A Level Set Method for Inverse Problems

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Abstract

This paper is devoted to the solution of shape reconstruction problems by a level set method. The basic motivation for the setup of this level set algorithm is the well-studied method of asymptotic regularization, which has been developed for ill-posed problems in Hilbert spaces. Using analogies to this method, the convergence analysis of the proposed level set method is established, it is shown that the evolving level set converges to a solution in the symmetric difference metric as the artificial time evolves to infinity. Furthermore, the regularizing properties of the level set method are shown, if the discrepancy principle is used as a stopping rule.

The numerical implementation of the level set method is discussed and applied to some examples in order to compare the numerical results with theoretical statements. The numerical results demonstrate the power of the level-set, in particular for examples where the number of connected components the solution consists of is not known a-priorly.

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1 Introduction

Level set methods have become an important tool for the computation of evolving boundaries and interfaces since they have were introduced by Osher and Sethian (cf. [24]). Their basic motivation, namely the Eulerian view on the motion of curves, enabled the construction of efficient numerical methods with less topological limitations compared to classical methods for curve evolution. We refer to the monograph by Sethian [28] and the survey paper by Osher and Fedkiw [25] for an overview of this topic.

A *shape reconstruction problem* is an inverse problem, whose solution is a bounded subset of $\mathbf{R}^N$. In many applications this domain and its complement represent the sets where some parameter takes different (constant) values, e.g., due to different material properties. The shape to be reconstructed can be an obstacle inside the domain (such as in inverse scattering, cf. e.g. [7]) or a region separated by an interface (such as in crack detection, cf. e.g. [1, 4]), which motivates the frequently used notions *inverse obstacle problems* and *inverse interface problems*. Since our approach does not rely on the separating curve between the domain and

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its complement set (which is closed in the first and open in the second case), there is no need to distinguish between these types of problems in our case and therefore we will use the general notion of *shape reconstruction problems*.

Recently, the use of level set methods for inverse obstacle problems of the form

$$A(u) = f, \tag{1.1}$$

with linear or nonlinear operator A mapping $L^2(\Omega)$ to some Hilbert space and piecewise constant function u, has been investigated by several authors (cf. [9, 10, 16, 22, 23, 26]). So far, there are no rigorous results for the methods proposed, but numerical results indicate their success. Most of these methods are based on an evolution in descent direction (in artificial time). The aim of this paper is to construct a similar level set method for shape reconstruction problems based on a mathematically rigorous theory. We shall not only show that the developed method converges in absence of noise, but also that it is a regularization method for ill-posed problems if the stopping time is chosen appropriately in dependence of the noise level. For the sake of simplicity we restrict our attention to problems, where u is an indicator function, noticing that all problems where u takes two different (known) values can be transformed to this case.

A level-set method for problem (1.1) consists in an evolution for an artificial time t such that

$$u(.,t) = \chi_{D(t)}, \qquad D(t) = \{\phi(.,t) \geq 0\}, \tag{1.2}$$

where ϕ is an appropriate continuous level set function. The evolution of the level set function is defined by a Hamilton-Jacobi equation of the form

$$\frac{\partial \phi}{\partial t} + V.\nabla\phi = 0 \tag{1.3}$$

with initial value given by

$$\phi(.,0) = \phi_0 \tag{1.4}$$

for a suitable function ϕ_0. The function $V : \mathbf{R}^N \times \mathbf{R}^+ \to \mathbf{R}^N$ is the velocity with which the level sets are moving, the evolution of a point ξ on the level set $\{\phi = 0\}$ is given by

$$\dot{\xi}(t) = V(\xi(t), t). \tag{1.5}$$

Consequently, the crucial part of any level-set method is an appropriate choice of the velocity V. In many applications of interface motion, the velocity V is chosen in the normal direction of the curve, i.e,

$$V = vn = v\frac{\nabla\phi}{|\nabla\phi|} \tag{1.6}$$

with $v : \mathbf{R}^N \times \mathbf{R}^+ \to \mathbf{R}$. In this case the evolution of the level-set can be described by the nonlinear equation

$$\frac{\partial \phi}{\partial t} + v|\nabla\phi| = 0, \tag{1.7}$$

we refer to the monograph by Sethian [28] for further details. In our approach the velocity will be chosen in dependence of the residual $Au(t) - f$ in order to obtain a descent method. This idea was also used by Santosa [26] to construct a level-set method for inverse obstacle problems, but so far there exists no convergence result on the method he proposed.

The starting point of our analysis is to exploit similarities to the well-studied method of asymptotic regularization, which would consist of solving the differential equation

$$\frac{d}{dt}u(t) = -A'(u(t))^*(Au(t) - f) \tag{1.8}$$

for $t \in \mathbf{R}^+$. It can be shown under certain conditions on the nonlinearity of A that $u(t)$ converges to a solution of equation (1.1) as $t \to \infty$ (cf. [31]). Obviously, this method cannot be applied directly to construct an evolution of piecewise constant functions, since the time derivative of u is not defined in $L^2(\Omega)$. However, we will use a basic property of asymptotic regularization as a guideline for the construction of a level-set method, namely the identity

$$\frac{d}{dt}\|u(t) - \overline{u}\|^2 = -2\|Au - f\|^2, \tag{1.9}$$

where $\overline{u}$ is a solution of (1.1). This estimate on the evolution of the distance between $u(t)$ and $\overline{u}$ is the most fundamental one for the convergence analysis of (nonlinear) asymptotic regularization (cf. [31]); we shall see below that a level-set method satisfying (1.9) can be analyzed in a similar way to asymptotic regularization.

In addition, we shall translate the regularizing properties of (1.8) to our level-set approach in the case of noisy data, i.e., if we only know a perturbation f^δ of f with an estimate of the noise given by

$$\|f - f^\delta\| \leq \delta. \tag{1.10}$$

The regularizing effect comes from an early termination of the evolution at some final time T chosen in dependence of the noise level δ and the noisy data f^δ. The standard method for choosing the stopping time T^* in asymptotic regularization is the discrepancy principle, which takes T^* as the minimal time such that the residual is less than the noise level. We will see below that the discrepancy principle can be used for level set methods, too.

The paper is organized as follows: in Section 2 we study the fundamental properties of the underlying level set method for given velocity. In 3 we present our approach in the case of a linear operator equation and investigate some of its basic properties. In Section 4, we carry out the convergence analysis in the case of exact data as well as in presence of noise. The numerical implementation of the constructed level-set method is discussed in Section 5. Finally, we show how the method can be applied to some well-studied linear and nonlinear inverse problems in Section 6 and present results of related numerical test examples in Section 7.

2 The Linear Level Set Equation

In the following we want to discuss some basic properties of the level set equation (1.3) for given velocity V. The classical theory of Hamilton-Jacobi equations (cf. [8]) shows that a unique solution of the level-set equation (1.3), which is uniformly continuous with respect to x, exists, if the initial value is uniformly continuous on R^N and if

$$V \in C(\mathbf{R}_+; C^{0,1}(\mathbf{R}^N)), \tag{2.1}$$

i.e., if V is continuous on $\mathbf{R}^N \times \mathbf{R}_+$ and Lipschitz-continuous with respect to x.

We will see below that in our particular case we also have to consider velocities V that are not Lipschitz-continuous, but have a weak derivative div $V \in L^1(0,T;L^\infty(\mathbf{R}^N))$. Therefore we will first consider this case and prove the existence of a weak solution in the finite time interval $(0,T)$ under the weaker condition

$$V \in L^1(0,T;L^2(\Omega))^2, \qquad \text{div } V \in L^1(0,T;L^\infty(\mathbf{R}^N)) \tag{2.2}$$

by the vanishing viscosity method (cf. [8]).

Proposition 2.1. *Let the velocity V be such that* (2.2) *is fulfilled, then the problem*

$$\frac{\partial \phi}{\partial t} + V.\nabla \phi \quad = \quad 0 \tag{2.3}$$

$$\phi|_{t=0} = \phi_0, \tag{2.4}$$

with $\phi_0 \in L^2(\mathbf{R}^N)$, has a weak solution $\phi \in C(0,T;L^2(\Omega))$ satisfying

$$\|\phi\|_{L^\infty(0,T;L^2(\mathbf{R}^N))} \leq \|\phi_0\|_{L^2(\mathbf{R}^N)} e^{\|\text{div } V\|_{L^1(0,T;L^\infty(\mathbf{R}^N))}}. \tag{2.5}$$

Proof. We first consider the viscous problem

$$\frac{\partial \phi^\epsilon}{\partial t} + V.\nabla \phi^\epsilon \quad = \quad \epsilon \Delta \phi^\epsilon \tag{2.6}$$

$$\phi^\epsilon|_{t=0} = \phi_0 \tag{2.7}$$

for $\epsilon > 0$, with initial value $\phi_0 \in L^2(\mathbf{R}^N)$. The existence of a solution of (2.6), (2.7) with $\phi^\epsilon(.,t) \to 0$ as $|x| \to \infty$ follows from the standard theory of linear parabolic equations (cf. e.g. [12, 29]). Furthermore, the standard estimate

$$\begin{aligned} &\|\phi^\epsilon(.,t)\|^2_{L^2(\mathbf{R}^N)} + 2\epsilon \int_0^t \|\nabla \phi^\epsilon(.,s)\|^2_{L^2(R^N)}\, ds \\ &\quad \leq \|\phi_0\|^2_{L^2(R^N)} + \int_0^t \|\text{div } V\|_{L^\infty(R^N)} \|\phi^\epsilon(.,s)\|^2_{L^2(R^N)}\, ds \end{aligned}$$

holds for almost all $t \in (0,T)$ and by applying Gronwall's inequality we may deduce

$$\|\phi^\epsilon\|_{L^\infty(0,T;L^2(\mathbf{R}^N))} \leq \|\phi_0\|_{L^2(\mathbf{R}^N)} e^{\|div\ V\|_{L^1(0,T;L^\infty(\mathbf{R}^N))}}. \tag{2.8}$$

The estimate (2.8) is uniform in ϵ and therefore we can extract a weakly convergent subsequence of ϕ^ϵ in $L^2(\mathbf{R}^N \times (0,T))$. Now let $\rho \in C^\infty(\mathbf{R}^N \times (0,T))$ such that $\rho(.,T) = 0$ and

$$\frac{\partial \rho}{\partial t}, \text{div } (V\rho), \nabla \rho \in L^2(\mathbf{R}^N \times (0,T)),$$

then we have

$$\int_0^T \int_{\mathbf{R}^N} \phi^\epsilon \left(\frac{\partial \rho}{\partial t} + \text{div } (V\rho) \right) dx\ dt = \epsilon \int_0^T \int_{\mathbf{R}^N} \nabla \phi^\epsilon \nabla \rho\ dx\ dt$$

for all $\epsilon > 0$ and the term on the right-hand side converges to zero as $\epsilon \to 0$, since

$$|\epsilon \int_0^T \int_{\mathbf{R}^N} \nabla \phi^\epsilon \nabla \rho\ dx\ dt| \leq \epsilon \|\phi^\epsilon\|_{L^2(\mathbf{R}^N \times (0,T))} \|\rho\|_{L^2(\mathbf{R}^N \times (0,T))}$$

and $\sqrt{\epsilon}\|\phi^\epsilon\|_{L^2(\mathbf{R}^N \times (0,T))}$ is uniformly bounded. Hence, each weak limit of ϕ^ϵ is a weak solution ϕ of (2.3), (2.4) and by standard techniques one can show that ϕ satisfies (2.5) and is an element of $C(0,T;L^2(\Omega))$. $\square$

For a Lipschitz-continuous velocity V we can easily prove the following statement on the solution:

Lemma 2.2. *Let, in addition to the assumptions of Proposition 2.1, V satisfy* (2.1). *Furthermore, let $\phi_0 \in L^\infty(\Omega) \cap L^2(\Omega)$. Then the solution ϕ is unique and there exists a Lipschitz-continuous map $\xi : \mathbf{R}^N \to \mathbf{R}^N \times (0,T)$ such that*

$$\phi(\xi(x,t),t) = \phi_0(x), \qquad \forall\ (x,t) \in \mathbf{R}^N \times (0,T) \tag{2.9}$$

and therefore $\phi \in L^\infty(\mathbf{R}^N \times (0,T))$. If in addition ϕ_0 is Lipschitz-continuous, then ϕ is Lipschitz-continuous on $\mathbf{R}^N \times (0,T)$.

