Stability and Convergence Analysis of Tikhonov Regularization for Parameter Identification in a Parabolic Equation

Heinz W. Engl [1] and **Jun Zou** [2]

Abstract

In this paper we investigate stability and convergence rates of the Tikhonov regularization method for the identification of the diffusion parameter in a multi–dimensional parabolic equation. By choosing a problem–adapted approach, we obtain much better results than by applying the general theory of Tikhonov regularization of nonlinear inverse problems.

1 Introduction

We will consider throughout the paper the parabolic problem

$$\frac{\partial u}{\partial t} - \nabla \cdot (q(x)\nabla u) = f(x,t) \quad \text{in} \quad \Omega \times (0, \mathrm{T}) \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega \tag{1.2}$$

and the boundary condition

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.3}$$

[1]Institut für Industriemathematik, Johannes Kepler Universität, Altenberger Straße 69, A–4040 Linz, Austria. E-Mail: engl@indmath.uni-linz.ac.at. Supported in Part by Christian Doppler Forschungsgesellschaft (Vienna) and by the Austrian Fonds zur Förderung der wissenschaftlichen Forschung, project F013-08

[2]Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong. E-mail: zou@math.cuhk.edu.hk. The work of this author was partially supported by Hong Kong RGC Grants No. CUHK 338/96E and CUHK 4004/98P.

where Ω is an open bounded connected domain in R^d, $d \geq 1$, with a piecewise smooth boundary $\partial\Omega$. $f \in L^2(0,T;H^{-1}(\Omega))$ is a given source density.

By integration by parts we obtain the variational formulation for the initial-boundary value problem (1.1) - (1.3):

Find $u \in H_0^1(\Omega)$ such that

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega \tag{1.4}$$

and

$$\int_\Omega u_t\,\phi\,dx + \int_\Omega q(x)\nabla u \cdot \nabla\phi\,dx = \int_\Omega f(x,t)\,\phi\,dx, \quad \forall\phi \in H_0^1(\Omega) \tag{1.5}$$

for a.e. $t \in (0,T)$. If we wish to emphasize the fact that the solution u depends on the parameter q, we will use the notation $u(q)$, or $u(q;t)$ or $u(q)(x,t)$ to denote the solution of the system (1.4)-(1.5).

In this paper, we are interested in the following inverse problem for the above parabolic initial-boundary problem: Suppose we are given a terminal status observation $z(x,t)$ on the solution u of (1.1)-(1.3):

$$z(x,t) = u(x,t) \quad \text{for} \quad t \in [T-\sigma,\ T] \tag{1.6}$$

where $\sigma > 0$ is a small time period; then we want to recover the physical parameter $q(x)$ using the observation data $z(x,t)$, cf. e.g. [6].

Parameter identification problems are usually ill-posed in the sense that small perturbations in the observed data can have large effects on the solution $u(x,t)$ (cf. e.g. [1], [3]). For their numerical solution, some type of regularization has to be introduced. In this paper, we will study one of the most frequently used methods, the output–least–squares method with a Tikhonov regularization term. We will consider two cases for terminal observations:

Case (a) The terminal status observation data of $u(x,t)$ is available in a gradient form: $\nabla z(x,t) = \nabla u(x,t)$ for $t \in [T-\sigma,\ T]$. Depending on the physical meaning of (1.1)-(1.3), this means that we assume that distributed measurements of temperature gradients or pressure gradients are available. Then we use the following regularized output-least-squares formulation of the above parameter identification problem:

$$J(q) := \int_{T-\sigma}^{T}\int_\Omega q(x)\,|\nabla(u(q)-z)|^2\,dxdt + \beta\int_\Omega |\nabla(q-q^*)|^2\,dx \to \min, q \in K \tag{1.7}$$

with $u(q)$ solving the problem (1.4)-(1.5); i.e., the energy norm is used as measurement norm.

Case (b) The terminal status observation data of $u(x,t)$ is available in a pointwise form: $z(x,t) = u(x,t)$ for $t \in [T-\sigma,\ T]$. Then we use the L^2-norm as the measurement norm and formulate the regularized parameter identification problem as

$$J(q) := \int_{T-\sigma}^{T} \int_\Omega q(x)\, |u(q) - z|^2\, dxdt + \beta \int_\Omega |\nabla(q - q^*)|^2\, dx \to \min, q\, \in K \tag{1.8}$$

with $u(q)$ solving the problem (1.4)-(1.5).

In both cases **(a)** and **(b)** above, $q^* \in H^1(\Omega)$ is an a-priori estimate of the true parameter q (to be identified), and the constraint set K is given by

$$K = \Big\{ q \in H^1(\Omega)\ /\ \alpha_1 \le q(x) \le \alpha_2 \quad \text{for } a.e.\ \ x \in \Omega \Big\}, \tag{1.9}$$

where α_1 and α_2 are two positive constants. Note that q^* need not be in the constraint set K. As will be discussed below, q^* plays the role of a selection criterion, i.e., if q is not unique, the choice of q^* influences which of the possible parameters is approximated.

Note that in both cases, we use the unknown parameter q as a weight in the measurement norm, this requiring more accuracy in the residual $u(q) - z$ or $\nabla(u(q) - z)$, respectively, where the parameter is large. This approach has given good numerical results in [5]. As one can see from the proofs in Sections 4 and 5, the results about convergence rates remain unchanged if this weight q is omitted or replaced by q^2, although the regularized solutions (the minimizers of (1.7) or (1.8), respectively) are different. The factor q^2 would be appropriate if the available data were $q\nabla u(q)$, i.e., flux data.

In both cases, we use an H^1-seminorm for regularization. Remarks about the use of other regularization norms will be made later.

Later on, we will always use $q^\dagger \in K$ to denote a solution such that $\nabla u(q^\dagger)(x,t) = \nabla z(x,t)$ for $t \in [T-\sigma,\ T]$ in the case **(a)**, or $u(q^\dagger)(x,t) = z(x,t)$ for $t \in [T-\sigma,\ T]$ in the case **(b)**. We remark that $q^\dagger$ may not exist, even if it exists it may not be unique. We will always assume, however, that such a “true parameter“ $q^\dagger$ exists, i.e., that the exact data ∇z or z are attainable. By some abuse of notation (since we do not use a norm for regularization), we call such a $q^\dagger$ which minimizes $\|\nabla(q^\dagger - q^*)\|$ among all admissible parameters compatible with the data a “q^*-MNS(minimum norm solution)“ of our parameter identification problem (cf. [4]). Under our attainability assumption, a q^*-MNS also exists, which can be seen using Lemma 3.1 below.

In practical applications, the available observation data $z(x,t)$, $t \in [T-\sigma,\ T]$, of the solution $u(x,t)$ always contain some measurement error. So the actual available data are of the form

$$z^\delta(x,t) := z(x,t) + \text{noise}.$$

Let $q^\delta_\beta \in K$ be a minimizer of the least regularized problem (1.7) (or (1.8), respectively) with z replaced by z^δ.

While in [5], the regularized solutions were studied for fixed values of the regularization parameter $\beta > 0$, we are here interested in the behaviour of the regularized solutions as $\beta, \delta \to 0$, where δ is assumed to be a bound for the data noise of the form

$$\int_{T-\sigma}^{T} \|\nabla u(q^\dagger) - \nabla z^\delta\|^2\,dt \le \delta^2 \tag{1.10}$$

in the case **(a)**, and

$$\int_{T-\sigma}^{T} \|u(q^\dagger) - z^\delta\|^2\,dt \le \delta^2 \tag{1.11}$$

in the case **(b)**. Here and below, $\|\cdot\|$ stands for the L^2-norm, and the scalar product in $L^2(\Omega)$ or in $L^2(\Omega)^d$ will be denoted by $(\cdot,\ \cdot)$.

