Plastic Strain Computation

Project F1306

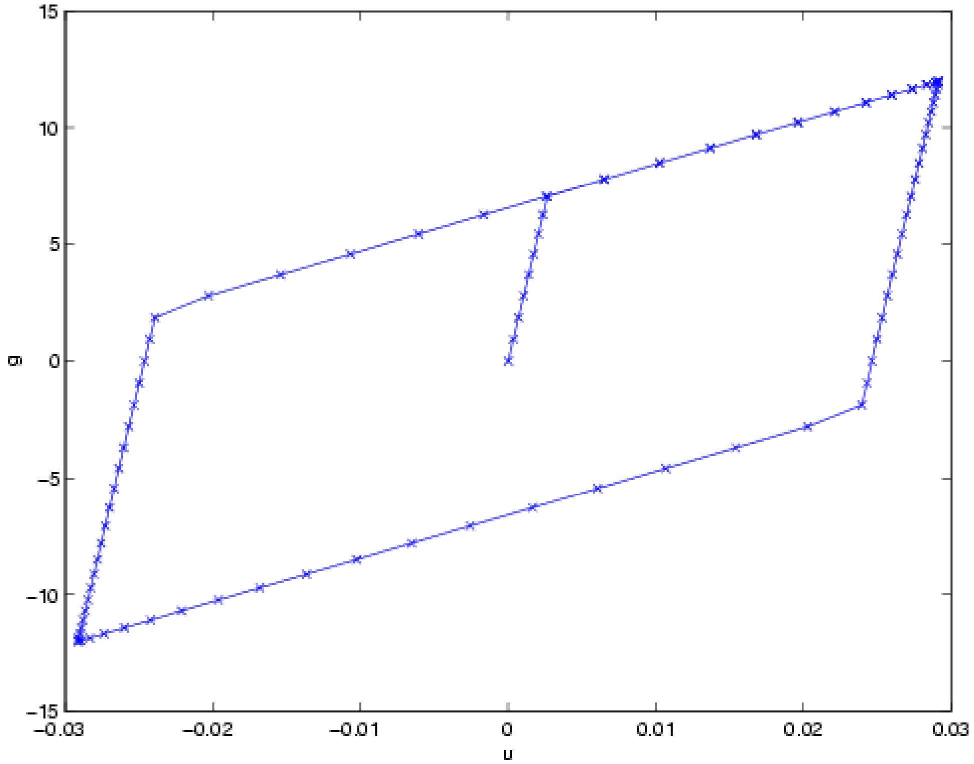
Jan Valdman
Jan.Valdman@sfb013.uni-linz.ac.at

Johanna Kienesberger

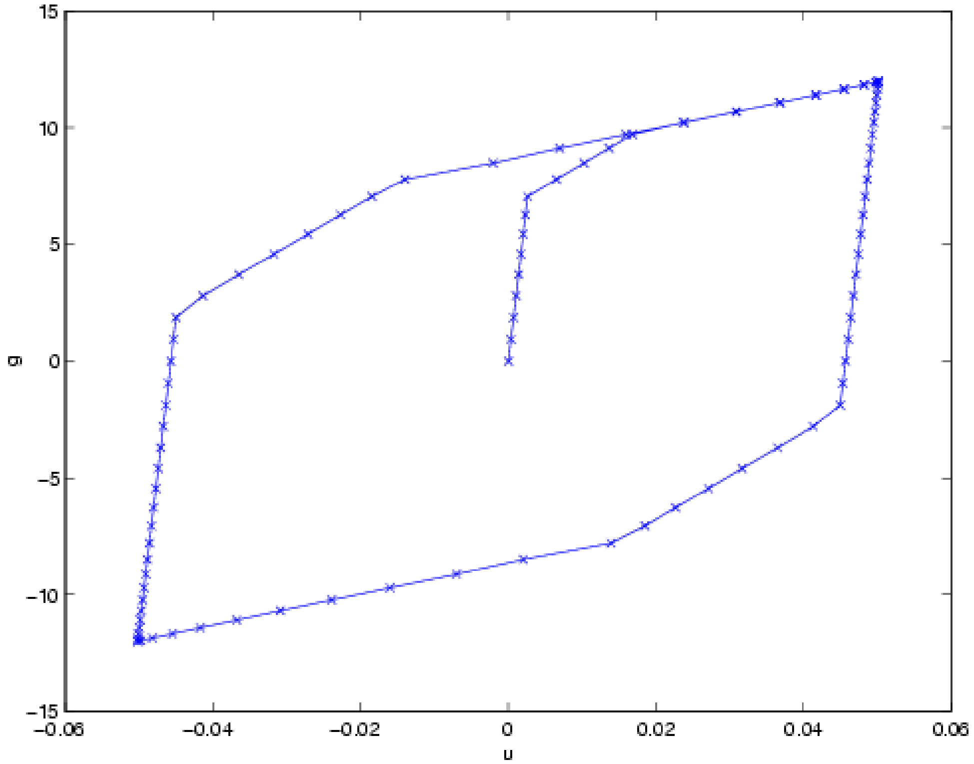
SFB F013
Johannes Kepler University
Linz, Austria