Proof. The existence and uniqueness of a solution for Lipschitz-continuous velocity V follows from a standard result on Hamilton-Jacobi equations (cf. [8, Theorem 1]). We define the map ξ via

$$\frac{\partial \xi}{\partial t}(x,t) = V(\xi(x,t),t), \quad \forall\ t \in (0,T), \qquad \xi(x,0) = x. \tag{2.10}$$

The Picard-Lindelöf Theorem (cf. [2]) implies that ξ is well-defined and from a standard result on the continuous dependence of solutions of ordinary differential equations (cf. [14, Theorem 13.1]) we may conclude that ξ is Lipschitz-continuous. Furthermore, we have

$$\begin{aligned} \frac{d}{dt}\phi(\xi(x,t),t) &= \frac{\partial \phi}{\partial t}(\xi(x,t),t) + \frac{\partial \xi}{\partial t}(x,t).\nabla\phi(\xi(x,t),t) \\ &= \frac{\partial \phi}{\partial t}(\xi(x,t),t) + V(\xi(x,t),t).\nabla\phi(\xi(x,t),t) = 0 \end{aligned}$$

due to (2.3), which implies (2.9) and consequently $\phi \in L^\infty(\mathbf{R}^N \times (0,T))$, respectively the Lipschitz-continuity of ϕ if ϕ_0 is Lipschitz-continuous. □

We note that due to Lemma 2.2 the Lipschitz-continuity of V implies that if ϕ_0 is an indicator function of a measurable set, then (2.3) has still a weak solution, and along the lines of the proof of Lemma 2.2 we observe that $\phi(.,t)$ is an indicator function for each $t \in [0,T]$.

Theorem 2.3. *Let the velocity V be such that* (2.2) *is fulfilled and let $V_n \in C(0,T;C^{0,1}(\mathbf{R}^N))$ be a sequence of velocities such that*

$$div\ V_n \to\ div\ V \qquad in\ L^1(0,T;L^\infty(\Omega)).$$

Then the corresponding weak solutions ϕ_n of (2.3),(2.4) *with velocity V_n and $\phi_0 \in L^\infty(\Omega)$ have a weakly convergent subsequence, whose limit ϕ is a weak solution of* (2.3), (2.4) *with velocity V. Moreover, if ϕ_0 is an indicator function of a measurable set, then $\phi(.,t)$ is the indicator function of a measurable set after modification on a subset of zero measure for almost all $t \in [0,T]$.*

Proof. Since ϕ_n is the unique solution of (2.3), (2.4), it is in particular a viscosity solution in the sense of Proposition 2.1 and therefore satisfies (2.5). In an analogous way to the proof of Proposition 2.1 we can now deduce the existence of a weakly convergent subsequence, whose limit is a weak solution of (2.3), (2.4) with velocity V.

If ϕ_0 is an indicator function, then we have seen that $\phi_n(.,t)$ is an indicator function of a measurable set for each $t \in [0,T]$ and consequently ϕ_n is the indicator function of a measurable subset of $\mathbf{R}^N \times [0,T]$. Without restriction of generality we assume that the sequence ϕ_n itself converges weakly to ϕ. Now suppose that there exist a set M of positive $(N+1)$-dimensional measure and an $\epsilon > 0$ such that

$$\phi(z) \notin (-\epsilon, \epsilon) \cup (1-\epsilon, 1+\epsilon) \qquad \text{for almost all } z \in M.$$

Moreover, suppose that there exists a subset $\tilde{M} \subset M$ of positive measure and a subsequence $(n_k) \subset \mathbf{N}$ such that

$$\int_{\tilde{M}} \phi_{n_k}(z)\ dz \to \mu(\tilde{M}),$$

then the weak convergence of ϕ_n implies that

$$\int_{\tilde{M}} \phi(z)\ dz = \int_{\mathbf{R}^N} \phi(z)\chi_{\tilde{M}}(z) = \lim \int_{\mathbf{R}^N} \phi_{n_k}(z)\chi_{\tilde{M}}(z)\ dz = \mu(\tilde{M}).$$

Consequently, there exists another subset $\tilde{\tilde{M}} \subset \tilde{M}$ of positive measure such that

$$\phi(z) \geq 1 + \epsilon \qquad \text{for almost all } z \in \tilde{\tilde{M}}$$

and hence

$$(1+\epsilon)\mu(\tilde{\tilde{M}}) \leq \int_{\mathbf{R}^N} \phi(z)\chi_{\tilde{\tilde{M}}}(z)\ dz = \lim \int_{\mathbf{R}^N} \phi_{n_k}(z)\chi_{\tilde{\tilde{M}}}(z)\ dz \leq \mu(\tilde{\tilde{M}}),$$

which is a contradiction.

Thus, we may conclude that $\int_M \phi_n(z)\ dz \to 0$ and therefore also $\int_M \phi(z)\ dz = 0$. Since in this case there exists a subset $\tilde{M} \subset M$ of positive measure such that

$$\phi(z) \leq -\epsilon \qquad \text{for almost all } z \in \tilde{M}$$

we obtain

$$-\epsilon\mu(\tilde{M}) \geq \int_{\mathbf{R}^N} \phi(z)\chi_{\tilde{M}}(z)\ dz = \lim \int_{\mathbf{R}^N} \phi_n(z)\chi_{\tilde{M}}(z)\ dz \geq 0,$$

which is again a contradiction. Hence, we have $\phi(z) \in \{0,1\}$ for almost all $z \in \mathbf{R}^N \times [0,T]$ and the assertion follows. $\square$

Finally, we investigate the connections of the linear level set equation with different initial values:

Theorem 2.4. *Let V_n and V be as in Theorem 2.3 and let ϕ_0 be the indicator function of a measurable set in $\mathbf{R}^N$. Suppose that $\tilde{\phi}_0 \in L^\infty(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$ is such that*

$$\{\phi_0 = 1\} \subset \{\tilde{\phi}_0 \geq 0\} \quad \text{and} \quad \{\phi_0 = 0\} \subset \{\tilde{\phi}_0 \leq 0\} \tag{2.11}$$

and denote by $\tilde{\phi}$ and ϕ the corresponding solutions of (2.3), (2.4) *with initial values $\tilde{\phi}_0$ and ϕ_0. Then after modification of $\tilde{\phi}$ on a set of zero measure in $\mathbf{R}^N \times [0,T]$ we obtain*

$$\{\phi = 1\} \subset \{\tilde{\phi} \geq 0\} \quad \text{and} \quad \{\phi = 0\} \subset \{\tilde{\phi} \leq 0\}. \tag{2.12}$$

Proof. We denote by $\tilde{\phi}_n$ and ϕ_n the solutions of (2.3), (2.4) with velocity V_n and initial values $\tilde{\phi}_0$ and ϕ_0, respectively. Without restriction of generality we assume that these sequences converge weakly to $\tilde{\phi}$ and ϕ. The proof of Lemma 2.2 shows that after modification on a set of zero measure, the inclusions

$$\{\phi_n = 1\} \subset \{\tilde{\phi}_n \geq 0\} \quad \text{ and } \quad \{\phi_n = 0\} \subset \{\tilde{\phi}_n \leq 0\}$$

hold and furthermore that

$$\|\tilde{\phi}_n\|_{L^\infty(\mathbf{R}^N \times [0,T])} \leq \|\tilde{\phi}_0\|_{L^\infty(\mathbf{R}^N)}, \qquad \forall\, n \in \mathbf{N}.$$

Let $M \subset \{\phi = 1\}$ be measurable, then we have

$$\begin{aligned} 0 \quad &\leq \quad \int_{M \cap \{\phi_n = 1\}} \tilde{\phi}_n(z)\; dz = \int_{\mathbf{R}^N} \tilde{\phi}_n(z)\chi_M(z)\; dz - \int_{M - \{\phi_n = 1\}} \tilde{\phi}_n(z)\; dz \\ &\to \quad \int_{\mathbf{R}^N} \tilde{\phi}(z)\chi_M(z)\; dz, \end{aligned}$$

noticing that

$$|\int_{M - \{\phi_n = 1\}} \tilde{\phi}_n(z)\; dz| \leq \|\tilde{\phi}_0\|_{L^\infty(\mathbf{R}^N)} \int_{\{\phi = 1\}} (\phi(z) - \phi_n(z))\; dz \to 0.$$

Hence, we have

$$\int_M \tilde{\phi}(z)\; dz \geq 0,$$

for all measurable subsets M of $\{\phi = 1\}$, which implies that after modification on a set of zero measure the inclusion

$$\{\phi = 1\} \subset \{\tilde{\phi} \geq 0\}$$

holds. The second inclusion in (2.12) can be shown in an analogous way. $\square$

Theorem 2.4 shows that the level sets of a solution of (2.3) with smooth initial values can be modified such that they cover the level sets of the solution of (2.3) with an indicator function as initial value. If in particular the level set function $\tilde{\phi}$ is such that $\{\tilde{\phi} = 0\}$ has zero measure (which is the usual case if $\tilde{\phi}_0$ is chosen appropriately), then we can modify it on a set of zero measure such that

$$\{\phi = 1\} = \{\tilde{\phi} > 0\} \quad \text{ and } \quad \{\phi = 0\} = \{\tilde{\phi} \leq 0\}, \tag{2.13}$$

which means that ϕ can be computed from the level set function $\tilde{\phi}$ by a simple application of the Heaviside function.

We note that so far we have only considered the level set equation for given velocity V. In the following section we shall investigate the appropriate choice of V in dependence of the level set and in the appendix we will show that the arising coupled problem has indeed a solution.

3 Choice of the Velocity

In the following we consider linear operator equations of the form

$$Au = f, \tag{3.1}$$

where $A : L^2(\Omega) \to Y$ is a bounded linear operator with unbounded generalized inverse. We seek the support of a solution u, which is piecewise constant, which we assume to be an indicator function. In addition, we will assume that there exists a function $\overline{u} = \chi_D$ that solves (3.1) for exact data.

For the reconstruction of obstacles in inverse problems, Santosa [26] proposed to use a level set method with a velocity chosen such that a flow in descent direction for the residual $\|Au - f\|^2$ is achieved. The choice of the velocity in this case was still corresponding to (1.6) and included some heuristic arguments and formal computations. In the following we will put the idea of a descent flow via the level set method into a strict functional-analytic framework.

The difference between the evolving indicator function and the solution at time t in the L^p-norm ($1 \leq p < \infty$) can be computed as

$$E(t) = \|u(.,t) - \overline{u}\|^p_{L^p(\Omega)} = \int_{D(t)\Delta D} 1 \; dx = d_{\mathcal{S}}(D(t), D), \tag{3.2}$$

where $d_{\mathcal{S}}$ denotes the distance of two sets in the symmetric difference metric, i.e,

$$d_{\mathcal{S}}(D(t), D) = |D(t) - D| + |D - D(t)|. \tag{3.3}$$

In order to compute the time derivative of the error functional $E(t)$ and the corresponding integrals we need the following classical result frequently used in mathematical mechanics and structural optimization:

Proposition 3.1. *[30, Proposition 2.45] Let $\psi \in C^1(\mathbf{R}^N)$ and $D(t) = F_t(D)$, where D is a given measurable set and F_t is the transformation of sets according to a velocity $V \in C(0,T;C^1(\mathbf{R}^N))$, then the functional*

$$t \mapsto \int_{D(t)} \psi(x) \; dx$$

is differentiable with respect to t and its derivative is given by

$$\frac{\partial}{\partial t}\left(\int_{D(t)} \psi(x) \; dx\right) = \int_{D(t)} div_x(\psi(x)V(x,t)) \; dx \tag{3.4}$$

Since we will need a slightly more general class of velocities we need the following extension of Proposition 3.1:

Proposition 3.2. *Let $V_n \in C(0,T;C^1(\mathbf{R}^N)) \cap C(0,T;L^2(\mathbf{R}^N))$ be a sequence of velocities such that $V_n \to V$ and div $V_n \to$ div V in $L^2(\mathbf{R}^N \times (0,T))$, for some $V \in L^\infty(0,T;L^2(\mathbf{R}^N))$ with div $V \in L^1(0,T;L^\infty(\mathbf{R}^N)) \cap L^\infty(0,T;L^2(\mathbf{R}^N))$. Then the functional*

$$\ell : t \mapsto \int_{\mathbf{R}^N} u(x,t) \; dx,$$

where the indicator function u is a weak solution of

$$\frac{\partial u}{\partial t} + V.\nabla u = 0, \qquad \text{in } \mathbf{R}^N \times (0,T) \tag{3.5}$$

with initial value $u(.,0) = \chi_D(.)$ for some measurable set D, is weakly differentiable with

$$\frac{d\ell}{dt}(t) = \int_{\mathbf{R}^N} u(x,t)\ \mathit{div}\, V(x,t)\ dx \tag{3.6}$$

for almost all $t \in (0,T)$.

