Multi-yield elastoplasticity: Analysis and Numerics

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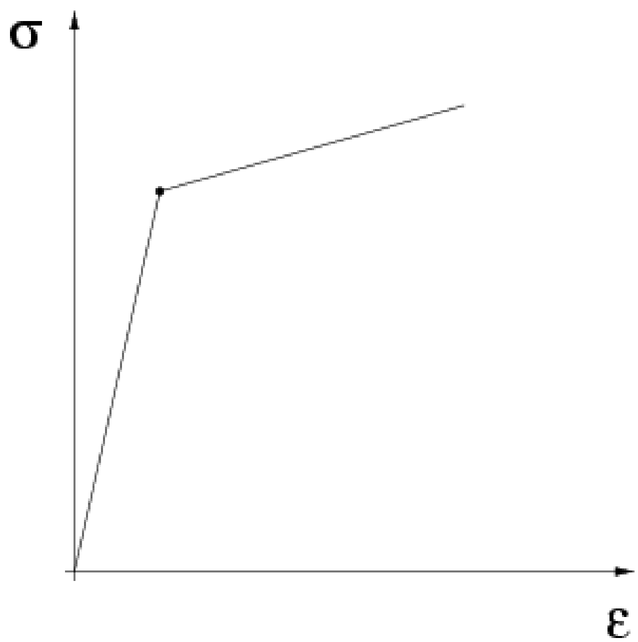
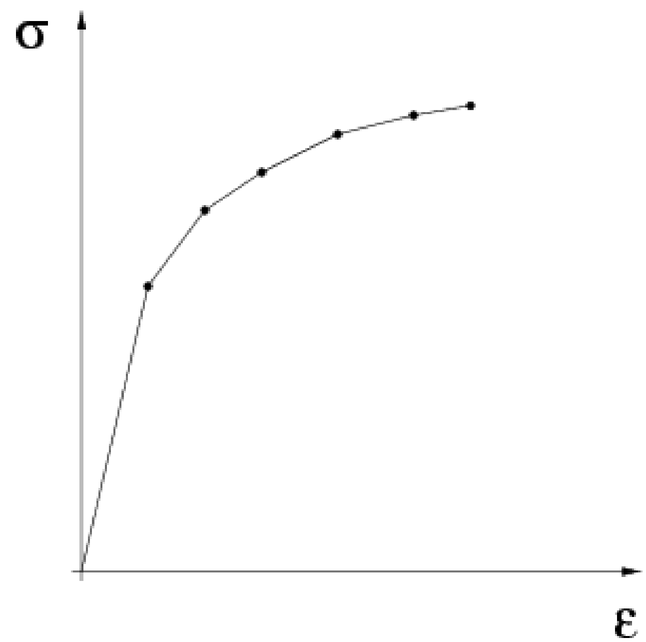
- *Why multi-yield plasticity?*
- *Mathemat. and numeric. modeling*
- *Symbolic computing problems*
- *Computational examples*
- *Future work*

Why multi-yield plasticity?



Single-yield model



Multi-yield model

- *Carstensen&Alberty '00, Han&Reddy '95, Johnson '76*
- *Brokate '98, Krejčí '96, Visintin '94*

Rheological model

$\varepsilon = e + \sum_{r=1}^{M} p_r,$ add. decomp. of strain

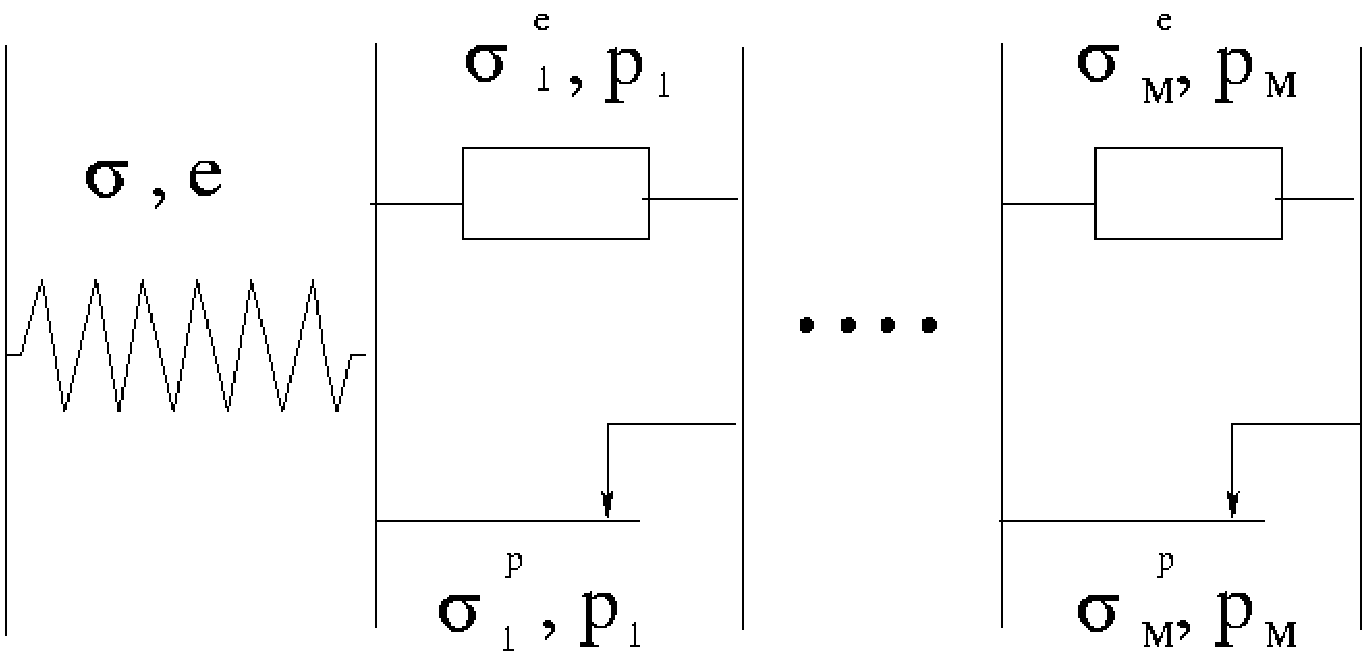
$\sigma = \sigma_r^e + \sigma_r^p$ for all $r = 1, \ldots, M,$

$\sigma_r^p \in Z_r,$

$\langle \dot{p}_r, q_r - \sigma_r^p \rangle \leq 0$ for all $q_r \in Z_r, \quad r = 1, \ldots, M,$

$\sigma = \mathbb{C} e,$ Hook's law

$\sigma_r^e = \mathbb{H}_r p_r, \quad r = 1, \ldots, M,$ Hardening's laws



Prandtl-Ishlinskii model of play type.

$Z = \{\sigma \in \mathbf{R}^{d\times d}_{sym} : ||\mathrm{dev}\,\sigma|| \leq \sigma^y\}$ von Mises

Mathematical model

Problem (PI): For $l \in H^1(0,T;\mathcal{H}^*), l(0) = 0$, find $w = (u, p_1, \ldots, p_M) : [0,T] \to \mathcal{H}, w(0) = 0$ s. t.

$$\langle l(t), z - \dot{w}(t) \rangle \le a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t))$$

$\forall z = (v, \tau_1, \ldots, \tau_M) \in \mathcal{H}$.

Notation:

$$\mathcal{H} = H^1_D(\Omega) \times \underbrace{L^2(\Omega)^{d\times d}_{sym} \times \cdots \times L^2(\Omega)^{d\times d}_{sym}}_{M \text{ times}},$$

$$a(w,z) = \int_\Omega \left(\mathbb{C}(\varepsilon(u) - \sum_{i=1}^M p_i) \right) : \left(\varepsilon(v) - \sum_{i=1}^M \tau_i \right) \mathrm{d}x + \sum_{i=1}^M \int_\Omega \mathbb{H}_i p_i : \tau_i \, \mathrm{d}x,$$

$$\langle l(t), z \rangle = \int_\Omega f(t) \cdot v \, \mathrm{d}x + \int_{\Gamma_N} g(t) \cdot v \, \mathrm{d}x,$$

$$j(z) = \int_\Omega \sum_{i=1}^M D_i(\tau_i) \, \mathrm{d}x,$$

$$D_i(x) = \begin{cases} \sigma_i^y \|x\| & \text{if } \mathrm{tr}\, x = 0, \\ +\infty & \text{otherwise.} \end{cases} \qquad \text{(von Mises)}$$

Existence and uniqueness

pos. definit elastic and hardening operators:

$$\mathbb{C}\xi : \xi \geq c||\xi||^2 \ \ \forall \xi \in \mathbf{R}^{d,d}$$
$$\mathbb{H}_i\xi : \xi \geq h_i||\xi||^2 \ \ \forall \xi \in \mathbf{R}^{d,d}, i = 1, \ldots, M$$

1) $a(\cdot,\cdot)$ - *bounded, elliptic bilinear form in* $\mathcal{H}$,

$$|a(w,z)| \leq \left((M+1)||\mathbb{C}|| + \max_{i=1,\ldots,M} ||\mathbb{H}_i||\right)||w||_{\mathcal{H}}||z||_{\mathcal{H}},$$
$$a(w,w) \geq \left(k \min_{i=1,\ldots,M}\{c, h_i\} \min\{1, K\}\right)||w||^2_{\mathcal{H}},$$

where $K > 0$ *(Korn's first inequality) and*

$$k = k(M) = 1 + \frac{M}{2} - \frac{1}{2}\sqrt{M(M+4)}$$

2) $j(\cdot)$ - *nonnegative, positive homogeneous and Lipschitz continuous functional in* $\mathcal{H}$

$$|j(z_1) - j(z_2)| \leq \left(\max_{i=1,\ldots,M}\{\sigma_i^y\} \mathit{meas}(\Omega)^{\frac{1}{2}} M^{\frac{1}{2}}\right) ||z_1 - z_2||_{\mathcal{H}}.$$

Theorem: Let $l \in H^1(0,T;\mathcal{H}')$ with $l(0) = 0$. $\exists!$ $w = (u, p_1, \ldots, p_M)(t) \in H^1(0,T;\mathcal{H})$ of Problem $(\mathbf{PI})$.

