Multigrid Methods for a Class of Parameter Dependent Problems in Primal Variables

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Abstract

This paper deals with the construction and analysis of multigrid methods for parameter dependent problems including nearly incompressible materials and Reissner Mindlin plate models. The considered algorithms can be applied equivalently for the positive definite system in primal variables, or for the indefinite saddle point system. We formulate hypothesis, on which the multigrid analysis based. Finally, we verify these hypothesis for the specific examples.

1 Introduction

In this paper we consider a class of problems depending in a certain way on a small parameter ε. Several such examples stem from computational mechanics, where material parameters (e.g. the Poisson ratio) may cause difficulties, or the shape of the body degenerates and lead to a model of structural mechanics.

A straight forward finite element discretization of these problems gives bad results. This is called locking effect, [4], [1]. One way out is the reformulation as a mixed variational problem, where the finite element discretization is done [13]. Often it is possible to retain to a positive definite problem in the primal variables, what is advantageously for practical computations.

As the discretization error, also the convergence rate of iterative methods like multigrid with standard components degenerates. So robust multigrid methods for the indefinite linear system have been designed, see [5], [11], [6] and the references therein. In these papers the norms stemming from the saddle point system are used. The approximation property is essentially the same, but the type of smoother varies. The technique from [6] is 'u - dominated', so it can also be applied in primal variables [22]. In [24] the smoothing property for the approximative version of the Braess-Sarazin smoother is verified.

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The present paper is an extension to [19], where the case of nearly incompressible materials was considered. Here, we first formulate conditions to a generic problem and prove multigrid convergence. Then the conditions are checked for the case of nearly incompressible materials with conforming or non-conforming discretization and for the Reissner Mindlin plate model with the finite element discretization from [14]. The limit problems with $\varepsilon = 0$, e.g. Stokes, can be handled by the saddle point version of the algorithm.

The key features of the algorithm are stable grid transfer operators and block smoothers capturing base functions of the kernel of the up-scaled term. One essential difference to the papers mentioned before, is that our analysis is based on the norms related to the primal problem. The main idea is the following. For the primal variable, the duality technique can be applied to achieve better convergence in a weaker norm. The dual variable, which corresponds to the constraints, is decomposed orthogonally into a coarse grid part and a local part. The coarse grid constraints are inherited from the coarse grid solution. The fine grid constraints are achieved by local projections in the grid transfer operator. It seems to be interesting that the considered problems need different decompositions of the dual variable.

The rest of the paper is organized as follows. In Section 2 the problem formulation and discretization is given, Section 3 defines the necessary multigrid ingredients. In these sections, the necessary hypothesis are formulated and will be marked with a preceding *Hyp* in the number of equation. In Sections 4, 5 and 6 multigrid methods are formulated and analyzed based only on the hypothesis of the former sections. In Section 7 we check the assumptions for nearly incompressible materials and Stokes, and in Section 8 we verify them for the Reissner Mindlin plate model. The author recommends to switch reading between the abstract Sections 2 and 3, and Section 7 applying the abstract theory to the problem of nearly incompressible materials.

2 Formulation and Discretization

We consider variational problems find $u \in V$ such that

$$A(u,v) = f(v) \qquad \forall\, v \in V \tag{1}$$

with the symmetric bilinear form $A(.,.)$ and the linear functional $f(.)$ on the Hilbert space V. The bilinear form can be written as

$$A(u,v) = a(u,v) + \varepsilon^{-1}\, c(\Lambda u, \Lambda v) \tag{2}$$

with the symmetric and non-negative bilinear forms $a(.,.)$ and $c(.,.)$ defined on V and L_2, respectively, the linear operator $\Lambda : V \to L_2$, and the small positive parameter ε. The problem is of special interest, if Λ has a non-trivial kernel V_0.

We assume that $a(u,v) + c(\Lambda u, \Lambda v)$ is elliptic and continuous on V. This implies that $A(.,.)$ is elliptic and continuous too, and a unique solution $u \in V$ of (1) exists. But, in the interesting case of non-trivial V_0, the ratio of the corresponding constants degenerates as ε becomes small.

By the definition of the dual variable

$$p = \varepsilon^{-1} \Lambda u \tag{3}$$

in $Q := L_2$ we obtain the equivalent mixed problem to find $(u, p) \in X := V \times Q$ such that

$$B((u, p), (v, q)) = f(v) \qquad \forall\, (v, q) \in X. \tag{4}$$

The block bilinear form $B(.,.) : X \times X \to \mathbb{R}$ is defined as

$$B((u, p), (v, q)) = a(u, v) + c(\Lambda u, q) + c(\Lambda v, p) - \varepsilon\, c(p, q). \tag{5}$$

We will need the stronger norms $\|.\|_{V^2}$ and $\|.\|_{Q^1}$ on dense subspaces $V^2 \subset V$ and $Q^1 \subset Q$ and the weaker norm $\|.\|_{V^0}$ on the larger space $V^0 \supset V$. For the Stokes problem the indices coincide with the according Sobolev space indices. If the linear functional f is in the more regular space $(V^0)^*$, the solution is more regular as well and is bounded by

$$\|u\|_{V^2} + \|p\|_{Q^1} \preceq \|f\|_{(V^0)^*}. \tag{6}$$

The symbol $a \preceq b$ does mean there exists a constant c independent of a and b, as well as ε and the level of refinement l, such that $a \leq c\, b$. We write $a \succeq b$ for $b \preceq a$, and $a \simeq b$ for $a \preceq b$ and $b \preceq a$.

The discretization of (4) is based on a sequence of possibly non-nested non-conforming finite element spaces

$$V_1, V_2, \ldots, V_L$$

and nested spaces

$$Q_1 \subset Q_2 \subset \ldots Q_L \subset Q$$

with norms $\|.\|_{V,l}$ and $\|.\|_{Q,l}$. The norms may and will depend on the level as well as on the parameter. We set $X_l = V_l \times Q_l$ with the product norm $\|(u, p)\|_{X,l} := (\|u\|_{V,l}^2 + \|p\|_{Q,l}^2)^{1/2}$. The index l denotes the level and takes the values 1 to L, unless explicitely defined differently. On each level l we need a symmetric bilinear form

$$B_l((u_l, p_l), (v_l, q_l)) = a_l(u_l, v_l) + b_l(u_l, q_l) + b_l(v_l, p_l) - \varepsilon\, c_l(p_l, q_l) \tag{7}$$

defined on $X_l \times X_l$. The part $c_l(.,.)$ is assumed to be positive and thus it leads to the norm $\|q_l\|_{c_l} := (c(q_l, q_l))^{1/2}$. We assume that $a_l(.,.)$ is non-negative, i.e.

$$a_l(u_l, u_l) \geq 0 \qquad \forall\, u_l \in V_l. \tag{8}$$

We will not need ellipticity of $a_l(.,.)$ on the space V_l. The bilinear form $B_l(.,.)$ is assumed to be continuous

$$B_l((u_l, p_l), (v_l, q_l)) \preceq \|(u_l, p_l)\|_{X,l}\, \|(v_l, q_l)\|_{X,l} \tag{9}$$

and stable

$$\sup_{(v_l, q_l) \in X_l} \frac{B_l((u_l, p_l), (v_l, q_l))}{\|(v_l, q_l)\|_{X,l}} \succeq \|(u_l, p_l)\|_{X,l} \qquad \forall\, (u_l, p_l) \in X_l. \tag{10}$$

On each space V_l the inverse estimate

$$\|u_l\|_{V,l} \preceq h_l^{-1} \|u_l\|_{V^0} \tag{11}$$

are fulfilled with coefficient h_l. Usually, h_l is the mesh-size on multigrid level l. For problems with different regularity properties, one can set the parameter h_l to some power of the mesh-size.

We require interpolation operators

$$I_l^X = (I_l^V, I_l^Q) : (V^2 \times Q^1) \to X_l \tag{12}$$

fulfilling

$$h_l^{-1} \|v - I_l^V v\|_{V^0} + \|(v,q) - I_l^X(v,q)\|_{X,l} \preceq h_l \left(\|v\|_{V^2} + \|q\|_{Q^1} \right) \tag{13}$$

for given $(v,q) \in V^2 \times Q^1$.

Let $(u_l, p_l) \in X_l$ be the solution of the mixed finite element problem

$$B_l((u_l,p_l),(v_l,q_l)) = f(v_l) \qquad \forall\, (v_l,q_l) \in X_l. \tag{14}$$

Then the a priori error estimate to the solution (u,p) of (4)

$$h_l^{-1} \|u - u_l\|_{V^0} + \|u - u_l\|_{V,l} + \|p - p_l\|_{Q,l} \preceq h_l\, \|f\|_{(V^0)^*} \tag{15}$$

is valid.

By the definiteness of $c_l(.,.)$ on the finite dimensional spaces Q_l we can define the operators

$$\Lambda_l : V_l \to Q_l : \qquad c_l(\Lambda_l u_l, q_l) = b_l(u_l, q_l) \qquad \forall\, u_l \in V_l,\ \forall\, q_l \in Q_l. \tag{16}$$

It is of special interest, if Λ_l can be computed locally. Then we can assemble the symmetric and elliptic form

$$A_l(u_l, v_l) := a_l(u_l, v_l) + \varepsilon^{-1} c_l(\Lambda_l u_l, \Lambda_l v_l) \tag{17}$$

corresponding to the original form (2). The finite element problem in primal variables find $u_l \in V_l$ such that

$$A_l(u_l, v_l) = f(v_l) \qquad \forall\, v_l \in V_l \tag{18}$$

provides the same solution u_l as (14).

3 Multigrid ingredients

In this section we will formulate the hypothesis related to the building blocks of multigrid. The actual multigrid algorithms will be given in the next section.

