"Approximate" Logic - An Overview

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May 20, 1998

Abstract

By "approximate" logics we mean extensions and modifications of classical (first order) predicate logic which provide means for representing and reasoning with knowledge that are in some sense finer than the classical two- valued approach. We give a classification of such logics and an overview of the literature on the subject.

1 Introduction

Logics which extend or modify classical (first order) predicate logic to provide means for representing and reasoning with knowledge that are in some sense finer than the classical two- valued approach usually consist of two parts (cf. [94]):

1. the knowledge representation part
2. the uncertainty management part

For the first part we already have fixed classical predicate logic [28], [104], possibly with some extension. In the following we will study how various different uncertainty management mechanisms can be used for the second part, obtaining different approximate logics. Also here classical mathematics provided the first basis with probability theory [34]. Later, research in Artificial Intelligence showed the need for various other uncertainty calculi [87], [54], [76], [60], [63], [107], [109], [108], [111]. In this context often the term "logic" is used, when concepts and tools from classical predicate logic are used, however often without the precision and rigor that would qualify them as logics in the classical mathematical sense.

There are various parameters along which approximate logics can be classified. The next section will describe these parameters. The following sections show how various forms of uncertainty management can be used to obtain different approximate logics and the last section shows how such logics have been used in logic programming and automated theorem proving.

The paper contains no original results but to our knowledge there is no other overview that takes such a broad viewpoint and systematization of approximate logics.

2 A Classification of Approximate Logics

The most important parameter along which approximate logics can be classified is, of course, the uncertainty management part used. Depending on this, the logic will be either truth-functional or not. Truth-functionality means that the truth value of a (propositional) formula is a function of the truth values of its immediate subformulas (i.e. ignoring the detailed structure of the subformulas). This means that we can build truth-tables for the connectives (conjunction, disjunction and negation) in this logic. Thus, in a truth- functional logic, information about the relation between subformulas (for example the equivalence of subformulas), can be lost (compare this to interval arithmetic, e.g. [73], [74], [4], [90], [91], [77], [46] and [89], where also the information about equal terms is lost).

Another categorization of approximate logics comes from the place where the uncertainty calculus is applied:

- To the truth of sentences, or
- to the solution set of a (open) formulas.

In classical predicate logic the first one is done by the notions of validity and satisfiability of a sentence while the second one is done by the existential and universal quantifiers. We will call the first approach degree of belief uncertainty and the second approach statistical uncertainty.

These examples also show another categorization: One can either do the measurement on the meta- level (validity/satisfiability) or within the language (existential and universal quantifiers).

The above parameters from a three- dimensional grid s.t. each of the following logics covers a certain part of this grid.

3 Modal Logic

This is not yet an approximate logic in the above sense because it still uses the classical boolean truth values for uncertainty management, but it is an extension of the knowledge representation part which is often used in combination with approximate logics and for similar applications. Basic modal logic [22, 51] provides operators for reasoning about the validity and satisfiability of a formula *within* the language (using the necessity operator $\Box$, the possibility operator $\Diamond$). Extensions into the direction of a general logic of knowledge (epistemic logic) include:

- A relation on the set of models (now called possible worlds, represented by so called *Kripke structures* [61]). Necessity is defined by truth in all models related to a "current" model.

- More than one set of knowledge about world (agents), represented by different such relations.

- Reasoning about common knowledge (every agent knows that every agent knows, that ...)

An overview from point of view of recent developments is given in [45].

4 Generalized Quantifiers

The theory of generalized quantifiers started from the study of quantifiers like "for infinitely many" or "for uncountable many" [75] in classical predicate logic and was later extended for the study of natural language semantics [66] and computer science (for example for expressing the Hamiltonicity of graphs in first order predicate logic [24]), see [114] and [115] for overviews.

5 Many- Valued Logic

Many (or Multiple-) Valued Logic [69, 68, 92, 113] generalizes the boolean operators from two to more values and builds logics upon these. Examples include:

- A three- valued logic, with the traditional truth values $\mathbf{T}$ and $\mathbf{F}$ plus the additional truth value $\mathbf{U}$ (for undefined), where for any logical connective its application to truth values is equal to the according boolean application if all the truth values are boolean, otherwise the result is $\mathbf{U}$. This logic models the occurrence of exceptions.

- A four valued logic, where the truth values consist of the subsets of $\{\mathbf{T}, \mathbf{F}\}$ and the logical operations are defined as the application of the according boolean operators to all the set elements. Here $\{\}$ models incomplete and $\{\mathbf{T}, \mathbf{F}\}$ models inconsistent knowledge [16].