Proof: [Han, Reddy '99.]

Discretization

- *in time:* *net* $0 = t_0 < t_1 < \cdots < t_N = T$,

  and implicit Euler scheme

$$\dot{X}(t_j) = \frac{X(t_j) - X(t_{j-1})}{t_j - t_{j-1}} \quad \textit{for } j = 1, \ldots, N.$$

- *in space: regular triangulation* $\mathcal{T}$ *of* Ω

  - *displacement* u : $H^1_D(\Omega)$ *approximated by*

$$\mathcal{S}^1_D(\mathcal{T}) := \{v \in H^1_D(\Omega) : \forall T \in \mathcal{T}, v|_T \in \mathbf{P}_1(T)^d\}$$

  - *plastic strains* p_1, p_2 : $L^2(\Omega)$ *approximated by*

$$\mathcal{S}^0(\mathcal{T}) := \{a \in L^2(\Omega) : \forall T \in \mathcal{T}, a|_T \in \mathbf{R}\}$$

Discretization of two-yield material model ($M=2$)

Problem ($PI_{discrete}$): Given $P_1^0, P_2^0 \in \text{dev}\, \mathcal{S}^0(\mathcal{T})^{d\times d}_{sym}$, find $U^1 \in \mathcal{S}^1_D(\mathcal{T})$ s. t.

$$\int_\Omega \mathbb{C}(\epsilon(U^1) - P_1^1 - P_2^1) : \epsilon(V)\,\mathrm{d}x - \int_\Omega f(t) V\,\mathrm{d}x - \int_{\Gamma_N} gV\,\mathrm{d}x = 0 \quad \forall V \in S^1_D(\mathcal{T}),$$

where $P = (P_1, P_2)^T = (P_1^1, P_2^1)^T - (P_1^0, P_2^0)^T$ minimizes on every element $T \in \mathcal{T}$

$$\min_Q \frac{1}{2}(\widehat{\mathbb{C}} + \widehat{\mathbb{H}})Q : Q - A|_T : Q + ||Q||_{\sigma^y},$$

$\forall Q = (Q_1, Q_2)^T, Q_1, Q_2 \in \mathbf{R}^{d\times d}_{sym}, \text{tr}\, Q_1 = \text{tr}\, Q_2 = 0$.

Notation:

$$A = \begin{pmatrix} \mathbb{C}\epsilon(U^1) \\ \mathbb{C}\epsilon(U^1) \end{pmatrix} - (\widehat{\mathbb{C}} + \widehat{\mathbb{H}}) \begin{pmatrix} P_1^0 \\ P_2^0 \end{pmatrix},$$

$$\widehat{\mathbb{C}} + \widehat{\mathbb{H}} = \begin{pmatrix} \mathbb{C} + \mathbb{H}_1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C} + \mathbb{H}_2 \end{pmatrix},$$

$$||(Q_1, Q_2)||_{\sigma^y} = \sigma_1^y ||Q_1|| + \sigma_2^y ||Q_2||.$$

Lemma (two-yield plastic dependence): Given $\hat{A} = (A_1, A_2)^T, A_1, A_2 \in \mathbf{R}^{d\times d}_{sym}$, then $\exists!$ $P = (P_1,\ P_2)^T,\ P_1,\ P_2 \in \mathbf{R}^{d\times d}_{sym}$ with tr $P_1 =$ tr $P_2 = O$ s. t.

$$(\hat{A} - (\widehat{\mathbb{C}} + \widehat{\mathbb{H}})P) : (Q - P) \leq ||Q||_{\sigma^y} - ||P||_{\sigma^y}$$

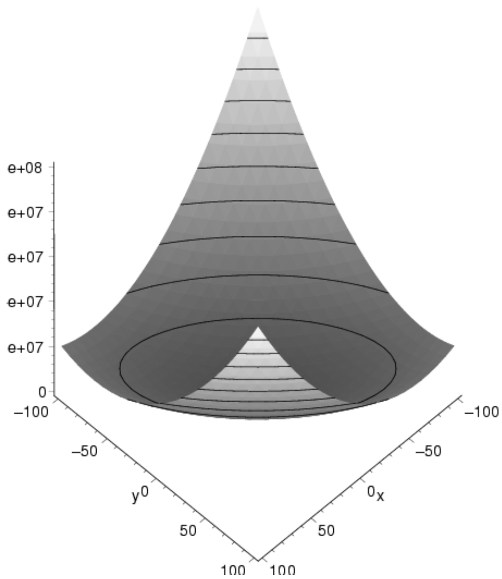
$\forall Q = (Q_1, Q_2)^T, Q_1, Q_2 \in \mathbf{R}^{d\times d}_{sym}$ with tr $Q_1 =$ tr $Q_2 = O$.

P is minimizer of

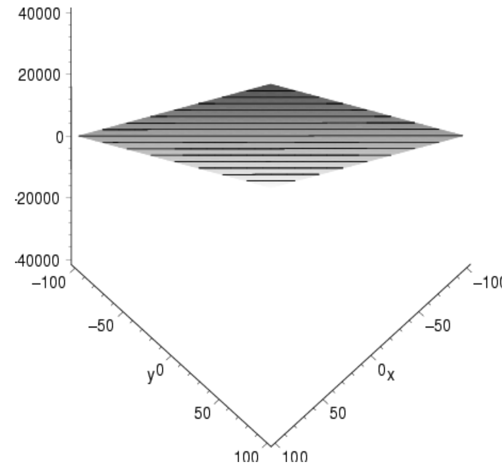
$$f(Q) = \frac{1}{2}(\widehat{\mathbb{C}} + \widehat{\mathbb{H}})Q : Q - Q : A + ||Q||_{\sigma^y}$$

amongst trace-free symmetric $d \times d$ matrices Q_1, Q_2.

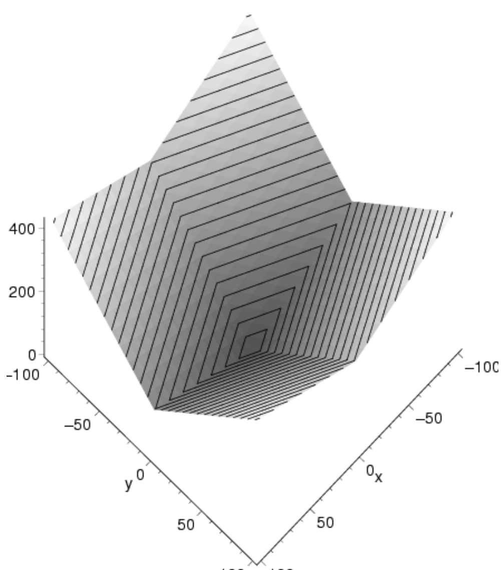
How to calculate P from continuous but non-smooth convex functional $f(P)$?



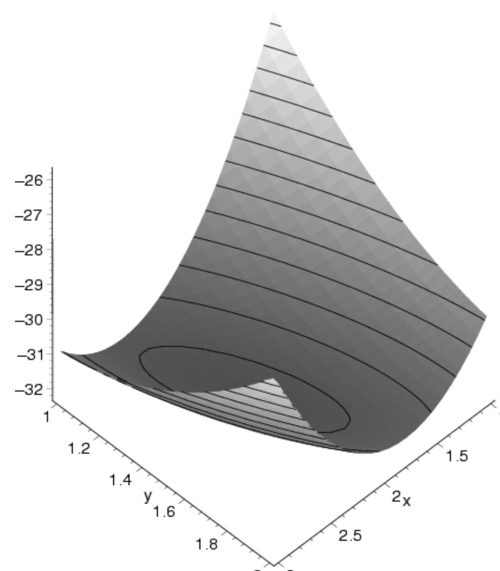
$\frac{1}{2}(\widehat{\mathbb{C}}+\widehat{\mathbb{H}})P : P$



$P : A$



$||P||_{\sigma_y}$



$\frac{1}{2}(\widehat{\mathbb{C}}+\widehat{\mathbb{H}})P : P$
$-P : A + ||P||_{\sigma_y}$

Functionals in argument P, where $P = (P_1, P_2)^T$, $P_1 = (x, 0; 0, -x), P_2 = (y, 0; 0, -y)$.

Analytical approach: one-yield

$$f(Q) = \frac{1}{2}(\mathbb{C} + \mathbb{H})Q : Q - Q : A + \sigma^y ||Q|| \to \text{min}$$

Lemma:(ACZ99) Let $f(P) = \min_Q f(Q)$.

Then $P = \frac{(||\text{dev}\, A|| - \sigma^y)_+}{2\mu + h} \frac{\text{dev}\, A}{||\text{dev}\, A||}$.

