Application of Preconditioned Coupled FETI/BETI Solvers to 2D Magnetic Field Problems *

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Abstract

This paper presents the implementation of preconditioned FETI/BETI solvers for two-dimensional problems in electromagnetics and discusses the results of our numerical experiments. As model problem, we consider the potential equation resulting from the Maxwell's equations in the two-dimensional case. In the air subdomains we take advantage of the boundary element discretization, whereas ferromagnetic subdomains and subdomains with prescribed currents are discretized by the finite element method. The resulting coupled system of finite and boundary element equations is solved via some FETI/BETI domain decomposition technique. In particular, we study the numerical behaviour of some preconditioners with respect to homogeneous and highly heterogeneous behaviour of the permeability. The scaled hypersingular preconditioner originating from the boundary element discretization works fine for boundary element as well as finite element equations and is robust with respect to large jumps in the permeability.

Keywords FETI, BETI, Steklov-Poincaré operator, potential equation, Schur complement, preconditioners, hypersingular operator

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1 Introduction

The classical Finite Element Tearing and Interconnecting (FETI) was proposed by Farhat and Roux in 1991 (see [3]). The principle of FETI is to assume conform triangulation of the whole computational domain and the Finite Element spaces are given on each subdomain including its boundary separately. The global continuity is then enforced by Lagrange multipliers, resulting in a saddle point problem that can be solved iteratively via its dual Schur complement problem. Once we have recovered the Lagrange multipliers, the computation of the primal variables can be easily done.

Since then this method has been applied to many problems. New FETI versions as FETI 2 and FETI-DP had appeared (Farhat et al. 2000 [2] [3], Mandel and Tezaur 2001 [13] [12], Widlund and Klawon, [7] [8]) as well as FETI for mortars (Stefanica [15]). Today, the FETI method represents one of the most powerful and frequently used Domain Decomposition (DD) methods. The possibility of parallel programming offered by this method (see [5] [9]), the moderate dependence of the iteration number on the complexity of the problem, as well as scalability and robustness were among the factors which contributed to the success and the wide spreading of FETI in present.

In 2002 Boundary Element Tearing and Interconnecting (BETI) was introduced by Langer and Steinbach [10]. The main idea of BETI is to use the discrete version obtained by the boundary element discretization, instead of the discrete version of the Steklov-Poincaré operator offered by the finite element discretization. The Dirichlet preconditioners are replaced by preconditioners derived from the hypersingular operator.

As a logical continuation of the BETI technique, Langer and Steinbach [11] introduced the Coupled Finite and Boundary Element Tearing and Interconnecting Methods (FETI/BETI). In dependence on each subdomain problem data both discrete Steklov-Poincaré operator versions can be used,i.e. the one originating from the finite element side as well as the one descending from boundary element.

Various preconditioners were proposed in [11] and a rigorous analysis was done. The scope of this paper is to present the application of the coupled FETI/BETI to a two dimensional electromagnetic field problem resulting from Maxwell's equations. We apply various preconditioners proposed in [10] and [11] and we obtain some convincing numerical results in concordance with the analysis.

The rest of the paper is organised as follows: In the next section we present the coupled FETI/BETI formulation. Section 3 is dedicated to the presentation of some preconditioners. Section 4 deals with a two dimensional model problem originating from Maxwell's equations and displays the corresponding numerical results. Finally, in Section 5 some conclusions are drawn about the use of the presented preconditioners and about the robustness of the scaled hypersingular preconditioner. It turns out that the boundary element preconditioners fit well even to the FETI domains, as can be seen in theory (Lemma 3.1 in Section 3)

2 Coupled FETI/BETI Formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. We consider the following Dirichlet boundary value problem (BVP):

$$-div[\alpha(x)\nabla u(x)] = f(x) \quad \text{for } \ x \in \Omega, \quad u(x) = 0 \quad \text{for } \ x \in \Gamma. \tag{1}$$