Proof. First we define $D_n(t) := F_t^n(D)$, where F_t^n is the transformation of sets according to the velocity V_n, and $u_n(.,t) := \chi_{D_n(t)}(.)$. Then it follows with Proposition 3.1 that u_n is the weak solution of

$$\frac{\partial v}{\partial t} + V_n.\nabla v = 0, \qquad \text{in } \mathbf{R}^N \times (0,T)$$

with initial value $v(.,0) = \chi_D(.)$. In Section (2) we will show that $u_n \rightharpoonup u$ in $L^2(\mathbf{R}^N \times (0,T)$, where u is an indicator function and the unique weak solution of (3.5) with initial value $u(.,0) = \chi_D(.)$.

If we now denote by ℓ_n the functional

$$\ell_n : t \mapsto \int_{\mathbf{R}^N} u_n(x,t)\ dx,$$

then $\ell_n \in C^1([0,T])$ due to Proposition 3.1 and $\ell_n \rightharpoonup \ell$ in $L^2([0,T])$ and furthermore,

$$\frac{d\ell_n}{dt} \rightharpoonup \ell' \qquad \text{in } L^2([0,T])$$

with

$$\ell'(t) := \int_{\mathbf{R}^N} u(x,t)\ \text{div}\ V(x,t)\ dx.$$

Hence, ℓ_n is weakly convergent in $H^1([0,T])$ and therefore we have $\ell \in H^1([0,T])$ and $\frac{d\ell}{dt} = \ell'$. Since we have $u \in L^\infty(0,T;L^2(\mathbf{R}^N))$ (see again Section 2) and div $V \in L^\infty(0,T;L^2(\mathbf{R}^N))$, we obtain that $\frac{d\ell}{dt} \in L^\infty([0,T])$ and thus, the identity (3.6) holds for almost all $t \in [0,T]$. $\square$

As a direct consequence of Proposition 3.2 we obtain

$$\frac{d}{dt}\int_{D(t)-\Omega} 1\ dx = \int_{D(t)-\Omega} \text{div}\ V(x,t)\ dx = 0 \quad \text{for almost all } t \in [0,T], \tag{3.7}$$

if the support of div V is a subset of Ω. Hence, after modifying the indicator function u on a set of zero measure, we can construct sets $D(t) \subset \Omega$ for almost all $t \in [0,T]$. In the following we will always denote by u such an indicator function such that $D(t) \subset \Omega$ without further noting.

Using Proposition 3.2 we can compute the time derivatives

$$\begin{aligned}\frac{\partial}{\partial t}\left(\int_{D(t)-D} 1\ dx\right) &= \int_{D(t)-D} \text{div}\ V(x,t)dx \\ &= \int_{D(t)} (u(x,t)-\overline{u}(x))\ \text{div}\ V(x,t)dx \qquad (3.8)\\ \frac{\partial}{\partial t}\left(\int_{D-D(t)} 1\ dx\right) &= \int_{D-D(t)} \text{div}\ V(x,t)dx \\ &= \int_{\Omega-D(t)} (\overline{u}(x)-u(x,t))\ \text{div}\ V(x,t)dx \qquad (3.9)\end{aligned}$$

and hence, we have derived the following result:

Proposition 3.3. *Let the function g be defined by*

$$g(x,t) := (-1+2u(x,t))\ \ div\ V(x,t), \qquad \forall\ x\in\Omega, \tag{3.10}$$

with V as in Proposition 3.2, let the level set function ϕ evolve according to (1.3) *and let u be chosen according to* (1.2)*. Then the time derivative of the error functional E is given by*

$$\frac{dE}{dt}(t) = \int_\Omega (u(x,t)-\overline{u}(x))g(x,t)dx, \tag{3.11}$$

for all $t\in\mathbf{R}^+$.

Proof. From (3.2), (3.8) and (3.9) we may conclude

$$\begin{aligned}\frac{dE}{dt}(t) &= \frac{\partial}{\partial t}\left(\int_{D(t)-D} 1\ dx\right) + \frac{\partial}{\partial t}\left(\int_{D(t)-D} 1\ dx\right)\\ &= \int_{D(t)} (u(x,t)-\overline{u}(x))g(x,t)\ dx + \\ &\quad \int_{\Omega-D(t)} (u(x,t)-\overline{u}(x))g(x,t)\ dx\\ &= \int_\Omega (u(x,t)-\overline{u}(x))g(x,t)\ dx.\end{aligned}$$

□

Consequently, in order to obtain a flow in descent direction that satisfies the estimate (1.9) (possibly with different scaling), we have to choose the velocity such that

$$\begin{aligned}-\ \text{div}\ V(x,t) &= -g(x,t)\,(-1+2u(x,t))\\ &= A^*(Au-f)(x,t)\,(-1+2u(x,t))\,. \qquad (3.12)\end{aligned}$$

holds in $\Omega\times\mathbf{R}^+$. Since the support of u should remain a subset of $\overline{\Omega}$ during the evolution, it seems advisable to choose

$$-\ \text{div}\ V(x,t) = 0 \qquad \forall\ x\in\mathbf{R}^N-\Omega, \tag{3.13}$$

which implies that $D(t) \subset \Omega$ as we have seen above. If we denote by P_Ω the extension operator from $L^2(\Omega)$ to $L^2(\mathbf{R}^N)$ defined via

$$P_\Omega[v](x) = \begin{cases} v(x) & \text{if } x \in \Omega \\ 0 & \text{else} \end{cases}, \tag{3.14}$$

we can rewrite (3.12), (3.13) as

$$-\text{ div } V = P_\Omega\left[A^*(Au - f)\right](-1 + 2u)\,. \tag{3.15}$$

We note that a function V with the property (3.15) always exists, e.g., we can choose

$$\nabla\psi - V \;=\; 0 \tag{3.16}$$

in $\mathbf{R}^N$ with $\psi(x) \to 0$ as $|x| \to 0$, which means that ψ is just the unique solution of the Dirichlet problem for the Laplace equation with right-hand side as in (3.15) on $\mathbf{R}^N$, and V is its gradient. In the remainder of the paper we will always write $A^*(Au - f)$ instead of $P_\Omega[A^*(Au - f)]$, i.e., we continue A^* to an operator mapping to $L^2(\mathbf{R}^N)$.

The velocity V defined by (3.15) does indeed not satisfy the assumptions of Proposition (3.1) in general, since $V \in C(0, T; C^1(\mathbf{R}^N))$ would imply that div V is continuous, but this is fulfilled by (3.15) only if $A^*(Au(t) - f)$ is continuous and vanishes on ∂D. However, the assumptions of Proposition 3.2 are satisfied if A^* maps Y continuously to $L^\infty(\Omega)$, since then we can define

$$-\text{ div } V_n \;=\; g_n * (A^*(Au - f)(-1 + 2u(t)))\,, \tag{3.17}$$

where $*$ denotes the Fourier convolution with respect to x and g_n is a sequence of smooth, nonnegative convolution kernels such that $g_n * v \to v$ for all $v \in L^\infty(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$ as $n \to \infty$. If we choose $V_n = \nabla\psi_n$, then ψ_n solves the Laplace equation with continuous right-hand side and hence, the standard elliptic regularity theory implies that $\psi_n \in C(0, T; C^2(\mathbf{R}^N))$ and consequently $V_n = \nabla\psi \in C(0, T; C^1(\mathbf{R}^N))$ and

$$\text{div } V_n \to \text{ div } V \qquad \text{in } L^\infty(\mathbf{R}^N \times [0, T]) \hookrightarrow L^1(0, T; L^\infty(\mathbf{R}^N)).$$

In the appendix we shall show that coupled problem for u and V has indeed a solution in an arbitrary time interval $[0, T]$ and that it is possible to consider the limit $T \to \infty$, which is crucial for the convergence analysis in the following section.

An immediate observation in (3.15) is that the modification

$$-\text{ div } V \;=\; \lambda A^*(Au - f)\,(-1 + 2u) \tag{3.18}$$

would yield the estimate

$$\frac{dE}{dt}(t) = -\lambda\|Au(t) - f\|^2, \tag{3.19}$$

i.e., a "steeper" descent. A simple calculation shows that with the new time variable $s = \lambda t$, we obtain again the original estimate for $\frac{dE}{ds}$. Hence, the factor λ corresponds to a scaling of the (artificial) time from t to λt, which means that this steeper descent can only be achieved in a larger time interval. We shall show in Section 5 that a choice of λ different from one also yields no advantages in the numerical solution of the problem and therefore we will restrict our attention to the case of $\lambda = 1$ in the remainder of the paper.

We finally note that an extension of (3.15) to the case of nonlinear operators A is possible if A is continuously Frèchet-differentiable on $L^2(\Omega)$. In this case we can choose

$$- \operatorname{div} V \;=\; A'(u(t))^*(A(u(t)) - f)\,(-1 + 2u(t))\,, \tag{3.20}$$

where $A'(u(t))^*$ denotes the adjoint of the Frèchet-derivative. If A satisfies the so-called *tangential cone condition*

$$\|A(v) - A(u^*) - A'(u)(v - u^*)\| \leq \eta \|A(v) - A(u^*)\|, \tag{3.21}$$

with $\eta < \frac{1}{2}$, for all v in appropriate neighborhood of the solution u^*, then we have

$$\begin{aligned} \frac{dE}{dt}(t) &= -\langle A'(u)(\overline{u} - u), f - A(u)\rangle \\ &\leq -(1 - \eta)\|A(u) - f\|^2, \end{aligned}$$

which allows an analogous analysis as in the linear case.

4 Convergence Analysis

In the following we turn our attention to the convergence analysis of the level set method as $t \to \infty$ and $\delta \to 0$. We start with the case of exact data, i.e., $\delta = 0$, for which we may consider the asymptotics for $t \to \infty$. For noisy data ($\delta > 0$) we have to stop the iteration at some finite time T, which should depend on δ and f^δ. The choice of the stopping time we consider is the *generalized discrepancy principle*, i.e.,

$$T(\delta, f^\delta) = \inf\{\, t \in \mathbf{R}^+ \mid \|Au(.,t) - f^\delta\| \leq \tau\delta \,\} \tag{4.1}$$

for some $\tau > 1$. The rationale behind this stopping rule is that with the knowledge of the noisy data only, it makes no sense to distinguish between approximations u which yield a residual less than the noise level.

