Some Aspect of computational mechanics: From elasticity to plasticity

Students project

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Outline

Time: 3 x 90 minutes
Schedule:

- Elasticity: Modeling (C. Carstensen et al.) + exercises
- Elasticity: Matlab software - C. Carstensen et al.
- Elastoplasticity: Modeling + exercises
- Elastoplasticity: Matlab software - J. Valdman
- Elastoplasticity: Netgen/NGSOLVE package demonstration

Elasticity: Modeling

Paper: *Matlab Implementation of the Finite Element Method in Elasticity* - J. Alberty, Kiel, C. Carstensen, Vienna, S.A. Funken, Kiel and R. Klose, Kiel

$$
\begin{aligned}
(\lambda + \mu)(\nabla \operatorname{div} u)^T + \mu \triangle u &= -f \quad \Omega, && (1) \\
(\lambda \operatorname{tr}(\varepsilon(u)) I + 2\mu\varepsilon(u)) \cdot n &= g \quad \text{on } \Gamma_N, && (2) \\
M \cdot u &= w \quad \text{on } \Gamma_D && (3)
\end{aligned}
$$

Exercise: $\sigma := 2\mu\varepsilon + \lambda(\operatorname{tr}\varepsilon)I, \varepsilon(u) := (\nabla u + (\nabla u)^T)/2 \Rightarrow \operatorname{div}\sigma =?$

Weak formulation

Find $u \in H^1(\Omega)$

$$\int_\Omega \varepsilon(u) : \mathbb{C}\varepsilon(v)\,\mathsf{d}x = \int_\Omega f \cdot v\,\mathsf{d}x + \int_{\Gamma_N} g \cdot v\,\mathsf{d}x \tag{4}$$

$\forall v \in H^1_D(\Omega) := \{v \in H^1(\Omega) : Mv = 0 \text{ on } \Gamma_D\}$

Exercise: Lax-Milgram Lemma $a(u,v) = f(v) \Rightarrow$ existence?

$$a(u,v) \geq c_e||u||||v||? \tag{5}$$

$$a(u,v) \leq c_b||u||||v||? \tag{6}$$

Finite Element Discretization

$$A_{kl} := \int_\Omega \varepsilon(\eta_k) : \mathbb{C}\varepsilon(\eta_l)\,\mathsf{d}x, \quad b_k = \int_\Omega f \cdot \eta_k \,\mathsf{d}x + \int_{\Gamma_N} g \cdot \eta_k \,\mathsf{d}x$$

Exercise: What are the properties of A matrix?

Numerical example: Elasticity in Matlab

Matlab software: Carstensen et al.

- Understanding the software structure:
  - assembly of the stiffness matrix
  - incorporating of BC
  - linear solver
  - postprocessing

- Students contribution
  - Creating the simple 2D geometry - triangles, rectangles
  - Various BC conditions representing various loads

Elastoplasticity: Modeling

(from J. Kienesberger)

Find $u \in W^{1,2}(0,T;\ H_0^1(\Omega)^n)$, $p \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{n\times n}))$, $\sigma \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{n\times n}))$, $\alpha \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{m}))$ such that

$$\begin{aligned}
-\operatorname{div}\sigma &= b \\
\sigma &= \sigma^T \\
\varepsilon(u) &= \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) \\
\varepsilon(u) &= \mathbb{C}^{-1}\sigma + p \\
\varphi(\sigma,\alpha) &< \infty \\
\dot{p} : (\tau - \sigma) - \dot{\alpha} : (\beta - \alpha) &\leq \varphi(\tau,\beta) - \varphi(\sigma,\alpha)
\end{aligned}$$

are satisfied in the variational sense with $(u,p,\sigma,\alpha)(0) = 0$ for all (τ,β). b and $\mathbb{C}^{-1}$ are given, $b(0) = 0$.