Let us also assume that Ω splits into subdomains Ω_i

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i,\ \Omega_i \cap \Omega_j = \emptyset \ \text{ for } \ i \neq j,\ \Gamma_i = \partial\Omega_i,\ \Gamma_{ij} = \Gamma_i \cap \Gamma_j,\ \Gamma_S = \bigcup_{i=1}^{p} \Gamma_i.$$

In the following we will make an extra assumption, namely that the coefficient function α is piecewise constant, i.e.

$$\alpha(x) = \alpha_i > 0 \quad \text{ for } \ x \in \Omega_i,\ i = 1, \ldots, p.$$

Thus instead of the global BVP (1) we have to solve now the local problems

$$-\alpha_i \Delta u_i(x) = f(x) \quad \text{for } \ x \in \Omega_i, \quad u_i(x) = 0 \quad \text{for } \ x \in \Gamma_i \cap \Gamma \tag{2}$$

along with the transmission conditions on the internal boundaries

$$u_i(x) = u_j(x), \quad \alpha_i \frac{\partial}{\partial n_i} u_i(x) + \alpha_j \frac{\partial}{\partial n_j} u_j(x) = 0 \quad \text{for } \ x \in \Gamma_{ij}, \tag{3}$$

where n_i is the unit outward normal vector with respect to Γ_i.

The solution of the Dirichlet BVP

$$-\alpha_i \Delta u_i(x) = f(x) \quad \text{for } \ x \in \Omega_i, \quad u_i(x) = g_i(x) \quad \text{for } \ x \in \Gamma_i \tag{4}$$

leads us to the Dirichlet-Neumann map

$$t_i(x) := \alpha_i \frac{\partial}{\partial n_i} u_i(x) = (S_i u_i)(x) - (N_i f)(x) \quad \text{for } \ x \in \Gamma_i, \tag{5}$$

where

$$\begin{aligned} &S_i : H^{1/2}(\Gamma_i) \longrightarrow H^{-1/2}(\Gamma_i) \quad \text{denotes the Steklov-Poincaré operator,} \\ &N_i : \tilde{H}^{-1}(\Omega) \longrightarrow H^{-1/2}(\Gamma_i) \quad \text{denotes the Newton potential.} \end{aligned}$$

For the definition of the spaces $\tilde{H}^{-1}(\Omega)$, $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$ we refer the readers to [17].

The coupled BVP (2)–(3) is then equivalent to the following problem:

$$\begin{array}{ll} t_i(x) = (S_i u_i)(x) - (N_i f)(x) & \text{for } \ x \in \Gamma_i, \\ u_i(x) = 0 & \text{for } \ x \in \Gamma_i \cap \Gamma, \\ t_i(x) + t_j(x) = 0 & \text{for } \ x \in \Gamma_{ij}, \\ u_i(x) = u_j(x) & \text{for } \ x \in \Gamma_{ij}. \end{array} \tag{6}$$

Let us now consider the trace space $H^{1/2}(\Gamma_S) := \{u \,|_{\Gamma_s} \colon u \in H^1(\Omega)\}$ on the skeleton Γ_S and its subspace

$$H_0^{1/2}(\Gamma_S, \Gamma) := \{v \in H^{1/2}(\Gamma_S) \ : \ v(x) = 0 \quad \text{for } x \in \Gamma\}.$$

So we have to solve now the skeleton problem : Find a function $u \in H_0^{1/2}(\Gamma_S, \Gamma)$ such that the equations

$$(S_i u_i)(x) + (S_j u_j)(x) = (N_i f)(x) + (N_j f)(x) \quad \text{for } x \in \Gamma_{ij} \tag{7}$$

are satisfied on all local coupling boundaries Γ_{ij} and $u_i := u$ on Γ_i. This leads us to the variational problem: Find $u \in H_0^{1/2}(\Gamma_s, \Gamma)$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} (S_i u)(x) v(x) ds_x = \sum_{i=1}^{p} \int_{\Gamma_i} (N_i f)(x) v(x) ds_x \quad \forall\, v \in H_0^{1/2}(\Gamma_S, \Gamma). \tag{8}$$

Due to the implicit definition of the local Dirichlet-Neumann map (5) it is in general not possible to discretize problem (8) in an exact manner. Thus we have to approximate the local Dirichlet problems which occur in the definition of the local Neumann-Dirichlet map. We can do that by using either finite elements or boundary elements, see [11].