Why Multi-yield (Two-yield) model?

- More realistic hysteresis stress-strain relation in materials!



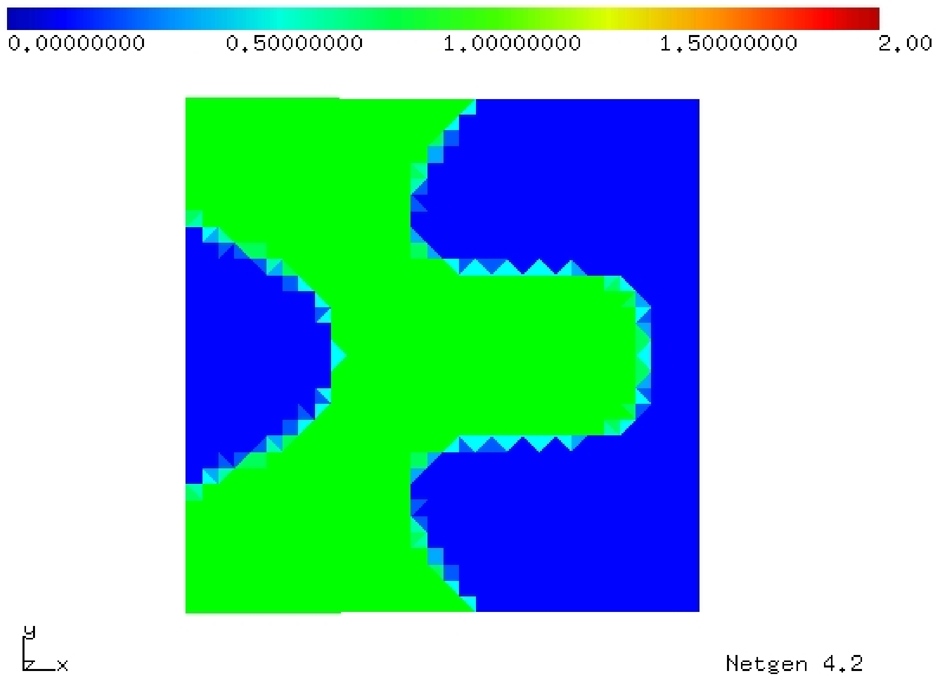
Kinematic hardening model.



Two-yield hardening model.

NETGEN/NGSOLVE calculations

Elastoplastic domains (blue - elastic, green - first plastic, red - second plastic)