Note that without information on the noise level, convergent regularization methods cannot be constructed (cf. [3]).

Starting from [11], [4] and [7], there exists a large amount of literature on the convergence theory including rate estimates for the Tikhonov regularization method for nonlinear ill-posed problems, c.f. e.g. [3] for an overview over the general theory.

The key ingredients of this theory for a nonlinear ill-posed operator equation

$$Tx = y \tag{1.12}$$

are the following: Tikhonov regularization is defined (with $\|y - y^\delta\| \le \delta$) via

$$\|Tx - y^\delta\|^2 + \beta\|x - x^*\|^2 \to \min. \tag{1.13}$$

over some admissible set. For stability (for fixed $\beta > 0$) and convergence as $\beta \to 0$, the "weak closedness" of T is needed:

$$x_n \rightharpoonup x \wedge Tx_n \rightharpoonup y \Longrightarrow Tx = y. \tag{1.14}$$

For a convergence rate

$$\|x_\beta^\delta - x^\dagger\| = O(\sqrt{\delta}) \tag{1.15}$$

of regularized solutions x_β^δ (i.e., minimizers of (1.13)) to an x^*-MNS $x^\dagger$ for $\beta \sim \delta$, a "source condition"

$$x^\dagger - x^* = T'(x^\dagger)^* w \tag{1.16}$$

and a "smallness condition"

$$L \cdot \|w\| < 1 \tag{1.17}$$

are needed, where L is a Lipschitz constant for the Fréchet derivative T' of T, which is also assumed to exist.

The source condition (1.16) is usually some type of a-priori smoothness condition. For "seminorm–regularization"

$$\|Tx - y^\delta\|^2 + \beta\|D(x - x^*)\|^2 \to \min. \tag{1.18}$$

it takes the form

$$D^* D(x^\dagger - x^*) = T'(x^\dagger)^* w, \tag{1.19}$$

see [7]. An important special case is $D = \nabla$ (cf. (1.7), (1.8)). A more flexible variant is "regularization in Hilbert scales" (cf. [3, Section 10.4]), where the smallness assumption is not needed.

The general theory has been applied to many specific inverse problems including parameter identification for elliptic and parabolic equations (see [3, Section 10.4], and especially [10] for a parabolic problem). It could in principle also be applied to our problem. However, especially the source conditions (1.16) and (1.19) would turn out rather complicated. Our problem has so much special structure that it is no surprise that we can get better results by using this special structure instead of just applying the general theory. This is the approach we will take in this paper; we will thereby get rid of the smallness condition and, by using a special adjoint related to the variational formulation (1.4)-(1.5), obtain a convergence rate $O(\sqrt{\delta})$ under a much weaker source condition. The proofs are related to those in the general theory but use special features of our problem in many places. There seems to be a possibility to also make the general theory more flexible using the ideas of this paper; we will comment on this in Section 7.

2 Preliminaries

Unless otherwise specified, we will assume throughout the paper that

$$f \in L^2(Q_T) \qquad \text{and} \qquad u_0 \in H^1(\Omega), \tag{2.1}$$

where $Q_T = \Omega \times (0,T)$. With the assumption (2.1), we know from the standard parabolic theory that for each $q \in K$, there exists a unique solution $u(q;t)$ to the variational problem (1.4)-(1.5) and that it has the following regularities:

$$u(q) \in L^2(0,T;H_0^1(\Omega)), \quad u(q) \in H^1(0,T;L^2(\Omega)), \quad u(q) \in C(0,T;L^2(\Omega)).$$

We first quote a result from [5] about the existence of solutions to the minimization problems (1.7) and (1.8):

Theorem 2.1 *There exists at least one minimizer to the optimization problem* (1.7) *and* (1.8)*, respectively.*

In the general theory, the weak closedness (1.14) is used to prove this result; here, the role of the weak closedness is taken over by Lemma 3.1 below.

Now we present some preliminary results related to the solution of the parabolic system (1.4)-(1.5) which will be used later for the convergence analysis.

As opposed to the general theory, we will not need the Fréchet differentiability of the "forward operator" $q \to u(q)$, but only the Gateaux differential $u'(q)p$; the remainder term $u(p+q)-u(q)-u'(q)p$, which is estimated via the definition of the Fréchet derivative in the general theory, will be estimated differently here. For any $q \in K$ and $p \in H^1(\Omega)$, $u'(q)\,p \in H_0^1(\Omega)$ satisfies a homogeneous initial condition and solves

$$\begin{aligned}\int_\Omega (u'(q)\,p)_t\,\phi\,dx + \int_\Omega q\,\nabla(u'(q)\,p)\cdot\nabla\phi\,dx = \\ -\int_\Omega p\nabla u(q)\cdot\nabla\phi dx \;\forall\phi \in H_0^1(\Omega)\end{aligned} \tag{2.2}$$

for a.e. $t \in (0,T)$.

In the next Lemma, we derive the variational problem satisfied by the remainder term:

Lemma 2.1 *For any $q \in K$ and $p \in H^1(\Omega)$ such that $q+p \in K$, the remainder $w = u(q+p) - u(q) - u'(q)\,p$ belongs to $H_0^1(\Omega)$ and solves*

$$\int_\Omega w_t\,\phi\,dx + \int_\Omega q\,\nabla w \cdot \nabla\phi\,dx = \int_\Omega p\,\nabla(u(q) - u(q+p)) \cdot \nabla\phi\,dx \quad \forall \phi \in H_0^1(\Omega) \tag{2.3}$$

for a.e. $t \in (0,T)$.

Proof. By (1.5), $u(q+p)$ satisfies

$$\int_\Omega u(q+p)_t\,\phi\,dx + \int_\Omega (q+p)\,\nabla u(q+p) \cdot \nabla\phi dx = \int_\Omega f\,\phi dx;$$

subtracting this from (1.5), we get

$$\begin{aligned}&\int_\Omega (u(q+p) - u(q))_t\,\phi dx + \int_\Omega q\,\nabla(u(q+p) - u(q)) \cdot \nabla\phi dx\\ &= -\int_\Omega p\,\nabla u(q+p) \cdot \nabla\phi dx.\end{aligned}$$

Now (2.3) follows by subtracting (2.2) from the above equation. $\square$

We now introduce a linear operator $F(q) : L^2(0,T;L^2(\Omega)^d) \to L^2(\Omega)$ which will appear in our source condition: For any $q \in K$, $F(q)$ is defined by

$$F(q)\,\Phi := -\int_{T-\sigma}^T \nabla u(q) \cdot \Phi\,dt \quad \forall\,\Phi \in L^2(0,T;L^2(\Omega)^d) \tag{2.4}$$

where $u(q)$ is the solution of (1.4)-(1.5). Using the equation (2.2), we immediately see that for any $p \in H^1(\Omega)$ and any

$$\phi \in H_0^1(T-\sigma,T;L^2(\Omega)) \cap L^2(T-\sigma,T;H_0^1(\Omega)),$$

the following holds:

$$\begin{aligned}(F(q)\nabla\phi,\ p) &= -\int_{T-\sigma}^T \int_\Omega p\,\nabla u(q) \cdot \nabla\phi\,dxdt\\ &= \int_{T-\sigma}^T \int_\Omega \Big\{(u'(q)\,p)_t\,\phi + q\,\nabla(u'(q)\,p) \cdot \nabla\phi\Big\}\,dxdt.\end{aligned} \tag{2.5}$$