Due to the non-nestedness of the spaces we have to define prolongation operators

$$R_l^{V,0} : V_{l-1} \to V_l, \tag{19}$$

with $2 \leq l \leq L$. They are called trivial prolongation operators and they are supposed to fulfill the continuity and approximation estimates

$$\begin{aligned} \|R_l^{V,0} u_{l-1}\|_{V,l} + h_l^{-1} \|u_{l-1} - R_l^{V,0} u_{l-1}\|_{V^0} &\preceq \|u_{l-1}\|_{V,l-1}, \\ \|I_l u - R_l^{V,0} I_{l-1} u\|_{V^0} &\preceq h_l^2 \|u\|_{V^2} \end{aligned} \tag{20}$$

for all $u_{l-1} \in V_{l-1}$ and $u \in V^2$. Since $Q_{l-1} \subset Q_l$, we can combine them with the identity to the trivial mixed prolongation operator $R_l^{X,0} = (R_l^{V,0}, id) : X_{l-1} \to X_l$. We assume norm equivalence across two levels

$$\|p_{l-1}\|_{Q,l-1} \simeq \|p_{l-1}\|_{Q,l} \qquad \forall\, p_{l-1} \in Q_{l-1}. \tag{21}$$

The actual grid transfer operation requires the space

$$X_{l,T} = V_{l,T} \times Q_{l,T} \subset X_l \tag{22}$$

with the properties defined below. We have to solve a variational problem with bilinear form $B_l(.,.)$ on the space $X_{l,T}$. To be efficient it is necessary that $X_{l,T}$ splits into subspaces of small dimension which are orthogonal with respect to $B_l(.,.)$. The space Q_l is decomposed as

$$Q_l = Q_{l,T} \oplus \widetilde{Q}_{l-1} \tag{23}$$

with $\widetilde{Q}_{l-1} \subset Q_{l-1}$. The decomposition is assumed to be orthogonal with respect to the inner product $c_l(.,.)$. The space $\widetilde{Q}_{l-1}$ corresponds to coarse grid constraints which will be inherited on the fine grid. The complement $Q_{l,T}$ characterizes the constraints which can be fulfilled by local projections. We define the c_l ortho-projector $I_{l-1}^{\widetilde{Q}} : Q_l \to \widetilde{Q}_{l-1}$. It is assumed to be continuous with norm $\|.\|_{Q_l \to Q_{l-1}}$. The bilinear-from $B_l(.,.)$ is assumed to be stable on $X_{l,T}$, namely for all $(u,p) \in X_{l,T}$ there holds

$$\sup_{(v,q) \in X_{l,T}} \frac{B_l((u,p),(v,q))}{\|(v,q)\|_{X,l}} \succeq \|(u,p)\|_{X,l}. \tag{24}$$

We assume the inclusion

$$\Lambda_l V_{l,T} \subset Q_{l,T}. \tag{25}$$

The inclusion ensures that corrections in $V_{l,T}$ will not disturb constraints corresponding to $\widetilde{Q}_{l-1}$. The trivial prolongation is supposed to satisfy

$$B_{l-1}((u_{l-1}, p_{l-1}), (0, \tilde{q}_{l-1})) = B_l((R_l^{V,0} u_{l-1}, p_{l-1}), (0, \tilde{q}_{l-1})) \tag{26}$$

for all $(u_{l-1}, p_{l-1}) \in X_{l-1}$ and $\tilde{q}_{l-1} \in \widetilde{Q}_{l-1}$. This condition guaranties that constraints corresponding to $\widetilde{Q}_{l-1}$ are preserved by the grid transfer. On the space $V_{l,T}$ not only the inverse estimate (11), but also the norm equivalence

$$\|u_{l,T}\|_{V^0} \simeq h_l \|u_{l,T}\|_{V,l} \qquad \forall\, u_{l,T} \in V_{l,T} \tag{27}$$

shall be fulfilled.

The construction of the smoother needs the decomposition

$$V_l = \sum_{i=1}^{N_i} V_{l,i}, \qquad Q_l = \sum_{i=1}^{N_i} Q_{l,i}. \tag{28}$$

into subspaces of small dimension. The decomposition may be overlapping. The suggested smoothers are block smoothers with respect to these subspaces. We set $X_{l,i} = V_{l,i} \times Q_{l,i}$ and demand stability of $B_l(.,.)$ on each of the spaces, namely for all $(u,p) \in X_{l,i}$ there holds

$$\sup_{(v,q)\in X_{l,i}} \frac{B_l((u,p),(v,q))}{\|(v,q)\|_{X,l}} \succeq \|(u,p)\|_{X,l}. \tag{29}$$

Additionally, the inclusions

$$\Lambda_l V_{l,i} \subset Q_{l,i} \tag{30}$$

must be fulfilled for all $1 \le i \le N_l$.

Based on the energy norm $\|u_l\|_{A_l} := A_l(u_l,u_l)^{1/2}$ we define the splitting norm

$$\|u_l\|^2_{D_l} := \inf_{\substack{u_l=\sum u_{l,i}\\ u_{l,i}\in V_{l,i}}} \|u_{l,i}\|^2_{A_l}. \tag{31}$$

We will prove the approximation property for the norm

$$\|u_l\|^2_{V,l,\tilde{0}} := h_l^{-2}\|u\|^2_{V^0} + \varepsilon^{-1}\|\Lambda_l u_l\|^2_{c_l} + \varepsilon^{-2}\|I^{\tilde{Q}}_{l-1}\Lambda_l u_l\|^2_{Q,l}. \tag{32}$$

The norm combines three different terms. The first one is the improved convergence in a weaker norm steming from the usual duality argument. The second term is essential to reflect continuity in energy norm. The last term bounds the coarse grid part of the dual variable $\varepsilon^{-1}\Lambda_l u_l$.

Let $\|.\|_{[A_l,D_l]_\alpha}$ be the norm obtained by interpolation with index $\alpha \in (0,1]$ of the Hilbert spaces $(V_l, \|.\|_{A_l})$ and $(V_l, \|.\|_{D_l})$, see [17], and also [10]. The proof of the smoothing property requires the estimate

$$\|u_l\|^2_{[A_l,D_l]_\alpha} \preceq \|u_l\|^2_{V,l,\tilde{0}}, \tag{33}$$

which has to be checked for the specific problem at hand. These are the ingredients we will use in the following three sections to formulate and analyze multigrid methods and we will verify for the considered problems in the final two sections.

4 Multigrid Formulation

In this section we formulate the multigrid methods which will be analyzed in the following sections. We suggest one family of algorithms in primal variables, which can be performed for positive ε. An other family of methods in mixed variables is well defined also for the limit problem with $\varepsilon = 0$. Later, we will see that both algorithms produce the same iterates.

Let $\langle .,. \rangle_l$ be an additional inner product on V_l. Usually, it is the Euclidean inner product of nodal values. We define the linear operator $A_l : V_l \to V_l$ by

$$\langle A_l u_l, v_l \rangle_l = A_l(u_l, v_l) \qquad \forall\, u_l, v_l \in V_l,$$

and the representant $f_l \in V_l$ of the linear functional by

$$\langle f_l, v_l \rangle_l = f(v_l) \qquad \forall\, v_l \in V_l.$$

If one identifies the finite element space V_l with the Euklidean space of nodal values, A_l is the system matrix and f_l is the load vector. The variational problem is equivalent to the operator equation find $u_l \in V_l$ such that

$$A_l u_l = f_l. \tag{34}$$

Let us define the projections

$$P^A_{l,i} : V_l \to V_{l,i} : \quad A_l(P^A_{l,i} u_l, v_{l,i}) = A_l(u_l, v_{l,i}) \qquad \forall\, u_l \in V_l,\ \forall\, v_{l,i} \in V_{l,i} \tag{35}$$

onto the spaces $V_{l,i}$ of (28). We define the smoother $S^A_l : V_l \to V_l$ as additive Schwarz method corresonding to this space splitting, i.e.

$$S^A_l := I - \tau \sum_{i=1}^{N_l} P^A_{l,i}. \tag{36}$$

This is the block Jacobi smoother with the blocks given by the subspaces $V_{l,i}$. The damping parameter τ is chosen such that S^A_l has a positive spectrum. We mention that τ depends only on the number of overlapping subspaces. The splitting norm $\|.\|_{D_l}$ from (31) is a Hilbert space norm. Thus it provides the quadratic form $D_l(.,.)$ and the linear operator $D_l : V_l \to V_l$ defined by

$$\langle D_l u_l, v_l \rangle_l = D_l(u_l, v_l) \qquad \forall\, u_l, v_l \in V_l.$$

The additive Schwarz theory [23], [18] allows the explicit representation of D_l by

$$D_l^{-1} = \sum P^A_{l,i} A_l^{-1}.$$

Thus, the smoother can be rewritten as

$$S^A_l = I - \tau D_l^{-1} A_l.$$

The other multigrid component is the grid transfer. For each l, $2 \le l \le L$, we define the projection

$$P^A_{l,T} : V_l \to V_{l,T} : \quad A_l(P^A_{l,T} u_l, v_{l,T}) = A_l(u_l, v_{l,T}) \qquad \forall\, u_l \in V_l,\ \forall\, v_{l,T} \in V_{l,T} \tag{37}$$

onto the subspace $V_{l,T}$ defined in (22). We mention that the projection $P_{l,T}$ can be computed efficiently, because $V_{l,T}$ splits $A_l(.,.)$ orthogonal into local spaces of small dimension. The suggested prolongation operator is

$$R_l^V := (I - P_{l,T}^A) R_l^{V,0}. \tag{38}$$

The meaning of the prolongation is the following. First, we lift the function from the coarse grid to the fine grid. Then the post processing step $(I - P_{l,T}^A)$ is performed. For the considered examples, the post processing requires the solution of a local Dirichlet problem inside of each of the coarse grid elements. This procedure is also known as discrete harmonic extension.

The restriction operator $[R_l^V]^T$ is the $\langle .,. \rangle_l$ - $\langle .,. \rangle_{l-1}$ adjoint of R_l^V. This gives

$$[R_l^V]^T := [R_l^{V,0}]^T (I - P_{l,T}^A)^T, \tag{39}$$

where $[R_l^{V,0}]^T : V_l \to V_{l-1}$ is defined by

$$\left\langle [R_l^{V,0}]^T u_l, v_{l-1} \right\rangle_{l-1} = \left\langle u_l, R_l^{V,0} v_{l-1} \right\rangle_l \qquad \forall\, u_l \in V_l, \;\; \forall\, v_{l-1} \in V_{l-1},$$

and $(I - P_{l,T}^A)^T = A_l (I - P_{l,T}^A) A_l^{-1}$ is the $\langle .,. \rangle_l$ adjoint of $(I - P_{l,T}^A)$.