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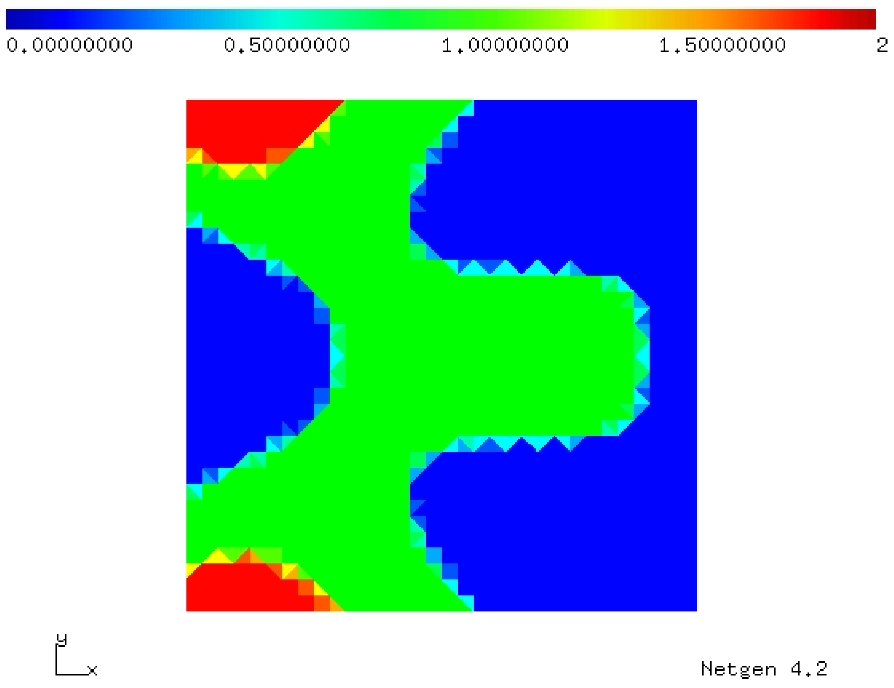

Two-yield hardening model.

Direct minimization problem in $\tilde{p}$

Kinematic hardening model:

$$f(Q) = \frac{1}{2}(\mathbb{C} + \mathbb{H})Q : Q - Q : A + \sigma^y ||Q|| \rightarrow \text{min}$$

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$$f\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbb{C} + \mathbb{H}_1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C} + \mathbb{H}_2 \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} : \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} - \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} : \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \sigma_1^y ||Q_1|| + \sigma_2^y ||Q_2|| \rightarrow \text{min}$$

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minimizer $(\tilde{p}_1, \tilde{p}_2) =?$

Direct minimization problem: Two-yield model - analytical approach

Lemma: Let $f(P) = \min_{Q} f(Q), P = (P_1, P_2),$ If $P_1 \neq 0, P_2 \neq 0 \Rightarrow ||P_2||$ is a root of a 8-th degree polynomial.

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Proof: f has a subdifferential, i.e., $\partial f(P) = (\hat{\mathbb{C}} + \hat{\mathbb{H}})P - A + \partial || \cdot ||_{\sigma^y}(P)$

Mininum condition on P: $0 \in \partial f(P) \Leftrightarrow A - (\hat{\mathbb{C}} + \hat{\mathbb{H}})P \in \partial || \cdot ||_{\sigma^y}(P)$

In case $P_1 \neq 0, P_2 \neq 0$ is $\partial || \cdot ||_{\sigma^y}(P) = \{\sigma_1^y \frac{P_1}{||P_1||}, \sigma_2^y \frac{P_2}{||P_2||}\}$

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Nonlinear system in $P_1, P_2 \in \operatorname{dev} \mathbb{R}^{d\times d}_{sym}$ with $\mu, h_1, h_2, \sigma_1^y, \sigma_2^y > 0, \operatorname{dev} A_1, \operatorname{dev} A_2 \in \operatorname{dev} \mathbb{R}^{d\times d}_{sym}$

$$\begin{pmatrix} \operatorname{dev} A_1 \\ \operatorname{dev} A_2 \end{pmatrix} - \begin{pmatrix} (2\mu + h_1)\mathbb{I} & 2\mu\mathbb{I} \\ 2\mu\mathbb{I} & (2\mu + h_2)\mathbb{I} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} \sigma_1^y \frac{P_1}{||P_1||} \\ \sigma_2^y \frac{P_2}{||P_2||} \end{pmatrix}$$

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Subst. $\xi_1 = ||P_1||, \xi_2 = ||P_2||$ with $A, B, C, D, E, F, G, H, I, J \in \mathbb{R}$

$$A + B\xi_1 + C\xi_1^2 - (G + H\xi_1 + I\xi_2 + J\xi_1\xi_2)^2 = 0$$
$$D + E\xi_2 + F\xi_2^2 - (G + H\xi_1 + I\xi_2 + J\xi_1\xi_2)^2 = 0$$