The starting point of the convergence analysis is a rather general result about the monotonicity of the iterations in the noisy and noise-free case:

Proposition 4.1. *For all $t \in \mathbf{R}^+$ with $\|Au(.,t) - f^\delta\| > \delta$, the error is decreasing, i.e.,*

$$\frac{dE^\delta}{dt}(t) < 0. \tag{4.2}$$

Moreover, for $\delta = 0$, we have

$$\int_0^\infty \|Au(.,t) - A\overline{u}\|^2 \; dt < \infty. \tag{4.3}$$

Proof. From Proposition 3.3 we obtain

$$\begin{aligned} \frac{dE^\delta}{dt}(t) &= -\int_\Omega (u(.,t) - \overline{u})A^*(Au(.,t) - f^\delta)dx \\ &= -\int_\Omega (u(.,t) - \overline{u})A^*(A(u(.,t) - \overline{u}) + f - f^\delta)dx \\ &\leq -\|A(u(.,t) - \overline{u})\|^2 + \delta\|A(u(.,t) - \overline{u})\|, \end{aligned}$$

and hence, (4.2) holds for $\|Au(.,t) - f^\delta\| \geq \delta$. In the particular case of $\delta = 0$ we obtain by integration with respect to time

$$\int_0^s \|A(u(.,t) - \overline{u})\|^2 \; dt \leq E^\delta(0) - E^\delta(s) \leq E^\delta(0)$$

for all $s \in \mathbf{R}^+$ and thus, (4.3) holds. □

Above we have estimated the decrease of the error between u and $\overline{u}$, which can be interpreted as the L^2-inner product of $u - \overline{u}$ with itself. In the following Lemma we will show how the inner product of u with an arbitrary indicator function evolves in time:

Lemma 4.2. *Let χ_C denote the indicator function of the domain C, then the function*

$$t \mapsto \langle u(.,t), \chi_C \rangle = \int_{D(t)\cap C} 1 \; dx \tag{4.4}$$

is continuously differentiable with derivative

$$\frac{d}{dt}\langle u(.,t), \chi_C \rangle = \langle A(\chi_C - u), Au - f^\delta \rangle. \tag{4.5}$$

Proof. We rewrite the function as

$$\langle u(.,t), \chi_C \rangle = \int_C 1 \; dx - \int_{C-D(t)} 1 \; dx - \int_{D(t)-C} 1 \; dx.$$

Due to Proposition 3.2 all three terms are continuously differentiable with derivatives

$$\begin{aligned}
\frac{d}{dt}\int_C 1 \; dx \quad &= \quad 0 \\
\frac{d}{dt}\int_{C-D(t)} 1 \; dx \quad &= \quad \int_{C-D(t)} \operatorname{div} V \; dx = \int_{\Omega-D(t)} \chi_C \operatorname{div} V \; dx \\
&= \quad -\int_{\Omega-D(t)} (\chi_C - u(.,t)) A^*(Au - f^\delta) \; dx \\
\frac{d}{dt}\int_{D(t)-C} 1 \; dx \quad &= \quad \int_{D(t)-C} \operatorname{div} V \; dx = \int_{D(t)} (u(.,t) - \chi_C) \operatorname{div} V \; dx \\
&= \quad -\int_{D(t)} (\chi_C - u(.,t)) A^*(Au - f^\delta) \; dx,
\end{aligned}$$

and hence, the function $t \mapsto \langle u(.,t), \chi_C \rangle$ is continuously differentiable with derivative

$$\begin{aligned}
\frac{d}{dt}\langle u(.,t), \chi_C \rangle \quad &= \quad \int_\Omega (\chi_C - u(.,t)) A^*(Au - f^\delta) \; dx \\
&= \quad \langle A(\chi_C - u(.,t)), Au - f^\delta \rangle.
\end{aligned}$$

□

In an analogous way to the monotonicity of the error in Proposition 4.1 we can show the monotonicity of the residual:

Lemma 4.3. *The function $t \mapsto \|Au(.,t) - f^\delta\|$ is monotonically decreasing.*

Using the preliminary results obtained above we can now prove the convergence of the sets in the symmetric difference metric:

Theorem 4.4 (Convergence for Exact Data). *Let $\delta = 0$ and let V satisfy (3.15), then $u(.,t) \to \hat{u}$, where $\hat{u}$ solves (3.1), as $t \to \infty$.*

Proof. First of all, the monotonicity and continuity with respect to t of the residual $\|Au(.,t) - f\|$ together with (4.3) implies that

$$\|Au(.,t) - f\| \to 0 \qquad \text{as } t \to \infty.$$

Now we show that $u(.,t)$ converges and therefore we consider $u(.,t) - u(.,s)$ for arbitrary $t > s$.

$$\begin{aligned}\|u(.,t) - u(.,s)\|^2 &= \|u(.,s) - \overline{u}\|^2 - \|u(.,t) - \overline{u}\|^2 + \\ &\quad 2\langle u(.,t) - u(.,s), u(.,t) - \overline{u}\rangle.\end{aligned}$$

From the monotonicity and boundedness of $E(t)$ we conclude the existence of an $\epsilon \geq 0$ such that $\|u(.,t) - \overline{u}\| \to \epsilon$ as $t \to \infty$. Due to Lemmas 4.2 and 4.3 we have

$$\begin{aligned}|\langle u(.,t) - u(.,s), u(.,t) - \overline{u}\rangle| &= \left| \int_s^t \langle A(u(.,t) - \overline{u}), Au(.,\tau) - f\rangle \, d\tau \right| \\ &\leq \int_s^t \|Au(.,t) - f\| \|Au(.,\tau) - f\| \, d\tau \\ &\leq \int_s^\infty \|Au(.,\tau) - f\|^2 \, d\tau.\end{aligned}$$

The last term converges to zero as $s \to \infty$ because of (4.3) and hence, for $s \to \infty$ and arbitrary $t > s$ we have

$$\|u(.,t) - u(.,s)\|^2 \to \epsilon^2 - \epsilon^2 + 0 = 0,$$

and hence, $u(.,t)$ converges to some function $\tilde{u}$, which must satisfy (3.1) since $\|Au(.,t) - f\| \to 0$. □

4.1 Convergence for Noisy Data

Now we investigate the behavior of the iterations in presence of noise. In order to express the specific noise level, we will denote the evolution with noisy data f^δ by $D^\delta(t)$ and $u^\delta(t)$ in this section. We first show that the stopping time $T(\delta, f^\delta)$ is always finite if δ is nonzero:

Lemma 4.5. *Let $\delta > 0$ and $\tau > 1$, then the stopping time $T(\delta, y^\delta)$ defined by (4.1) is finite.*

Proof. Suppose $T(\delta, y^\delta) = \infty$, then for all $t \in \mathbf{R}_+$ the estimate

$$\|Au^\delta(.,t) - f^\delta\| > \tau\delta$$

holds. From the proof of Proposition 4.1 we obtain

$$\begin{aligned}\frac{dE^\delta}{dt}(t) &= -\langle Au^\delta(.,t) - f, Au^\delta(.,t) - f^\delta\rangle \\ &\leq -\|Au^\delta(.,t) - f^\delta\| \left(\|Au^\delta(.,t) - f^\delta\| - \delta\right) \\ &\leq -\tau(\tau-1)\delta^2\end{aligned}$$

and hence,

$$E^\delta(t) = E^\delta - \tau(\tau-1)\delta^2 t,$$

but this would imply the existence of a $t_* \in \mathbf{R}^+$ with $E^\delta(t_*) < 0$ if $\delta > 0$, which is a contradiction. Thus, the stopping index $T(\delta, f^\delta)$ is finite. □

Having verified the finiteness of the stopping time, it only remains to show that the reconstructed sets obtained at this stopping time $T(\delta, f^\delta)$ really converge to a solution as the noise level δ tends to zero, which will be carried out in the following:

Theorem 4.6 (Convergence for Noisy Data). *Let the stopping time T be chosen according to the discrepancy principle* (4.1) *with $\tau > 1$ and let D^δ denote the evolution of sets for noise level δ, then $D^\delta(T(\delta, f^\delta)) \to \hat{D}$ in the symmetric difference metric as $\delta \to 0$, where $\chi_{\hat{D}}$ solves* (3.1).

Proof. With Proposition 4.1 and Theorem 4.4 we have the same preliminaries as needed for the proof of Theorem 4 in [31] and therefore the proof carries over from asymptotical regularization. □

5 Numerical Methods

In the following we shall discuss numerical methods for the level set approach developed above. For the sake of simplicity we restrict ourselves again to the special choice (3.16) for the velocity, then we have to solve the system (3.5), (3.15), (3.16). In the level-set framework we do not solve (3.5) directly for u but use a smooth function ϕ_0 such that $\phi_0 > 0$ in $D(0)$ and $\phi_0 \leq 0$ outside. Hence, we have to solve the coupled problem (1.2), (1.3), (3.15), (3.16), supplemented by the initial condition (1.4).

The numerical implementation involves several subproblems, which we will discuss in the following. First of all, we have to solve the level set equation (1.3). The common choice for such a hyperbolic equation of first order is an explicit method in time on some grid $\{(x_i, y_j)\}_{i=1,\ldots,n_x, j=1,\ldots,n_y}$. Although one has to solve equation (1.3) on the whole domain $\mathbf{R}^N$ in principle, it suffices if the grid covers a sufficiently large domain, since the information propagates with finite speed in a hyperbolic differential equation, and the solutions will therefore not differ in a domain containing the zero-level set. The behavior of the level set function far away from this level set is of no interest any way.

The time discretization will be denoted by

$$0 = t_0 < t_1 < \ldots < t_m = T \tag{5.1}$$

in the following, furthermore we use the notation $\tau_k = t_{k+1} - t_k$ for the time steps. The standard explicit methods for a first-order hyperbolic equation are stable if

$$\tau_k \max_{(i,j)} |V(x_i, y_j, t_k)| < h, \qquad \forall\, k \in \{0, \ldots, m-1\} \tag{5.2}$$

where h is the typical size for the spatial discretization, i.e.,

$$h = \max \left\{ \max_i \min_{p \neq i} |x_i - x_p|, \max_j \min_{q \neq j} |y_j - y_q| \right\}. \tag{5.3}$$

This condition can be guaranteed by an adaptive choice of the time step, i.e., one can choose

$$\tau_k = \eta \frac{h}{\max_{(i,j)} V(x_i, y_j, t_k)}, \tag{5.4}$$

for some $\eta \in (0,1)$. We note that in the case of (3.18) the velocity is multiplied by a factor λ and therefore the stability bound for the time step τ_k^λ becomes

$$\tau_k^\lambda \leq \tau_k,$$

where τ_k is the time step for the problem with $\lambda = 1$. Therefore, the higher speed for $\lambda > 1$ is compensated by a smaller time step, which leads to the same numerical effort.

A simple and efficient discretization method is the following upwind scheme proposed by Sethian (cf. [28])

$$\begin{aligned}
\phi(x_i, y_j, t_{k+1}) &= \phi(x_i, y_j, t_k) - \tau_k (A_{ijk} + B_{ijk}) \\
A_{ijk} &= \max\{V_1(x_i, y_j, t_k), 0\} \frac{\phi(x_i, y_j, t_k) - \phi(x_{i-1}, y_j, t_k)}{x_i - x_{i-1}} + \\
&\quad \min\{V_1(x_i, y_j, t_k), 0\} \frac{\phi(x_{i+1}, y_j, t_k) - \phi(x_i, y_j, t_k)}{x_{i+1} - x_i} \\
B_{ijk} &= \max\{V_2(x_i, y_j, t_k), 0\} \frac{\phi(x_i, y_j, t_k) - \phi(x_i, y_{j-1}, t_k)}{y_j - y_{j-1}} + \\
&\quad \min\{V_2(x_i, y_j, t_k), 0\} \frac{\phi(x_i, y_{j+1}, t_k) - \phi(x_i, y_j, t_k)}{y_{j+1} - y_j}
\end{aligned}$$

where we used the notation $V = (V_1, V_2)$ for the velocity. This scheme is motivated by the propagation of information through characteristics in a first-order hyperbolic equation. E.g., if $V_1 > 0$ at some point, then the characteristics evolve in positive direction with respect to x and hence it seems natural to use a backward difference quotient. For analogous reasons a forward difference quotient should be used if $V_1 < 0$, and this is exactly realized by the upwind scheme. The efficient computation of the terms A_{ijk} and B_{ijk} differs from the formulas given above, since one only needs to compute one of the two difference quotients in dependence of the signs of V_1 and V_2. The upwind scheme presented above can be implemented efficiently, but it is only of first order in time. Schemes on triangular grids or stable higher order methods for Hamilton-Jacobi equation can be developed using special terms providing a non-oscillatory behavior of the numerical solutions. We do not go into further details but refer to the monograph by Sethian [28] and to the recent paper by Jiang and Peng [17].

We note that in our case an explicit time step in the level set equation also leads to a decoupling of the system, since we can first perform a time step for (1.3) from t_k to t_{k+1} using the values of the velocity V at time step t_k. Then we can proceed step by step to compute the values of u, ψ and V at time t_{k+1}. The indicator function u can be computed easily on a set of grid points, namely via

$$u(x_i, y_j, t_k) = H(\phi(x_i, y_j, t_k)),$$

where H is the Heaviside-function. The boundary of the level set can be approximated by those grid points (x_i, y_j) such that $u(x_i, y_j, t_k) = 1$ and the indicator function takes the value zero at an adjacent grid point. However, such an approximation is of rather low quality and looses a lot of information on the geometric properties of the level set. A more accurate approximation of the level set can be obtained by piecewise polynomial approximations of ϕ in each cell and an appropriate interpolation technique to determine its zero level set.