Proof: f has a subdifferential, i.e.,

$$\partial f(P) = (\mathbb{C} + \mathbb{H})P - A + \sigma^y \partial || \cdot ||(P)$$

Mininum condition on P

$$0 \in \partial f(P) \Leftrightarrow A - (\mathbb{C} + \mathbb{H})P \in \sigma^y \partial || \cdot ||(P)$$

In case $P \neq 0$ is

$$\partial || \cdot ||(P) = \{\frac{P}{||P||}\}$$

Nonlinear system in P_1, P_2

$$\text{dev}\, A = (\sigma^y + (2\mu + h)||P||)\frac{P}{||P||}$$

Subst. $\xi = ||P||$,

$$\xi = \frac{(||\text{dev}\, A|| - \sigma^y)}{2\mu + h},$$

$$P = \frac{(||\text{dev}\, A|| - \sigma^y)_+}{2\mu + h} \frac{\text{dev}\, A}{||\text{dev}\, A||}.$$

Analytical approach: two-yield

$$f(Q) = \frac{1}{2}(\hat{\mathbb{C}} + \hat{\mathbb{H}})Q : Q - Q : A + ||Q||_{\sigma^y} \to \min$$

Lemma: Let $f(P) = \min_Q f(Q), P = (P_1, P_2)$,
If $P_1 \neq 0, P_2 \neq 0 \Rightarrow ||P_2||$ is a root of a 8-th degree polynomial.

Proof: f has a subdifferential, i.e.,

$$\partial f(P) = (\hat{\mathbb{C}} + \hat{\mathbb{H}})P - A + \partial || \cdot ||_{\sigma^y}(P)$$

Mininum condition on P

$$0 \in \partial f(P) \Leftrightarrow A - (\hat{\mathbb{C}} + \hat{\mathbb{H}})P \in \partial || \cdot ||_{\sigma^y}(P)$$

In case $P_1 \neq 0, P_2 \neq 0$ is

$$\partial || \cdot ||_{\sigma^y}(P) = \{\sigma_1^y \frac{P_1}{||P_1||}, \sigma_2^y \frac{P_2}{||P_2||}\}$$

Nonlinear system in P_1, P_2

$$\begin{pmatrix} \text{dev}\, A_1 \\ \text{dev}\, A_2 \end{pmatrix} - \begin{pmatrix} (2\mu + h_1)\mathbb{I} & 2\mu\mathbb{I} \\ 2\mu\mathbb{I} & (2\mu + h_2)\mathbb{I} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} \sigma_1^y \frac{P_1}{||P_1||} \\ \sigma_2^y \frac{P_2}{||P_2||} \end{pmatrix}$$

Subst. $\xi_1 = ||P_1||, \xi_2 = ||P_2||$

$$A + B\xi_1 + C\xi_1^2 - (G + H\xi_1 + I\xi_2 + J\xi_1\xi_2)^2 = 0$$
$$D + E\xi_2 + F\xi_2^2 - (G + H\xi_1 + I\xi_2 + J\xi_1\xi_2)^2 = 0$$

MAPLE $\Rightarrow$ 8-th degree polynomial in $\xi_2 \Rightarrow$ no analytical formula!

Analytical approach: polynomial in ξ_2

$$
\begin{aligned}
&\left(J^4F^2\right)\xi_2^8\\
&+\left(2\%4J^2F\right)\xi_2^7\\
&+\left(2\%3J^2F+\%4^2\right)\xi_2^6\\
&+\left(2\%2J^2F+2\%3\%4\right)\xi_2^5\\
&+\left(2\%1J^2F+2\%2\%4+\%3^2-F(BJ+2IC)^2\right)\xi_2^4\\
&+\Big(-E(BJ+2IC)^2-2F(2CG+BH)(BJ+2IC)+2\%1\%4\\
&\qquad+2\%2\%3\Big)\xi_2^3\\
&+\Big(-\mathrm{D}(BJ+2IC)^2-2E(2CG+BH)(BJ+2IC)-F(2CG+BH)^2\\
&\qquad+2\%1\%3+\%2^2\Big)\xi_2^2\\
&+\left(-2\mathrm{D}(2CG+BH)(BJ+2IC)-E(2CG+BH)^2+2\%1\%2\right)\xi_2\\
&+\left(\%1^2-\mathrm{D}(2CG+BH)^2\right)=0,
\end{aligned}
$$

where

$\%1 := H^2\,\mathrm{D} - C\,G^2 - A\,H^2 - B\,G\,H - C\,\mathrm{D},$

$\%2 := -B\,G\,J - 2\,H\,J\,A - C\,E - 2\,I\,C\,G + H^2\,E - I\,B\,H + 2\,H\,J\,\mathrm{D},$

$\%3 := -C\,F - J^2\,A + 2\,H\,J\,E - I\,B\,J + C + J^2\,\mathrm{D} + H^2\,F,$

$\%4 := 2\,H\,J\,F + J^2\,E.$

Analytical Approach: example

Given $\mu = 1, \sigma_1^y = 1, \sigma_2^y = 2, h_1 = 1, h_2 = 1$ and

$$A_1 = A_2 = \begin{pmatrix} 20 & 0 \\ 0 & 0 \end{pmatrix}.$$

The direct calculation shows

$$l_1 = \begin{pmatrix} 10 + 10\xi_1 & 0 \\ 0 & -10 - 10\xi_1 \end{pmatrix},$$
$$l_2 = \begin{pmatrix} 20 - 10\xi_2 & 0 \\ 0 & -20 - 10\xi_2 \end{pmatrix},$$
$$r = 5\,\xi_1\,\xi_2 + 6\,\xi_1 + 3\,\xi_2 + 2.$$

The nonlinear system of equation for $\xi_1, \xi_2 > 0$!

$$200 + 400\,\xi_1 + 200\,\xi_1^2 - (2 + 3\,\xi_2 + 6\,\xi_1 + 5\,\xi_1\,\xi_2)^2 = 0,$$
$$800 + 800\,\xi_2 + 200\,\xi_2^2 - (2 + 3\,\xi_2 + 6\,\xi_1 + 5\,\xi_1\,\xi_2)^2 = 0.$$

ξ_1 solved from the second equation

$$\xi_1 = -\frac{1}{2}\frac{24 + 56\xi_2 + 30\xi_2^2 \pm 20\sqrt{2}\,(12 + 16\xi_2 + 5\xi_2^2)}{(6 + 5\,\xi_2)^2},$$

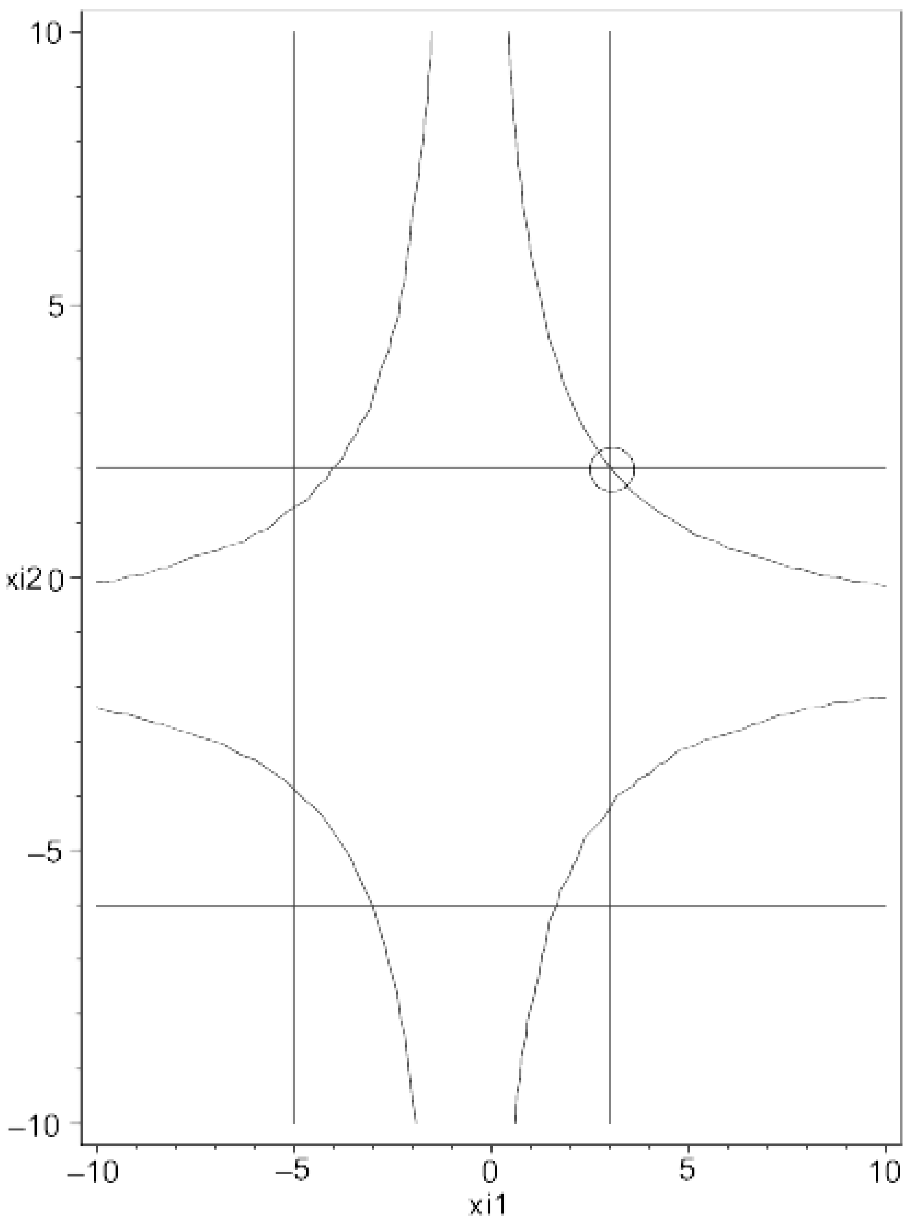
subst. ($-$ term only!) in the first eq.

