First attempt to interface prediction in plasticity via level set methods

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F1306 - Adaptive Multilevel Methods for
Nonlinear 3D Mechanical Problems

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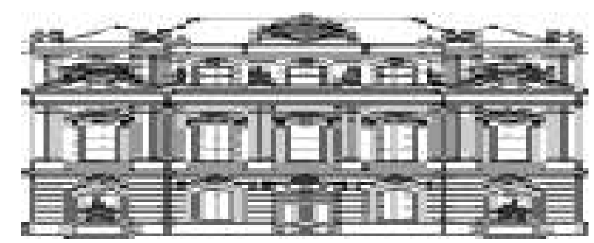
F1308 - Computational Inverse Problems and Applications



SFB Statusseminar, March 1 - April 2, 2005, Strobl

Outline

- Modeling (F1306)
- h-adaptivity (interfaces) (F1306)
- Level set approach (F1308)
- Outlook (F1308)

Modeling

Find $u \in W^{1,2}(0,T;\ H^1_0(\Omega)^n)$, $p \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{n\times n}))$,
$\sigma \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{n\times n}))$, $\alpha \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{m}))$ such that

$$\begin{aligned}
-\operatorname{div}\sigma &= b \\
\sigma &= \sigma^T \\
\sigma &= \mathbb{C}(\varepsilon(u) - p) \\
\varepsilon(u) &= \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) \\
\varphi(\sigma,\alpha) &< \infty \\
\dot{p} : (\tau - \sigma) - \dot{\alpha} : (\beta - \alpha) &\leq \varphi(\tau,\beta) - \varphi(\sigma,\alpha)
\end{aligned}$$

are satisfied in the variational sense with $(u, p, \sigma, \alpha)(0) = 0$ for all (τ, β).
b and $\mathbb{C}^{-1}$ are given, $b(0) = 0$.

Numeric-analytic steps

- Time discretization: $t_1 = t_0 + \Delta t$
- Reformulation of the problem using functional-analytic arguments (switching arguments in variational inequalities using a dual functional)
- Equivalent minimization problem:

  Find the minimizer $(u, p, \alpha) \in H \times L^{n\times n}_{sym} \times L^m$ of

$$f(u,p,\alpha) := \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u)-p] : (\varepsilon(u)-p)dx + \frac{1}{2}\int_\Omega |\alpha|^2 dx + \Delta t \int_\Omega \varphi^*\left(\frac{p-p_0}{\Delta t}, \frac{\alpha_0-\alpha}{\Delta t}\right)dx - \int_\Omega b\, u\, dx \to \text{min}$$

with φ describing the hardening law.

Minimization problem for isotropic hardening

New variable: $\tilde{p} = p - p_0$

Find the minimizer $(u, p, \alpha) \in H \times L^{n \times n}_{sym}$ of

$$f(u, \tilde{p}) := \frac{1}{2} \int_\Omega \mathbb{C}(\varepsilon(u) - \tilde{p} - p_0) : (\varepsilon(u) - \tilde{p} - p_0)\,\mathsf{d}x + \frac{1}{2} \int_\Omega \alpha_0^2 \,\mathsf{d}x + \frac{1}{2} \int_\Omega \sigma_y^2 H^2 |\tilde{p}|^2 \,\mathsf{d}x$$
$$+ \int_\Omega \sigma_y (1 + \alpha_0 H) |\tilde{p}| \,\mathsf{d}x - \int_\Omega b u \,\mathsf{d}x \to \mathsf{min}$$

under the constraint $\mathrm{tr}(p - p_0) = 0$.

Regularization (of $|\tilde{p}|$) Approach

$$|p|_\epsilon := \begin{cases} |p| & \text{if } |p| \geq \epsilon \\ \frac{1}{2\epsilon}|p|^2 + \frac{\epsilon}{2} & \text{if } |p| < \epsilon \end{cases}$$

$\Rightarrow$ convex smooth problem

$$f(u,\tilde{p}) := \frac{1}{2}\int_\Omega \mathbb{C}(\varepsilon(u) - \tilde{p} - p_0) : (\varepsilon(u) - \tilde{p} - p_0)\,\mathrm{d}x + \frac{1}{2}\int_\Omega \alpha_0^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \sigma_y^2 H^2 |\tilde{p}|^2\,\mathrm{d}x$$
$$+ \int_\Omega \sigma_y(1+\alpha_0 H)|\tilde{p}|_\epsilon\,\mathrm{d}x - \int_\Omega bu\,\mathrm{d}x \to \min$$