  This can be generalized [39, 41] to the notion of a bilattice in which two lattices are combined. The elements of the first lattice denote the belief that a formula is true and the elements of the second lattice denote the belief that a formula is false. An operator can be defined which creates a bilattice out of an arbitrary lattice.

- A logic with truth values in the real interval $[0, 1]$ and conjunction defined as the minimum, disjunction defined as the maximum and negation defined as $\lambda x.1 - x$ of the arguments. Such logics are often called fuzzy logics (although they are different to the logics described the next section).

Here again one can use the bilattice construction to get truth values consisting of two real numbers. See the later section on Evidence theory for a similar approach.

Multi- valued logic is truth- functional (e.g. the truth value of a sentence $A \wedge B$ is a function of the truth values of A and B). The is a severe restriction. For example, contrary to intuition, in the second logic above, if A has the truth value $\{\mathbf{T}, \mathbf{F}\}$, then $A \leftrightarrow A$ and $A \wedge \neg A$ have the truth- values $\{\mathbf{T}, \mathbf{F}\}$, although using knowledge about the equality of the different A's one could infer $\{\mathbf{T}\}$ for the first formula and $\{\mathbf{F}\}$ for the second formula.

6 Fuzzy Logic

Fuzzy logic builds on fuzzy set theory [124, 126] which generalizes the membership function of classical set theory to a function to the real interval $[0,1]$. Fuzzy sets can be combined for example by intersection, union and negation operators, most of the times modelled by the minimum, maximum and inversion $(\lambda x.1 - x)$ of the membership functions.

The truth values of fuzzy logic are fuzzy sets. They are combined by their set- theoretic operations and by modifiers such as "very", "more or less", and so on. Only a countable set of such truth values are allowed. Reasoning amounts to approximately solving a set of equations on these truth values. This makes fuzzy logic hardly recognizable as a logic at all.

7 Probabilistic Logic

Probabilistic logics uses elements of the real interval $[0,1]$ as truth values and uses the axioms of probability theory as the basis for the inference of these truth values. So, different to many- valued logics with these truth-values, probabilistic logics are not truth- functional. There are approaches for degree of belief uncertainty and for statistical uncertainty (see for example [8]):

- Assigning probabilities to sentences (degree of belief uncertainty) is an old problem. Early studies came from a philosophical or informal point of view [57, 53, 21], later it has been studied from the viewpoint of mathematical logic and model theory (with the aim of extending the mathematical theory of probability). The first in this direction was Gaifman [40] for first- order languages. His work has been extended to infinitary logic [98] later. The approach of defining the probability of a sentence as the sum of the probabilities of its models has first been taken by Łoṡ [67], and has later been extended by Fenstad [35, 37]. See [36] for an overview.

Later the emphasis shifted to more practical points of view, especially for applications in artificial intelligence (expert systems). Nilsson [79] and, independently [43] show that assigning consistent probability values to sentences results in a linear programming problem. For example, given two sentences P and $P \to Q$ with associated probabilities the probability of Q lies in a certain interval.

Later languages were defined, which allowed reasoning about probabilities within the language (using a probability operator) and formal calculi, their completeness and decision procedures were studied [32, 1].

- Logical languages for reasoning about the size of solution sets (i.e. statistical uncertainty) correspond to the logic developed and used in this thesis. They also have been studied from the viewpoint of mathematical logic and model theory first. H. J. Keisler introduced probability quantifiers [55, 49, 56], where a formula $(Px \geq r)[\phi(x)]$ means that the solution set of ϕ is greater than a certain real constant r. He uses an infinitary language and allows no function symbols and only one measurable 2- ary relation symbol. The result is a language where all occurring solution sets are measurable (since no classical quantifiers are allowed). [50] gives an improved completeness theorem for this language.

  Also here, later approaches shifted the viewpoint to applications in artificial intelligence. Bacchus [6, 7, 8, 11] extended the first- order predicate logic with an operator for reasoning about statistical knowledge. Differently to Keisler's approach, also classical quantifiers and arbitrarily many function and relation symbols are allowed. Also the probabilities need not be fixed constants, but reasoning about probability values is a part of the language. This results is a two- sorted language (the domain of reasoning and a field for representing the probabilities).

  His approach has the following restrictions, which come from the fact it has been created with applications (from AI) in mind, where only finite sets (or at least no uncountable sets) occur:

  - It explicitly allows only structures, where no unmeasurable sets can be created by the language (without giving evidence about the form of such structures).
  - It does not require countable additivity of the used measures. This means that for uncountable structures, they are no measures in the mathematical sense any more.
  - It uses only the field axioms as the knowledge about probability values.

Hong [48] was the first to propose the use of a logic modeling statistical uncertainty in algebraic constraint solving over the reals.

