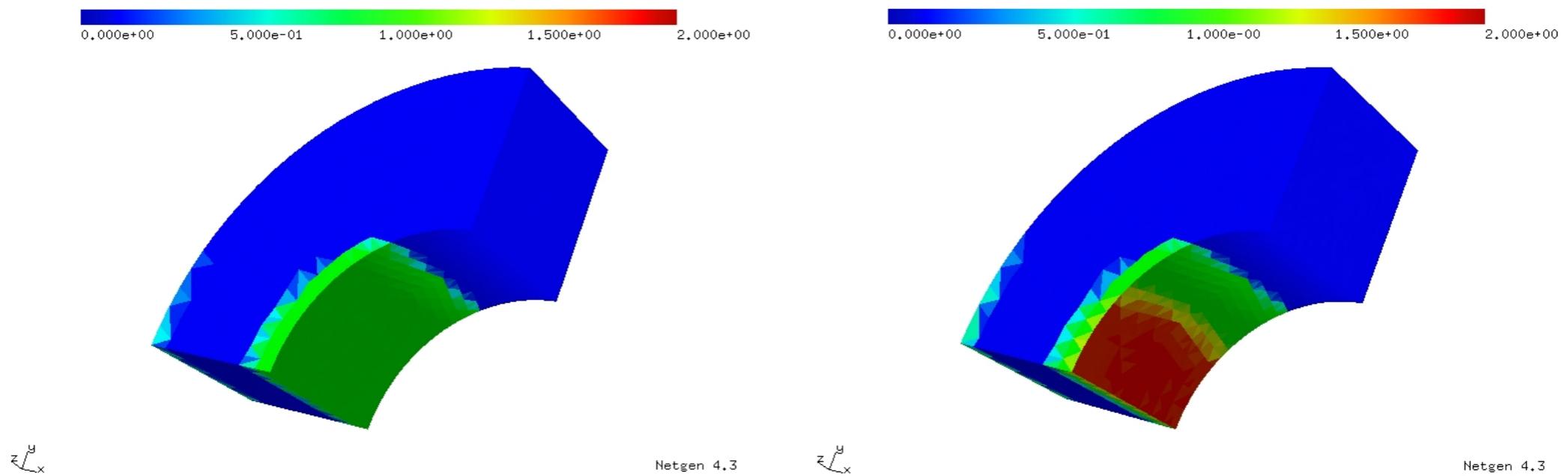
Kinematic hardening model.

Two-yield hardening model.

Outlook

Future Work:

• *Convergence proof*

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- *Minimize $\tilde{p}$ using symbolic methods (with RISC-Linz)*

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Download Netgen and NGSolve:

http://www.hpfem.jku.at/netgen/