The third expression, considered as a linear operator acting on p, can thus be viewed as some kind of adjoint of $F(p)$, not with respect to the standard inner product, but with respect to the bilinear form (2.2) defining the parabolic equation linearized with respect to the parameter. This problem–adapted adjoint will replace the standard adjoint in the right-hand side of the source condition (1.19); the operator D on the left–hand side in our setting is ∇, so that we also need the adjoint ∇^*, which requires some explanation, since it could be considered in different settings (cf. also [7]):

One could consider the bounded operator $\nabla : H^1(\Omega) \to L^2(\Omega)^d$ and define the adjoint with respect to this structure, which we will denote by $\nabla^{\#}$, via

$$(\nabla\phi, w)_{L^2(\Omega)^d} = \langle \phi, \nabla^{\#} w\rangle_{H^1(\Omega)} \quad \forall w \in L^2(\Omega)^d, \phi \in H^1(\Omega). \tag{2.6}$$

In our proofs below, we will need the L^2-variant of (2.6) only, i.e., we will need an adjoint ∇^* defined via

$$(\nabla\phi, w)_{L^2(\Omega)^d} = (\phi, \nabla^* w)_{L^2(\Omega)} \quad \forall w \in L^2(\Omega)^d, \phi \in H^1(\Omega). \tag{2.7}$$

Thus, we consider ∇ as an unbounded densely defined operator from $L^2(\Omega)$ into $L^2(\Omega)^d$ and thus ∇^* as an (also unbounded) operator from $L^2(\Omega)^d$ into $L^2(\Omega)$. Like in (1.19), we will also need $\nabla^*\nabla\psi$, which would normally be defined only for $\psi \in H^2(\Omega)$ fulfilling appropriate boundary conditions, e.g., for $\Omega = (0,1)$, the domain of $\nabla^*\nabla$ would be

$$\{\varphi \in H^2(0,1)/\varphi'(0) = \varphi'(1) = 0\}.$$

However, we will only need a weak form of $\nabla^*\nabla$ in the following sense: Our source condition will require the existence of an H^1-function ϕ (for precise requirements see later) such that

$$F(q^\dagger)\nabla\phi = \nabla^*\nabla(q^\dagger - q^*) \tag{2.8}$$

holds; normally, this would require that $q^\dagger - q^* \in H^2(\Omega)$. However, our proofs need only that

$$(F(q^\dagger)\nabla\phi, \psi) = (\nabla(q^\dagger - q^*), \nabla\psi) \quad \forall \psi \in H^1(\Omega), \tag{2.9}$$

which we call the weak form of (2.8); this condition makes sense also for $q^\dagger - q^* \in H^1(\Omega)$.

3 Stability and Convergence Without Rates

In this section we study the stability and the convergence of the Tikhonov regularization methods (1.7) and (1.8), respectively, without rates. In Sections 4-5, we will derive convergence rates.

We now first show that solving the regularized problems (1.7) and (1.8), respectively, is indeed a regularization in the sense of continuous dependence of the solutions q_β^δ on the data ∇z^δ and z^δ, respectively. Since neither the regularized solutions nor the q^*-MNS are in general unique, convergence here is always to be understood in a multivalued sense.
The following Lemma from [5] replaces the weak closedness used in the general theory:

Lemma 3.1 *For any sequence $\{q_n\}$ in K which converges to some $q \in K$ in $L^2(\Omega)$ as $n \to \infty$, we have (for a subsequence) $u(q_n) \rightharpoonup u(q)$ in $L^2(0,T;\ H_0^1(\Omega))$ and*

$$\lim_{n\to\infty} \int_{T-\sigma}^{T} \int_\Omega q_n\, |\nabla(u(q_n) - z)|^2 dxdt = \int_{T-\sigma}^{T} \int_\Omega q\, |\nabla(u(q) - z)|^2 dxdt.$$

Now, for the energy-norm formulation (1.7) of the regularized problem, we obtain

Theorem 3.1 *For any $\beta > 0$, let $\{\nabla z_n\}$ be a sequence such that $\nabla z_n \to \nabla z^\delta$ in $L^2(T-\sigma, T; L^2(\Omega))$, and $\{q_n\}$ be minimizers of the problem (1.7) with ∇z^δ replaced by ∇z_n. Then there exists a subsequence of $\{q_n\}$ that converges in $H^1(\Omega)$, and the limit of every such convergent subsequence is a minimizer of (1.7).*

Proof. By definition of $\{q_n\}$,

$$\begin{aligned}\int_{T-\sigma}^{T} \int_\Omega q_n\, |\nabla u(q_n) - \nabla z_n|^2 dxdt + \beta \int_\Omega |\nabla q_n - \nabla q^*|^2 dx \\ \le \int_{T-\sigma}^{T} \int_\Omega q\, |\nabla u(q) - \nabla z_n|^2 dxdt + \beta \int_\Omega |\nabla q - \nabla q^*|^2 dx, \quad \forall q \in K,\end{aligned}$$

from which we know that $\{\|\nabla q_n\|\}$ and $\{\int_{T-\sigma}^{T} \|\nabla u(q_n)\|^2 dt\}$ are bounded; since $q_n \in K$, also the L^2-norms of q_n are bounded, so that $\{q_n\}$ is bounded in $H^1(\Omega)$. Hence there exists a subsequence (again denoted by $\{q_n\}$) such that $q_n \to \bar{q}$ in $L^2(\Omega)$ and

$$q_n \rightharpoonup \bar{q} \ \text{ in } \ H^1(\Omega) \quad \text{and} \quad u(q_n) \rightharpoonup \bar{u} \ \text{ in } \ L^2(T-\sigma, T;\ H_0^1(\Omega)\,).$$

Using Lemma 3.1, we obtain $\bar{u} = u(\bar{q})$ and

$$\lim_{n\to\infty} \int_{T-\sigma}^{T} \int_\Omega q_n \, |\nabla u(q_n) - \nabla z_n|^2 dxdt = \int_{T-\sigma}^{T} \int_\Omega \bar{q} \, |\nabla u(\bar{q}) - \nabla z^\delta| \, dxdt. \quad (3.10)$$

Using this with $\nabla z_n \to \nabla z^\delta$, we derive

$$\begin{aligned} J(\bar{q}) &= \int_{T-\sigma}^{T} \int_\Omega \bar{q} \, |\nabla u(\bar{q}) - \nabla z^\delta|^2 dxdt + \beta \int_\Omega |\nabla \bar{q} - \nabla q^*|^2 dx \\ &\leq \liminf \Big\{ \int_{T-\sigma}^{T} \int_\Omega q_n \, |\nabla u(q_n) - \nabla z_n|^2 dxdt + \beta \int_\Omega |\nabla q_n - \nabla q^*|^2 dx \Big\} \\ &\leq \limsup \Big\{ \int_{T-\sigma}^{T} \int_\Omega q_n \, |\nabla u(q_n) - \nabla z_n|^2 dxdt + \beta \int_\Omega |\nabla q_n - \nabla q^*|^2 dx \Big\} \\ &\leq J(q) \quad \forall q \in K, \end{aligned} \quad (3.11)$$

therefore $\bar{q}$ is a minimizer of (1.7).