8 Possibility Theory / Possibilistic Logic

Possibility theory ([125], [27], chapter 8 in [126]) arises from the modeling of vague knowledge, e.g. the knowledge that a person is "tall" (see also the chapter on fuzzy logic). This leads to the notion of possibility measure. Given a universe Ω such a possibility measure P follows these laws:

$$P(\emptyset) = 0$$

$$P(\Omega) = 1$$

$$P(A \cup B) = \max(P(A), P(B))$$

$P(\Omega - A)$ is called the necessity of A ($N(A)$) and we have:

$$N(\emptyset) = 0$$

$$N(\Omega) = 1$$

$$N(A \cap B) = \min(N(A), N(B))$$

Replacing probability measures by possibility measures, most of the notions and constructions from probabilistic logic can be straightforwardly translated to possibility theory to get a possibilistic logic. Since the possibility measure of a set $A \cap B$ cannot directly be inferred from the possibility measures of A and B, the resulting logic is not truth- functional. We do not know of any approaches for using possibility theory for a logic of statistical uncertainty. This would result in a language with builtin maximum and minimum quantifiers.

9 Dempster- Shafer Theory of Evidence

This theory [99, 101, 42] generalizes both probability theory and possibility theory. In the finite case it assigns to all subsets of a universe a positive real number s.t. the sum of all assignments is one (this is called the basic probability assignment). Then the measure of a subset of the universe (the belief function) is the sum of all assigned numbers of subsets of this set. If one assigns non-zero numbers to singletons only, then one gets a probability measure. If the non- zero elements can be ordered s.t. we get a chain $q_0 \subseteq \cdots \subseteq q_i \subseteq \cdots \subseteq q_n$, then one gets a necessity measure (the dual of a possibility measure).

This results in the following axioms for the general (possibly infinite) case:

$$B(\emptyset) = 0$$

$$B(\Omega) = 1$$

$$B(A_1 \cup \cdots \cup A_k) \geq \sum_{I \subseteq \{1,\ldots,k\}, I \neq \emptyset} (-1)^{|I|+1} B(\bigcap_{i \in I} A_i)$$

The theory also provides a method for dealing with inconsistent evidence (the basic probability assignment does not sum up to 1, or even $B(A) + B(\Omega - A) > 1$). The resulting process, renormalization, re- distributes the values s.t. they again sum up to 1, by using a rule called Dempster's rule. So, interestingly, the theory normalizes inconsistent knowledge but does not normalize incomplete knowledge (e.g. when $B(A) = 0.8$ but $B(\neg A) = 0$). From or own subjective viewpoint a more elegant approach would do reasoning completely without renormalization but would provide a process for acquiring probabilities from the resulting (incomplete or inconsistent) information.

For the relation between the Dempster- Shafer theory of evidence and probability theory see the introduction to chapter 7 in [101], where the claim is that belief function's only correspond to lower probabilities (or inner measures in general) as long as no renormalization has to be done. Further discussion is given in [100]. [88] shows how DS- theory can be formulated in terms of a probabilistic propositional logic.

[30] discusses the relationship between belief functions and inner/outer measures alongside with developing a logic for reasoning about evidence, claiming their equivalence within the presented framework (which does not allow reasoning about inconsistent knowledge). Saffiotti [95] develops a first-order logic extended with evidence theory. It does assigns belief only to whole formulas but allows reasoning about inconsistent knowledge.

10 Rough Logic

Rough logic builds on the theory of Rough Sets [84, 85]. Let $\mathcal{U}$ be a universe. Let $\sim$ be an equivalence relation on $\mathcal{U}$ (e.g. constructed by taking a function ϕ on $\mathcal{U}$ and defining $x \sim y$ iff $\phi(x) = \phi(y)$). Given a set $X \subseteq \mathcal{U}$ one can define the $\sim$- lower approximation (or core) of X ($\underline{\sim}X$) as the union of all equivalence classes of $\sim$ for which all elements are in X, and the $\sim$- upper approximation (or envelope) of X ($\overline{\sim}X$) as the union of all equivalence classes of $\sim$ for which there is an element in X. Intuitively this means that $\underline{\sim}X$ contains all the elements which can possibly be classified as belonging to X using $\sim$, and $\overline{\sim}X$ contains all the elements which must necessarily be classified as belonging to X using $\sim$. Now we have: $\underline{\sim}X \subseteq X \subseteq \overline{\sim}X$ and we call such a pair ($\underline{\sim}X, \overline{\sim}X$) a rough set. Alternatively we call it the rough measure of the set X.