Due to Theorem (2.4), the reconstruction of the domain from the level set function ϕ with smooth initial value is equivalent to the direct computation of the indicator function if the set $\{\phi(., t) = 0\}$ is of zero measure in $\mathbf{R}^N$. From the numerical point of view, we can express a similar statement, namely the reconstruction becomes difficult if the level set function in too flat close to $\{\phi(., t) = 0\}$. A possible way to overcome this problem is to construct a new "steeper" level set function according to the level set obtained after some time steps and to continue the numerical evolution with this new level set function. This technique as well as efficient methods for constructing new level set functions are discussed in detail by Sethian [28].

The determination of the residual $Au - f$ and the "adjoint residual" $A^*(Au - f)$, which is needed for the computation of V, depends significantly on the structure of the operator A and will therefore not be discussed in detail here. In order to determine V and ψ we have to solve the saddle-point problem (3.15), (3.16) with given right-hand side. Since the value of V is of importance only close to the zero level set, we can approximate the problem in $\mathbf{R}^N$ by the homogeneous Dirichlet problem in a sufficiently large bounded domain $\tilde{\Omega}$. In our case, V itself is important for the algorithm and therefore it seems of advantage to use a finite element discretization directly for the mixed problem (3.15), (3.16) on $\tilde{\Omega}$. For basic background on mixed finite element discretization techniques we refer to [5].

6 Applications

In this section we apply the idea of the level set method to two standard examples arising in applications.

6.1 Deblurring

Deblurring problems in image treatment are usually of the form

$$Au = \int_\Omega k(x,y)u(y)\ dy. \tag{6.1}$$

A is a compact and bounded operator on $L^2(\Omega)$ if $k \in L^2(\Omega \times \Omega)$, the adjoint of A is given by

$$A^*v = \int_\Omega k(y,x)u(y)\ dy, \tag{6.2}$$

it is continuous to $L^\infty(\Omega)$ if $k \in L^infty(\Omega; L^2(\Omega))$. Of particular importance are deconvolution problems, where k is of the form

$$k(x,y) = g(x-y) \tag{6.3}$$

for some kernel $g \in L^1(\mathbf{R}^N)$.

6.2 An Inverse Source Problem

The following inverse source problem was studied by Kunisch and Pan [19]:

$$-\Delta v = u \qquad \text{in } \Omega \tag{6.4}$$

$$v = 0 \qquad \text{on } \partial\Omega \tag{6.5}$$

$$\frac{\partial v}{\partial n} = f \qquad \text{on } \partial\Omega. \tag{6.6}$$

with $u = \chi_D$ and $\Omega \subset \mathbf{R}^2$ sufficiently regular. In this case we can define a continuous linear operator $A : L^2(\Omega) \to L^2(\partial\Omega)$ that maps u to $\frac{\partial v}{\partial n}|_{\partial\Omega}$, where $v \in H_0^1(\Omega) \cap H^2(\Omega)$ is the unique weak solution of (6.4), (6.5). In the following we assume that $\partial\Omega$ is sufficiently regular such that $v \in H^2(\Omega)$ and therefore A is continuous as the concatenation of the continuous maps $u \mapsto v \in H^2(\Omega)$ and $v \mapsto \frac{\partial v}{\partial n}|_{\partial\Omega} \in L^2(\partial\Omega)$. A standard argument for trace operators (cf. e.g. [21]) shows that the second map is even compact and hence, A is compact, too. This implies that the inverse problem of solving (3.1) is ill-posed (cf. e.g. [11]).

The adjoint problem in this case is given by

$$-\Delta w = 0 \qquad \text{in } \Omega \tag{6.7}$$

$$w = -g \qquad \text{on } \partial\Omega, \tag{6.8}$$

more precisely, $w = A^* g$.

6.3 An Inverse Coefficient Problem

Recently, Kwang et. al. studied the identification of an indicator function u in the elliptic equation

$$-\Delta v + u\ v = 0 \qquad \text{in } \Omega, \tag{6.9}$$

from the overdetermined boundary data $v|_{\partial\Omega} = f$ and $\frac{\partial v}{\partial \nu}|_{\partial\Omega} = g$. If we define the forward operator $A : L^2(\Omega) \to L^2(\partial\Omega)$ by $A : u \mapsto v|_{\partial\Omega}$, where v is the solution of (6.9) subject to the Neumann boundary condition $\frac{\partial v}{\partial \nu}|_{\partial\Omega} = g$, then A is well-defined and continuously Fréchet-differentiable on the set

$$\mathcal{D}(A) = \{\ u \in L^2(\Omega) \mid u \geq 0 \text{ a.e. }\},$$

which includes all indicator functions. Furthermore, the operator A and its derivative A' satisfy the nonlinearity condition (3.21) (cf. [18]), i.e., the convergence theory developed above for nonlinear inverse problems is applicable to this case. For further details on this inverse problem and the computation of the derivative A' and its adjoint we refer to [18, 20].

7 Numerical Results

As a numerical test example we investigate the deconvolution with a Gaussian kernel on $\Omega = \mathbf{R}^2$, i.e.,

$$k(x, y) = G(x - y) := e^{-\frac{|x-y|^2}{2}}.$$

Fourier's Theorem implies that

$$\mathcal{F}(Au) = (\mathcal{F}G)(\mathcal{F}u), \tag{7.1}$$

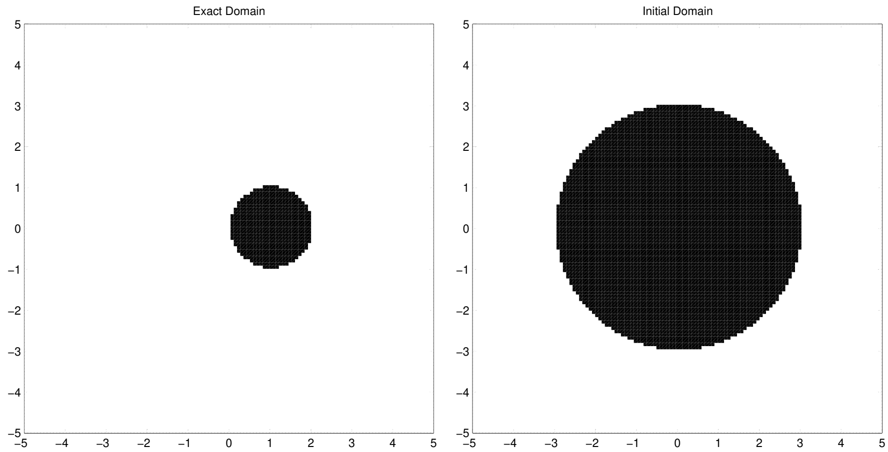

Figure 1: Exact solution $\overline{D}$ (left) and initial domain $D(0)$ in Example 1

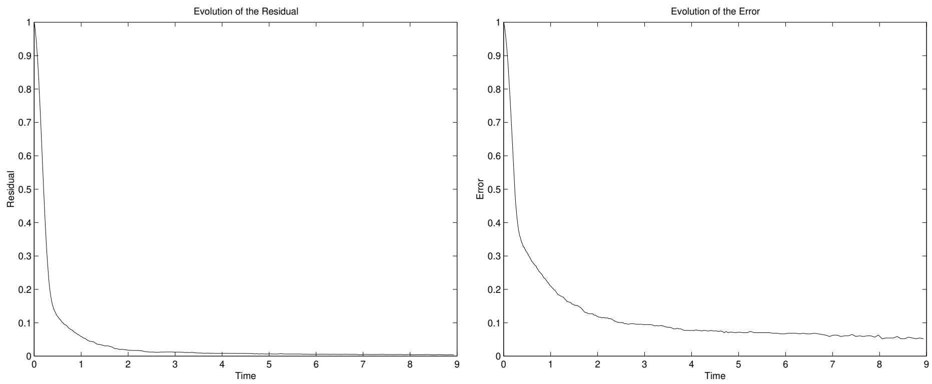

Figure 2: Evolution of the residual $\|Au(t) - f\|$ (left) and the error $d_{\mathcal{S}}(D(t), \overline{D})$ (right) plotted versus the artificial time.

where $\mathcal{F}$ denotes the Fourier transformation. As the domain $\tilde{\Omega}$ we use a large rectangle with 128×128 grid points, which allows to evaluate the operators A and A^* with low effort using FFT-techniques.

We carried out two different tests with known solution, one with a circle of radius 1 as the exact solutions and a second one with a set D consisting of two connected components. In both cases, the level set method was started from a large circle of radius 3 centered at the origin. The exact solution in the first example and the initial domain for the reconstruction are shown in Figure 1.

In the absence of noise, the theory developed above predicts convergence of the level set to the exact domain as $t \to \infty$; this convergence behavior is illustrated in Figure 2, which shows the evolution of the residual $\|Au(t) - f\|$ and the error $d_{\mathcal{S}}(D(t), \overline{D})$ in time. One observes that both error and residual decrease in a monotone way, which confirms Proposition 4.1 and Lemma 4.3 numerically. Furthermore, the figures show that the decrease is faster in the residual than in the error, which is a usual effect for an ill-posed problem. Since the artificial time is not the most important classification number for the efficiency of the method, the step

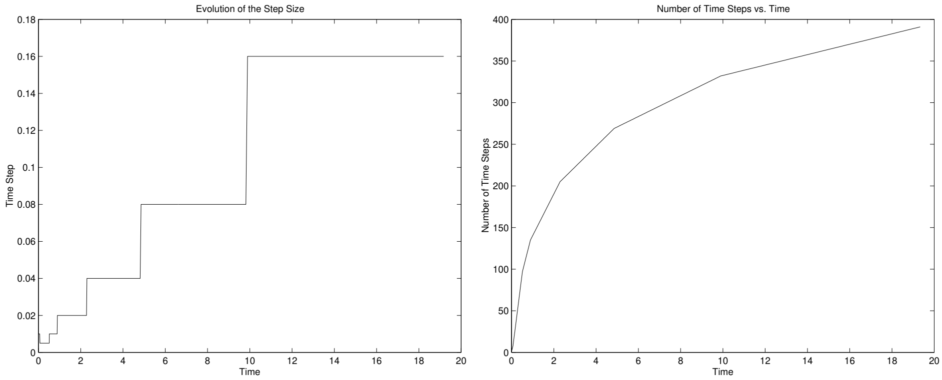

Figure 3: Evolution of the time step τ (left) and the number of time steps needed (right).

size and the number of time steps are plotted vs. time in Figure 3. One observes that the number of time steps needed for a reasonable approximation of the exact domain is relatively large (more than 200), but since each time step is explicit and very cheap the overall numerical effort is still acceptable. Moreover, the evolution of the step size τ^j shows a monotonically increasing behavior, which is caused by a choice of the form (5.4), more precisely we chose τ^k such that

$$0.1\frac{h}{\max_{(i,j)} V(x_i,y_j,t_k)} \leq \tau^k \leq 0.3\frac{h}{\max_{(i,j)} V(x_i,y_j,t_k)}. \tag{7.2}$$

The reason for the increase of the step size is the decrease of the residual and the choice (3.15), which implies that also V is decreasing.

The evolution of the domain $D(t)$ and the level set function ϕ is shown in Figures 4 and 5, respectively. One observes that the domain shrinks first in a very fast way and later adapts the shape to approximate the exact solution. This is a desirable property if one wants to know the size of a domain rather than its precise shape, since the size is approximated very well after few time steps.

Finally, we investigate the behavior of the level set method with respect to noise in the data. The evolution in artificial time seems to be very robust with respect to noise, which is again an analogy to the method of asymptotical regularization. The development of the error between the approximate solutions computed with the discrepancy principle as a stopping rule (here with $\tau = 1.5$) as $\delta \to 0$ is plotted in Figure 6. This plot and even more clearly the one vs. $\sqrt{\delta}$ indicate a convergence rate of order $\sqrt{\delta}$, which is the same as one would expect e.g. for asymptotical regularization. The same effect occurs for the stopping time $T_*(\delta, f^\delta)$, which is plotted vs. the noise level δ and vs. $\frac{1}{\sqrt{\delta}}$ in Figure 7, one observes that the increase of T_* is of order $\frac{1}{\sqrt{\delta}}$.