Then the local problem reads

$$f(u,\bar{p}) = \frac{1}{2}\begin{pmatrix} u \\ \bar{p} \end{pmatrix}^T \begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C}P \\ -P\mathbb{C}B & P^T(\mathbb{C}+\mathbb{H})P \end{pmatrix} \begin{pmatrix} u \\ \bar{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ P^T\mathbb{C}p_0 \end{pmatrix}^T \begin{pmatrix} u \\ \bar{p} \end{pmatrix}$$
$$+ \frac{1}{2}\mathbb{C}\tilde{p_0} : \tilde{p}_0 + \frac{1}{2}\alpha_0^2 \to \min,$$

where $\mathbb{H} = \mathbb{H}(\tilde{p}) = \sigma_y^2 H^2 b + 2\sigma_y(1+\alpha_0 H)\frac{b}{|p|_\epsilon}$.

No Regularization - Yoshida Moreau Approach

Functional $f(u, \tilde{p})$ can be written in a more abstract way

$$f(u,p) = \frac{1}{2}\|\varepsilon(u) - p(\varepsilon(u))\|^2_{\mathbb{C}} + \varphi(p(\varepsilon(u))) - L(u) \;\longrightarrow\; \min$$

with φ convex and L linear.

$\Rightarrow$ Yoshida Moreau theorem from convex analysis guarantees the differentiability of $f(u) = f(u, p(u))$:

$$\mathcal{D}f(u,v) = \int_\Omega \mathbb{C}\left(\varepsilon(u) - \tilde{p}(\varepsilon(u))\right) : \varepsilon(v)\,\mathsf{d}x - \int_\Omega f v\,\mathsf{d}x - \int_{\Gamma_N} t v\,\mathsf{d}s$$

Nonlinear system

$$\mathcal{D}f(u,v) = 0 \quad \forall v \in H$$

Work in progress: P. Gruber, J. Valdman

Minimization in $\tilde{p}$: Regularization + No Regularization

The objective in each integration point writes as

$$F(\tilde{p}) = \frac{1}{2}\tilde{p}^T \mathbb{C}\tilde{p} + p_0^T \mathbb{C}\tilde{p} - \tilde{p}^T \mathbb{C}\varepsilon(u) + \frac{1}{2}\sigma_y^2 H^2 |\tilde{p}|^2 + \sigma_y(1+\alpha_0 H)|\tilde{p}|_\epsilon$$

No regularization ($\epsilon = 0$) a unique solution

$$\tilde{p} = \frac{(||\operatorname{dev} A|| - a)_+}{2\mu + \sigma_y^2 H^2} \frac{\operatorname{dev} A}{||\operatorname{dev} A||}, . \qquad (1)$$

where

$$A = \mathbb{C}[\varepsilon(u) - p_0], \quad a = \sigma_y(1+\alpha_0 H).$$

Regularization: Newton method

$$P^T F''(\tilde{p}) P \Delta\bar{p} = -P^T F'(\tilde{p}). \qquad (2)$$

Minimization in u: Regularization Approach

Simplification: $\mathbb{H} = \mathbb{H}(\tilde{p})$ dependence frozen $\Rightarrow f(u, \tilde{p})$ is perfectly quadratic functional.
A necessary condition of the minima of (1) is $f'(u, \tilde{p}) = 0$, i.e.,

$$\begin{pmatrix} B^T \mathbb{C} B & -B^T \mathbb{C} P \\ -P \mathbb{C} B & P^T (\mathbb{C} + \mathbb{H}) P \end{pmatrix} \begin{pmatrix} u \\ \bar{p} \end{pmatrix} + \begin{pmatrix} -b - B^T \mathbb{C} p_0 \\ P^T \mathbb{C} p_0 \end{pmatrix} = 0. \tag{3}$$