We now prove that $q_n \to \bar{q}$ in $H^1(\Omega)$ by contradiction: Assume $q_n \not\to \bar{q}$, then, since $q_n \rightharpoonup \bar{q}$ in $H^1(\Omega)$ and $q_n \to \bar{q}$ in $L^2(\Omega)$, $\nabla q_n \not\to \nabla \bar{q}$. Since $\nabla q_n \rightharpoonup \nabla \bar{q}$ in $L^2(\Omega)$, by the weak lower semicontinuity of the norm, we have

$$\|\nabla \bar{q} - \nabla q^*\| \leq \liminf_{n\to\infty} \|\nabla q_n - \nabla q^*\|.$$

Hence,

$$\mu := \limsup_{n\to\infty} \|\nabla q_n - \nabla q^*\|^2 > \|\nabla \bar{q} - \nabla q^*\|^2,$$

which implies that there exists a subsequence $\{q_m\}$ such that

$$\lim_{m\to\infty} \|\nabla q_m - \nabla q^*\|^2 = \mu; \quad (3.12)$$

but from (3.11), we know by taking $q = \bar{q}$ that

$$\begin{aligned} &\lim_{m\to\infty} \{ \int_{T-\sigma}^{T} \int_\Omega q_m \, |\nabla u(q_m) - \nabla z_m|^2 dxdt + \beta \int_\Omega |\nabla q_m - \nabla q^*|^2 dx \} \\ &\quad = \int_{T-\sigma}^{T} \int_\Omega \bar{q} \, |\nabla u(\bar{q}) - \nabla z^\delta|^2 dxdt + \beta \int_\Omega |\nabla \bar{q} - \nabla q^*|^2 dx. \end{aligned}$$

Combining this with (3.12), we obtain

$$\begin{aligned}
&\lim_{m\to\infty}\int_{T-\sigma}^{T}\int_{\Omega} q_m\,|\nabla u(q_m)-\nabla z_m|^2 dxdt \\
&= \int_{T-\sigma}^{T}\int_{\Omega}\bar q\,|\nabla u(\bar q)-\nabla z^\delta|^2 dxdt + \beta\Big(\int_{\Omega}|\nabla\bar q-\nabla q^*|^2 dx-\mu\Big) \\
&< \int_{T-\sigma}^{T}\int_{\Omega}\bar q\,|\nabla u(\bar q)-\nabla z^\delta|^2 dxdt,
\end{aligned}$$

which is a contradiction to (3.10). Thus, $q_n \to \bar q$ in $H^1(\Omega)$. □

For the L^2-norm formulation (1.8), we have the following stability result whose proof is analogous to the one of Theorem 3.1:

Theorem 3.2 *For any $\beta > 0$, let $\{z_n\}$ be a sequence in $L^2(T-\sigma, T; H_0^1(\Omega))$ such that $z_n \to z^\delta$ in $L^2(\Omega)$, and $\{q_n\}$ be minimizers of* (1.8) *with z^δ replaced by z_n. Then there exists a subsequence of $\{q_n\}$ that converges in $H^1(\Omega)$, and the limit of every such convergent subsequence is a minimizer of* (1.8).

After these stability results, we end the section with the following general convergence results of the Tikhonov regularization method for our problem:

Theorem 3.3 *For any positive sequence $\{\delta_k\}$, let $\beta_k := \beta(\delta_k)$ be such that $\beta_k \to 0,\ \delta_k^2/\beta_k \to 0$ as $k\to\infty$, and $z^{\delta_k} \in L^2(0,T;\ H_0^1(\Omega))$ satisfy (1.10) (or (1.11), respectively). Then the sequence $\{q_{\beta_k}^{\delta_k}\}$ formed by solutions of (1.7) (or (1.8), respectively) with z and β replaced by z^{δ_k} and β_k, respectively, has a subsequence which converges in $H^1(\Omega)$; all limits of such convergent subsequences are minimizers of* (1.7) *(or* (1.8), *respectively) for $\beta = 0$ which minimize, among all such minimizers, $\|\nabla q - \nabla q^*\|$, i.e., are q^*-MNS of our problem for case* (**a**) *or* (**b**), *respectively.*

Proof. We prove this only for the L^2-norm formulation, i.e., case **(b)**. The proof of case **(a)** is similar.

As $q_{\beta_k}^{\delta_k}$ is a minimizer, we have $J(q_{\beta_k}^{\delta_k}) \le J(q) \ \ \forall\, q\in K$. By taking $q = q^\dagger$ and using (1.11) we get, with α_1, α_2 as in (1.9),

$$\alpha_1\int_{T-\sigma}^{T}\int_{\Omega}|u(q_{\beta_k}^{\delta_k})-z^{\delta_k}|^2\,dxdt+\beta_k\,\|\nabla q_{\beta_k}^{\delta_k}-\nabla q^*\|^2 \le \alpha_2\,\delta_k^2+\beta_k\,\|\nabla q^\dagger-\nabla q^*\|^2,$$

thus

$$\int_{T-\sigma}^{T}\int_{\Omega}|u(q_{\beta_k}^{\delta_k})-z^{\delta_k}|^2\,dxdt\to 0 \quad \text{as} \quad k\to\infty,$$

which with (1.11) implies

$$u(q_{\beta_k}^{\delta_k})\to z \quad \text{in} \quad L^2(T-\sigma,T;\ L^2(\Omega)) \tag{3.13}$$

and

$$\limsup_{k\to\infty}\|\nabla q_{\beta_k}^{\delta_k}-\nabla q^*\|\le\|\nabla q^\dagger-\nabla q^*\|. \tag{3.14}$$

Therefore $\{q_{\beta_k}^{\delta_k}\}$ is bounded in $H^1(\Omega)$ and there exists a subsequence, again denoted by $\{q_{\beta_k}^{\delta_k}\}$, such that

$$q_{\beta_k}^{\delta_k}\rightharpoonup\bar q \quad \text{in} \quad H^1(\Omega), \qquad q_{\beta_k}^{\delta_k}\to\bar q \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad k\to\infty.$$

By Lemma 3.1,

$$u(q_{\beta_k}^{\delta_k})\rightharpoonup u(\bar q) \quad \text{in} \quad L^2(0,T;\ H_0^1(\Omega)),$$

which with (3.13) shows that $u(\bar q)=z$ for $a.e.$ $t\in[T-\sigma,T]$, so that $\bar q$ is a minimizer of (1.8) with $\beta=0$. By the weak lower semicontinuity of the norm and the fact that $\nabla q_{\beta_k}^{\delta_k}\rightharpoonup\nabla\bar q$ in $L^2(\Omega)$,

$$\|\nabla\bar q-\nabla q^*\|\le\liminf_{k\to\infty}\|\nabla q_{\beta_k}^{\delta_k}-\nabla q^*\|,$$

which implies with (3.14) that $\bar q$ is a q^*-MNS.

We next show that $q_{\beta_k}^{\delta_k}$ converges to $\bar q$ strongly in $H^1(\Omega)$. Similar to (3.14), by using $J(q_{\beta_k}^{\delta_k})\le J(\bar q)$ we obtain that

$$\limsup_{n\to\infty}\|\nabla q_{\beta_k}^{\delta_k}-\nabla q^*\|\le\|\nabla\bar q-\nabla q^*\|;$$

This along with the identity $\|\nabla q_{\beta_k}^{\delta_k}-\nabla\bar q\|^2=\|\nabla q_{\beta_k}^{\delta_k}-\nabla q^*\|^2+\|\nabla q^*-\nabla\bar q\|^2+2\,(\nabla q_{\beta_k}^{\delta_k}-\nabla q^*,\ \nabla q^*-\nabla\bar q)$ yields $\limsup_{k\to\infty}\|\nabla q_{\beta_k}^{\delta_k}-\nabla\bar q\|^2\le 0$. Hence, $\|\nabla q_{\beta_k}^{\delta_k}-\nabla\bar q\|\to 0$ as $k\to\infty$, which completes the proof. □