8 Conclusions and Open Problems

We have seen above that a level set method based on the property (3.15) leads to a convergent reconstruction method for linear and nonlinear inverse interface problems. In addition, this algorithm is a regularization method for ill-posed problems, if the stopping index is chosen appropriately, which is a theoretical advantage to other level set approaches for inverse

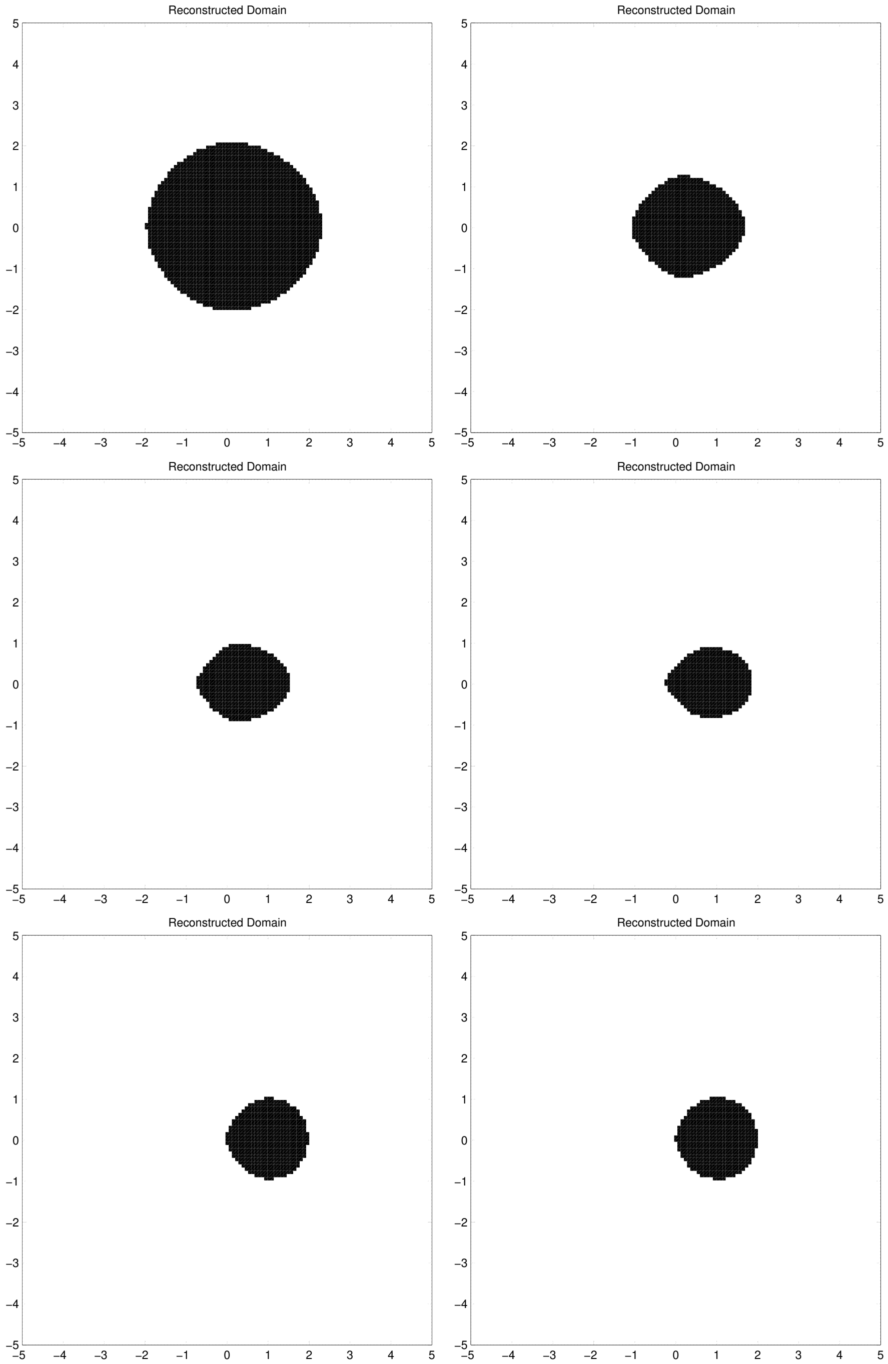

Figure 4: Evolution of the domain $D(t)$ in time, pictures taken at $t = 0.18, 0.33, 0.48, 1.195, 3.695, 9.735$ in the absence of noise ($\delta = 0$).

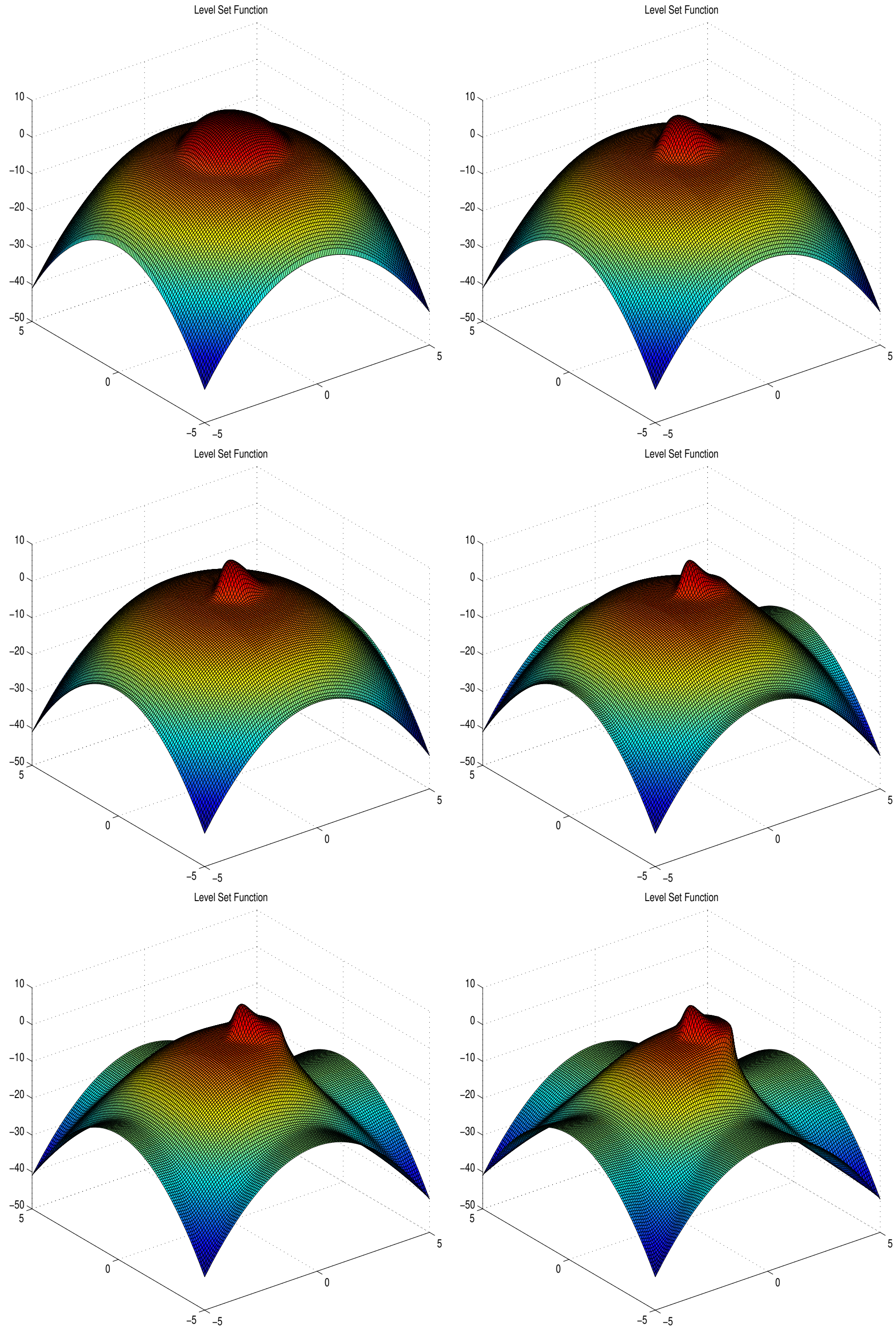

Figure 5: Evolution of the level set function $\phi(., t)$ in time, pictures taken at $t = 0.18, 0.33, 0.48, 1.195, 3.695, 9.735$ in the absence of noise ($\delta = 0$).

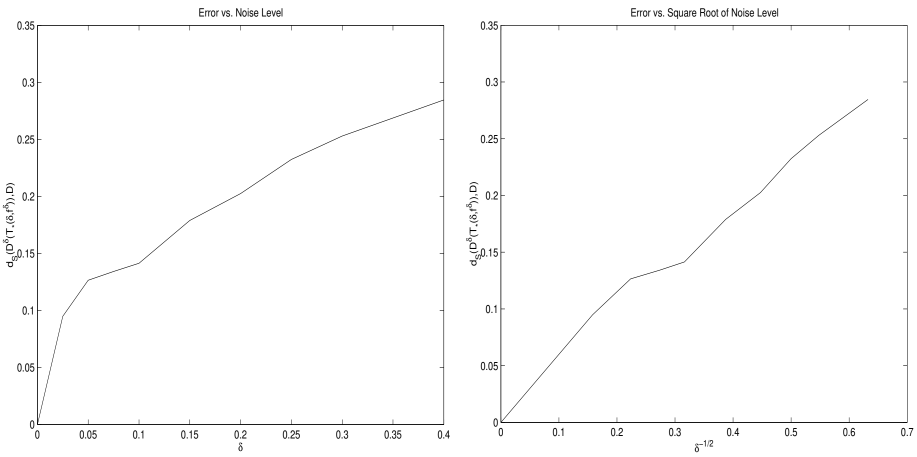

Figure 6: Plot of the error between $D^{\delta}(T_*(\delta, f^{\delta}))$ and the exact solution D vs. noise level δ (left) and vs. $\sqrt{\delta}$ (right)

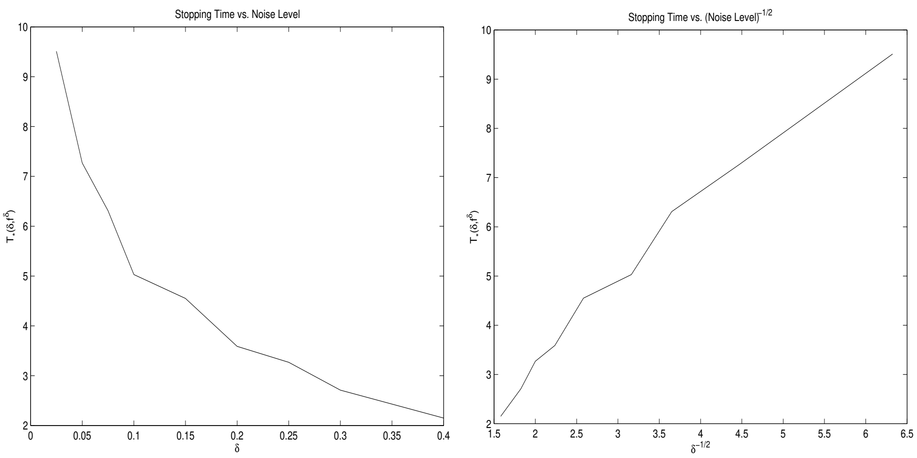

Figure 7: Plot of the stopping time $T_*(\delta, f^{\delta})$ vs. noise level δ (left) and vs. $\frac{1}{\sqrt{\delta}}$ (right).