The Galerkin discretization of the problem (8) with boundary or finite elements approximations of the local Dirichlet problems yields the linear system

$$\sum_{i=1}^{p} A_i^\top S_{i,h}^{FEM/BEM} A_i \underline{u} = \sum_{i=1}^{p} A_i^\top f_{i,h}^{FEM/BEM}, \tag{9}$$

which is uniquely solvable due the positive definiteness of the assembled stiffness matrix. The matrices $S_{i,h}^{FEM/BEM}$ are nothing else than the discretized version of the Steklov-Poincaré operator by FEM or BEM. A_i denote the connectivity matrices that map the vectors $\underline{v}$ originating from the global discretization of Ω in their local components $\underline{v}_i$ corresponding to the local discretization of Ω_i, $\underline{v}_i = A_i \underline{v}$. For more details see [11]. Then we introduce the local vectors $\underline{u}_i = A_i \underline{u}$. The continuity of the primal variables across the interfaces can be enforced by the constraint

$$\sum_{i=1}^{p} B_i \underline{u}_i = 0\,. \tag{10}$$

Each row of the matrix $B = (B_1, \ldots, B_p)$ is connected with a pair of matching nodes across the interface. The entries of such a matrix are 1 and -1 for the indices corresponding to the matching nodes and 0 elsewhere. By introducing the Lagrange multiplier $\underline{\lambda} \in \mathbb{R}^m$ (where m is the number of equations in (10)) we have to solve the linear system:

$$\begin{pmatrix} S_{1,h}^{BEM/FEM} & & & B_1^\top \\ & \ddots & & \vdots \\ & & S_{p,h}^{BEM/FEM} & B_p^\top \\ B_1 & \ldots & B_p & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_p \\ 0 \end{pmatrix}. \tag{11}$$

For i=1,...,p, we consider now the solvability of the local systems

$$S_{i,h}^{FEM/BEM} \underline{u}_i = \underline{f}_i^{FEM/BEM} - B_i^\top \underline{\lambda}. \tag{12}$$

Let us define

$$\tilde{S}_{i,h}^{FEM/BEM} = S_{i,h}^{FEM/BEM} + \beta_i \underline{e}_i \underline{e}_i^\top , \tag{13}$$

with $\beta_i = 0$ if Ω_i is nonfloating (i.e. $\Gamma_i \cap \Gamma \neq \emptyset$) and some regularization constant $\beta_i > 0$ for the floating subdomains (i.e. $\Gamma_i \cap \Gamma = \emptyset$). We also have to impose the solvability condition for the floating domains:

$$\underline{e}_i^\top \left[\underline{f}_i^{FEM/BEM} - B_i^\top \underline{\lambda} \right] = 0. \tag{14}$$

This leads us to the systems :

$$\tilde{S}_{i,h}^{FEM/BEM} \underline{u}_i = \underline{f}_i^{FEM/BEM} - B_i^\top \underline{\lambda}, \tag{15}$$

which are equivalent to (12) and uniquely solvable. However for floating domains we must insert also the constant kernel functions. Thus the general solution of (15) has the form

$$\underline{u}_i = \left[\tilde{S}_{i,h}^{FEM/BEM} \right]^{-1} \left[\underline{f}_i^{FEM/BEM} - B_i^\top \underline{\lambda} \right] + \gamma_i \underline{e}_i \, , \tag{16}$$

with $\gamma_i = 0$ for all non floating subdomains.