4 Convergence Rates for the Energy-Norm Formulation

For the energy-norm formulation (1.7), we will prove the following Theorem, where the source condition (4.15) is to be understood in the weak sense (2.9):

Theorem 4.1 *Assume that there exists a function*

$$\phi \in H_0^1(T-\sigma, T; L^2(\Omega)) \cap L^2(T-\sigma, T; H_0^1(\Omega))$$

such that, with $F(q^\dagger)$ defined by (2.4),

$$F(q^\dagger)\,\nabla\phi = \nabla^*\nabla(q^\dagger - q^*). \tag{4.15}$$

Then, with $\beta \sim \delta$, we have

$$\|\nabla q_\beta^\delta - \nabla q^\dagger\| = O(\sqrt{\delta}) \tag{4.16}$$

and

$$\int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|^2 dt = O(\delta^2). \tag{4.17}$$

Proof. As q_β^δ is a minimizer of (1.7), we have $J(q_\beta^\delta) \le J(q^\dagger)$, which with (1.10) implies (with α_1, α_2 as in (1.9))

$$\alpha_1 \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2 dt + \beta\,\|\nabla(q_\beta^\delta - q^*)\|^2 \le \alpha_2\,\delta^2 + \beta\,\|\nabla(q^\dagger - q^*)\|^2,$$

from which we obtain

$$\begin{aligned}
&\alpha_1 \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2 dt + \beta\,\|\nabla(q^\dagger - q_\beta^\delta)\|^2 \\
&\le\ \alpha_2\,\delta^2 + \beta\|\nabla(q^\dagger - q^*)\|^2 + \beta\Big\{\|\nabla(q^\dagger - q_\beta^\delta)\|^2 - \|\nabla(q_\beta^\delta - q^*)\|^2\Big\} \\
&=\ \alpha_2\,\delta^2 + 2\beta\,(\nabla(q^\dagger - q^*),\ \nabla(q^\dagger - q_\beta^\delta)\,).
\end{aligned} \tag{4.18}$$

Using the assumption (4.15)(note that only its weak form (2.9) is needed) and the relation (2.5), we deduce for the last term in (4.18) that

$$\begin{aligned}
&(\nabla(q^\dagger - q^*),\ \nabla(q^\dagger - q_\beta^\delta)\,) = (F(q^\dagger)\,\nabla\phi,\ q^\dagger - q_\beta^\delta) \\
&=\ \int_{T-\sigma}^{T}\int_\Omega \Big\{(u'(q^\dagger)(q^\dagger - q_\beta^\delta))_t\,\phi + q^\dagger\,\nabla(u'(q^\dagger)(q^\dagger - q_\beta^\delta))\cdot\nabla\phi\Big\}dxdt.
\end{aligned}$$

Now let

$$r_\beta^\delta := u(q_\beta^\delta) - u(q^\dagger) - u'(q^\dagger)(q_\beta^\delta - q^\dagger); \tag{4.19}$$

by Lemma 2.1, $r_\beta^\delta \in H_0^1(\Omega)$. Using this notation, we get

$$\begin{aligned}
&\beta\,(\nabla(q^\dagger - q^*),\ \nabla(q^\dagger - q_\beta^\delta)\,) = \\
&= \ -\beta\int_{T-\sigma}^{T}\int_\Omega (u(q_\beta^\delta) - u(q^\dagger))_t\,\phi\,dxdt \\
&+ \ \beta\int_{T-\sigma}^{T}\int_\Omega \Big\{(r_\beta^\delta)_t\,\phi + q^\dagger\,\nabla r_\beta^\delta\cdot\nabla\phi\Big\}dxdt \\
&- \ \beta\int_{T-\sigma}^{T}\int_\Omega q^\dagger\,\nabla(u(q_\beta^\delta) - u(q^\dagger))\cdot\nabla\phi dxdt \\
&=: \ (\mathrm{I})_1 + (\mathrm{I})_2 + (\mathrm{I})_3.
\end{aligned} \tag{4.20}$$

We next estimate $(\mathrm{I})_1, (\mathrm{I})_2, (\mathrm{I})_3$ separately. First for $(\mathrm{I})_1$, using integration by parts with respect to t and the boundary condition, we derive

$$(\mathrm{I})_1 = \beta\int_{T-\sigma}^{T}\int_\Omega (u(q_\beta^\delta) - u(q^\dagger))\phi_t\,dxdt;$$

applying the Cauchy–Schwarz, the Poincaré and triangle inequalities, we obtain that

$$\begin{aligned}
|(\mathrm{I})_1| \ &\le \ C\,\beta\int_{T-\sigma}^{T}\|\nabla(u(q_\beta^\delta) - u(q^\dagger))\|\,\|\phi_t\|\,dt \\
&\le \ C\,\beta\int_{T-\sigma}^{T}\|\nabla u(q_\beta^\delta) - \nabla z^\delta\|\,\|\phi_t\|\,dt + C\beta\int_{T-\sigma}^{T}\|\nabla z^\delta - \nabla u(q^\dagger)\|\,\|\phi_t\|\,dt.
\end{aligned}$$

Using (1.10) and Young's inequality in the form

$$a\cdot b \le \varepsilon\,a^2 + \frac{b^2}{4\,\varepsilon} \tag{4.21}$$

for any $\varepsilon > 0$ yields

$$|(\mathrm{I})_1| \le C\varepsilon\int_{T-\sigma}^{T}\|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2 dt + \frac{C\beta^2}{2\varepsilon}\int_{T-\sigma}^{T}\|\phi_t\|^2 dt + C\,\varepsilon\,\delta^2.$$

To estimate $(\mathrm{I})_2$, we use (2.3) to get

$$(\mathrm{I})_2 = \beta\int_{T-\sigma}^{T}\int_\Omega (q_\beta^\delta - q^\dagger)(\nabla u(q^\dagger) - \nabla u(q_\beta^\delta))\cdot\nabla\phi\,dxdt.$$

Then using the Cauchy-Schwarz and Young's inequalities, we get

$$\begin{aligned}
|(\mathrm{I})_2| &\le \beta \int_{T-\sigma}^{T} \|\nabla u(q^\dagger) - \nabla z^\delta\| \, \|(q_\beta^\delta - q^\dagger)\nabla\phi\| \, dt \\
&\quad + \beta \int_{T-\sigma}^{T} \|\nabla z^\delta - \nabla u(q_\beta^\delta)\| \, \|(q_\beta^\delta - q^\dagger)\nabla\phi\| \, dt \\
&\le \varepsilon \, \delta^2 + \frac{2\beta^2\alpha_2^2}{\varepsilon} \int_{T-\sigma}^{T} \|\nabla\phi\|^2 dt + \varepsilon \int_{T-\sigma}^{T} \|\nabla z^\delta - \nabla u(q_\beta^\delta)\|^2 dt,
\end{aligned}$$

where we have used the boundedness of q_β^δ and $q^\dagger$ in K.