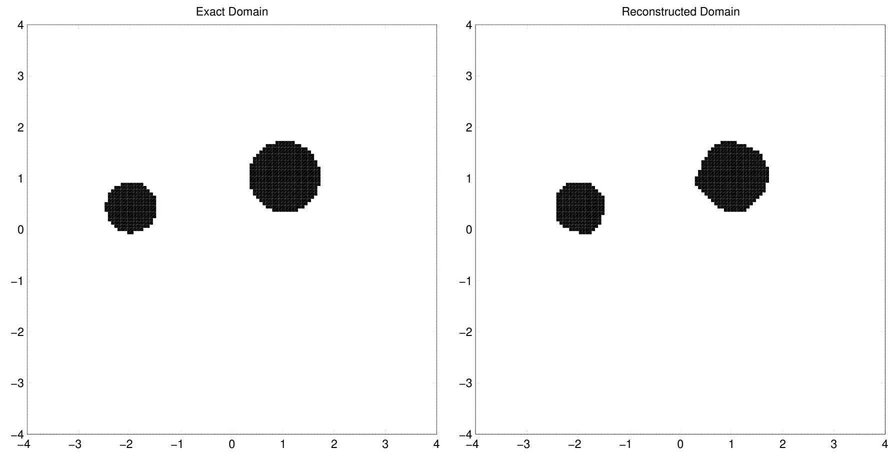

Figure 8: Exact solution $\overline{D}$ (left) and reconstruction $D(T)$ for $T = 200$ in Example 2

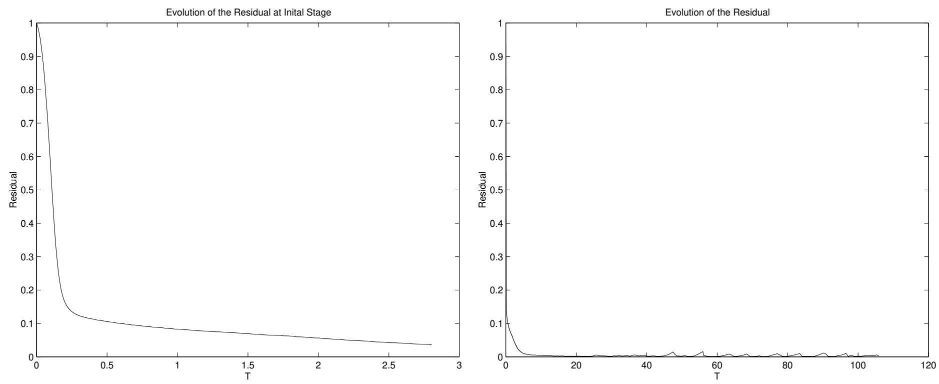

Figure 9: Evolution of the residual $\|Au(t) - f\|$ plotted versus the artificial time during the initial stage (left) and during a large time interval (right)

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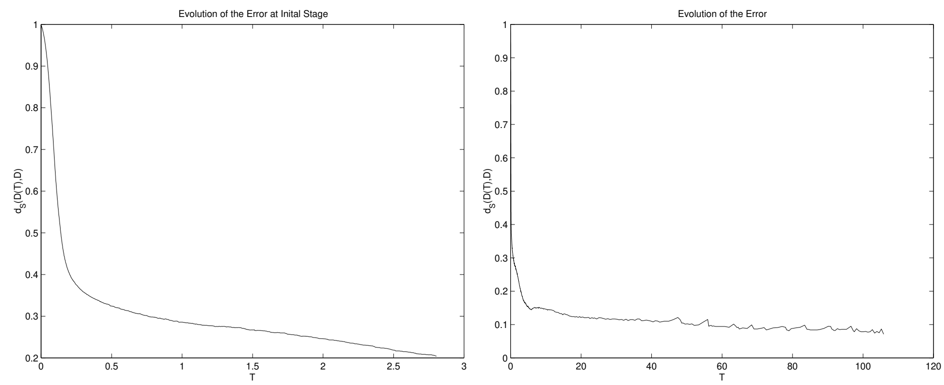

Figure 10: Evolution of the error $d_{\mathcal{S}}(D(t), \overline{D})$ plotted versus the artificial time during the initial stage (left) and during a large time interval (right).

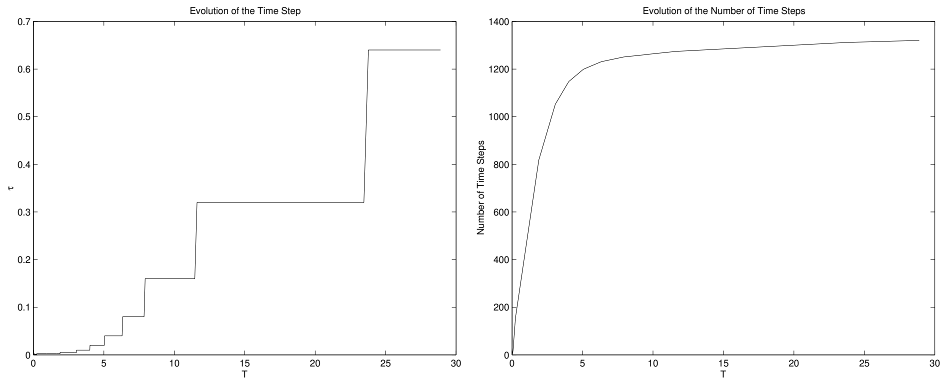

Figure 11: Evolution of the time step τ (left) and the number of time steps needed (right).

problems that can be found in literature.

The numerical results show that our level set method is able to reconstruct solutions consisting of several connected components without a-priori knowledge on there number, which is an important advantage of many level set methods compared to classical shape reconstruction methods based on curve parameterizations.

However, some questions related to the level set method (1.2)-(1.4), (3.15) are still open, we want to sum up the most important, which we plan to deal with in future research, in the following:

- **Convergence Rates:** For the standard regularization methods for inverse problems one can prove convergence rates results for the regularized solutions as $\delta \to 0$ if the exact solution satisfies certain smoothness conditions, e.g., if

$$\exists w \in Y : \overline{u} = A^* w \tag{8.1}$$

  for some $\nu > 0$. Since for an operator A^* that maps to $C(\Omega)$ (e.g. an integral operator with continuous kernel) the representation (8.1) would imply that $\overline{u}$ is continuous, such smoothness conditions seem to be unusable for shape reconstruction problems. It remains an open question how to set up appropriate smoothness conditions for such problems that can be used to prove convergence rate results for the proposed level set method.

- **Regularity:** So far we have not treated the regularity of the level set function and of the boundary ∂D during the evolution. It seems obvious that a certain regularity of the initial level set function and the initial shape should also yield to regularity during the evolution.

- **Convergence in other Metrics:** For the construction of the level set method we have used the symmetric difference metric as a notion for the convergence of sets. An interesting problem for the future is the construction of level set methods that lead to convergence in other metrics, e.g. in the Hausdorff metric of the boundaries that has been proposed by Berg and Holmstroem [3].

- **Complex Applications:** In the numerical examples presented above we have investigated two standard inverse problems that can be solved with relatively low effort. Of

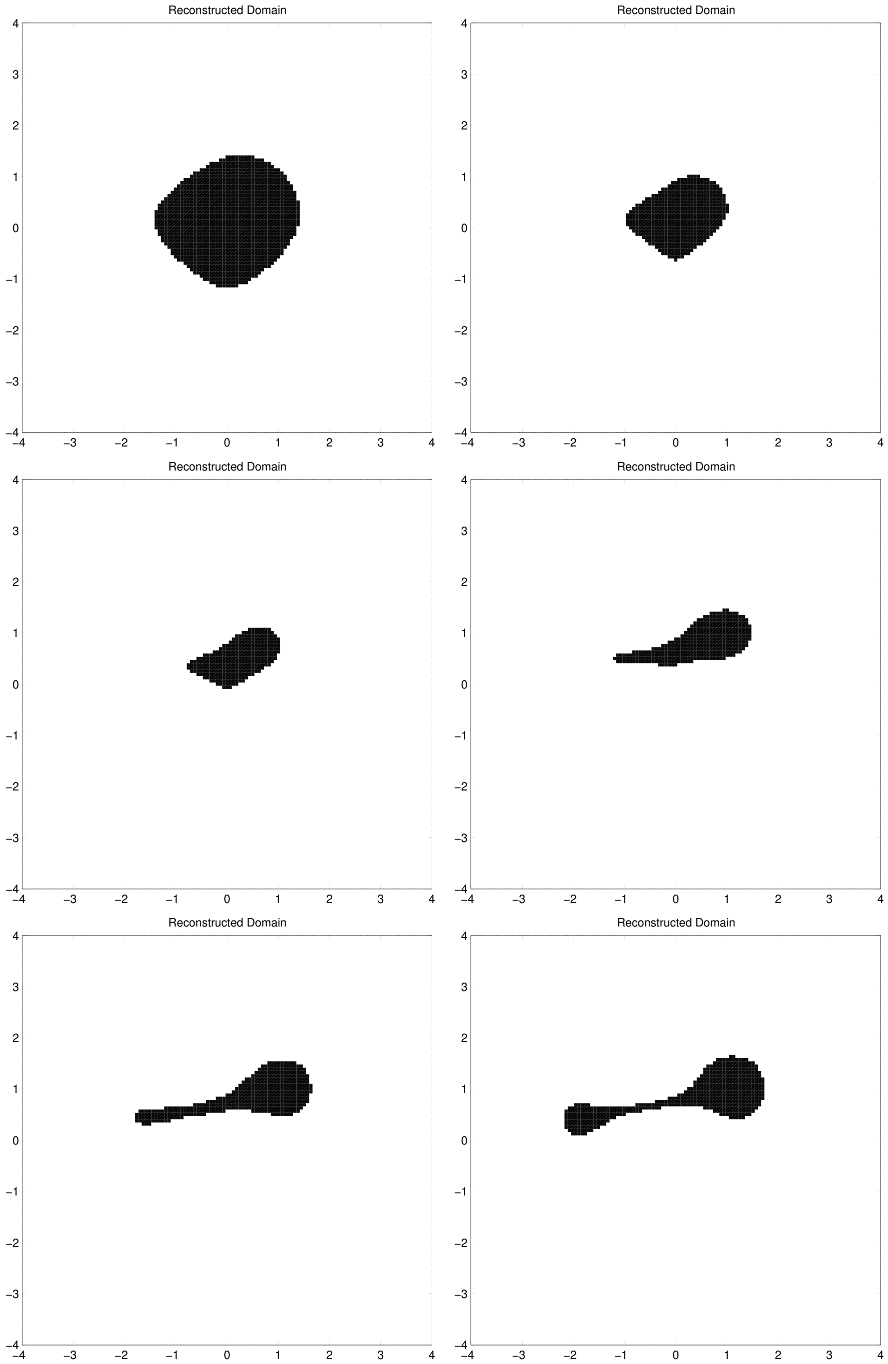

Figure 12: Evolution of the domain $D(t)$ in time during the initial stage, pictures taken at $t = 0.1438, 0.2188, 0.45, 1.1, 1.85, 2.905$ in the absence of noise ($\delta = 0$).