Substituting these local solutions into the equation (11), we obtain the Schur complement system

$$\sum_{i=1}^{p} B_i \left[\tilde{S}_{i,h}^{FEM/BEM} \right]^{-1} B_i^\top \underline{\lambda} - \sum_{i=1}^{p} \gamma_i B_i \underline{e}_i = \sum_{i=1}^{p} B_i \left[\tilde{S}_{i,h}^{FEM/BEM} \right]^{-1} \underline{f}_i^{FEM/BEM}. \tag{17}$$

Thus we obtain the linear system

$$\begin{pmatrix} F & -G \\ G^\top & 0 \end{pmatrix} \begin{pmatrix} \underline{\lambda} \\ \underline{\gamma} \end{pmatrix} = \begin{pmatrix} \underline{d} \\ \underline{e} \end{pmatrix} , \tag{18}$$

where

$$F := \sum_{i=1}^{p} B_i \left[\tilde{S}_{i,h}^{FEM/BEM} \right]^{-1} B_i^\top , \qquad G := (B_i \underline{e}_i)_{\{1 \leq i \leq p \,|\, \Gamma_i \cap \Gamma = \emptyset\}}$$

and the right hand side is defined as

$$d := \sum_{i=1}^{p} B_i \left[\tilde{S}_{i,h}^{FEM/BEM} \right]^{-1} \underline{f}_i^{FEM/BEM}, \quad \underline{e} := (\underline{e}_i^\top \underline{f}_i^{FEM/BEM})_{\{1 \leq i \leq p \,|\, \Gamma_i \cap \Gamma = \emptyset\}}.$$

Let us define the orthogonal projection

$$P := I - G \left(G^\top G \right)^{-1} G^\top : \Lambda := \mathbb{R}^M \longrightarrow \Lambda_0 := \ker G^\top = (\mathrm{Range}\, G)^\perp$$

with respect to the Euclidean scalar product. Now, we decouple the computation of λ from γ. Applying P to the first equation in (18) we obtain the equation

$$PF\underline{\lambda} = P\underline{d}, \tag{19}$$

Algorithm 1 Projected Preconditioned Conjugate Gradient Iteration (PPCG)

FETI/BETI (PPCG) Iteration

$\underline{\lambda}_0 = G\left(G^\top G\right)^{-1}\underline{e}$, $r_0 = P\underline{d} - PF\underline{\lambda}_0$, $n = 1$ { initialization step }
{ begin iterations }

while $(Mr_{n-1}, r_{n-1}) \geq \varepsilon^2(Mr_0, r_0)$ **do**

$w_{n-1} = Pr_{n-1}$ { projection step}

$z_{n-1} = Mw_{n-1}$ { precondition step}

$y_{n-1} = Pz_{n-1}$ { project the correction}

$\beta_n = (y_{n-1}, r_{n-1})/(y_{n-2}, r_{n-2})$, $(\beta_1 = 0)$

$p_n = y_{n-1} + \beta_n p_{n-1}$, $(p_1 = y_0)$ { update the search direction in Λ_0}

$\alpha_n = (y_{n-1}, r_{n-1})/(Fp_n, p_n)$

$\lambda_n = \lambda_{n-1} + \alpha_n p_n$ { update of the iterate}

$r_n = r_{n-1} - \alpha_n PFp_n$ { update the defect }

$n = n + 1$

end while

{ end iteration loop }

because $PG\,\underline{\gamma}=0$. As soon as we get λ from (19) we obtain

$$\underline{\gamma} = \left(G^\top G\right)^{-1} G^\top\left(F\underline{\lambda} - \underline{d}\right). \tag{20}$$

Finally in the end we get the vectors u_i from (16).

The dual problem (19) is solved by a preconditioned conjugate gradient subspace iteration with some preconditioner M. The matrix by vector multiplication with the F involves the application of the inverse modified discrete Steklov-Poincaré operators $\left[\tilde{S}_{i,h}^{FEM/BEM}\right]^{-1}$ to some vector $B_i^\top\underline{\lambda}$. This can be done by solving directly extended systems for the local Neumann problems as the usual technique in tearing and interconnecting methods.

3 Preconditioners

In this section we briefly present some possible options for the preconditioner M used in Algorithm 1 described above.