Now we are prepared to define the multigrid algorithm. One step of the iteration

$$\hat{u}_l = MG(l, u_l, f_l) \tag{40}$$

is given in Algorithm 1. It defines a family of multigrid methods. The parameter q defines the cycle. Usually, the V-cycle with $q = 1$ or the W-cycle with $q = 2$ is performed. The number of smoothing steps m_l may depend on the level. A V-cycle with a geometrically increasing number of smoothing steps on coarser levels is called variable V-cycle. The multigrid algorithm defines the multigrid operator $M_l^A : V_l \to V_l$ by

$$M_l^A u_l := MG(l, u_l, 0) \tag{41}$$

and the multigrid preconditioner $C_l : V_l \to V_l$ by

$$C_l^{-1} f_l := MG(l, 0, f_l). \tag{42}$$

Next, we define a multigrid method for the mixed formulation. Let B_l be defined by an inner product on X_l. The projections $P_{l,T}^B : X_l \to X_{l,T}$ and $P_{l,i}^B : X_l \to X_{l,i}$ are defined by

$$B_l(P_{l,T}^B(u_l, p_l), (v_{l,T}, q_{l,T})) = B_l((u_l, p_l), (v_{l,T}, q_{l,T})) \quad \forall\, (v_{l,T}, q_{l,T}) \in X_{l,T} \tag{43}$$

and

$$B_l(P_{l,i}^B(u_l, p_l), (v_{l,i}, q_{l,i})) = B_l((u_l, p_l), (v_{l,i}, q_{l,i})) \quad \forall\, (v_{l,i}, q_{l,i}) \in X_{l,i} \tag{44}$$

Procedure $MG(l, u_l, f_l)$
if $l = 1$

$$MG(l, u_l, f_l) = A_l^{-1} f_l$$

else

$$\begin{aligned}
&u_l^{1,0} = u_l \\
&\text{do } j = 1, \ldots, m_l \\
&\qquad u_l^{1,j} = u_l^{1,j-1} + \tau D_l^{-1}(f_l - A_l u_l^{1,j-1}) \\
&d_{l-1} = [R_l^V]^T (f_l - A_l u_l^{1,m_l}) \\
&u_{l-1}^{2,0} = 0 \\
&\text{do } j = 1, \ldots, q \\
&\qquad u_{l-1}^{2,j} = MG(l-1, u_{l-1}^{2,j-1}, d_{l-1}) \\
&u_l^{3,0} = u_l^{1,m_l} + R_l^V u_{l-1}^{2,q} \\
&\text{do } j = 1, \ldots, m_l \\
&\qquad u_l^{3,j} = u_l^{3,j-1} + \tau D_l^{-1}(f_l - A_l u_l^{3,j-1}) \\
&MG(l, u_l, f_l) = u_l^{3,m_l}
\end{aligned}$$

Algorithm 1: Multigrid Procedure

for all $(u_l, p_l) \in X_l$, respectively. Then the smoother and the prolongation operator are defined as

$$\begin{aligned}
S_l^B &= I - \tau \sum P_{l,i}^B, \\
R_l^X &= (I - P_{l,T}^B) R_l^{X,0}.
\end{aligned}$$

The smoother requires the solution of local saddle point problems. That is like a Vanka - smoother [21]. But usual versions of the family of Vanka smoothers do not fulfill our assumption (30). The multigrid method for the mixed problem is defined corresponding to Algorithm 1.

5 Multigrid Analysis

First, we establish the equivalence of the primal algorithm and the mixed one. We formulate the approximation property, but postpone its proof to the next section. Assuming its truth, we prove optimal and robust rate of convergence for the W-cycle and variable V-cycle multigrid algorithm.

We define the subspaces $X_{l,0} \subset X_l$

$$X_{l,0} = \{(u_l, p_l) \in X_l : \Lambda_l u_l = \varepsilon p_l\}. \tag{45}$$

For all $(u_l, p_l) \in X_{l,0}$ there holds

$$B_l((u_l, p_l), (0, q_l)) = 0 \qquad \forall\, q_l \in Q_l \tag{46}$$

as well as

$$A_l(u_l, v_l) = B_l((u_l, p_l), (v_l, 0)) \qquad \forall\, v_l \in V_l. \tag{47}$$

The multigrid components in mixed variables preserve the spaces $X_{l,0}$. On the subspace, they reduce to the components in primal variables. This is collected in the following lemma:

Lemma 1. *The multigrid components fulfill the following properties:*

1. *The smoother S_l^B preserves $X_{l,0}$. On the subspace it is equivalent to the smoother S_l^A in primal variables. This means for $(u_l, p_l) \in X_{l,0}$ and*
$$(\hat{u}_l, \hat{p}_l) = S_l^B(u_l, p_l)$$
there holds $(\hat{u}_l, \hat{p}_l) \in X_{l,0}$ and
$$\hat{u}_l = S_l^A u_l.$$

2. *The prolongation R_l^X maps $X_{l-1,0}$ into $X_{l,0}$. On the subspace it is equivalent to the prolongation R_l^V in primal variables. This means for $(u_{l-1}, p_{l-1}) \in X_{l-1,0}$ and*
$$(\hat{u}_l, \hat{p}_l) = R_l^X(u_{l-1}, p_{l-1})$$
there holds $(\hat{u}_l, \hat{p}_l) \in X_{l,0}$ and
$$\hat{u}_l = R_l^V u_{l-1}.$$

3. *The coarse grid solution operator maps $X_{l,0}$ into $X_{l-1,0}$. On the subspace it is equivalent to the coarse grid solution operator in primal variables. This means for $(u_l, p_l) \in X_{l,0}$ and*
$$(\hat{u}_{l-1}, \hat{p}_{l-1}) = B_{l-1}^{-1}[R_l^X]^T B_l\,(u_l, p_l)$$
there holds $(\hat{u}_{l-1}, \hat{p}_{l-1}) \in X_{l-1,0}$ and
$$\hat{u}_{l-1} = A_{l-1}^{-1}[R_l^A]^T A_l u_l.$$

Proof. 1. It is sufficient to prove the corresponding properties for the projections $P_{l,i}^B$. Let $(u_l, p_l) \in X_{l,0}$ and $(\hat{u}_l, \hat{p}_l) = P_{l,i}^B(u_l, p_l)$. From (30), $\Lambda_l V_{l,i} \subset Q_{l,i}$ there follows $\Lambda_l \hat{u} - \varepsilon \hat{p} \in Q_{l,i}$. Using the definition of the projection we get

$$\|\Lambda_l \hat{u}_l - \varepsilon \hat{p}_l\|_{c_l} = \sup_{q_{l,i} \in Q_{l,i}} \frac{(\Lambda_l \hat{u}_l - \varepsilon \hat{p}_l, q_{l,i})_{c_l}}{\|q_{l,i}\|_{c_l}} = \sup_{q_{l,i} \in Q_{l,i}} \frac{(\Lambda_l u_l - \varepsilon p_l, q_{l,i})_{c_l}}{\|q_{l,i}\|_{c_l}} = 0,$$

i. e. $(\hat{u}_l, \hat{p}_l) \in X_{l,0}$. By $(\hat{u}_l, \hat{p}_l) \in X_{l,0} \cap X_{l,i}$ and (47) we obtain

$$A_l(\hat{u}_l, v_{l,i}) = B_l((\hat{u}_l, \hat{p}_l), (v_{l,i}, 0)) = B_l((u_l, p_l), (v_{l,i}, 0)) = A_l(u_l, v_{l,i}) \qquad \forall\, v_{l,i} \in V_{l,i},$$

what is the definition of $P_{l,i}^A$.

2. Let $(u_{l-1}, p_{l-1}) \in X_{l-1,0}$. We define

$$\begin{aligned} (\tilde{u}_l, \tilde{p}_l) &= R_l^{X,0}(u_{l-1}, p_{l-1}), \\ (u_{l,T}, p_{l,T}) &= P_{l,T}^B(\tilde{u}_l, \tilde{p}_l), \\ (\hat{u}_l, \hat{p}_l) &= (\tilde{u}_l, \tilde{p}_l) - (u_{l,T}, p_{l,T}). \end{aligned}$$

By the definition of the projection $P_{l,T}^B$ we have

$$(\Lambda_l u_{l,T} - \varepsilon p_{l,T}, q_{l,T})_{c_l} = (\Lambda_l \tilde{u}_l - \varepsilon \tilde{p}_l, q_{l,T})_{c_l} \qquad \forall\, q_{l,T} \in Q_{l,T},$$

and by $\Lambda_l u_{l,T} - \varepsilon p_{l,T} \in Q_{l,T} \perp_{c_l} \widetilde{Q}_{l-1}$ we have

$$(\Lambda_l u_{l,T} - \varepsilon p_{l,T}, \tilde{q}_{l-1})_{c_l} = 0 \qquad \forall\, \tilde{q}_{l-1} \in \widetilde{Q}_{l-1}.$$

Thus we get

$$(\Lambda_l \hat{u}_l - \varepsilon \hat{p}_l, q_l)_{c_l} = (\Lambda_l \tilde{u}_l - \varepsilon \tilde{p}_l, I_{l-1}^{\widetilde{Q}} q_l)_{c_l} \qquad \forall\, q_l \in Q_l.$$

We use assumption (26), $I_{l-1}^{\widetilde{Q}} q_l \in Q_{l-1}$, and (46) to show

$$(\Lambda_l \hat{u}_l - \varepsilon \hat{p}_l, q_l)_{c_l} = B_l((\tilde{u}_l, \tilde{p}_l), (0, I_{l-1}^{\widetilde{Q}} q_l)) = B_{l-1}((u_{l-1}, p_{l-1}), (0, I_{l-1}^{\widetilde{Q}} q_l)) = 0,$$

i.e. $(\hat{u}_l, \hat{p}_l) \in X_{l,0}$. We apply (47) to $(\hat{u}_l, \hat{p}_l)$, namely

$$A_l(\hat{u}_l, v_l) = B_l((\hat{u}_l, \hat{p}_l), (v_l, 0)) \qquad \forall\, v_l \in V_l.$$

For $v_{l,T} \in V_{l,T}$ this gives

$$A_l(\tilde{u}_l - u_{l,T}, v_{l,T}) = 0,$$

i.e. $u_{l,T} = P_{l,T}^A \tilde{u}_l$, and

$$\hat{u}_l = (I - P_{l,T}^A)\tilde{u}_l = (I - P_{l,T}^A) R_l^{V,0} u_{l-1},$$

and the equivalence is proved.