MAPLE 5 $\Rightarrow$ 8-th degree polynomial in ξ_2 ($\Rightarrow$ no analytical formula!):

$$
\begin{aligned}
&\left(J^4F^2\right)\xi_2^8+\left(2\%4J^2F\right)\xi_2^7+\left(2\%3J^2F+\%4^2\right)\xi_2^6+\left(2\%2J^2F+2\%3\%4\right)\xi_2^5\\
&+\left(2\%1J^2F+2\%2\%4+\%3^2-F(BJ+2IC)^2\right)\xi_2^4\\
&+\left(-E(BJ+2IC)^2-2F(2CG+BH)(BJ+2IC)+2\%1\%4+2\%2\%3\right)\xi_2^3\\
&+\left(-\mathrm{D}(BJ+2IC)^2-2E(2CG+BH)(BJ+2IC)-F(2CG+BH)^2+2\%1\%3+\%2^2\right)\xi_2^2\\
&+\left(-2\mathrm{D}(2CG+BH)(BJ+2IC)-E(2CG+BH)^2+2\%1\%2\right)\xi_2\\
&+\left(\%1^2-\mathrm{D}(2CG+BH)^2\right)=0,
\end{aligned}
$$

where $\%1 := H^2\,\mathrm{D} - C\,G^2 - A\,H^2 - B\,G\,H - C\,\mathrm{D},$

$\%2 := -B\,G\,J - 2\,H\,J\,A - C\,E - 2\,I\,C\,G + H^2\,E - I\,B\,H + 2\,H\,J\,\mathrm{D},$

$\%3 := -C\,F - J^2\,A + 2\,H\,J\,E - I\,B\,J + C + J^2\,\mathrm{D} + H^2\,F,$

$\%4 := 2\,H\,J\,F + J^2\,E.$

Analytical Approach: example

Given $\mu = 1, \sigma_1^y = 1, \sigma_2^y = 2, h_1 = 1, h_2 = 1$ and $A_1 = A_2 = \begin{pmatrix} 20 & 0 \\ 0 & 0 \end{pmatrix}$.

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The direct calculation shows

$$l_1 = \begin{pmatrix} 10 + 10\xi_1 & 0 \\ 0 & -10 - 10\xi_1 \end{pmatrix},$$

$$l_2 = \begin{pmatrix} 20 - 10\xi_2 & 0 \\ 0 & -20 - 10\xi_2 \end{pmatrix},$$

$$r = 5\,\xi_1\,\xi_2 + 6\,\xi_1 + 3\,\xi_2 + 2.$$

The nonlinear system of equation for $\xi_1, \xi_2 > 0$:

$$200 + 400\,\xi_1 + 200\,\xi_1^2 - (2 + 3\,\xi_2 + 6\,\xi_1 + 5\,\xi_1\,\xi_2)^2 = 0,$$

$$800 + 800\,\xi_2 + 200\,\xi_2^2 - (2 + 3\,\xi_2 + 6\,\xi_1 + 5\,\xi_1\,\xi_2)^2 = 0.$$

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ξ_1 solved from the second equation

$$\xi_1 = \frac{-3\xi_2 - 2 \pm 10\sqrt{10}(2 + \xi_2)}{5\xi_2 + 6}$$

Subst. (− term only!) in the first eq. gives

$$\frac{(5\xi_2 + 8 - 10\sqrt{2})(5\xi_2 + 4 - 10\sqrt{2})(\xi_2 + 2)}{(6 + 5\,\xi_2)} = 0.$$

Roots

$$\xi_2 = \{-\frac{4}{5} + 2\sqrt{2}, -\frac{8}{5} - 2\sqrt{2}, -2, -2\}$$

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Subtitution of positive solution $\xi_2 = -\frac{4}{5} + 2\sqrt{2} \approx 2.028427124$ into the second eq. gives

$$(995\xi_1 + 801 - 50\sqrt{2})(5\xi_1 - 1 - 10\sqrt{2}) = 0$$

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Roots

$$\xi_1 = \{-\frac{1}{5}\frac{-1 + 40\sqrt{2}}{1 + 10\sqrt{2}}, \frac{1}{5}\frac{201 + 2\sqrt{2}}{1 + 10\sqrt{2}}\},$$

Positive solution

$$\xi_1 = -\frac{1}{5}\frac{-1 + 40\sqrt{2}}{1 + 10\sqrt{2}} \approx 3.028427125$$

Symbolic techniques

- Resolvent in Maple: original 8-th degree polynomial contains a second degree polynomial factor with complex solutions only $\Rightarrow$ 6-th degree polynomial in ξ_2 (longer expression)

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New perspectives/cooperations:

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New perspectives/cooperations:

- Homotopy method for numerical aproximation of a polynomials (with F1303 - J. Schicho, T. Beck)
- Lagrange inversion formula leading to Melin's serie in polynomial coefficients (with F1305 - P. Paule)

Direct minimization problem: Two-yield model - iterative approach

Algorithm (*): Given $\mu, h_1, h_2, \sigma_1^y, \sigma_2^y, \operatorname{dev} A_1, \operatorname{dev} A_2$ and $tolerance \geq 0$.