[82] and [81] give a logic based on rough sets based for expressing degree of belief uncertainty. Given an equivalence relation on the set of worlds $\mathcal{U}$, the set of models of a sentence ϕ defines a rough set $R(\phi)$. This can be taken as the (rough) truth value of the sentence. One can also coarsen this approach by defining ϕ to be necessarily valid iff $R(\phi)$ is ($\mathcal{U}$,$\mathcal{U}$), possibly valid iff the envelope of $R(\phi)$ is $\mathcal{U}$, possibly invalid iff the core of $R(\phi)$ is empty and necessarily invalid iff $R(\phi)$ is $(\emptyset, \emptyset)$ (the terminology used here is different from the one in the above papers). Now one can define a calculus for reasoning in this logic independent from a certain partitioning of $\mathcal{U}$. Since the resulting logic is truth- functional and has finitely many truth values, it is just a certain many valued logic.

There are also approaches that try to express statistical uncertainty in logics based on rough sets. They define rough existential and universal quantifiers where rough existence means that there is a whole equivalence class and rough universal quantification means that in every equivalence class there is at least one element (see e.g. [83], [62] and [15], which also investigates connections to modal logic).

In [105] and [86] the relation between rough set theory and the theory of evidence is discussed.

11 Supervaluations

The method of supervaluations arose from research on free logics, i.e. logics which doubt sentences like $(\exists x)[x = t]$. Here t is called a non-denoting constant. This method was initiated by van Fraassen [118, 119]), and further developed for example in [106], [72], [18], [19] and [123]. For overviews see [113] and [17]. In this method interpretations leave the values of atomic formulas containing non-denoting constants undefined. Then one uses extensions of an interpretation I which give values to the undefined atomic formulas. A supervaluation over I is a function that assigns "true" to every sentence which is true in all extensions of I, "false" to every sentence which is false in all extensions of I and "undefined" otherwise.

In this logic $\phi \vee \neg\phi$ is true even if both ϕ and $\neg\phi$ have the truth value "undefined". So the logic is not truth- functional.

12 Combinations and General Approaches

Halpern [44] tries to combine the notions of degree of belief and statistical probability quantifiers into one language. The combined language is incomplete and it is not yet completely clear how degrees of belief can be generated from statistical information. Work on this is going on (see [11], [10], [59], [9] and [12]).

Another active research area is the combination of epistemic logics with reasoning about degrees of belief [31], [120], [121], [3], [33], [116].

Ruspini [93] gives a framework for reasoning about knowledge (modal logic) and evidence, and discusses connections to probability theory. Saffiotti [94] gives a general framework for representing uncertain knowledge by identifying a part for knowledge representation (e.g. first order logic) and a part for managing uncertainty (e.g. probability theory).

13 Incorporation into Logic Programming and Automated Theorem Proving

For approaches to automated modal theorem proving see for example [38, 2, 80], for modal logic extensions of logic programming [25, 13, 96].

For many- valued logic [29], [102], [97] allow to annotate clauses with functions which compute the certainty of the head using the certainties of the literals - see also the notion of quantitative Logic Programming [117]. [23] gives a generalization of SLD resolution for symbolic approximate reasoning by supporting five different forms of inference (collected in an ad-hoc manner).

[20] defines a framework for dealing with inconsistencies in logic programming with negation. They use a four- valued logic (originally proposed in [16]) over the lattice { unknown, true, false, inconsistent } (see [58] for a general calculus). In [39] this approach is generalized using bilattices. Kifer and Subrahmanian [58] provide a general framework for logic programming with annotations in lattices, which generalizes the approaches above.

There are various approaches to incorporating fuzzy logic into logic programming (see e.g. [5], [70], [110], [65], [103], [52], [47], [112]). Weigert [122] gives a generalization of the resolution method for such a (multi- valued/fuzzy) logic. Matyska [71] presents a constraint logic programming system over fuzzy sets.

Ng and Subrahmanian [78] present a framework for incorporating probability theory into logic programming by assigning probabilities to subformulas and using the semantics of probabilistic logic. Executing such a program involves solving a linear programming problem. Referring to empirical (statistical) probabilities (as modelled by the system described in this thesis) they explicitely state: “It is not clear how to use Bacchus‘ system as a basis for logic programming".

[64] allows annotations in logic programs that describe the scenarios in which clauses hold. Then from a probability distribution on the scenarios belief and doubt probabilities of answers can be inferred.

[26] generalizes the resolution rule to possibilistic logic.

In evidential support logic programming [14] programs consist of certain first order formulas where evidence values are assigned whole formulas only. Two values ε_1 and ε_2 can be written down telling that the evidence for a sentence is at least ε_1 and at most ε_2. Reasoning is done using Dempster's rules of combination [99].

14 Conclusion

We have given a classification of "approximate logics" and an overview of the literature on the subject. Research is widely scattered among different groups and further steps would be needed for the systematization of the area.

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