3. Let $(u_l, p_l) \in X_l$. The definition of $(\hat{u}_{l-1}, \hat{p}_{l-1})$ is equivalent to find $(\hat{u}_{l-1}, \hat{p}_{l-1}) \in X_{l-1}$ such that

$$B_{l-1}((\hat{u}_{l-1}, \hat{p}_{l-1}), (v_{l-1}, q_{l-1})) = B_l((u_l, p_l), R_l^{X,0}(v_{l-1}, q_{l-1})) \qquad \forall\, (v_{l-1}, q_{l-1}) \in X_{l-1}.$$

We set $(v_{l-1}, q_{l-1}) = (0, q_{l-1})$ and obtain $(\hat{u}_{l-1}, \hat{p}_{l-1}) \in X_{l-1,0}$. Thus we can apply (47) on both sides to get

$$A_{l-1}(\hat{u}_{l-1}, v_{l-1}) = A_l(u_l, R_l^{V,0} v_{l-1}) \qquad \forall\, v_{l-1} \in V_{l-1},$$

what is equivalent to $\hat{u}_{l-1} = A_{l-1}^{-1} [R_l^{V,0}]^T A_l u_l$. $\square$

Theorem 1 (Equivalence of algorithms). *The multigrid algorithm in mixed variables preserves the space $X_{l,0}$. On this subspace it is equivalent to the multigrid algorithm in primal variables. This means for $(u_l, p_l) \in X_{l,0}$ and $(\hat{u}_l, \hat{p}_l) = M_l^B(u_l, p_l)$ there holds $(\hat{u}_l, \hat{p}_l) \in X_{l,0}$ and*

$$\hat{u}_l = M_l^A u_l.$$

Proof. The multigrid operator M_l^A fulfills the recursion

$$\begin{aligned} M_1^A &= 0, \\ M_l^A &= (S_l^A)^{m_l} \left(I - R_l^V (I - (M_{l-1}^A)^q) A_{l-1}^{-1} [R_l^V]^T A_l\right) (S_l^A)^{m_l}, \end{aligned}$$

and the mixed operator M_l^B fulfills a corresponding one. We apply Lemma 1 and the theorem is proved by induction. □

The multigrid analysis is based on the estimate of the two-grid operator

$$\hat{M}_l^A = (S_l^A)^{m_l} \left(I - R_l^V A_{l-1}^{-1} [R_l^V]^T A_l\right). \tag{48}$$

The framework is the classical theory based on the approximation property and the smoothing property [16].

Theorem 2 (Approximation property). *Let $u_1 \in V_l$, and compute*

$$u_5 = \left(I - R_l^V A_{l-1}^{-1} [R_l^V]^T A_l\right) u_1. \tag{49}$$

Then the estimate

$$\|u_5\|_{V,l,\tilde{0}} \preceq \|u_1\|_{A_l} \tag{50}$$

holds. The norm $\|.\|_{V,l,\tilde{0}}$ is defined in (32).

The proof is postponed to the next section.

Theorem 3 (Smoothing Property). *The smoothing operator S_l^A fulfills the estimate*

$$\|(S_l^A)^{m_l} u_l\|_{A_l} \preceq m_l^{-\alpha/2} \|u_l\|_{V,l,\tilde{0}}. \tag{51}$$

The power α is the index of the interpolation estimate (33).

Proof. By construction, the spectrum of the smoother fulfils $\sigma(S_l^A) \subset [0, 1)$. The estimate

$$\|(S_l^A)^{m_l} u_l\|_{A_l} = \|(I - \tau D_l^{-1} A_l)^{m_l} u_l\|_{A_l} \preceq m_l^{-1/2} \|u_l\|_{D_l}$$

is standard in multigrid theory. The smoother is non expansive in energy norm, i.e.

$$\|(S_l^A)^{m_l} u_l\|_{A_l} \leq \|u_l\|_{A_l}.$$

We apply operator interpolation to the linear operator $(S_l^A)^{m_l}$ to obtain

$$\|(S_l^A)^{m_l} u_l\|_{A_l} \preceq m^{-\alpha/2} \|u_l\|_{[A_l, D_l]_\alpha}.$$

The combination with the problem specific estimate (33) proves the theorem. □

The following theorem follows by standard multigrid techniques [16], [9], [8].

Theorem 4 (Optimal convergence rate of multigrid in primal variables). *Let $\varepsilon > 0$. The following multigrid cycles lead to optimal iterative solvers for the problem in primal variables:*

- *The energy norm of the W-cycle operator is bounded uniformly in L and ε if the number of smoothing steps m_l is sufficiently large.*
- *The variable V-cycle operator with $m^l \geq \beta^{L-l}$, $\beta > 1$ leads to a preconditioner C_L with condition number $\kappa(C_L^{-1}A_L)$ bounded uniformly in L and ε.*

We define the norm

$$\|(u_l, p_l)\|_{B_l}^2 := a_l(u_l, u_l) + \varepsilon c_l(p_l, p_l). \tag{52}$$

On the space $X_{l,0}$ it is identic to the primal energy norm $\|.\|_{A_l}$, i.e.

$$\|u_l\|_{A_l} = \|(u_l, p_l)\|_{B_l} \qquad \forall\, (u_l, p_l) \in X_{l,0}. \tag{53}$$

For $\varepsilon = 0$ the norm $\|.\|_{B_l}$ degenerates to a semi-norm.

Theorem 5 (Optimal convergence rate of multigrid in mixed variables). *Let $\varepsilon > 0$. The initial error is assumed to be in $X_{l,0}$. Perform either a W-cycle with sufficiently many smoothing steps or a variable V-cycle with sufficient damping for the system in mixed variables. Then the iterates converge with rates uniformly in L and ε with respect to the norm $\|.\|_{B_L}$.*

Assume additionally $B_{l,\varepsilon}(.,.) \to B_{l,\varepsilon=0}(.,.)$, and the limit form $B_{l,\varepsilon=0}(.,.)$ is stable with respect to a limit norm $\|.\|_{X,l,\varepsilon=0}$ on the subspaces defined above. Then the algorithm in mixed variables can be performed for $\varepsilon = 0$ and the primal variable converges with respect to the norm $\|.\|_{a_l}$ uniformly in L and ε.

Proof. First, we assume $\varepsilon > 0$. The errors of the iterates stay in $X_{l,0}$. Thus, the algorithm is equivalent to the corresponding one in primal variables. The last theorem stated uniform convergence rates with respect to the primal energy norm. Due to (53), this is equivalent to convergence in the norm $\|.\|_{B_l}$.

If the additional assumption is fulfilled, we can apply continuity arguments and pass to the limit $\varepsilon = 0$. □

6 Approximation Property

In this section we will prove Theorem 2. The coarse grid correction step (49) in primal variables is equivalent to the coarse grid correction in mixed variables. We set $p_1 = \varepsilon^{-1}\Lambda_l u_1$, and Lemma 1 privides

$$(u_5, p_5) = (I - R_l^X B_{l-1}^{-1}[R_l^X]^T B_l)(u_1, p_1). \tag{54}$$

On X_l, we define the local norm in mixed variables as

$$\|(u_l, p_l)\|^2_{X,l,\tilde{0}} = h^{-2}\|u_l\|^2_{V^0} + \varepsilon\|p_l\|^2_{c_l} + \|I^{\widetilde{Q}}_{l-1} p_l\|^2_{Q,l}.$$

On the subspace $X_{l,0}$ it reduces to $\|.\|_{V,l,\tilde{0}}$, i.e.

$$\|(u_l, p_l)\|_{X,l,\tilde{0}} = \|u_l\|_{V,l,\tilde{0}} \qquad \forall\, (u_l, p_l) \in X_{l,0}.$$

The property to be proven is (50), which reads in mixed variables as

$$\|(u_5, p_5)\|_{X,l,\tilde{0}} \preceq \|(u_1, p_1)\|_B.$$

We recall the definition of R^X_l and evaluate its adjoint

$$\begin{aligned} R^X_l &= (I - P^B_{l,T}) R^{X,0}_l, \\ [R^X_l]^T &= [R^{X,0}_l]^T B_l (I - P^B_{l,T}) B_l^{-1}. \end{aligned}$$

The operation (54) is split into the steps

$$(u_2, p_2) = (I - P^B_{l,T})(u_1, p_1), \tag{55}$$

$$(u_3, p_3) = B^{-1}_{l-1} [R^{X,0}_l]^T B_l (u_2, p_2), \tag{56}$$

$$(u_4, p_4) = (I - P^B_{l,T}) R^{X,0}_l (u_3, p_3), \tag{57}$$

$$(u_5, p_5) = (u_1, p_1) - (u_4, p_4).$$

The first step is called preprocessing step. The third step (57) is similar, it is called postprocessing. Both modify the functions by local operations, such that the classical coarse grid correction (56) becomes efficient.

We apply the triangle inequality in the form

$$\begin{aligned} \|(u_5, p_5)\|_{X,l,\tilde{0}} &= \|(u_1, p_1) - (u_4, p_4)\|_{X,l,\tilde{0}} \\ &\leq \|(u_1, p_1) - (u_2, p_2)\|_{X,l,\tilde{0}} + \|(u_2, p_2) - R^{X,0}_l (u_3, p_3)\|_{X,l,\tilde{0}} + \\ &\quad \|R^{X,0}_l (u_3, p_3) - (u_4, p_4)\|_{X,l,\tilde{0}} \end{aligned}$$

Each one of the three lemmata below estimates one of the three terms. We start with the preprocessing step.