Finally we estimate $(\mathrm{I})_3$: We have

$$\begin{aligned}
|(\mathrm{I})_3| &\le \alpha_2\beta \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\| \, \|\nabla\phi\| \, dt \\
&\quad + \alpha_2\beta \int_{T-\sigma}^{T} \|\nabla z^\delta - \nabla u(q^\dagger)\| \, \|\nabla\phi\| \, dt \\
&\le \varepsilon \, \alpha_2^2\delta^2 + \frac{\beta^2}{2\,\varepsilon} \int_{T-\sigma}^{T} \|\nabla\phi\|^2 dt + \varepsilon \, \alpha_2^2 \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2 dt.
\end{aligned}$$

Substituting the above estimates for $(\mathrm{I})_1, (\mathrm{I})_2$ and $(\mathrm{I})_3$ into (4.18) and (4.20) we obtain

$$\begin{aligned}
&\alpha_1 \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2 dt + \beta \|\nabla(q^\dagger - q_\beta^\delta)\|^2 \le \\
&\le \alpha_2\delta^2 + 2\varepsilon(C + 1 + \alpha_2^2) \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2 dt + \\
&+ \frac{C\beta^2}{\varepsilon} \int_{T-\sigma}^{T} \|\phi_t\|^2 dt + 2\varepsilon(C + 1 + \alpha_2^2)\delta^2 + \\
&+ \beta^2 \int_{T-\sigma}^{T} \|\nabla\phi\|^2 dt \cdot \left[\frac{4\alpha_2^2}{\varepsilon} + \frac{1}{\varepsilon} \right].
\end{aligned}$$

From this, we get

$$[\alpha_1 - 2\varepsilon(C + 1 + \alpha_2^2)] \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2 dt +$$

$$+ \quad \beta\|\nabla(q_\beta^\delta - q^\dagger)\|^2 \le \delta^2[\alpha_2 + 2\varepsilon(C+1+\alpha_2^2)] +$$
$$+ \quad \frac{C\beta^2}{\varepsilon}\int\limits_{T-\sigma}^{T}\|\phi_t\|^2 dt + \beta^2\left[\frac{4\alpha_2^2}{\varepsilon} + \frac{1}{\varepsilon}\right]\int\limits_{T-\sigma}^{T}\|\nabla\phi\|^2 dt.$$

From this, with $\varepsilon < \frac{\alpha_1}{2(C+1+\alpha_2^2)}$, we obtain that, for $\beta \sim \delta$, (4.16) and (4.17) hold. $\square$

We will discuss the source condition (4.15) in Section 6; note that no smallness condition is needed.

The constants in the $O-$terms in (4.16) and (4.17) can be derived from the proof. They grow with α_2^2, $\frac{1}{\alpha_1}$, and with $\int\limits_{T-\sigma}^{T}\|\nabla\phi\|^2 dt$ and $\int\limits_{T-\sigma}^{T}\|\phi_t\|^2 dt$.

5 Convergence Rates for the L^2-Norm Formulation

We now carry over the convergence results of Section 4 to the L^2-norm formulation (1.8); again, the source condition is to be understood in the weak sense (2.9).

Theorem 5.1 *Assume that there exists a function*

$$\phi \in H_0^1(T-\sigma, T; L^2(\Omega)) \cap L^2(T-\sigma, T; H_0^1(\Omega)) \tag{5.1}$$

such that (4.15) holds. Furthermore, assume that

$$\phi \in L^2(T-\sigma, T; W^{1,\infty}(\Omega)) \quad \textit{and} \quad \Delta\phi \in L^2(T-\sigma, T;\ L^2(\Omega)). \tag{5.2}$$

Then, with $\beta \sim \delta$, we have the estimates (4.16) and

$$\int_{T-\sigma}^{T}\|u(q_\beta^\delta) - u(q^\dagger)\|^2 dt = O(\delta^2). \tag{5.3}$$

Proof. Using the same techniques as used in deriving (4.18) and (4.20), we obtain

$$\begin{aligned}\alpha_1\int_{T-\sigma}^{T}\|u(q_\beta^\delta) - z^\delta\|^2 dt + \beta\,\|\nabla(q^\dagger - q_\beta^\delta)\|^2 \\ \le \quad \alpha_2\,\delta^2 + 2\beta\,(\nabla(q^\dagger - q^*),\ \nabla(q^\dagger - q_\beta^\delta))\end{aligned} \tag{5.4}$$

and

$$
\begin{aligned}
&\beta\,(\nabla(q^\dagger - q^*),\ \nabla(q^\dagger - q^\delta_\beta)\,)\\
&\quad= \quad -\beta \int_{T-\sigma}^{T}\int_\Omega (u(q^\delta_\beta) - u(q^\dagger))_t\,\phi\,dxdt\\
&\quad+ \quad \beta \int_{T-\sigma}^{T}\int_\Omega \Big\{(r^\delta_\beta)_t\,\phi + q^\dagger\,\nabla r^\delta_\beta\cdot\nabla\phi\,\Big\}dxdt\\
&\quad- \quad \beta \int_{T-\sigma}^{T}\int_\Omega q^\dagger\,\nabla(u(q^\delta_\beta) - u(q^\dagger))\cdot\nabla\phi dxdt\\
&\quad=: \quad (\mathrm{II})_1 + (\mathrm{II})_2 + (\mathrm{II})_3. \qquad (5.5)
\end{aligned}
$$

Note that (5.5) is still the same as (4.20), but we now have to estimate the terms differently, since we have to aim at $\int\limits_{T-\sigma}^{T}\|u(q^\delta_\beta) - z^\delta\|^2dt$ instead of $\int\limits_{T-\sigma}^{T}\|\nabla(u(q^\delta_\beta) - z^\delta)\|^2dt.$

We now estimate $(\mathrm{II})_1$, $(\mathrm{II})_2$ and $(\mathrm{II})_3$: $(\mathrm{II})_1$ can be bounded readily by means of integration by parts with respect to t and Young's inequality (4.21):

$$
|(\mathrm{II})_1| \ \le\ \varepsilon\int_{T-\sigma}^{T}\|u(q^\delta_\beta) - z^\delta\|^2dt + \frac{\beta^2}{2\varepsilon}\int_{T-\sigma}^{T}\|\phi_t\|^2dt + \varepsilon\delta^2,
$$

while $(\mathrm{II})_3$ can be estimated similarly using integration by parts with respect to x and the Cauchy–Schwarz and Young's inequalities:

$$
\begin{aligned}
|(\mathrm{II})_3| \ &=\ \beta\,\Big|\int_{T-\sigma}^{T}\int_\Omega (u(q^\delta_\beta) - u(q^\dagger))\,\nabla\cdot(q^\dagger\nabla\phi)dxdt\Big|\\
&\le\ \varepsilon\int_{T-\sigma}^{T}\|u(q^\delta_\beta) - z^\delta\|^2dt + \varepsilon\,\delta^2 + \frac{\beta^2}{2\varepsilon}\int_{T-\sigma}^{T}\|\nabla\cdot(q^\dagger\nabla\phi)\|^2dt.
\end{aligned}
$$

What remains is to bound $(\mathrm{II})_2$. As in the treatment of $(\mathrm{I})_2$ in the proof of Theorem 4.1 we obtain with integration by parts with respect to x that

$$
\begin{aligned}
(\mathrm{II})_2 \ &=\ -\beta\int_{T-\sigma}^{T}\int_\Omega (q^\delta_\beta - q^\dagger)(\nabla u(q^\delta_\beta) - \nabla u(q^\dagger))\cdot\nabla\phi dxdt\\
&=\ -\beta\int_{T-\sigma}^{T}\int_\Omega (u(q^\delta_\beta) - u(q^\dagger))\nabla\cdot((q^\delta_\beta - q^\dagger)\nabla\phi)dxdt.
\end{aligned}
$$

Using Young's inequality again, we get

$$
|(\mathrm{II})_2| \le \varepsilon\,\delta^2 + \frac{\beta^2}{2\varepsilon}\int_{T-\sigma}^{T}\|\nabla\cdot((q^\delta_\beta - q^\dagger)\nabla\phi)\|^2dt + \varepsilon\int_{T-\sigma}^{T}\|u(q^\delta_\beta) - z^\delta\|^2dt.
$$