$$\frac{P_4(\xi_2)}{(6+5\,\xi_2)^2} = 0.$$

Condition $(6+5\,\xi_2) > 0!$ $\Rightarrow$ $P_4(\xi_2) = 0.$

$$\xi_2 = \{-4.428427124, -2, -2, 2.028427124\}.$$

Analytical Approach: example - geometrical interpretation



(ξ_1, ξ_2) = Two perpend. lines and a hyberbole intersection

$$200 + 400\,\xi_1 + 200\,\xi_1^2 - (2 + 3\,\xi_2 + 6\,\xi_1 + 5\,\xi_1\,\xi_2)^2 = 0,$$
$$800 + 800\,\xi_2 + 200\,\xi_2^2 - (2 + 3\,\xi_2 + 6\,\xi_1 + 5\,\xi_1\,\xi_2)^2 = 0.$$

Iterative Approach

Algorithm (*): Given $tolerance \geq 0$.
(a) Choose $(P_1^0, P_2^0) \in \mathrm{dev}\,\mathbf{R}_{sym}^{d\times d} \times \mathrm{dev}\,\mathbf{R}_{sym}^{d\times d}$, set $i := 0$.
(b) Find $P_2^{i+1} \in \mathrm{dev}\,\mathbf{R}_{sym}^{d\times d}$ s. t.

$$f(P_1^i, P_2^{i+1}) = \min_{Q_2 \in \mathrm{dev}\,\mathbf{R}_{sym}^{d\times d}} f(P_1^i, Q_2).$$

(c) Find $P_1^{i+1} \in \mathrm{dev}\,\mathbf{R}_{sym}^{d\times d}$ s. t.

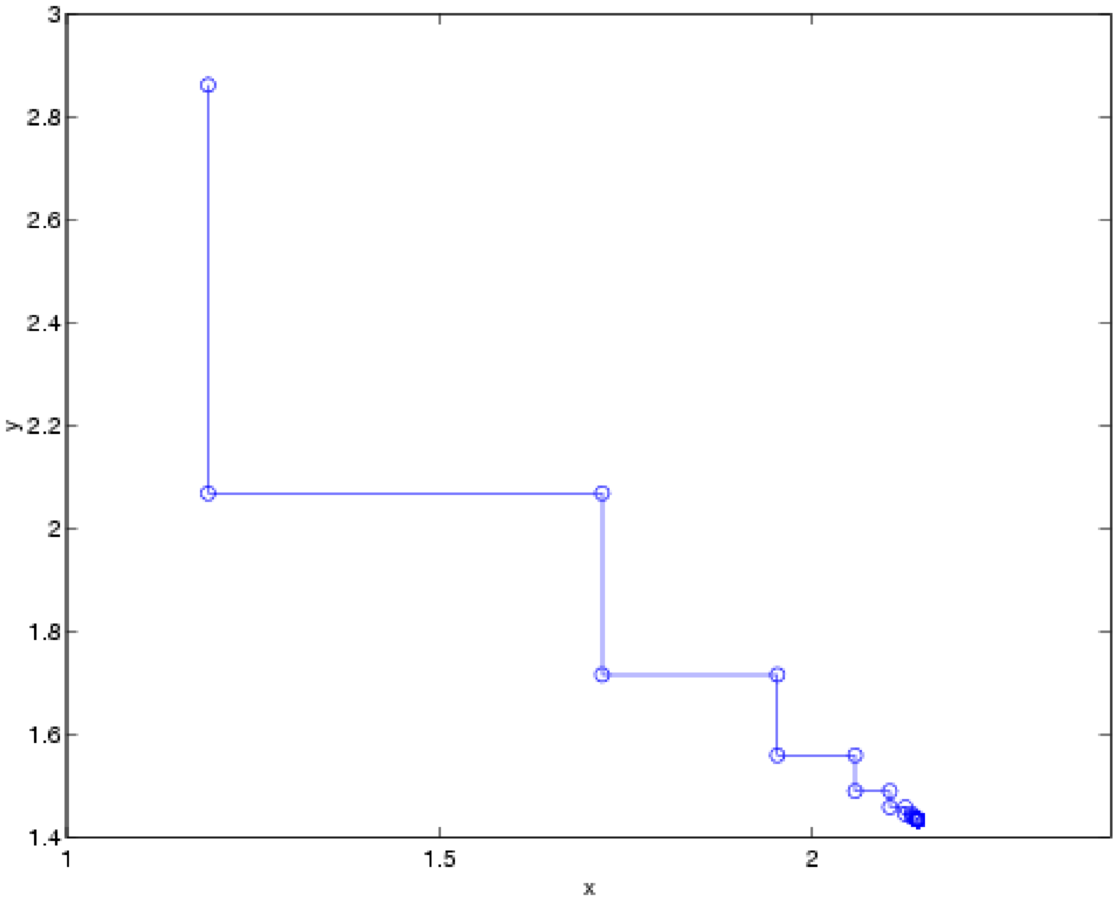
$$f(P_1^{i+1}, P_2^{i+1}) = \min_{Q_1 \in \mathrm{dev}\,\mathbf{R}_{sym}^{d\times d}} f(Q_1, P_2^{i+1}).$$

(d) If $\frac{||P_1^{i+1} - P_1^i|| + ||P_2^{i+1} - P_2^i||}{||P_1^{i+1}|| + ||P_1^i|| + ||P_2^{i+1}|| + ||P_2^i||} > tolerance$ set $i := i + 1$ and goto (b), otherwise output (P_1^{i+1}, P_2^{i+1}).

- global convergence with the rate 1/2:

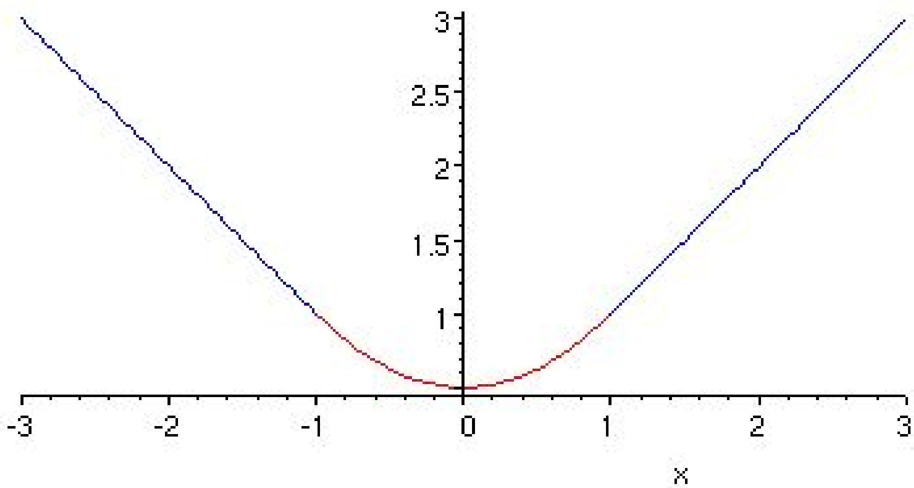
$$||P_1^i - P_1||^2 + ||P_2^i - P_2||^2 \leq C_0 \cdot q^i$$

Iterative Approach - example



The approximations $P_1^i = (x^i, 0; 0, -x^i), P_2^i = (y^i, 0; 0, -y^i), i = 0, 1, \ldots$ of Algorithm (*) displayed as the points (x^i, y^i) in the $x - y$ coordinate system.