By eliminating $\tilde{p}$ from (3), we get a linear system for u only

$$S\, u = b + B^T (\mathbb{C} - \mathbb{C} P (P^T (\mathbb{C} + \mathbb{H}) P)^{-1} P^T \mathbb{C})\, p_0, \tag{4}$$

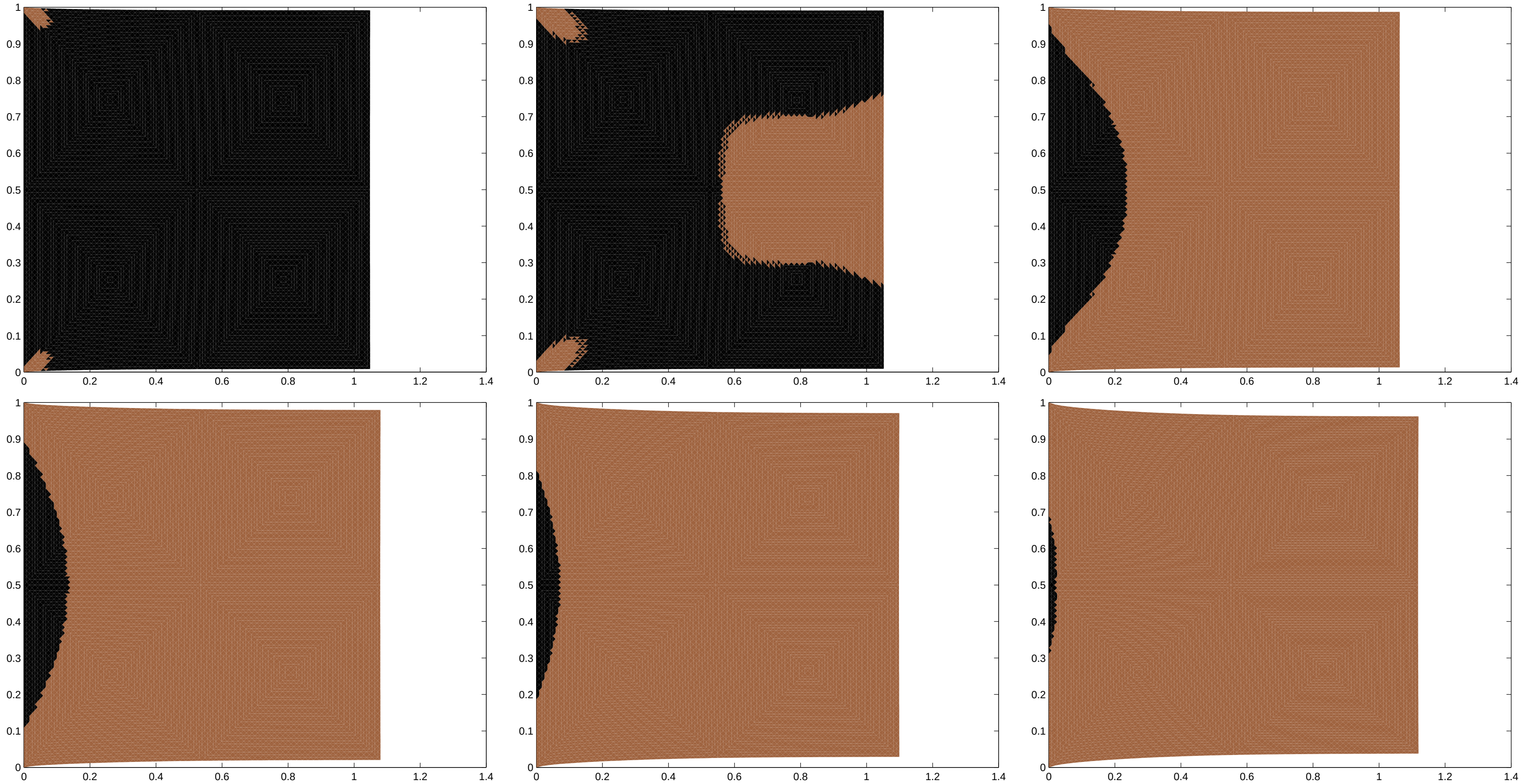
where $S := B^T (\mathbb{C} - \mathbb{C} P (P^T (\mathbb{C} + \mathbb{H}) P)^{-1} P^T \mathbb{C}) B$ represents the Schur-complement.