The second term in this bound can be further estimated:

$$\begin{aligned}
&\frac{\beta^2}{2\varepsilon}\int_{T-\sigma}^{T}\int_{\Omega}|\nabla\cdot((q_\beta^\delta-q^\dagger)\nabla\phi)|^2dxdt\\
&= \frac{\beta^2}{2\varepsilon}\int_{T-\sigma}^{T}\int_{\Omega}\left|\nabla(q_\beta^\delta-q^\dagger)\cdot\nabla\phi+(q_\beta^\delta-q^\dagger)\Delta\phi\right|^2dxdt\\
&\le \beta\,\|\nabla(q_\beta^\delta-q^\dagger)\|^2\left\{\frac{\beta}{\varepsilon}\int\limits_{T-\sigma}^{T}\|\nabla\phi\|^2_{L^\infty(\Omega)}\,dt\right\}\\
&\quad +\frac{\beta^2}{\varepsilon}\int_{T-\sigma}^{T}\|\Delta\phi\|^2_{L^2(\Omega)}\,dt.
\end{aligned}$$

Since, unlike in the proof of Theorem 4.1, the term $\beta\|\nabla(q_\beta^\delta-q^\dagger)\|^2$ now also appears on the right–hand side, one has first to take $\varepsilon<\frac{\alpha_1}{12}$ and then take β small enough so that $\frac{\beta}{\varepsilon}\int\limits_{T-\sigma}^{T}\|\nabla\phi\|^2_{L^\infty(\Omega)}dt<1$. Then we proceed as in the proof of Theorem 4.1. □

6 Discussion of the Source Condition

First, we note that due to Theorem 3.1, the parameter $q^\dagger$ to which the regularized solutions in Theorems 4.1 and 5.1, respectively, converge must be a q^*-MNS ; in the proof of the relevant part of Theorem 3.1, only the convergence of $\nabla q_{\beta_k}^{\delta_k}$, not of the full H^1-norm, is needed. There might be more than one q^*-MNS, but only one can fulfill the conditions of Theorems 4.1 and 5.1, respectively.

The crucial assumption in Theorems 4.1 and 5.1 is the existence of a function

$$\phi\in H_0^1(T-\sigma,T;\ L^2(\Omega))\cap L^2(T-\sigma,T;H_0^1(\Omega)) \tag{6.1}$$

(with the additional assumption (5.1) in Theorem 5.1) such that (4.15), i.e.,

$$\int_{T-\sigma}^{T}\nabla u(q^\dagger)\cdot\nabla\phi dt=\nabla^*\nabla(q^*-q^\dagger), \tag{6.2}$$

holds in the weak form (2.9), i.e.,

$$\int_{T-\sigma}^{T}(p\,\nabla\phi,\ \nabla u(q^\dagger)\,)\,dt=(\nabla(q^*-q^\dagger),\ \nabla p)\quad\forall\,p\in H^1(\Omega). \tag{6.3}$$

In (6.3), $q^* - q^\dagger \in H^1(\Omega)$ suffices, so that the source condition does not impose or require any more regularity on q^* and $q^\dagger$. This is quite remarkable. While usually, such source conditions are quite restrictive and impose severe a–priori conditions on the unknown solution of the inverse problem (cf. [3]), we can now even construct functions ϕ such that the source condition is fulfilled for the one–dimensional case:

Lemma 6.1 *Let $\Omega = (0,1)$, $\psi_1, \psi_2 \in H^1(T-\sigma, T)$ and $\phi \in H^1(\Omega)$ be arbitrary functions. Let u be another given function defined on $\Omega \times (T-\sigma, T)$ such that*

$$\frac{q-w}{u_2} \in L^2(0,1) \quad \text{and} \quad \int_0^1 \frac{q(x)-w(x)}{u_2(x)} dx = 0, \tag{6.4}$$

where w, u_1 and u_2 are given by

$$\begin{aligned} w(x) :=&\ u_1(x)\Big\{(1-2x)\phi_1(x) + (x-x^2)\,\phi_1'(x)\Big\}, \\ u_i(x) :=&\ \int_{T-\sigma}^{T} (T-t)(T-\sigma-t)\, u_x(x,t)\,\psi_i(t)\,dt, \quad i = 1,2. \end{aligned}$$

Then, for any $q \in L^2(\Omega)$, the equation

$$\int_{T-\sigma}^{T} u_x(x,t)\,\phi_x(x,t)dt = q(x) \quad \forall x \in (0,1) \tag{6.5}$$

has the solutions

$$\phi(x,t) := (T-t)(T-\sigma-t)\Big\{(x-x^2)\,\phi_1(x)\psi_1(t) + \psi_2(t)\int_0^x \frac{q(\xi)-w(\xi)}{u_2(\xi)} d\xi\Big\}, \tag{6.6}$$

and (6.1) holds.

Proof. By definition,

$$\phi(x,T) = \phi(x, T-\sigma) = 0 \quad \forall x \in \Omega$$

and

$$\phi(0,t) = \phi(1,t) = 0 \quad \forall t \in (T-\sigma,\ T),$$

so ϕ fulfills the boundary conditions required by (6.1) for both t and x. To show that ϕ satisfies (6.5), we make the simple calculation

$$\begin{aligned} \phi_x(x,t) \;=\;& (T-t)(T-\sigma-t)\Big\{(1-2x)\phi_1(x)\,\psi_1(t) + (x-x^2)\phi_1'(x)\psi_1(t) \\ & +\psi_2(t)\frac{q(x)-w(x)}{u_2(x)}\Big\}, \end{aligned}$$

from which and the definitions of w and u_2 we derive

$$\begin{aligned}
&\int_{T-\sigma}^{T} u_x(x,t)\,\phi_x(x,t)dt \\
&= \{(1-2x)\phi_1(x) + (x-x^2)\phi_1'(x)\}u_1(x) + \{q(x)-w(x)\} \\
&= q(x).
\end{aligned}$$

Finally, we can easily verify that

$$\phi_x, \quad \phi_t \in L^2(T-\sigma, T;\; L^2(\Omega)\,),$$

which completes the proof of Lemma 6.1. □

Remark 6.1 *Lemma 6.1 provides a general way of constructing solutions ϕ for the equation* (6.2) *(in the weak form (6.3)) in the case $d=1$. For any $\psi_1, \psi_2 \in H^1(T-\sigma, T)$ with $\psi_1 \neq \psi_2$ satisfying (with u_i defined by Lemma 6.1)*

$$\int_0^1 \frac{(1-2x)u_1(x)}{u_2(x)} \neq 0, \tag{6.7}$$

we can find a constant γ_0 such that

$$\gamma_0 \int_0^1 \frac{(1-2x)\,u_1(x)}{u_2(x)}dx = \int_0^1 \frac{q(x)}{u_2(x)}\,dx;$$

we put $\phi_1(x) \equiv \gamma_0$. This results in

$$\int_0^1 \frac{q(x)-w(x)}{u_2(x)}dx = 0,$$

and ϕ as defined in (6.6) is the required solution of (6.2) *provided that $\frac{q-w}{u_2} \in L^2(0,1)$, which is the only remaining condition. This condition is natural: It contains an expression involving u_x in the denominator; if u_x vanishes in a whole region between times $T-\sigma$ and T, it would be imposible to determine q there.*

Also for $d>1$, (6.3) imposes no obvious added regularity, since all expressions are in $L^2(\Omega)$. However, the strong form of (6.2) would require that $q^* - q^\dagger \in H^2(\Omega)$.