Lemma 2. *There holds the approximation estimate*

$$\|(u_2, p_2) - (u_1, p_1)\|_{X,l,\tilde{0}} \preceq \|(u_1, p_1)\|_{B_l}.$$

Additionally, the stability estimate

$$\|u_2\|_{V,l} + \|p_2 - I^{\widetilde{Q}}_{l-1} p_2\|_{Q,l} \preceq \|(u_1, p_1)\|_{B_l}.$$

is fulfilled.

Proof. We use $(u_2, p_2) \in X_{l,0}$, and thus

$$\|(u_2, p_2)\|_{B_l} = \|u_2\|_{A_l} = \|(I - P^A_{l,T})u_1\|_{A_l} \le \|u_1\|_{A_l} = \|(u_1, p_1)\|_{B_l}.$$

This bounds the first term of the stability estimate. Next, we will establish the estimate

$$\|w\|_{V,l} \preceq \|w\|_{A_l}. \tag{58}$$

We use stability (10) and continuity (9) of $B_l(.,.)$ on X_l to obtain

$$\begin{aligned}
\|w_l\|_{V,l} &\preceq \sup_{(v_l,q_l)\in X_l} \frac{B_l((w_l,0),(v_l,q_l))}{\|(v_l,q_l)\|_{X,l}} = \sup_{(v_l,q_l)\in X_l} \frac{a_l(w_l,v_l) - c_l(\Lambda_l w_l, q_l)}{\|(v_l,q_l)\|_{X,l}} \\
&\preceq \sup_{(v_l,q_l)\in X_l} \frac{[a_l(w_l,w_l) + \varepsilon^{-1} c_l(\Lambda_l w_l, \Lambda_l w_l)]^{1/2}\,[a_l(v_l,v_l) + \varepsilon\, c_l(q_l,q_l)]^{1/2}}{\|(v_l,q_l)\|_{X,l}} \\
&= \sup_{(v_l,q_l)\in X_l} \frac{A_l(w_l,w_l)^{1/2}\, B_l((v_l,q_l),(v_l,-q_l))^{1/2}}{\|(v_l,q_l)\|_{X,l}} \preceq \|w_l\|_{A_l}.
\end{aligned}$$

For $(u_l, p_l) \in X_{l,0}$, inequality (58) can be rewritten as

$$\|u_l\|_{V,l} \preceq \|(u_l, p_l)\|_{B_l}.$$

To bound the approximation term we use $(u_2 - u_1, p_2 - p_1) \in X_{l,T} \cap X_{l,0}$, the norm equivalence (27), the orthogonal decomposition (23), and the estimate above to obtain

$$\begin{aligned}
&\|(u_2,p_2) - (u_1,p_1)\|^2_{X,l,\tilde{0}} = h_l^{-2}\|u_2 - u_1\|^2_{V^0} + \varepsilon\,\|p_2 - p_1\|^2_{c_l} + \|I^{\tilde{Q}}_{l-1}(p_2 - p_1)\|^2_{Q,l} \\
&\preceq \|u_2 - u_1\|^2_{V,l} + \varepsilon\,\|p_2 - p_1\|^2_{c_l} \preceq \|(u_2 - u_1, p_2 - p_1)\|^2_{B_l} \preceq \|(u_1,p_1)\|^2_{B_l}.
\end{aligned}$$

From $p_2 - I^{\tilde{Q}}_{l-1}p_2 \in Q_{l,T}$, stability (24) of $X_{l,T}$, inclusion (25), orthogonality (23), and the definition of $P^B_{l,T}$ we obtain

$$\begin{aligned}
&\|p_2 - I^Q_{l-1}p_2\|_{Q,l} \preceq \sup_{(v,q)\in X_{l,T}} \frac{B_l((0, p_2 - I^{\tilde{Q}}_{l-1}p_2),(v,q))}{\|(v,q)\|_{X,l}} \\
&= \sup_{(v,q)\in X_{l,T}} \frac{B_l((0,p_2),(v,q)) - c_l(I^{\tilde{Q}}_{l-1}p_2, \Lambda_l v - q)}{\|(v,q)\|_{X,l}} = \sup_{(v,q)\in X_{l,T}} \frac{B_l((0,p_2),(v,q))}{\|(v,q)\|_{X,l}} \\
&= \sup_{(v,q)\in X_{l,T}} \frac{B_l((-u_2,0),(v,q))}{\|(v,q)\|_{X,l}} \preceq \|u_2\|_{V,l} \preceq \|(u_2,p_2)\|^2_{B_l}.
\end{aligned}$$

□

Lemma 3. *The classical coarse grid correction fulfills the approximation inequality*

$$\|R^{X,0}_l(u_3, p_3) - (u_2, p_2)\|_{X,l,\tilde{0}} \preceq \|(u_1, p_1)\|_{B_l}.$$

Additionally, the stability estimate

$$\|u_3\|_{V,l-1} + \|p_3 - I^{\tilde{Q}}_{l-1}p_3\|_{Q,l} \preceq \|(u_1, p_1)\|_{B_l}.$$

is fulfilled.

Proof. From the variational definition of $(u_3, p_3) \in X_{l-1}$

$$B_{l-1}((u_3, p_3), (v, q)) = B_l((u_2, p_2), R_l^{X,0}(v, q)) \qquad \forall\, (v, q) \in X_{l-1}, \tag{59}$$

and property (26) we get

$$B_{l-1}((u_3, p_3 - I_{l-1}^{\widetilde{Q}} p_2), (v, q)) = B_l((u_2, p_2 - I_{l-1}^{\widetilde{Q}} p_2), R_l^{X,0}(v, q)) \qquad \forall\, (v, q) \in X_{l-1}.$$

Stability (10) on X_{l-1}, continuity (9) of $B_l(.,.)$, continuity (20)+(21) of the prolongation operator $R_l^{X,0}$, and Lemma 2 give

$$\begin{aligned}
\|u_3\|_{V,l-1} + \|p_3 - I_{l-1}^{\widetilde{Q}} p_2\|_{Q,l-1} &\preceq \sup_{(v,q)\in X_{l-1}} \frac{B_{l-1}((u_3, p_3 - I_{l-1}^{\widetilde{Q}} p_2), (v, q))}{\|(v, q)\|_{X,l-1}} \\
&= \sup_{(v,q)\in X_{l-1}} \frac{B_l((u_2, p_2 - I_{l-1}^{\widetilde{Q}} p_2), R_l^{X,0}(v, q))}{\|(v, q)\|_{X,l-1}} \\
&\preceq \|u_2\|_{V,l} + \|p_2 - I_{l-1}^{\widetilde{Q}} p_2\|_{Q,l} \preceq \|(u_1, p_1)\|_{B_l}.
\end{aligned}$$

Norm equivalence (21) on Q_{l-1} and Q_l, and Lemma 2 give

$$\|p_3 - p_2\|_{Q,l} \preceq \|p_3 - I_{l-1}^{\widetilde{Q}} p_2\|_{Q,l-1} + \|p_2 - I_{l-1}^{\widetilde{Q}} p_2\|_{Q,l} \preceq \|(u_1, p_1)\|_{B_l}$$

and the estimate

$$\|p_3 - I_{l-1}^{\widetilde{Q}} p_3\|_{Q,l} \leq \|p_3 - p_2 + I_{l-1}^{\widetilde{Q}}(p_2 - p_3)\|_{Q,l} + \|p_2 - I_{l-1}^{\widetilde{Q}} p_2\|_{Q,l} \preceq \|(u_1, p_1)\|_{B_l}.$$

We use continuity (9) of $B_l(.,.)$ to bound the term $\varepsilon\|.\|_{c_l}^2$ of the approximation estimate, namely

$$\begin{aligned}
\varepsilon\, \|p_2 - p_3\|_{c_l}^2 &\preceq \varepsilon\, \|p_2 - I_{l-1}^{\widetilde{Q}} p_2\|_{c_l} + \varepsilon\, \|p_3 - I_{l-1}^{\widetilde{Q}} p_2\|_{c_l} \\
&\preceq \|p_2 - I_{l-1}^{\widetilde{Q}} p_2\|_{Q_l} + \|p_3 - I_{l-1}^{\widetilde{Q}} p_2\|_{Q_l} \\
&\preceq \|(u_1, p_1)\|_{B_l}
\end{aligned}$$

The only term left is the improved estimate in the weaker norm, which we will bound by

$$\|u_2 - R_l^{V,0} u_3\|_{V^0} \preceq h_l\, \|(u_2, p_2 - I_{l-1}^{\widetilde{Q}} p_2)\|_{X,l}. \tag{60}$$

We define the dual problem

$$B((\varphi, \psi), (v, q)) = (u_2 - R_l^{V,0} u_3, v)_{V^0}.$$

The right hand side is a linear functional in $(V^0)^*$ with norm $\|u_2 - R_l^{V,0} u_3\|_{V^0}$. We define the finite element problems

$$B_l((\varphi_l, \psi_l), (v_l, q_l)) = (u_2 - R_l^{V,0} u_3, v_l)_{V^0}, \tag{61}$$

$$B_{l-1}((\varphi_{l-1}, \psi_{l-1}), (v_{l-1}, q_{l-1})) = (u_2 - R_l^{V,0} u_3, v_{l-1})_{V^0}, \tag{62}$$

$$B_{l-1}((\widetilde{\varphi}_{l-1}, \widetilde{\psi}_{l-1}), (v_{l-1}, q_{l-1})) = (u_2 - R_l^{V,0} u_3, R_l^{V,0} v_{l-1})_{V^0}. \tag{63}$$

with test functions $(v_l, q_l) \in X_l$ and $(v_{l-1}, q_{l-1}) \in X_{l-1}$.