Analytical Approach - regularized one-yield and two-yield problems



the absolute value regularizator: $a \in \mathbf{R}_0^+, \epsilon > 0$,

$$a_\epsilon = \begin{cases} a & \text{if } a \geq \epsilon, \\ \frac{1}{2\varepsilon}a^2 + \frac{\epsilon}{2} & \text{if } a < \epsilon. \end{cases}$$

One-yield problem

$$f(Q) = \frac{1}{2}(\mathbb{C} + \mathbb{H})Q : Q - Q : A + \sigma^y ||Q||_\varepsilon \to \text{min}$$

minimizer $P = ?$

Two-yield problem

$$f(Q) = \frac{1}{2}(\widehat{\mathbb{C}} + \widehat{\mathbb{H}})Q : Q - Q : A + (||Q||_{\sigma^y})_\varepsilon \to \text{min}$$

minimizer $P = (P_1, P_2)?$

Numerical Algorithms

Matlab (J. Valdman)
NGSOLVE (J. Kienesberger)

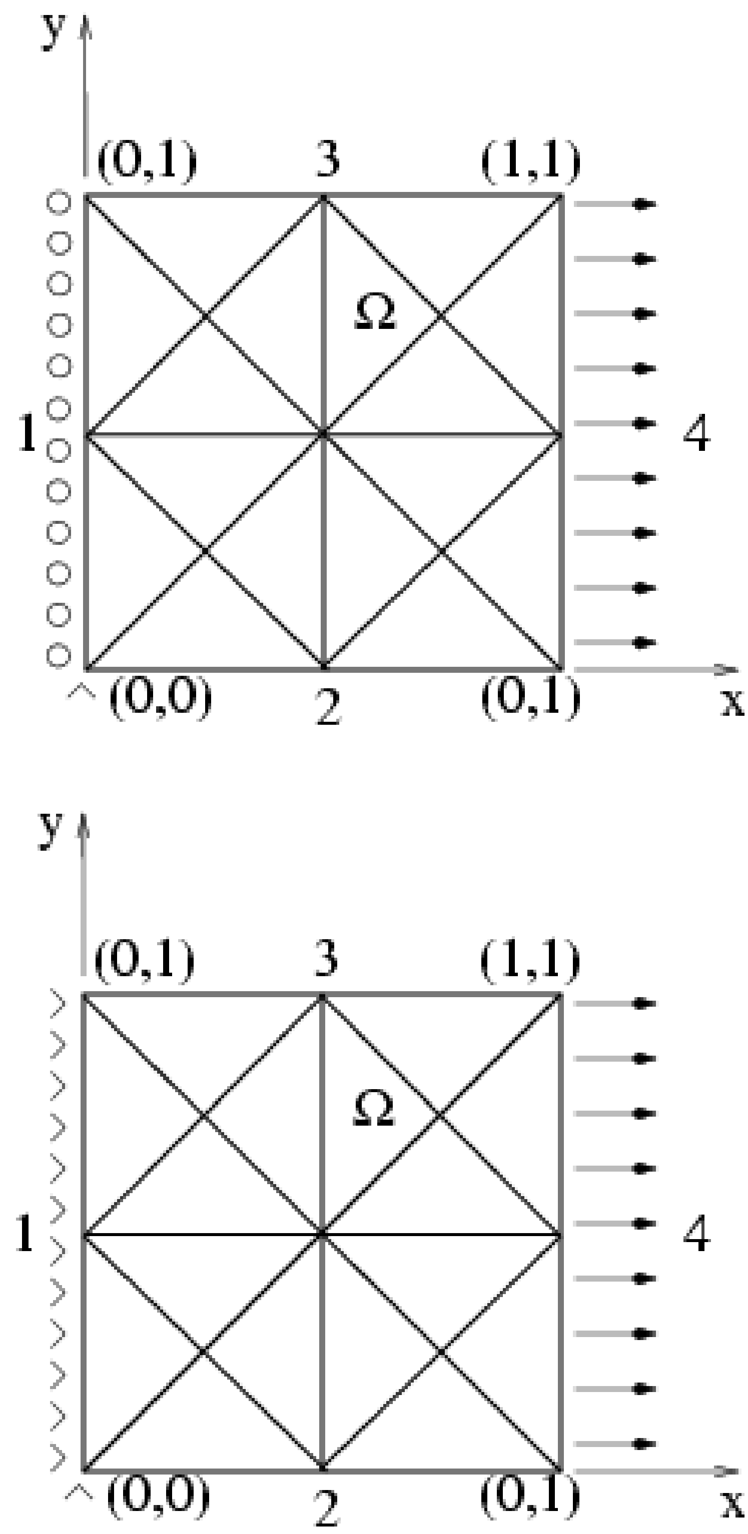
- *Alternating direction's method for P solving*
  Newton method
  Exact analytical formula?

- *Newton-Raphson method for U solving*

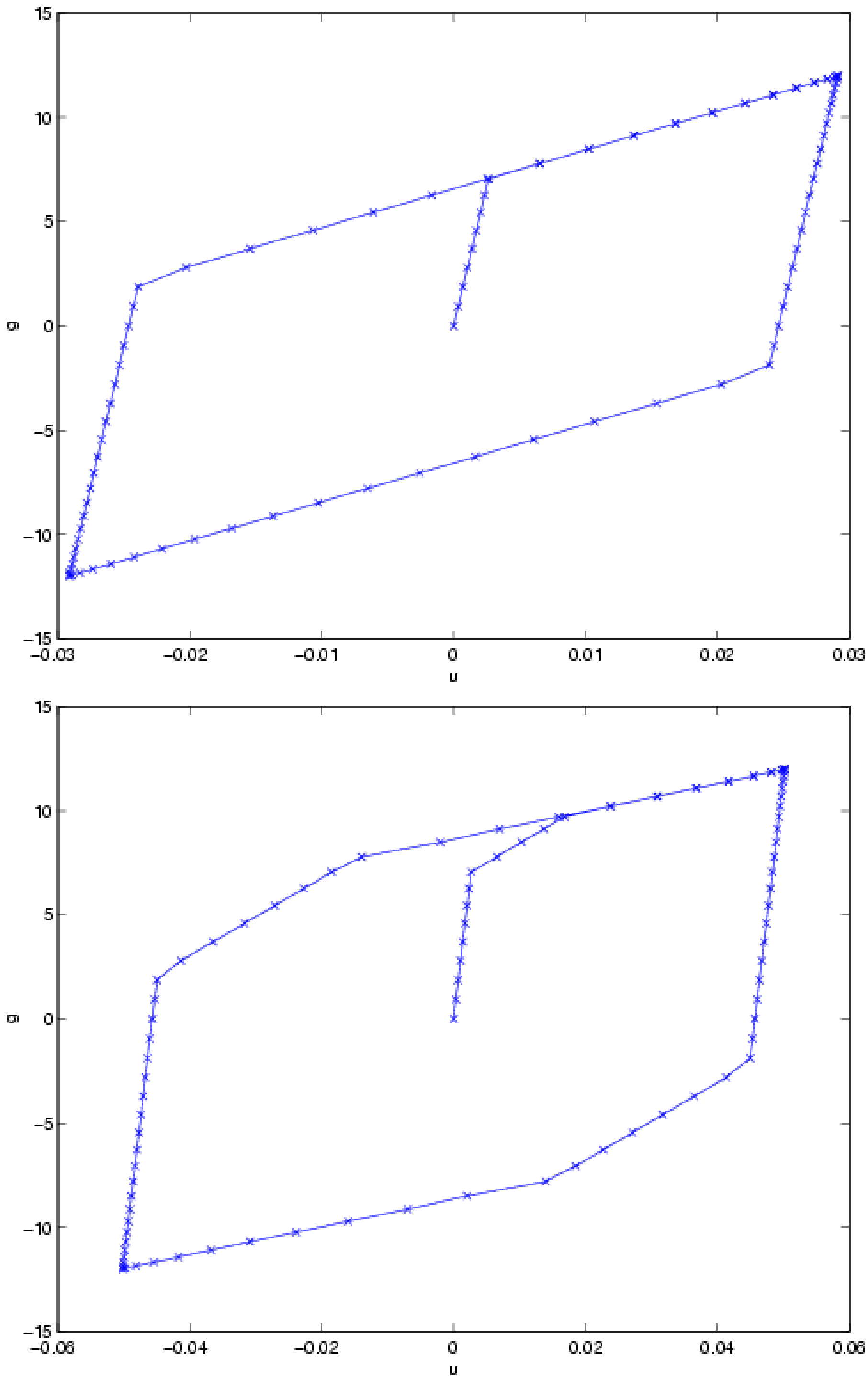
$$\begin{pmatrix} D\mathbf{F}(\mathbf{U^1_{k-1}}) & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \triangle\mathbf{U^1_k} \\ \lambda \end{pmatrix} = \begin{pmatrix} -\mathbf{F}(\mathbf{U^1_{k-1}}) \\ \mathbf{0} \end{pmatrix}$$