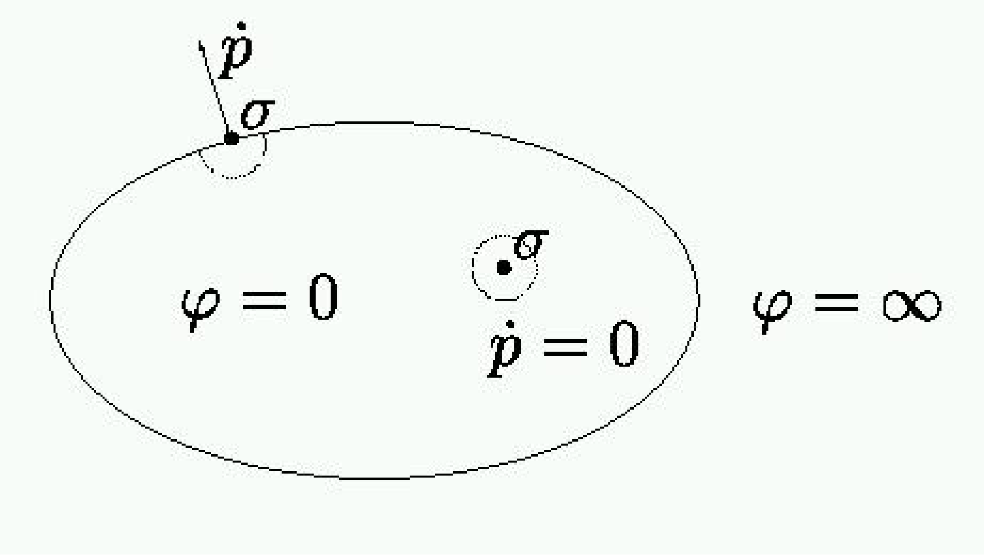
Exercise: What happens if $p = 0$?

Normality law

(from J. Kienesberger)

Formulas without α (perfect plasticity)

$$\begin{aligned} \varphi(\sigma) &< \infty \\ \dot{p} : (\tau - \sigma) &\leq \varphi(\tau) - \varphi(\sigma) \end{aligned}$$



Some convex analysis

Definition 1 (indicator function, conjugate function). *Let $Y \subset X$ be a convex set, $x \in Y$. Then For any set $S \subset X$, the* indicator function *I_S of S is defined by*

$$I_S(x) = \begin{cases} 0 & \text{if } x \in S, \\ +\infty & \text{if } x \notin S. \end{cases} \tag{7}$$

For a function $f : X \to [-\infty, \infty]$ we define the conjugate function *$f^* : X^* \to [-\infty, \infty]$ by*

$$f^*(x^*) = \sup_{x \in X} (\langle x^*, x \rangle - f(x)). \tag{8}$$

Exercise: von Mises yield condition

$$S = \{\sigma \in \mathbb{R}^{d \times d}_{sym} : ||\operatorname{dev}\sigma||_F \leq \sigma_y\}, \tag{9}$$

Calculate I_S and I_S^*?

Some convex analysis

Definition 2 (subdifferential). *Let f be a convex function on X. For any $x \in X$ the* subdifferential *$\partial f(x)$ of x is the possibly empty subset of X^* defined by*

$$\partial f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq f(y) - f(x) \quad \forall y \in X\}. \tag{10}$$

Exercise: What is $\partial|x|$?
Exercise: Show that $\frac{x}{||x||} \in \partial||\cdot||(x)$

Numerical example: Elastoplasticity in Matlab

Matlab software: J. Valdman - software to PhD. thesis: Mathematical and Numerical Analysis of Elastoplastic Material with Multi-Surface Stess-Strain Relation

- Understanding of additional software features (compare to elasticity)
  - solving of the nonlinear system
  - elastoplastic zones
- Students contribution
  - Testing of prepared models: Cook's membrane, plate with a hole

Netgen/NGSOLVE

Explanation of some new features

- 3D geometry
- Multigrid solver
- Elastoplasticity