First, we estimate the difference of the solutions of (62) and (63) by

$$
\begin{aligned}
&\|(\varphi_{l-1} - \widetilde{\varphi}_{l-1}, \psi_{l-1} - \widetilde{\psi}_{l-1})\|_{X,l-1} \qquad (64)\\
&\preceq \sup_{(v_{l-1},q_{l-1})\in X_{l-1}} \frac{B_{l-1}((\varphi_{l-1} - \widetilde{\varphi}_{l-1}, \psi_{l-1} - \widetilde{\psi}_{l-1}), (v_{l-1}, q_{l-1}))}{\|(v_{l-1}, q_{l-1})\|_{X,l-1}} \\
&= \sup_{(v_{l-1},q_{l-1})\in X_{l-1}} \frac{(u_2 - R_l^{V,0} u_3, v_{l-1} - R_l^{V,0} v_{l-1})_{V^0}}{\|(v_{l-1}, q_{l-1})\|_{X,l-1}} \\
&\leq \|u_2 - R_l^{V,0} u_3\|_{V^0} \sup_{v_{l-1}\in V_{l-1}} \frac{\|v_{l-1} - R_l^{V,0} v_{l-1}\|_{V^0}}{\|v_{l-1}\|_{V,l-1}} \preceq h\, \|u_2 - R_l^{V,0} u_3\|_{V^0}.
\end{aligned}
$$

We have used the approximation (20). Using the variational problems (61) and (63), and the variational specification of the coarse grid correction (59) we get

$$
\begin{aligned}
\|u_2 - R_l^{V,0} u_3\|_{V^0}^2 &= (u_2 - R_l^{V,0} u_3, u_2)_{V^0} - (u_2 - R_l^{V,0} u_3, R_l^{V,0} u_3)_{V^0} \\
&= B_l((\varphi_l, \psi_l), (u_2, p_2)) - B_{l-1}((\widetilde{\varphi}_{l-1}, \widetilde{\psi}_{l-1}), (u_3, p_3)) \\
&= B_l((\varphi_l, \psi_l), (u_2, p_2)) - B_l(R_l^{X,0}(\widetilde{\varphi}_{l-1}, \widetilde{\psi}_{l-1}), (u_2, p_2)) \\
&= B_l((\varphi_l, \psi_l) - R_l^{X,0}(\widetilde{\varphi}_l, \widetilde{\psi}_l), (u_2, p_2)).
\end{aligned}
$$

Next, we use (26) and observe that

$$
\begin{aligned}
&B_l((\varphi_l, \psi_l) - R_l^{X,0}(\widetilde{\varphi}_{l-1}, \widetilde{\psi}_{l-1}), (0, I_{l-1}^{\widetilde{Q}} p_2)) \qquad (65)\\
&= B_l((\varphi_l, \psi_l), (0, I_{l-1}^{\widetilde{Q}} p_2)) - B_{l-1}((\widetilde{\varphi}_{l-1}, \widetilde{\psi}_{l-1}), (0, I_{l-1}^{\widetilde{Q}} p_2)) = 0.
\end{aligned}
$$

We continue and obtain

$$
\begin{aligned}
\|u_2 - R_l^{V,0} u_3\|_{V^0}^2 &= B_l((\varphi_l, \psi_l) - R_l^{X,0}(\widetilde{\varphi}_l, \widetilde{\psi}_l), (u_2, p_2 - I_{l-1}^{\widetilde{Q}} p_2)) && (66)\\
&\leq \|(\varphi_l, \psi_l) - R_l^{X,0}(\widetilde{\varphi}_l, \widetilde{\psi}_l)\|_{X,l}\ \|(u_2, p_2 - I_{l-1}^{\widetilde{Q}} p_2)\|_{X,l}. && (67)
\end{aligned}
$$

We use the continuity of the prolongation $R_l^{X,0}$, and (64) to get

$$
\begin{aligned}
&\|(\varphi_l, \psi_l) - R_l^{X,0}(\widetilde{\varphi}_l, \widetilde{\psi}_l)\|_{X,l} \\
&\preceq \|(\varphi_l, \psi_l) - R_l^{X,0}(\varphi_{l-1}, \psi_{l-1})\|_{X,l} + \|R_l^{X,0}(\varphi_{l-1}, \psi_{l-1}) - R_l^{X,0}(\widetilde{\varphi}_{l-1}, \widetilde{\psi}_{l-1})\|_{X,l} \\
&\preceq \|\varphi_l - R_l^{V,0} \varphi_{l-1}\|_{V,l} + \|\psi_l - \psi_{l-1}\|_{Q,l} + h\, \|u_2 - R_l^{V,0} u_3\|_{V^0}.
\end{aligned}
$$

If the spaces V_{l-1} and V_l are nested, and $R_l^{V,0}$ is the embedding operator, we can apply the a priori estimate (15) and we are done. We did not use the regularity estimate (6) explicitely for this case. If $R_l^{V,0}$ is not the embedding, we need the intermediate steps

$$
\|\varphi_l - R_l^{V,0} \varphi_{l-1}\|_{V,l} \leq \|\varphi_l - I_l^V \varphi\|_{V,l} + \|I_l^V \varphi - R_l^{V,0} I_{l-1}^V \varphi\|_{V,l} + \|R_l^{V,0} I_{l-1}^V \varphi - R_l^{V,0} \varphi_{l-1}\|_{V,l}.
$$

Now we apply the approximation estimate (13) of the interpolation operator, the a priori estimate (15), the approximation estimate of the prolongation operator (20), and the regularity estimate (6) to obtain

$$\|\varphi_l - R_l^{V,0}\varphi_{l-1}\|_{V,l} + \|\psi_l - \psi_{l-1}\|_{Q,l} \preceq h\,\|u_2 - R_l^{V,0}u_3\|_{V^0}.$$

Using these estimates in (66) and canceling one factor $\|u_2 - R_l^{V,0}u_3\|_{V^0}$ gives the result.

□

Lemma 4. *The postprecessing step (57) fulfills the approximation estimate*

$$\|(u_4,p_4) - R_l^{X,0}(u_3,p_3)\|_{X,l,\tilde{0}} \preceq \|(u_1,p_1)\|_{B_l}. \tag{68}$$

Proof. Norm equivalence (27) on $V_{l,T}$, orthogonality (23), stability (24) of $X_{l,T}$ and the definition of the projection (43) give

$$\begin{aligned}
&\|(u_4,p_4) - R_l^{X,0}(u_3,p_3)\|_{X,l,\tilde{0}} \simeq \|u_4 - R_l^{V,0}u_3\|_{V,l} + \varepsilon^{1/2}\|p_4 - p_3\|_{c_l} \\
\preceq\ & \|(u_4,p_4) - R_l^{X,0}(u_3,p_3)\|_{X,l} \preceq \sup_{(v,q)\in X_{l,T}} \frac{B_l((R_l^{V,0}u_3 - u_4, p_3 - p_4),(v,q))}{\|(v,q)\|_{X,l}} \\
=\ & \sup_{(v,q)\in X_{l,T}} \frac{B_l((R_l^{V,0}u_3, p_3),(v,q))}{\|(v,q)\|_{X,l}} = \sup_{(v,q)\in X_{l,T}} \frac{B_l((R_l^{V,0}u_3, p_3 - I_{l-1}^{\tilde{Q}}p_3),(v,q))}{\|(v,q)\|_{X_l}} \\
\preceq\ & \|(R_l^{V,0}u_3, p_3 - I_{l-1}^{\tilde{Q}}p_3)\|_{X,l} \preceq \|(u_3, p_3 - I_{l-1}^{\tilde{Q}}p_3)\|_{X,l-1} \preceq \|(u_1,p_1)\|_{B_l},
\end{aligned}$$

and the proof if complete.

□

7 Application to nearly incompressible materials and Stokes

In [19] the case of nearly incompressible materials was investigated. We reformulate the results in the present framework. The problem is to find $u \in V := [H_0^1(\Omega)]^2$ such that

$$2\mu \int_\Omega e(u) : e(v)\,\mathrm{dx} + \lambda \int_\Omega \operatorname{div} u \operatorname{div} v\,\mathrm{dx} = \int_\Omega \tilde{f}^T v\,\mathrm{dx}, \tag{69}$$

with the positive constants λ and μ of Lamé, the strain operator $e(u) := 0.5(\nabla u + (\nabla u)^T)$, and the volume force $\tilde{f} \in [L_2(\Omega)]^2$. We are interested in the nearly incompressible case, i.e. the Poisson ration ν is close to 0.5. Then the parameter $\varepsilon := 2\mu/\lambda$ becomes small. After dividing by 2μ, the primal bilinear form $A(.,.)$, the operator Λ and the mixed bilinear form are

$$\begin{aligned}
A(u,v) &= (e(u),e(v))_0 + \varepsilon^{-1}(\operatorname{div} u, \operatorname{div} v)_0, \\
\Lambda &= \operatorname{div}, \\
B((u,p),(v,q)) &= (e(u),e(v))_0 + (\operatorname{div} u, q)_0 + (\operatorname{div} v, p)_0 - \varepsilon\,(p,q)_0.
\end{aligned}$$

The spaces with stronger and weaker norms are $V^2 = H^2(\Omega)$, $Q^1 = H^1(\Omega)$ and $V^0 = L_2$. The regularity assumption (6) does hold on convex polygonal domains [12].

In [19] the stable pair of $P_2 - P_0$ finite element spaces was used, and the bilinear forms $B_l(.,.)$ were set to $B(.,.)$. The norms $\|.\|_{V,l}$ and $\|.\|_{Q,l}$ the norms of H^1 and L_2.

The parameter h_l is the mesh size, the inverse inequality (11) and the interpolation operator (13) is standard. The a priori estimate (15) follows by the Aubin-Nitsche technique [13].

The primal bilinear form on level l evaluates to

$$A_l(u,v) = (e(u), e(v))_0 + \varepsilon^{-1}(\overline{\operatorname{div} u}^l, \overline{\operatorname{div} v}^l)_0,$$

where the $\bar{\cdot}$ is element-wise averaging on level l.

The multigrid components are defined as follows. The trivial prolongation operator $R_l^{V,0}$ is the embedding operator. The spaces $V_{l,T}$ and $Q_{l,T}$ are defined as

$$\begin{aligned} V_{l,T} &:= \{v_l \in V_l : v_l|_{\partial T_{l-1}} = 0 \ \ \forall T_{l-1} \in \mathcal{T}_{l-1}\}, \\ Q_{l,T} &:= \{q_l \in Q_l : \overline{q_l}^{l-1} = 0\}, \end{aligned}$$

where $\mathcal{T}_l$ is the set of triangles at level l. The connected nodes of these spaces are drawn in Figure 1. The splitting (23) is done with $\widetilde{Q}_{l-1} = Q_{l-1}$.