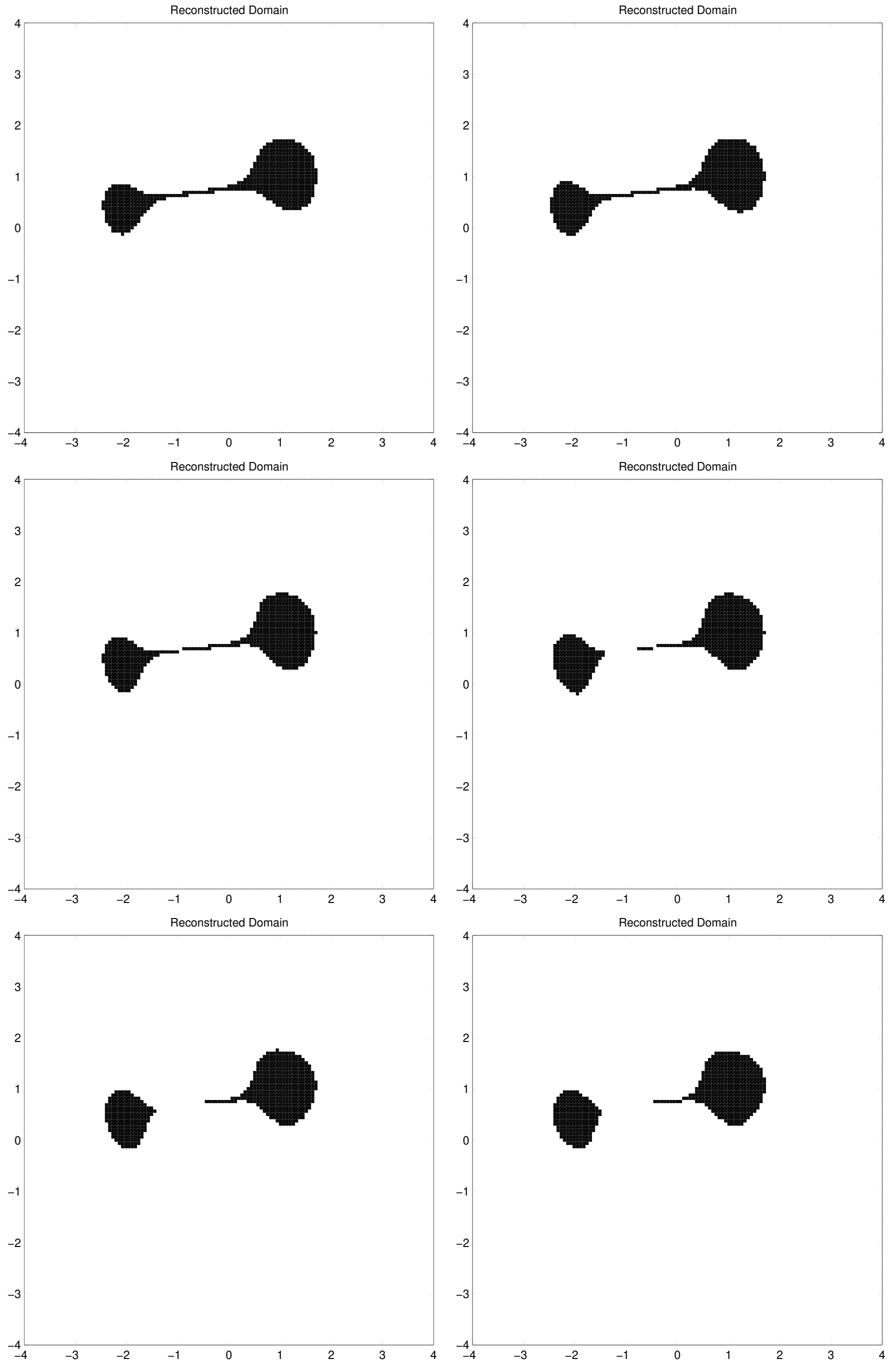

Figure 13: Evolution of the domain $D(t)$ in time, pictures taken at $t = 5.89, 7.055, 9.375, 13.535, 19.935, 28.895$ in the absence of noise ($\delta = 0$).

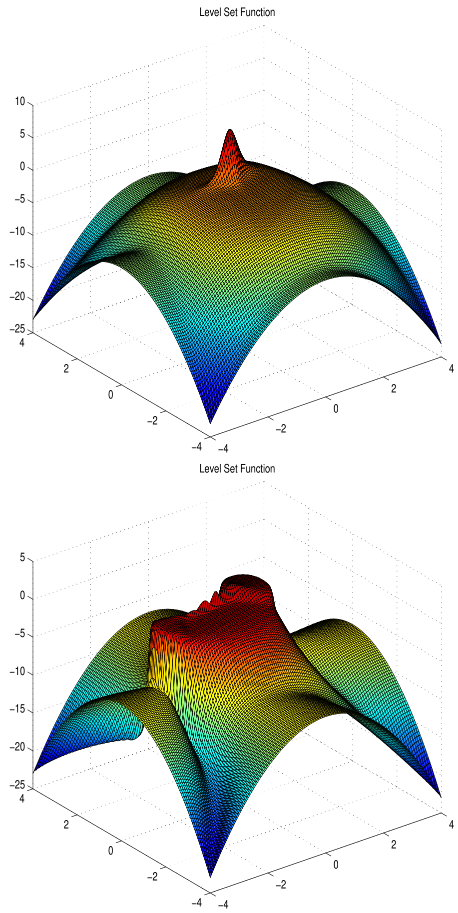

Figure 14: Evolution of the domain $D(t)$ in time, pictures taken at $t = 0.45, 1.85, 5.89, 19.935$ in the absence of noise ($\delta = 0$).

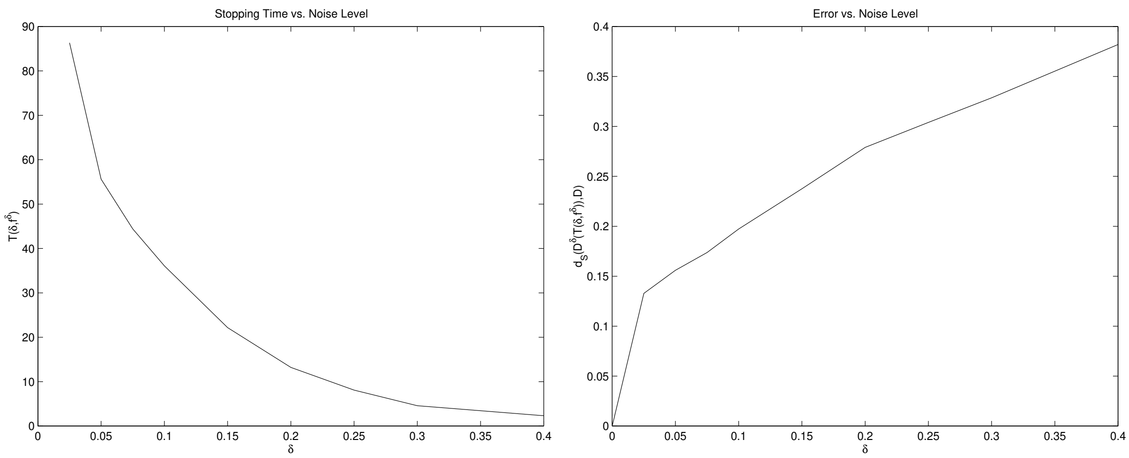

Figure 15: Plot of the stopping time $T_*(\delta, f^\delta)$ vs. noise level δ (left) and of the error $D^\delta(T_*(\delta, f^\delta))$ and the exact solution D vs. noise level δ (right) in example 2.

course there are much more complex applications of shape reconstruction problems in practice, such as the *inverse conductivity problem* (cf. e.g. [6, 15]) or *inverse scattering* (cf. [7] and the references quoted there), whose efficient solution by the level set method is an important and challenging task for future research.

- **Identification of Discontinuities:** In some applications one is not interested in parameters that are piecewise constant functions, but rather general functions containing some discontinuity. Since the level set could also represent such a domain separated by a discontinuity, the level set method could possibly adapted for this case, too.

Appendix: Existence of a Solution

We shall now show that the level set equation (1.3) coupled to the specific choice (3.15) of the velocity V has a solution. For the sake of simplicity we restrict our attention to the model (3.15), (3.16), i.e., we have to solve a Hamilton-Jacobi equation for u coupled to a Laplace equation for ψ.

In order to show the existence of a solution of the level set equation coupled to the condition (3.15) for the velocity V, we restrict ourselves to a particular case, namely the choice (3.16), which leads to the coupled system

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla\psi.\nabla u &= 0 \qquad (A.1)\\ -\Delta\psi - A^*(Au-f)\,(-1+2u) &= 0 \qquad (A.2)\end{aligned}$$

in $\mathbf{R}^N \times (0,T)$, supplemented by

$$\begin{aligned} u|_{t=0} &= u_0, \qquad (A.3)\\ \psi|_{|x|\to\infty} &\to 0. \qquad (A.4)\end{aligned}$$

Furthermore, we need the following assumptions on the operator A:

(A) The operator A is linear, bounded and compact; its norm shall be denoted by C_A.

(B) The L^2-adjoint A^* is also a continuous operator from Y to $L^\infty(\Omega)$ with $\|A^*y\|_{L^\infty(\Omega)} \leq C_*\|y\|$ for all $y \in Y$.

The outline of our existence proof is the following: we first consider the approximation

$$-\Delta\psi = g * \big(A^*(Au-f)(-1+2u)\big) \qquad (A.5)$$

of (A.2) where g is a continuous convolution kernel. The continuity of g implies that the right-hand side in (A.5) is continuous and therefore we have $\psi \in C^2(\mathbf{R}^N)$ and consequently V is Lipschitz-continuous with respect to x. Hence, we can apply Lemma 2.2 in this situation and we will conclude the existence of a solution by a fixed-point argument. Finally, we will show the existence of a solution of the original system by a weak convergence technique using an appropriate sequence g_n of convolution kernels.

Proposition A.1. *Let $u_0 \in L^2(\mathbf{R}^N)$ be the indicator function of a bounded measurable set $D(0) \subset \Omega$. Then for each $g \in C_b(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$, there exists a weak solution $(u,\psi) \in L^\infty(0,T;L^2(\mathbf{R}^N)) \times L^1(0,T;C_b^2(\mathbf{R}^N))$ of* (A.1), (A.3), (A.4) (A.5).

Sketch of Proof. Let $B(0,R)$ be a ball in $\mathbf{R}^N$ centered at the origin, with radius R sufficiently large such that $D(0) \subset B(0,R)$. Then the indicator function u at time t can be computed equivalently from the knowledge of the flow ξ defined by (2.10) only, namely via

$$u(y,t) = \begin{cases} 1 & \text{if } y \in \xi(D(0),t) \\ 0 & \text{else} \end{cases} \tag{A.6}$$

Hence, we can consider the coupled system (2.10), (A.4), (A.5), (A.6) with $V = \nabla\psi$. This system can be transformed to a fixed problem of the form

$$\xi = \mathcal{F}(\xi),$$

where $\mathcal{F}$ consists of the concatenation of the fixed point maps

$$\begin{array}{ll} \xi \mapsto u & \text{from } C(B(0,R) \times (0,T)) \text{ to } C(0,T;L^2(\mathbf{R}^N)) \\ u \mapsto V & \text{from } C(0,T;L^2(\mathbf{R}^N)) \text{ to } C(0,T;C^{0,1}(\mathbf{R}^N)) \\ V \mapsto \xi & \text{from } C(0,T;C^{0,1}(\mathbf{R}^N)) \text{ to } C(B(0,R) \times (0,T)). \end{array}$$

One can show that $\mathcal{F}$ is continuous and that it maps the compact set

$$\mathcal{K} = \{\ \xi \in C(\Omega_0 \times (0,T) \mid \|\xi\|_{C^{0,1}} \leq \kappa(1 + \|g\|_{C(\mathbf{R}^N)})\ \}$$

into itself for some $\kappa > 0$. Consequently, Schauder's fixed point theorem implies the existence of a fixed point $\xi \in \mathcal{K}$.

Since the evolution of $\xi(x,.)$ does not affect the velocity V for $x \notin B(0,R)$ we can show that we even have a solution $\xi \in C^{0,1}(\mathbf{R}^N \times (0,T)$ and consequently a solution (u,ψ) of the original system is well-defined by (A.6) and (A.4), (A.5). $\square$

Now we are able to prove the main result of this section:

Theorem A.2. *For each $T \in \mathbf{R}^+$, there exists a weak solution $(u,\psi) \in L^\infty(0,T;L^2(\Omega)) \times L^2(0,T;H_0^1(\mathbf{R}^N))$ of* (A.1)-(A.4).

Sketch of Proof. Let g_n be a sequence of continuous convolution kernels, such that $\|g_n\|_{L^1(\mathbf{R}^N)}$ is uniformly bounded and

$$g_n * f \to f, \qquad \forall\ f \in L^\infty(\mathbf{R}^N) \cap L^2(\mathbf{R}^N).$$

By (u_n,ψ_n) we denote the according sequence of solutions of (A.1), (A.3), (A.4), (A.5), which exist because of Proposition A.1. One can show that Proposition A.1 implies that (u_n,ψ_n) is uniformly bounded in the Hilbert space norm of $L^2(\mathbf{R}^N \times [0,T]) \times L^2(0,T;H_0^1(\mathbf{R}^N))$ and hence, there exists a weakly convergent subsequence. By standard methods one can prove that the weak limit (u,ψ) of this subsequence is a weak solution of (A.1)-(A.4). $\square$

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