(a) **Dirichlet preconditioner**. This is the adapted coupled FETI/BETI version of the first known preconditioner used in FETI methods, adequate for moderately changing coefficients and no crosspoints. (For technical details and implementation see [6]).

$$M = BS_hB^\top = \sum_{i=1}^{p} B_i S_{i,h}^{FEM/BEM} B_i^\top \tag{21}$$

(b) **Hypersingular preconditioner I**.It is known as a BETI modification of the Dirichlet FETI preconditioner based on the spectral equivalence between the matrices generated by the discrete hypersingular operator and the discrete Steklov-Poincaré operator.

$$M = BD_hB^\top = \sum_{i=1}^{p} B_iD_{i,h}B_i^\top \tag{22}$$

with $D_h = diag(D_{i,h})_{i=1:p}$ and $D_{i,h}$ are the boundary element matrices corresponding to the discrete hypersingular boundary operators D_i (see [10]).

(c) **Hypersingular preconditioner II**. The slightly modified version of the first hypersingular preconditioner, adapted for moderate changing coefficients with crosspoints looks as follows:

$$M = (BB^\top)^{-1}BD_hB^\top(BB^\top)^{-1} . \tag{23}$$

(d) **Scaled hypersingular BETI preconditioner**. After a closer look this preconditioner turns out to be a brilliant combination between the well known Jacobi preconditioner and Hypersingular preconditioner II. Very stable with respect to large coefficient jumps. Despite its long and not very friendly look it is not very expensive to calculate as it could look at a first view. It also has a good behaviour in the case of crosspoints.

$$M = (BC_\alpha^{-1}B^\top)^{-1}BC_\alpha^{-1}D_hC_\alpha^{-1}B^\top(BC_\alpha^{-1}B^\top)^{-1} \tag{24}$$

where $C_\alpha = diag(C_{\alpha,i})_{i=1:p}$ and $C_{\alpha,i} = diag(c_l^i)_{l=1:m_i}$ with appropriately chosen weights (see [10] and [11]).

(e) **Scaled Schur complement FETI preconditioner**, the FETI counterpart of (d) introduced by Klawonn and Widlund [7] (see also [1] and [4]).

$$M = (BC_\alpha^{-1}B^\top)^{-1}BC_\alpha^{-1}S_hC_\alpha^{-1}B^\top(BC_\alpha^{-1}B^\top)^{-1} \tag{25}$$

At the first glance the inverting of the first and the last component of the last two preconditioners, namely $(BC_\alpha^{-1}B^\top)$ may raise some problems but looking more carefully we observe that this matrix has the form $\begin{pmatrix} D & 0 \\ 0 & T \end{pmatrix}$ that is a combination between a diagonal matrix D and a tridiagonal bandwidth matrix T which is relatively easy to invert. The tridiagonal bandwidth part corresponds to the Lagrange multipliers corresponding to the crosspoints.

The following lemma was proved in [11] and constitutes the theoretical foundation for the preconditioners introduced above.

Lemma 3.1. *The local boundary element Schur complement matrix $S_{i,h}^{BEM}$ and the local finite element Schur complement matrix $S_{i,h}^{FEM}$ are spectrally equivalent to the exact Galerkin matrix $S_{i,h}$ of the Steklov-Poincaré operator S_i and to the boundary element matrix $D_{i,h}$ of the local hypersingular boundary integral operator D_i, i.e.,*

$$S_{i,h}^{BEM} \simeq S_{i,h}^{FEM} \simeq S_{i,h} \simeq D_{i,h}$$

for all i=1, ..., p, where $A \simeq B$ means that the matrices A and B are spectrally equivalent (with spectral constants which are independent of discretization constants).

A detailed analysis of the preconditioners presented above can be found in [11].