The kernel of Λ_l has the base functions drawn in Figure 2. These leads to the definition of the n_l local spaces

$$V_{l,i} = [H_0^1(\Omega_i)]^2 \cap V_l \qquad \text{and} \qquad Q_{l,i} = [L_2(\Omega_i)/\mathbb{R}] \cap Q_l \tag{70}$$

assigned to the corner nodes $N_{l,i}$, of the triangulation. Ω_i is the union of elements in $\mathcal{T}_l$ adjacent to the node $N_{l,i}$. These space splitting is indicated in Figure 3.

The interpolation estimate (33) was proven in [19] with $\alpha = 1/2$.

Very popular for Stokes is the non-conforming $P_1 - P_0$ pairing [13]. It can be also used for nearly incompressible materials with pure Dirichlet boundary conditions [15].

The present analysis covers this element. The forms A_l and B_l are obtained by splitting the integrals over the triangles. The trivial prolongation $R_l^{V,0}$ is constructed by taking the mean value of the left and right values. The approximation (20) and property (26) are fulfilled. The spaces $X_{l,T}$ and $X_{l,i}$ for the prolongation and the smoother are defined analogue to the conforming case, and the analysis of the interpolation estimate (33) is similar to the conforming case.

8 Application to Reissner Mindlin plate

We use the Reissner Mindlin plate model with the discretization of [14]. In the primal space $V = H_0^1(\Omega) \times [H_0^1]^2$ we look for the transverse displacement w and the rotation vector β. The scaled problem is defined as

$$a(\beta, \eta) + \frac{k}{t^2}(\nabla w - \beta, \nabla v - \eta) = (g, v)_0 \qquad \forall\, (v, \eta) \in V$$

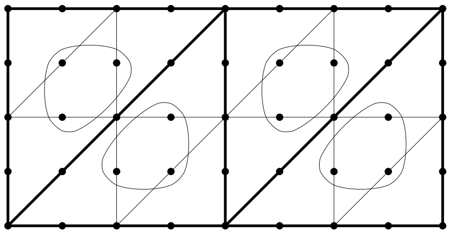


Figure 1: Subspace $V_{l,T}$ used in prolongation

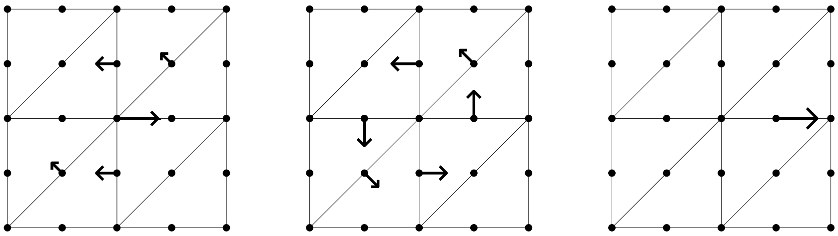


Figure 2: Basis functions for kernel of $I_l^Q \operatorname{div}$

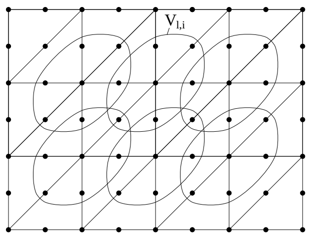


Figure 3: Subspaces containing div-free basis functions

with the bilinear form

$$a(\beta,\eta)=\frac{1}{6}\left[(e(\beta),e(\eta))_0+\frac{\nu}{1-\nu}(\operatorname{div}\beta,\operatorname{div}\eta)_0\right].$$

The small parameter is the plate thickness t. The transverse load is Gt^3g. Further, k is the shear correction factor, ν the poisson ratio, G the shear modulus and $e(.)$ is the linear strain operator. We extend the linear functional $(g,.)_0$ onto V and set

$$\langle f,(v,\eta)\rangle=\langle g,v\rangle+\langle\delta,\eta\rangle \qquad \forall\,(v,\eta)\in V.$$

With the definitions

$$\begin{aligned}\Lambda &= \nabla w-\beta,\\ \varepsilon &= t^2/k\end{aligned}$$

the bilinear form $A(.,.)$ has the structure of (2). The dual variable, the scaled shear force is

$$p=\varepsilon^{-1}\Lambda(w,\beta)=\frac{k}{t^2}(\nabla w-\beta).$$

The mixed form $B(.,.)$ evaluates to

$$B((w,\beta,p),(v,\eta,q))=a(\beta,\eta)+(\nabla v-\eta,p)_0+(\nabla w-\beta,q)_0-\frac{t^2}{k}(p,q)_0$$

Following [14], we define the norms

$$\begin{aligned}\|(w,\beta)\|_{V^2}^2=\|w\|_{V^{2,w}}^2+\|\beta\|_{V^{2,\beta}}^2 &= \inf_{w=w_0+w_r}\left\{\|w_0\|_3^2+t^{-2}\|w_r\|_2^2\right\}+\|\beta\|_2^2,\\ \|(w,\beta)\|_{V^0}^2=\|w\|_{V^{0,w}}^2+\|\beta\|_{V^{0,\beta}}^2 &= \inf_{w=w_0+w_r}\left\{\|w_0\|_1^2+t^{-2}\|w_r\|_0^2\right\}+\|\beta\|_0^2,\\ \|p\|_{Q^1} &= \|p\|_0^2+t^2\|p\|_1^2.\end{aligned}$$

We evaluate the dual norm

$$\begin{aligned}\|(g,\delta)\|_{(V^0)^*} &= \sup_{(v,\eta)\in V^0}\frac{\langle g,v\rangle+\langle\delta,\beta\rangle}{\left\{\inf\limits_{v=v_0+v_r}\|v_0\|_1^2+t^{-2}\|v_r\|_0^2+\|\beta\|_0^2\right\}^{1/2}}\\ &= \sup_{v_0,v_r,\beta}\frac{\langle g,v_0\rangle+\langle g,v_r\rangle+\langle\delta,\eta\rangle}{\{\|v_0\|_1^2+t^{-2}\|v_r\|_0^2+\|\beta\|_0^2\}^{1/2}}\end{aligned}$$

The supremum is taken at the solution (w_0,w_r,β) of the variational problem

$$(w_0,v_0)_1+t^{-2}(w_r,v_r)_0+(\beta,\eta)_0=\langle g,v_0\rangle+\langle g,v_r\rangle+\langle\delta,\eta\rangle \qquad \forall\,v_0,v_r,\eta$$

and is equal to $\{\|w_0\|_1^2+t^{-2}\|w_r\|_0^2+\|\beta\|_0^2\}^{1/2}=\{\|g\|_{-1}^2+t^2\|g\|_0^2+\|\delta\|_0^2\}^{1/2}$. Thus we computed

$$\|(g,\delta)\|_{(V^0)^*}^2=\|g\|_{-1}^2+t^2\|g\|_0^2+\|\delta\|_0^2. \tag{71}$$

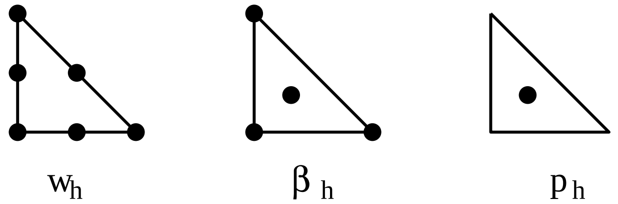


Figure 4: Finite elements for Reissner Mindlin

This is exactly the right hand side norm, for which [3] and [14] proved a solution in $V^2 \times Q^1$ and the regulaity estimate (6).

The discretization of [2] and [14] uses P_2 elements for the transverse displacement, P_1^+ for the rotations and P_0 for the shear, see Figure 4. The space P_1^+ are the piecewise linear functions enriched by the element bubbles $\lambda_1\lambda_2\lambda_3$ in barycentric coordinates. These spaces are non-nested due to the bubble functions. In [14], the form $B(.,.)$ is replaced by the mesh dependent bilinear form

$$\begin{aligned}B_l((w,\beta,p),(v,\eta,q)) &= B((w,\beta,p),(v,\eta,q)) \\ &+ \frac{\mu k}{t^2+h^2}\left(\nabla w - \beta - \frac{t^2}{k}p, \nabla v - \eta - \frac{t^2}{k}q\right)_0\end{aligned}$$

with some constant $\mu \in (0,1)$. In [14] a local mesh size is used instead of the global one $h = h(l)$. The discrete shear operator can be computed locally and evaluates to

$$p_l = \varepsilon^{-1}\Lambda_l(w_l,\beta_l) = \frac{k}{t^2}(\overline{\nabla w_l - \beta_l}),$$

where $\bar{\cdot}$ is the element-wise averaging operator. The reduction to a positive definite problem gives the form

$$\begin{aligned}A_l((w_l,\beta_l),(v_l,\eta_l)) &= a(\beta_l,\eta_l) + \frac{k\mu}{h^2+t^2}(\nabla w_l - \beta_l, \nabla v_l - \eta_l)_0 \\ &+ \left(\frac{k}{t^2} - \frac{k\mu}{h^2+t^2}\right)(\overline{\nabla w_l - \beta_l}, \overline{\nabla v_l - \eta_l})_0\end{aligned}$$

The discrete norms are

$$\begin{aligned}\|(w,\beta)\|_{V,l}^2 &= \|\beta\|_1^2 + \frac{1}{h^2+t^2}\|\nabla v - \beta\|_0^2, \\ \|p\|_{Q,l}^2 &= (h^2+t^2)\|p\|_0^2.\end{aligned}$$