  Multigrid preconditioned CG method

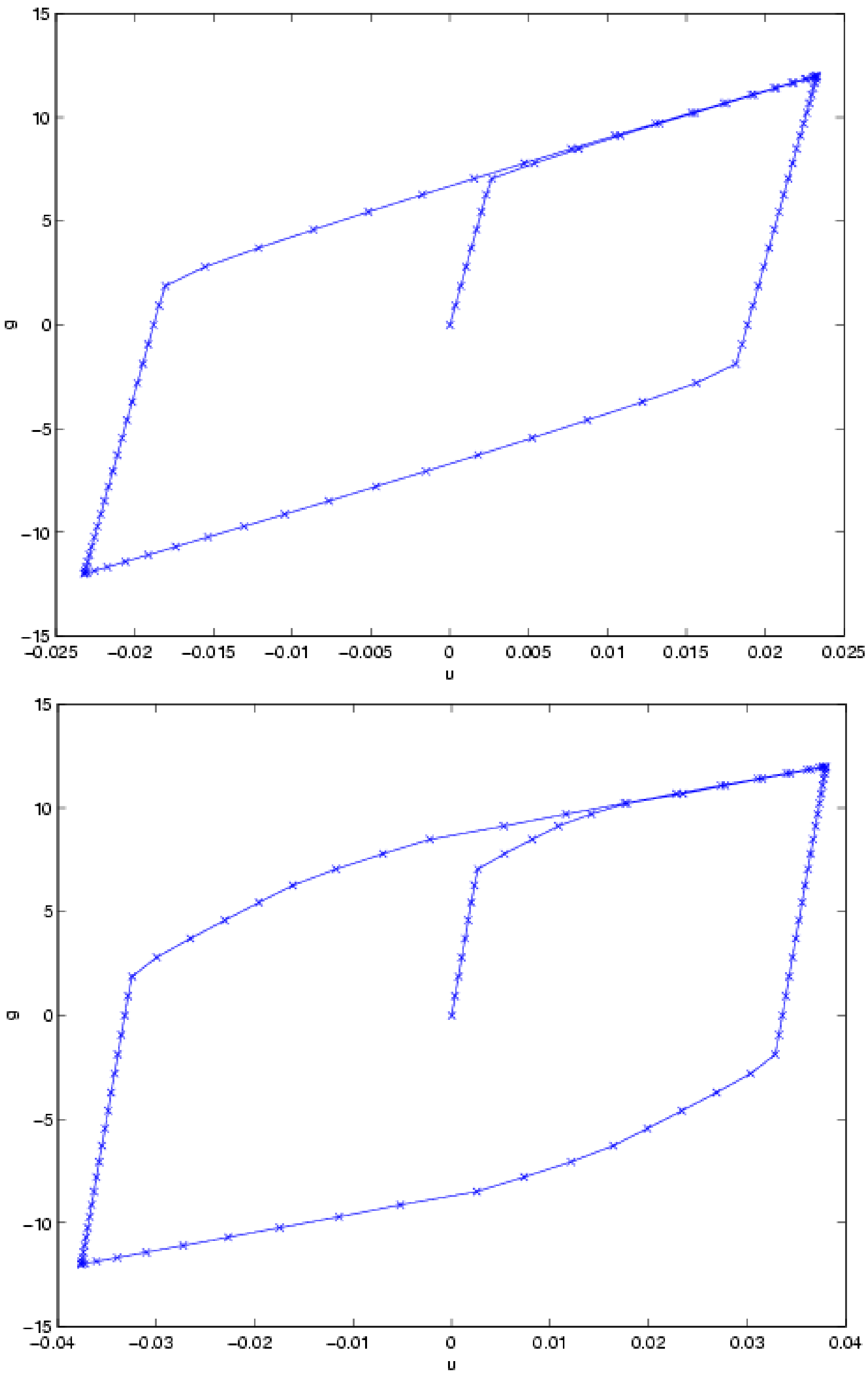
- *Adaptive mesh-refining (ZZ-estimator)*



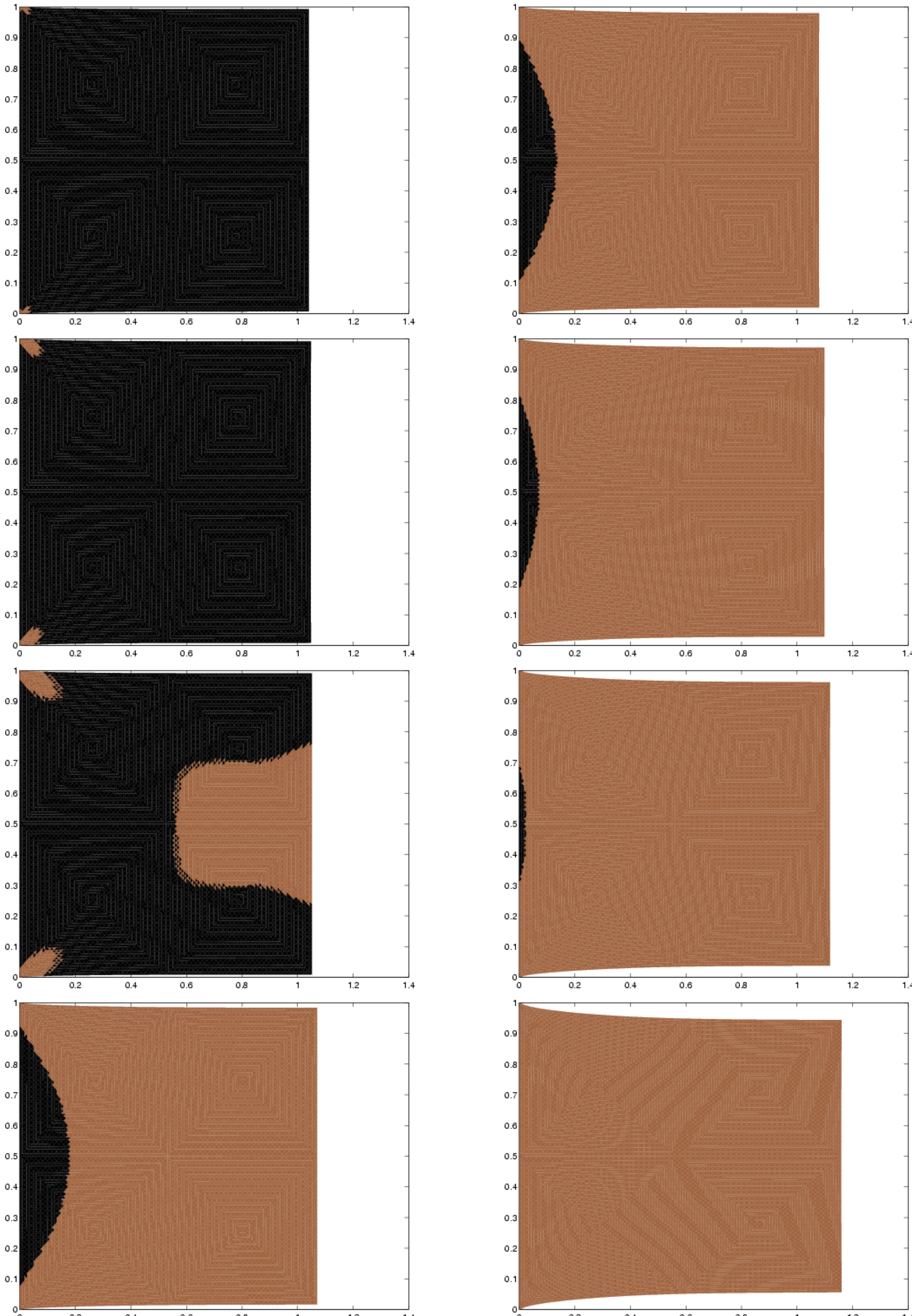
Geometry and coarse mesh $\mathcal{T}_0$ problems of beam with 1D effects (up) and 2D effects (down).



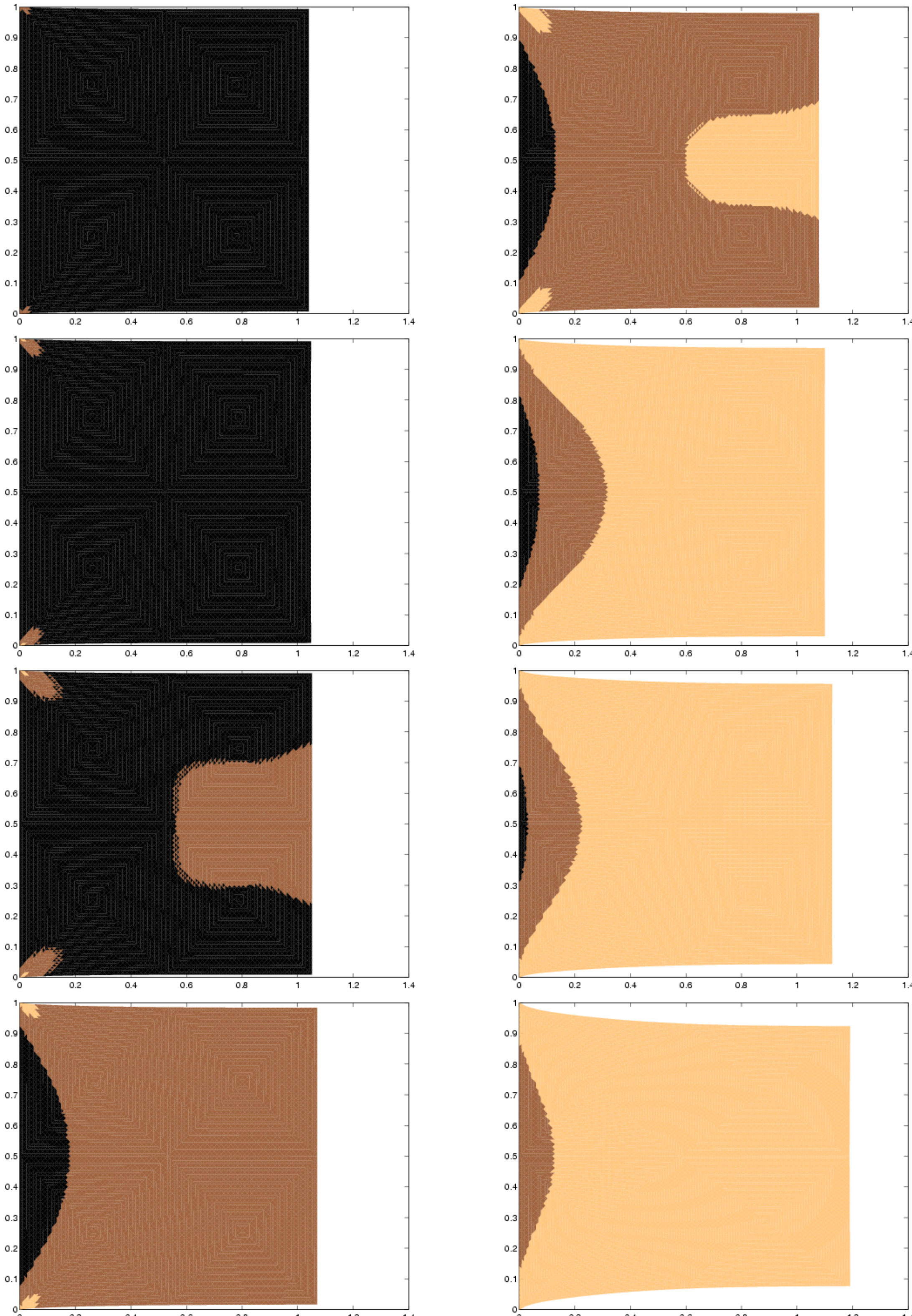
Loading-deformation relation: uniform surface loading $g_x(t)$ versus the x-displacement of the point $(0, 1)$ for problems of the single-yield (up) and multi-yield (down) beam with 1D effects.