4 A 2D Model Problem and Numerical Results

In this section we will apply various preconditioners to a magnetostatic field problem. Let us first consider a rectangular domain Ω divided into 15 square subdomains Ω_i, i=1,...,15 (see Fig. 1). On the non-floating subdomains we have air, in Ω_5 and Ω_{11} we have two electric coils and in middle (Ω_8) an iron core. Deriving from the Maxwell's equations (see [14]) we have to solve the following problem to compute the magnetic potential: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nu(x) \nabla u(x) \cdot \nabla v(x) dx = \int_\Omega f(x) v(x) dx \;\; \forall \;\; v \in H_0^1(\Omega) \tag{26}$$

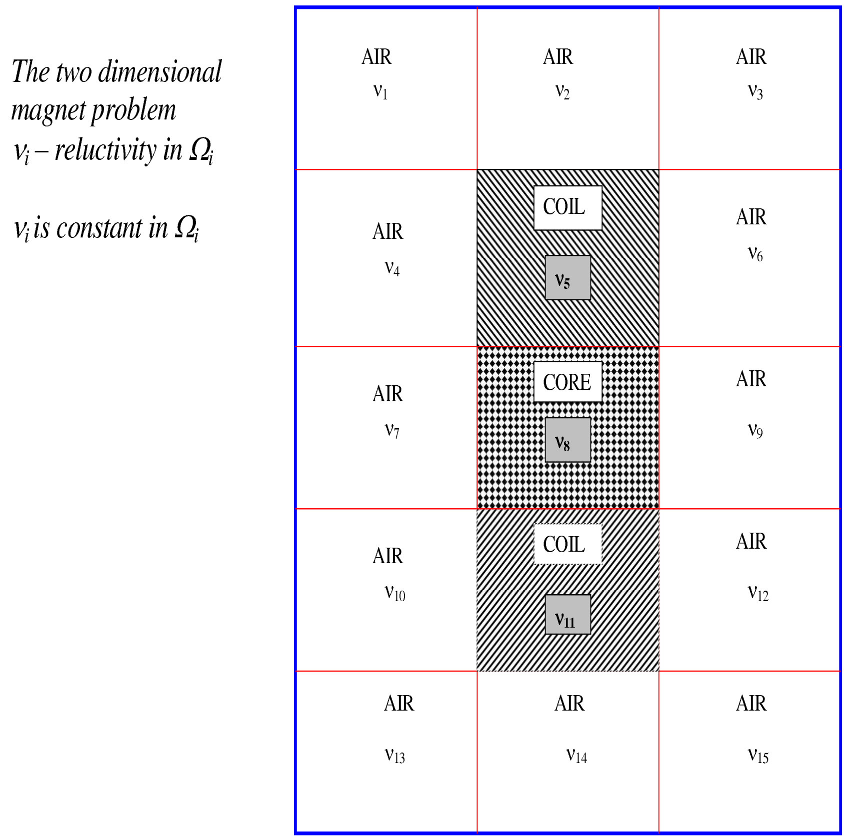

Figure 1: Magnetic field problem.

where the source (impressed current) is given by :

$$f(x) = \begin{cases} c\ Am^{-2} & \text{if } x\ \in \Omega_5, \\ -c\ Am^{-2} & \text{if } x\ \in \Omega_{11}, \\ 0 & \text{elsewhere,} \end{cases} \quad \text{where c is a positive real constant} \tag{27}$$

$\nu = \frac{1}{\mu}$ is the reluctivity, $\mu = \mu_0 \cdot \mu_r$ is the permeability, $\mu_0 = 1.245 \cdot 10^{-6} \frac{Vs}{Am}$ is the permeability in vacuum and μ_r is the relative permeability with

$$\mu_r(x) = \begin{cases} 1 & \text{for air} \\ 1.5 & \text{for coils} \\ 1000 & \text{in the core} \end{cases}$$