The fact that the source condition is so weak might be credited to several things: Firstly, we used a lot of special structure of our problem; secondly, we get convergence rates not for the full H^1-norm, but only for the L^2-norm of the gradients (which would imply by the Poincaré inequality H^1-convergence if we knew the parameter on a part of the boundary of Ω with positive measure and included this as a constraint into the admissible set K). Finally, it might be an indication that our problem is close to being well–posed, which is again conceivable since we use a lot of data and two–sided a–priori bounds.

7 Variants, Extensions

Our results are still in the infinite–dimensional setting, unrealistically assuming that we have distributed measurements of $(\nabla)u$. Of course, in reality only discrete measurements are available, which requires to extend the theory by adding discretization in solving the forward problem and in approximating the unknown parameter. For fixed $\beta > 0$, this has been considered in [5], for $\beta \to 0$ we could proceed by combining our arguments with those in [8], [9].

Both (1.7) and (1.8) involve gradients in the regularization term. Instead, we could also consider L^2–regularization, i.e.,

$$J(q) := \int_{T-\sigma}^{T} \int_{\Omega} q(x)|(\nabla)(u(q) - z^\delta)|^2 dx dt + \beta \int_{\Omega} |q - q^*|^2 dx \to \min, q \in K. \tag{7.1}$$

Results corresponding to Theorems 4.1 and 5.1 can be proven in a completely analogous way, where $\nabla^*\nabla(q^\dagger - q^*)$ in (4.15) is replaced by $(q^\dagger - q^*)$, and $\|\nabla q_\beta^\delta - \nabla q^\dagger\|$ in (4.16) is replaced by $\|q_\beta^\delta - q^\dagger\|$. For the analogue of Theorem 5.1, we would need to incorporate an a–priori bound on $|\nabla q|$ to take care of the term $\beta\|\nabla(q_\beta^\delta - q^\dagger)\|^2$ on the right–hand side of the estimate for $(\mathrm{II})_2$ in the proof.

The existence of minimizers of (7.1) cannot be guaranted by Theorem 2.1. This difficulty could either be resolved by using some weak- closedness argument or circumvented by incorporating a tolerance η into the minimization, i.e., replacing minimizers of (7.1) by elements $q_{\beta,\eta}^\delta \in K$ such that

$$J(q_{\beta,\eta}^\delta) \le \inf_{q\in K} J(q) + \eta;$$

as long as $\eta = O(\delta^2)$, all proofs carry over. This can of course also be done for (1.7) and (1.8).

We close by pointing out the abstract structure of our approach in comparison with the general theory outlined in Section 1, where we proceed only in a formal way without precise assumptions and omit, for simplicity of notation, the integration over $[T-\sigma, T]$ in our observations (cf. e.g. (1.7), (1.8), (1.10), (1.11)):

Our direct (forward) problem corresponding to (1.12) has the weak formulation

$$\langle u_t, \psi\rangle + B(N(q,u), \psi) = \langle f, \psi\rangle \quad \forall \psi \in H \tag{7.2}$$

on a Hilbert space H with inner product $\langle , \rangle$. In our case, the bilinear form B is

$$B(v, \psi) := (v, \nabla \psi) \tag{7.3}$$

and the nonlinear operator N is

$$N(q, u) := q\nabla u. \tag{7.4}$$

The linearization $u'(q)$ of u with respect to q has the weak form

$$\langle (u'(q)p)_t, \psi\rangle + B(N_u(q, u(q))u'(q)p, \psi) = -B(pN_q(q, u(q)), \psi) \tag{7.5}$$

as can be seen by implicit differentiation of (7.2). The classical source condition corresponding to (1.16) would be

$$(\exists \phi) \quad q^\dagger - q^* = u'(q^\dagger)^*\phi \tag{7.6}$$

or in weak form

$$(\exists \phi)(\forall \psi) \quad \langle q^\dagger - q^*, \psi\rangle = \langle \phi, u'(q^\dagger)\psi\rangle. \tag{7.7}$$

This is used in a crucial step of the convergence–rates proof (cf. [4] or [3, Theorem 10.4]) as follows:

$$\langle q^\dagger - q^*, q^\dagger - q^\delta_\beta\rangle = \langle u'(q^\dagger)^*\phi, q^\dagger - q^\delta_\beta\rangle = \langle \phi, u'(q^\dagger)(q^\dagger - q^\delta_\beta)\rangle.$$

In the analogous step in the proofs of Theorems 4.1 and 5.1, we used (4.15) instead, which corresponds to $\langle q^\dagger - q^*, \psi\rangle = -B(\psi N_q(q^\dagger, u(q^\dagger)), \phi) =$

$\langle (u'(q^\dagger)\psi)_t, \phi\rangle + B(N_u(q^\dagger, u(q^\dagger))u'(q^\dagger)\psi, \phi)$ with $\psi = q^\dagger - q_\beta^\delta$, where the first equality is the abstract version of the source condition, i.e.,

$$(\exists\phi)(\forall\psi) \quad \langle q^\dagger - q^*, \psi\rangle = -B(\psi N_q(q^\dagger, u(q^\dagger)), \phi) \tag{7.8}$$

or in the case of seminorm–regularization (1.18),

$$(\exists\phi)(\forall\psi) \quad \langle D(q^\dagger - q^*), D\psi\rangle = -B(\psi N_q(q^\dagger, u(q^\dagger)), \phi). \tag{7.9}$$

The coupling of the source condition with the weak form (7.5) of the linearized equation is crucial.

To proceed with the convergence–rate proof on this abstract level would now require specific assumptions about the interplay of the bilinear form B with the nonlinearity N and the operator D. We will not pursue this until we are convinced that there are other interesting problems that fall into this abstract setting. An obvious first candidate might be the identification of a parameter in a parabolic equation depending nonlinearly on the solution, i.e., B as in (7.3) and $N(q, u) := q(u) \cdot \nabla u$.

References

[1] H. Banks and K.Kunisch, Parameter Estimation Techniques for Distributed Systems, Birkhäuser, Boston 1989.

[2] Z. Chen and J. Zou, An augmented Lagrangian method for identifying discontinuous parameters in elliptic systems, SIAM J. Control Optim., accepted for publication.

[3] H.W. Engl, M. Hanke and A. Neubauer, Regularization of Inverse Problems, Kluwer Academic Publishers, Dordrecht 1996.

[4] H.W. Engl, K. Kunisch and A. Neubauer, Convergence rates for Tikhonov regularization of nonlinear ill–posed problems, Inverse Problems 5 (1989), 523–540.

[5] Y. Keung and J. Zou, Numerical identifications of parameters in parabolic systems, Inverse Problems 14 (1998), 83–100.

[6] C. Kravaris and H. Seinfeld, Identification of parameters in distributed parameter systems by regularization, SIAM J. Control Optim. 23 (1985), 217–241.

[7] K. Kunisch and W. Ring, Regularization of nonlinear illposed problems with closed operators, Numer. Funct. Anal. and Optimiz. 14 (1993), 389- 404.

[8] A. Neubauer, Tikhonov regularization for nonlinear ill–posed problems: optimal convergence and finite–dimensional approximation, Inverse Problems 5 (1989), 541–557.

[9] A. Neubauer and O. Scherzer, Finite–dimensional approximation of Tikhonov regularized solutions of nonlinear ill–posed problems, Numer. Funct. Anal. Optim. 11 (1990), 85–99

[10] O. Scherzer, H.W. Engl and R.S. Anderssen, Parameter identification from boundary measurements in a parabolic equation arising from geophysics, Nonlinear Anal. 20 (1993), 127–156.

[11] T.I. Seidman and C.R. Vogel, Well–posedness and convergence of some regularization methods for nonlinear ill–posed problems, Inverse Problems 5 (1989), 227-238.