Loading-deformation relation: uniform surface loading $g_x(t)$ versus the x-displacement of the point $(0, 1)$ for problems of the single-yield (up) and multi-yield (down) beam with 2D effects.



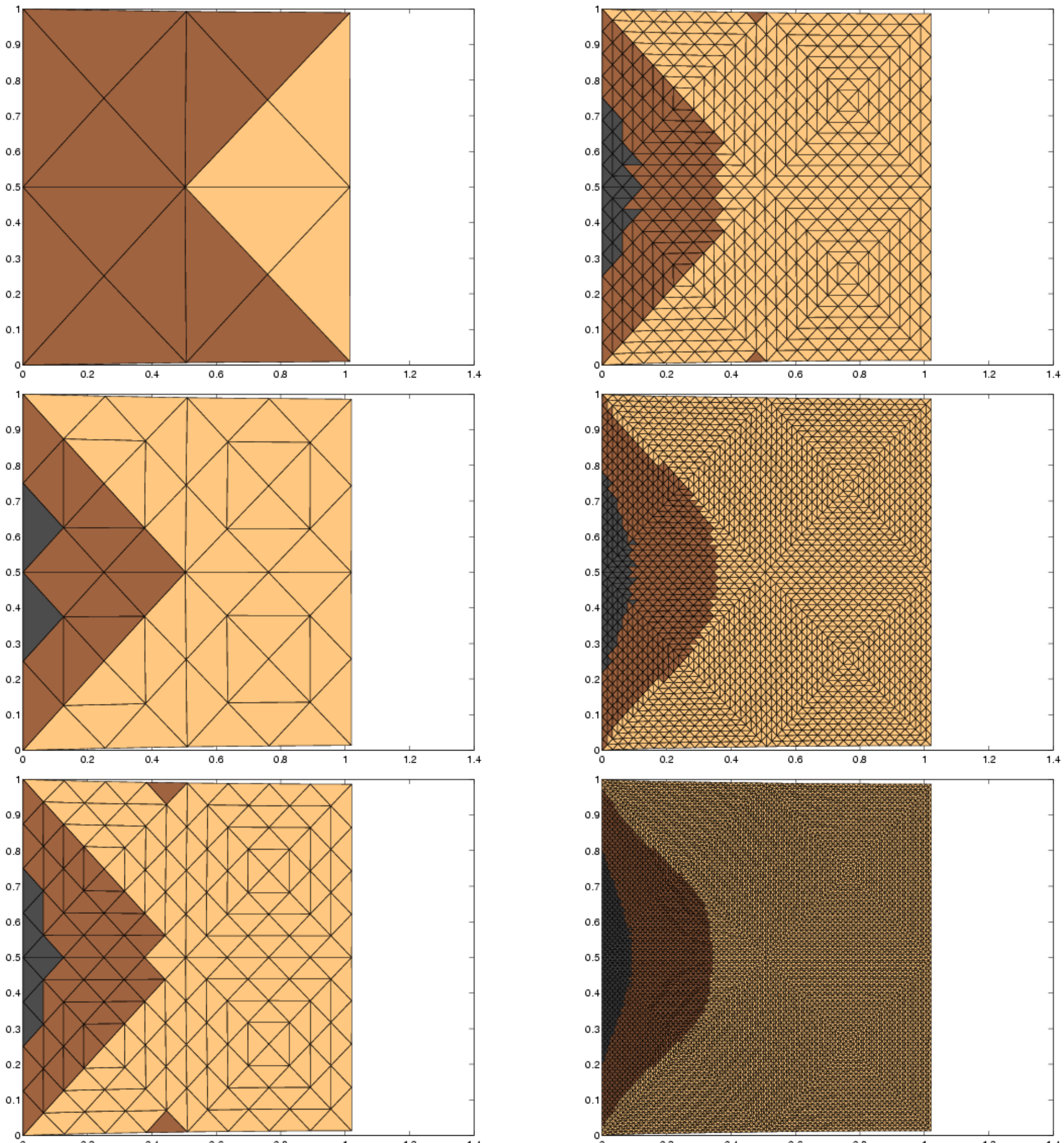
Evolution of elastoplastic zones at discrete times $t = 4.5, 5, 5.5, 6, 6.5, 7, 8, 9$ for the single-yield beam with 2D effects. 16334 elements, CPU time $= 15.59$ hours.



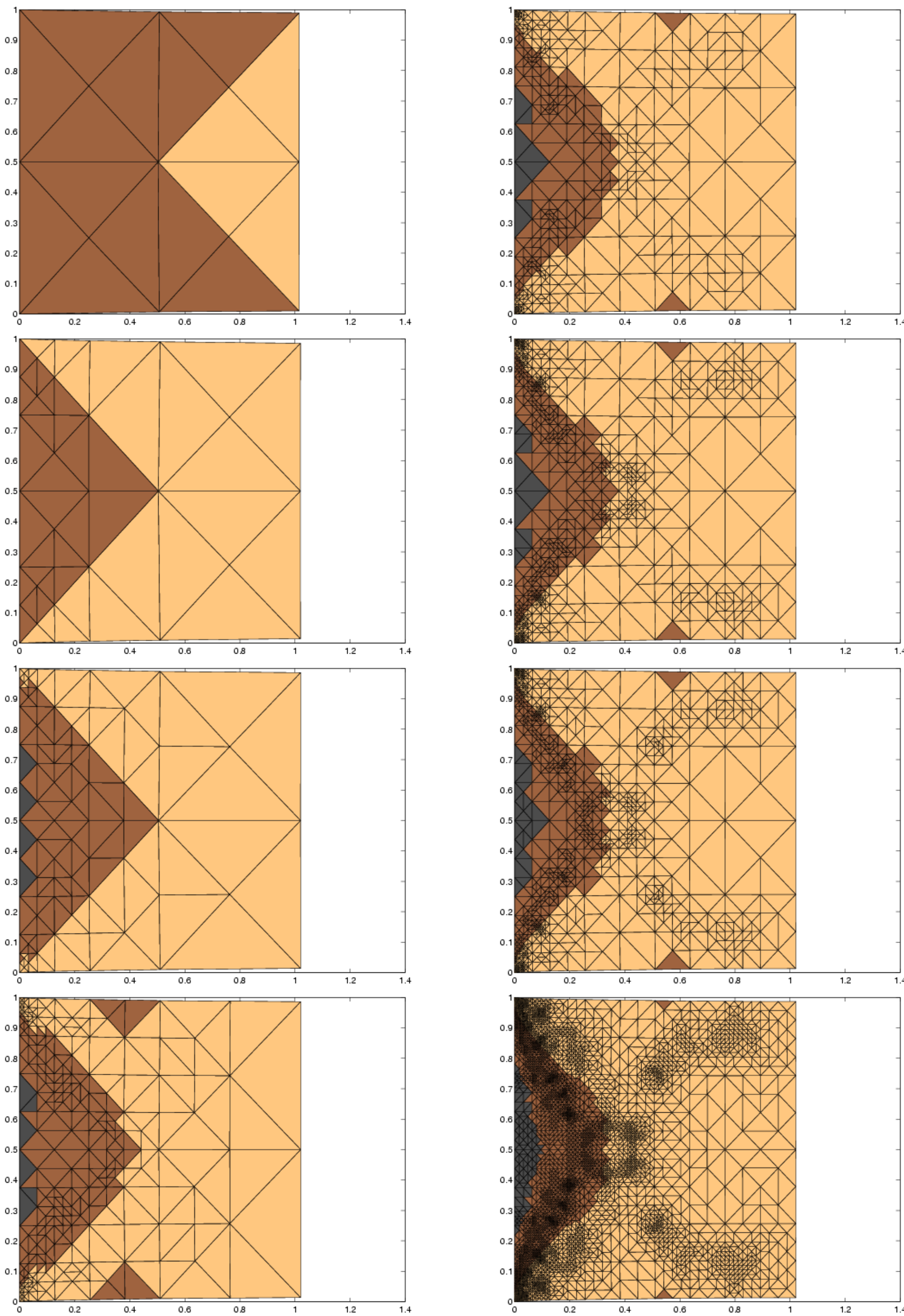
Evolution of elastoplastic zones at discrete times $t = 4.5, 5, 5.5, 6, 6.5, 7, 8, 9$ of the two-yield beam with 2D effects. 16334 elements, CPU time = 25.17 hours.

material model	CPU time (in hours)	total number of Newton steps	CPU time - Algorithm (*) (in %)
single-yield	15.59	126	3.5
two-yield	29.57	126	49.6

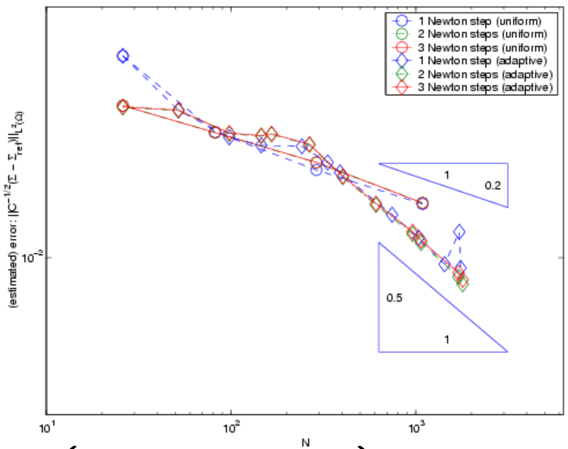
Problem of beam with 2D effects. Discrete times $t = \{0, 0.5, 1, \ldots, 10\}$, and uniform mesh $\mathcal{T}_5$ with 16384 elements.