Algorithm: Regularization Approach

Algorithm 1. *(One time step iteration) Given initial u.*

1. *Calculate local A, a from (1) and local $\tilde{\mathrm{p}}$ using Newton method (2).*
2. *Substitute $\tilde{\mathrm{p}}$ to $\mathbb{H}$ in local Schur-complement (4) and assemble the global Schur-complement.*
3. *Solve new u from the global stiffness matrix using CG-multigrid preconditioned method.*
4. *Repeat steps (1)-(3) until the convergence is reached.*
5. *Upgrade $p = \tilde{\mathrm{p}} + p_0$ and output u and p.*

Elastoplastic interface time development



Black - elastic, brown - plastic

Exponential convergence

Rank, Düster and Nübel $\Rightarrow$ the exact identification of the elastoplastic interface important for the convergence rate improvement!!!!

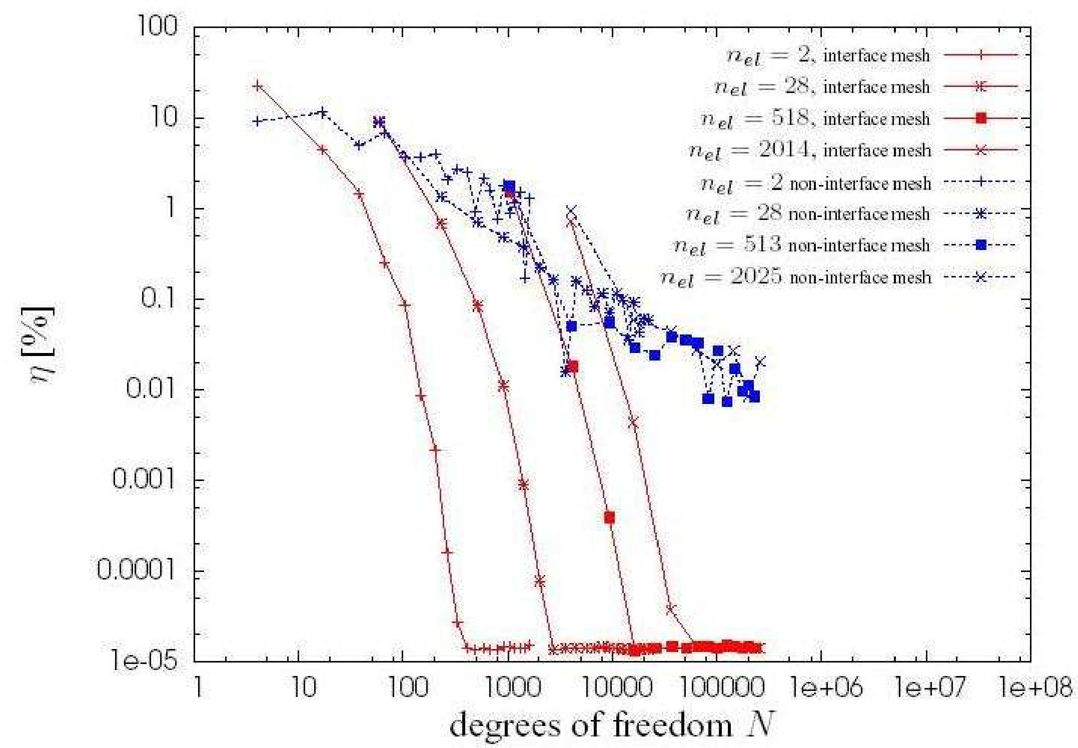
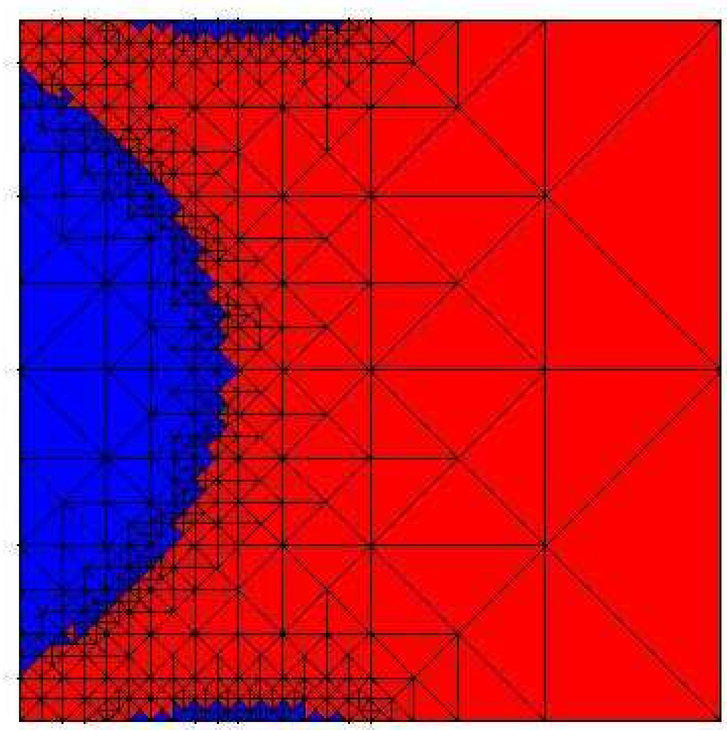
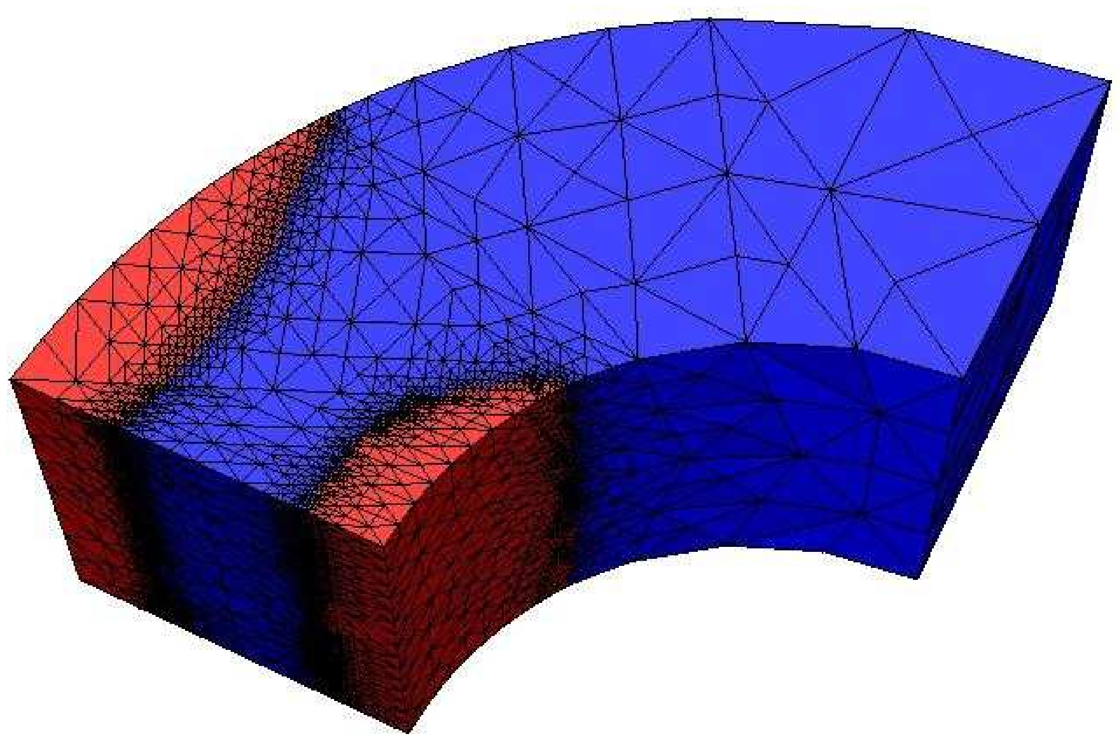


Figure 6: Relative error $\eta = \sqrt{\frac{|W_{ref} - W_{FE}|}{W_{ref}}}\, 100\, [\%]$ versus the number of degrees of freedom N

Elasto-plastic interface adaptivity for piecewise constant stresses: elastoplastic zones

Error estimator marks neighboring elements which have different phases and share a common edge (2D) or face (3D)





Blue - elastic, red - plastic