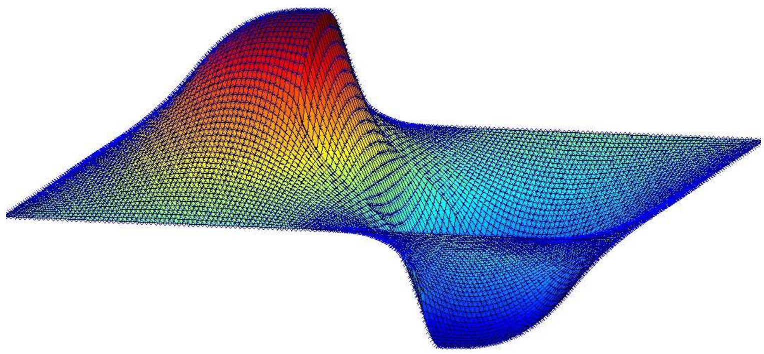

Figure 2: Solution of the magnetostatic field problem

We used various direct solvers for inverting the discrete local single layer potentials for the applying of the preconditioner matrix as well as for the projection steps. As stopping criteria for the global (PCG) algorithm we used a relative error $\varepsilon = 10^{-6}$(see PPCG algorithm). Table 1 shows the iteration numbers in the case when $\mu_r = 1$ everywhere which means no coefficients jumps. In Table 2 the iteration numbers are shown in the case of our model problem with a current density c = 1000. During the tests we changed the values of μ_r in order to observe how the various preconditioners behave with and without jumps.

In the tests, we used the Dirichlet preconditioner (DP), the hypersingular preconditioner (HP) and the scaled hypersingular preconditioner (SHP). For the FETI we used finite element discretization for all subdomains, and for the combined FETI/BETI we used finite element discretization for the floating domains (core and coils) and boundary elements for the nonfloating domains (surrounding air). The numbers appearing in the first row of each table represent the number of discretization nodes on the boundary of each subdomain and between the brackets appears the number of the Lagrange multipliers of the global system. In the Figures 2, 3 and 4 we can observe the solution of the problem and

No Jumps		40 (222)	60 (332)	80 (442)	104 (574)
FETI	HP	21	22	25	30
	DP	17	17	19	18
	SHP	16	16	16	16
FETI BETI	HP	32	33	36	41
	DP	36	31	35	38
	SHP	16	16	17	17

Table 1: The number of iterations steps for no jumps, μ_r =1 everywhere.

Large Jumps		40 (222)	60 (332)	80 (442)	104 (574)
FETI	HP	82	76	83	118
	DP	47	55	58	60
	SHP	13	13	13	13
FETI BETI	HP	89	98	105	134
	DP	68	101	115	142
	SHP	13	13	13	14

Table 2: The number of iterations steps for large jumps, μ_r =1000 in the core, μ_r=1.5 in the coils and μ_r=1 elsewhere.

the behaviour of relative error on a logarithmic scale for each preconditioner respectively. The numerical experiments were done using MATLAB. The BEM matrices were generated using the software package OSTBEM developed by Olaf Steinbach (see [16]).