(a) Choose $(P_1^0, P_2^0) \in \operatorname{dev} \mathbb{R}^{d\times d}_{sym} \times \operatorname{dev} \mathbb{R}^{d\times \longrightarrow \min! d}_{sym}$, set $i := 0$.

(b) Find $P_2^{i+1} \in \operatorname{dev} \mathbb{R}^{d\times d}_{sym}$ s. t.

$$f(P_1^i, P_2^{i+1}) = \min_{Q_2 \in \text{ dev } \mathbb{R}^{d\times d}_{sym}} f(P_1^i, Q_2).$$

(c) Find $P_1^{i+1} \in \operatorname{dev} \mathbb{R}^{d\times d}_{sym}$ s. t.

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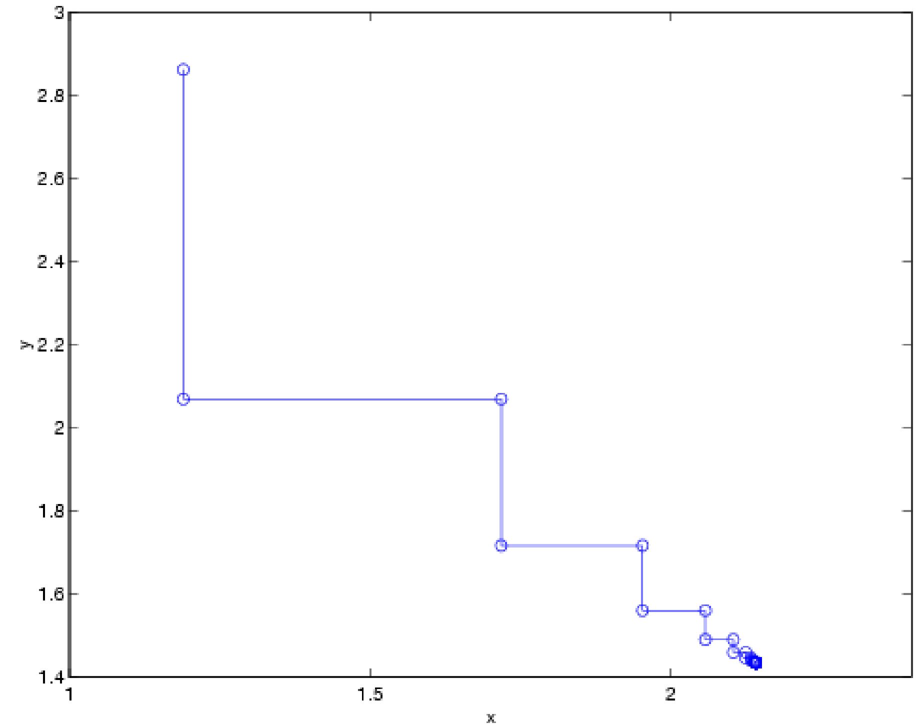
(d) If $\frac{||P_1^{i+1} - P_1^i|| + ||P_2^{i+1} - P_2^i||}{||P_1^{i+1}|| + ||P_1^i|| + ||P_2^{i+1}|| + ||P_2^i||} > tolerance$ set $i := i + 1$ and goto (b), otherwise output (P_1^{i+1}, P_2^{i+1}).



The approximations $P_1^i = (x^i, 0; 0, -x^i)$, $P_2^i = (y^i, 0; 0, -y^i)$, $i = 0, 1, \ldots$ of Algorithm (*) in the $x - y$ coordinate system.

- global convergence with the rate $1/2$:

$$||P_1^i - P_1||^2 + ||P_2^i - P_2||^2 \leq C_0 \cdot q^i$$