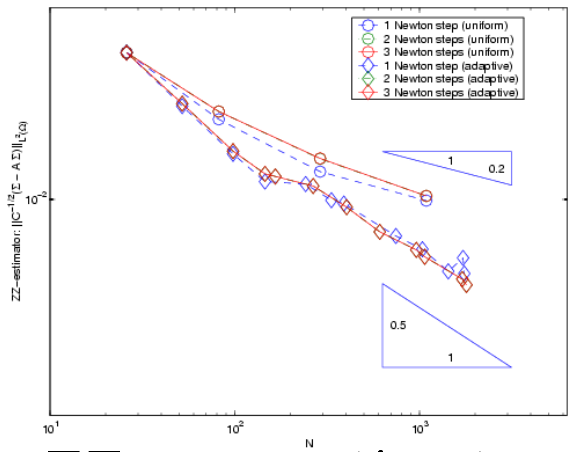
Uniformly refined meshes and elastoplastic zones, one time-step $t_0 = 0, t_1 = 8.5$, two-yield beam with 2D effects.



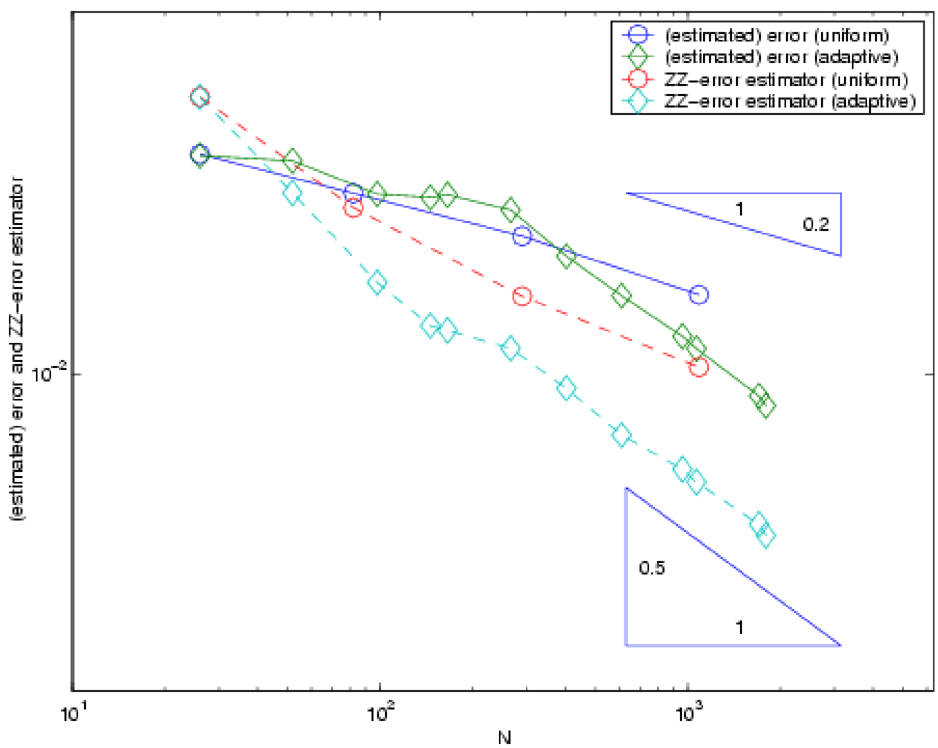
Adaptively refined meshes and elastoplastic zones, one time-step $t_0 = 0, t_1 = 8.5$, two-yield beam with 2D effects.



(Estimated) error



ZZ-error estimator

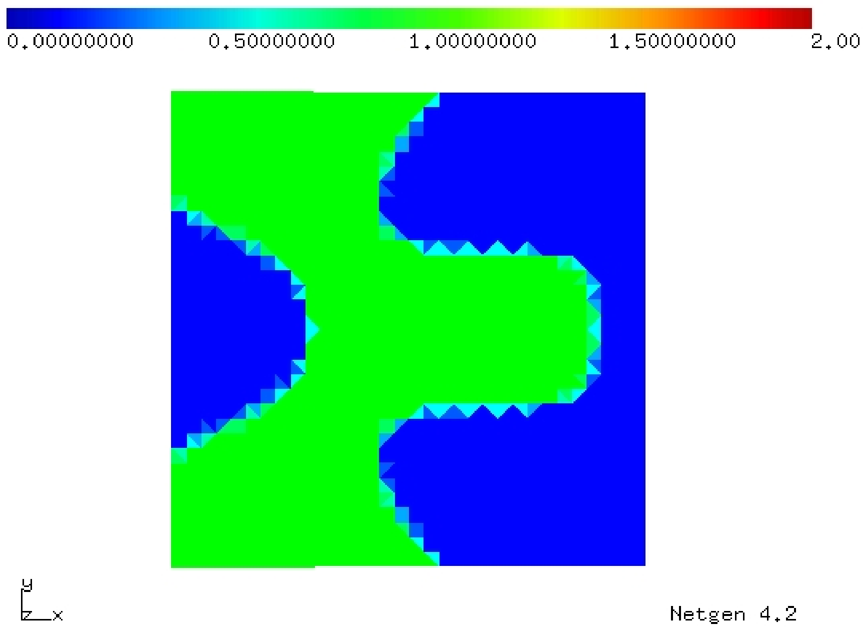


(Estimated) error and ZZ-error estimator

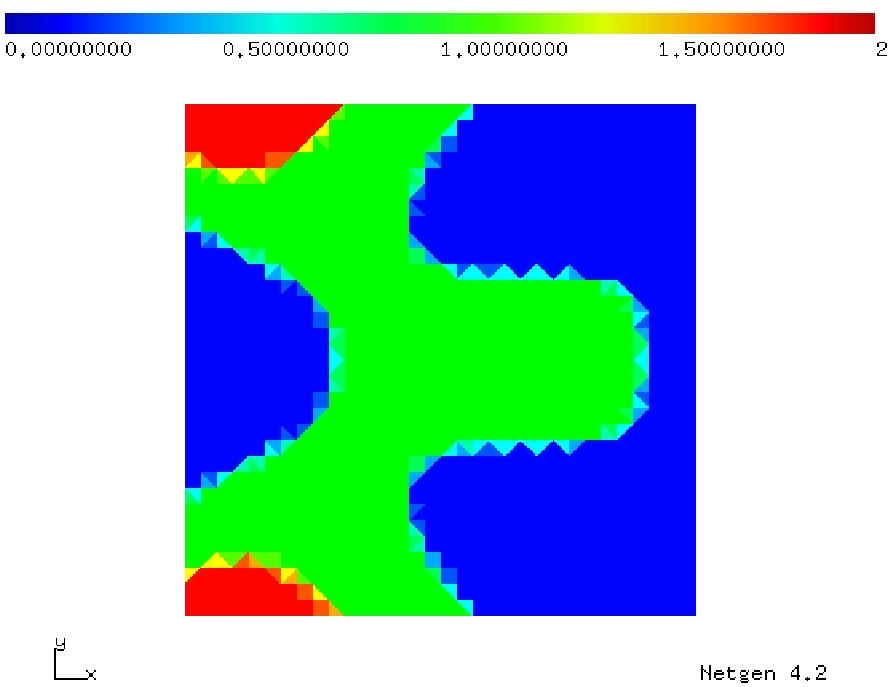
One time-step with $t_0 = 0, t_1 = 9$. (Estimated) error and ZZ-error estimator are displayed versus the degrees of freedom N.

NGSOLVE calculations:

Elastoplastic domains (blue -elastic, green - first plastic, red - second plastic)



Kinematic hardening model calculation.



Two-yield hardening model calculation.

Conclusions

Single-yield model versus two-yield model:

- *The same variational inequality ($\Rightarrow$ existence and uniqueness!)*
- *Plastic dependency for two-yield model not analytical anymore*

Numerical experiments:

- *The priority of adaptive mesh-refinements over uniform mesh-refinements*
- *One Newton iteration in the nested iteration technique is usually sufficient.*

Future work

- *Completion of NGSOLVE Two-yield plasticity package.*
- *Generalization to isotropic hardening or combined kinematic-isotropic hardening.*
- *Exact analytical formulas.*
- *Nonlinear hardening models, big deformations, temperature.*