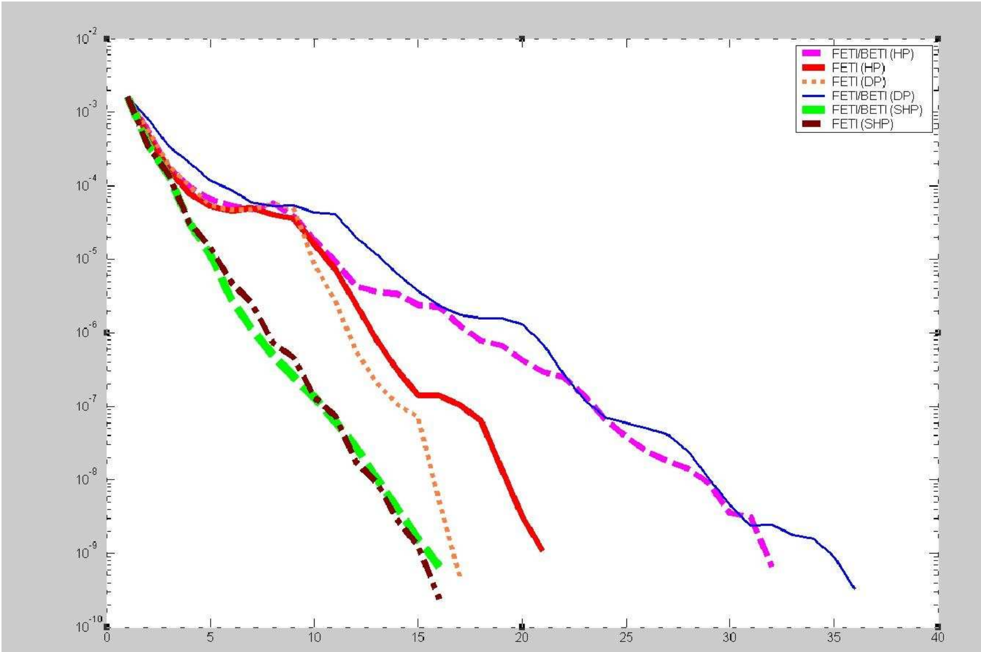

Figure 3: The norm of the error on logarithmic scale (no jumps)

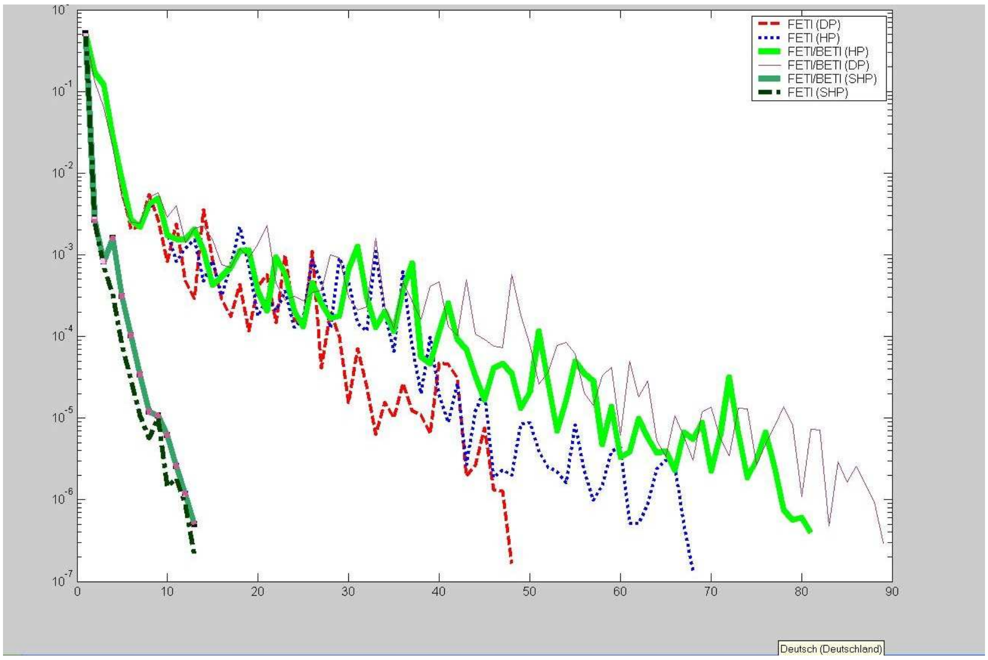

Figure 4: The norm of the error on logarithmic scale (with large jumps)

5 Concluding Remarks

Both FETI and BETI methods are using discrete versions of the Steklov-Poincaré operator. Therefore, the coupling of FETI and BETI techniques and the use of BETI preconditioners even in the FEM domains are very natural. Lemma 3.1 provides us with the theoretical support for this. The standard Dirichlet and hypersingular preconditioners have an acceptable convergence behaviour in the case of constant coefficients or moderate jumps in the coefficients (see Fig. 3). We observe from Fig. 4 that these preconditioners heavily suffer from the presence of large coefficients jumps. In addition to the reduction of the convergence rate some strange oscillations appear in convergence history. To overcome these drawbacks, an appropriate scaling must be introduced. The scaled hypersingular preconditioner that uses the scaling introduced by Klawonn and Widlund [7] ensures a fast and smooth reduction of the iteration error as in the case of constant coefficients (see again Fig. 3 and Fig. 4).

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