We verify the approximation inequality (13) for the component-wise nodal interpolation operator $I_l^V = (I_l^{V,w}, I_l^{V,\beta})$. Let $w = w_0 + w_r$ with $\|w_0\|_3 + t^{-1}\|w_r\|_2 \preceq \|w\|_{V^{2,w}}$. Then

$$\|w - I_l^{V,w}w\|_{V^{0,w}} + \|\beta - I_l^{V,\beta}\beta\|_{V^{0,\beta}}$$

$$
\begin{aligned}
&\preceq \|w_0 - I_l^{V,w} w_0\|_1 + t^{-1}\|w_r - I_l^{V,w} w_r\|_0 + \|\beta - I_l^{V,\beta}\beta\|_0 \\
&\preceq h^2\left\{\|w_0\|_3 + t^{-1}\|w_r\|_2 + \|\beta\|_2\right\} \preceq h^2\,\|(w,\beta)\|_{V^2},
\end{aligned}
$$

and similar

$$
\begin{aligned}
&\|(w - I_l^{V,w} w, \beta - I_l^{V,\beta}\|_{V,l} \\
&\quad\preceq \|\beta - I_l^{V,\beta}\beta\|_1 + (h+t)^{-1}\|\nabla(w - I_l^{V,w} w) - (\beta - I_\beta^{V,\beta})\|_0 \\
&\quad\leq \|\beta - I_l^{V,\beta}\beta\|_1 + h^{-1}\|\beta - I_l^{V,\beta}\beta\|_0 + \frac{1}{h+t}\left\{\|\nabla(w_0 - I_l^{V,w} w_0)\|_0 + \|\nabla(w_r - I_l^{V,w} w_r)\|_0\right\} \\
&\quad\leq h\,\|\beta\|_2 + \frac{h^2}{h+t}\|w_0\|_3 + \frac{ht}{h+t} t^{-1}\,\|w_r\|_2 \preceq h\,\|(w,\beta)\|_{V^2}.
\end{aligned}
$$

Define $P_{V,t}^w : V^w \to V_{l,t}^w$ as Clément interpolation operator into a finite element subspace of H_0^1 of mesh size t. It fulfills the approximation estimate

$$
\|w - P_{V,t}^w w\|_0 \preceq t\|w\|_1
$$

and the continuity and inverse estimates

$$
\|P_{V,t}^w w\|_k \preceq t^{m-k}\|w\|_m \qquad 0 \leq m \leq k \leq 1.
$$

It provides an explicit splitting equivalent to the norm $\|w\|_{V^0}$ on V^w

$$
\|P_{V,t}^w w\|_1 + t^{-1}\|w - P_{V,t}^w w\|_0 \simeq \|w\|_{V^{0,w}}. \tag{72}
$$

The left hand side dominates the right by definition, the other direction is proved by taking a splitting $w = w_0 + w_r$ such that $\|w_0\|_1 + t^{-1}\|w_r\|_0 \preceq \|w\|_{V^0}$ and

$$
\begin{aligned}
&\|P_{l,t}^w w\|_1 + t^{-1}\|w - P_{V,t}^w w\|_0 \\
&\quad\preceq \|P_{l,t}^w w_0\|_1 + \|P_{l,t}^w w_r\|_1 + t^{-1}\|w_0 - P_{V,t}^w w_0\|_0 + t^{-1}\|w_r - P_{V,t}^w w_r\|_0 \\
&\quad\preceq \|w_0\|_1 + t^{-1}\|w_r\|_0 + t^{-1} t\,\|w_0\|_1 + t^{-1}\|w_r\|_0 \preceq \|w\|_{V^{0,w}}.
\end{aligned}
$$

It is used for the verification of the inverse estimate (11). First, we estimate

$$
\begin{aligned}
\frac{1}{h+t}\|w_l\|_1 &\preceq \frac{1}{h+t}\|P_{l,t}^w w_l\|_1 + \frac{1}{h+t}\|(I - P_{l,t}^w) w_l\|_1 \\
&\preceq \frac{1}{h}\|P_{l,t}^w w_l\|_1 + \frac{1}{h+t}(h^{-1} + t^{-1})\|(I - P_{l,t}^w) w_l\|_0 \\
&= h^{-1}\left(\|P_{l,t}^w w_l\|_1 + t^{-1}\|(I - P_{l,t}^w) w_l\|_0\right) \preceq h^{-1}\|w\|_{V^{0,w}}
\end{aligned}
$$

using the inverse estimate $\|v\|_1 \preceq (h^1 + t^{-1})\|v\|_0$ on the space $V_l \cup V_{l,t}$. Then we get

$$
\|(w_l, \beta_l)\|_{V,l} \preceq \|\beta_l\|_1 + (h+t)^{-1}\|\nabla w_l - \beta_l\|_0 \preceq h^{-1}\|\beta_l\|_0 + (h+t)^{-1}\|w_l\|_1 \preceq \|(w_l, \beta_l)\|_{V^0}.
$$

Applying the duality trick in [14], Proposition 5, directly with the inner product from V_0 we obtain the a priori estimate (15).

Next, we start with the multigrid components. Let the trivial prolongation $R_l^{V,0}$ map the vertical displacement w_{l-1} to $w_l = w_{l-1}$, and map the rotations β_{l-1} to β_l by piecewise linear interpolation (and forget the bubbles). This prolongation is accurate enough for (20). The space splitting (23) is done by setting

$$Q_{l,T} = Q_l \qquad \text{and} \qquad \widetilde{Q}_{l-1} = \{0\}.$$

Stability (24) is obtained by using all bubble functions for the rotations, i.e.

$$V_{l,T} = \{(0, \beta_l) \in V_l : \beta_l|_{\partial T} = 0 \;\; \forall T \in \mathcal{T}_l\}.$$

The stability proof of [14], Proposition 3, directly applies to the space $X_{l,T}$. The inclusion (25) and property (26) are trivially fulfilled. The norm equivalence (27) follows from Friedrichs' inequality plus scaling.

The space splitting for the smoother is done as

$$V_{l,i} = [H_0^1(\Omega_i)]^2 \cap V_l \qquad \text{and} \qquad Q_{l,i} = L_2(\Omega_i) \cap Q_l, \tag{73}$$

where each index i is associated with one corner node $N_{l,i}$. The set Ω_i is the union of elements sharing the node $N_{l,i}$. Again, the stability condition (29) is proved as in [14], the inclusion (30) is trivial.

The norm $\|.\|_{V,l,\tilde{0}}$ used in the multigrid analysis evaluates to

$$\|(w_l, \beta_l)\|_{V,l,\tilde{0}}^2 := h^{-2} \inf_{w_l = w_0 + w_r} \left\{ \|w_0\|_1^2 + t^{-2} \|w_r\|_0^2 \right\} + h^{-2} \|\beta_l\|_0^2 + \frac{k}{t^2} \|\overline{\nabla w_l - \beta_l}\|_0^2, \tag{74}$$

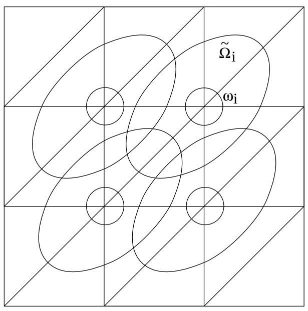
and the interpolation estimate (33) is the result of the following theorem:

Theorem 6. *The splitting estimate*

$$\|(w, \beta)\|_{[A,D]_{1/2}} \preceq \|(w, \beta)\|_{V,l,\tilde{0}} \tag{75}$$

is valid for all $(w, \beta) \in V_l$.

Proof. For the construction of the splitting we cover the domain with overlapping subdomains $\widetilde{\Omega}_i$ contained in the sets Ω_i of triangles sharing the corner node N_i. The sets ω_i contain the nodes N_i, have measure $|\omega_i| \succeq |\Omega_i|$ and fulfill $\tilde{\Omega}_j \cap \omega_i = \emptyset$ for $i \neq j$, see the figure below.



Let $\{\Psi_i\}$ be a partition of unity fulfilling

$$\sum \Psi_i = 1, \qquad \operatorname{supp} \Psi_i \subset \tilde{\Omega}_i,$$

$$h^2 \|\Psi_i\|_{2,\infty} + h \|\Psi_i\|_{1,\infty} + \|\Psi_i\|_{0,\infty} \preceq 1.$$

We follow [2] and construct the Fortin - operators

$$\begin{aligned} P_{V,l}^{w} &= P_{V,l}^{w,1} + P_{V,l}^{w,2}(id - P_{V,l}^{w,1}), \\ P_{V,l}^{\beta} &= P_{V,l}^{\beta,1} + P_{V,l}^{\beta,2}(id - P_{V,l}^{\beta,1}), \end{aligned}$$

where $P_{V,l}^{w,1}$ and $P_{V,l}^{\beta,1}$ are Scott-Zhang type projection operatos [20] based on the sets ω_i. The projector $P_{V,l}^{w,2}$ adjusts the edge bubbles such that

$$(P_{V,l}^{w,2} w)(N_i) = 0 \qquad \text{and} \qquad \int\limits_{E_j} w - P_{V,l}^{w,2} w = 0$$

for corner nodes N_i and edges E_j. The projector $P_{V,l}^{\beta,2}$ adjusts the element bubbles such that

$$(P_{V,l}^{\beta,2} w)(N_i) = 0 \qquad \text{and} \qquad \int\limits_{T_j} \beta - P_{V,l}^{\beta,2} \beta = 0$$

for corner nodes N_i and triangles T_j.

The constructed operators are projections, are continuous in the norms

$$\|P_{V,l}^{w}\|_1 \preceq 1, \qquad \|P_{V,l}^{\beta}\|_1 \preceq 1, \quad \|P_{V,l}^{\beta}\|_0 \preceq 1,$$

do approximate as good as

$$\begin{aligned} \|w - P_{V,l}^{w} w\|_k &\preceq h^{m-k} \|w\|_m \qquad \forall\, w \in H_0^m, \qquad 0 \le k < m \le 2, \\ \|\beta - P_{V,l}^{\beta} \beta\|_0 &\preceq h \|\beta\|_1 \qquad \forall\, \beta \in H_0^1, \end{aligned} \tag{76}$$

and fulfill

$$\overline{\nabla w - \beta} = \overline{\nabla P_{V,l}^{w} w - P_{V,l}^{\beta} \beta},$$

as in [2].

Define $P_l^{H^2}$ as Clément interpolation operator into some finite element sub-subspace of H_0^2 of the same mesh-size h fulfilling the approximation

$$\|w - P_l^{H^2} w\|_k \preceq h^{m-k} \, \|w\|_m \quad \forall\, w \in H_0^m, \quad 0 \le k < m \le 2,$$

and the continuity and inverse estimate

$$\|P_l^{H^2} w\|_k \preceq h^{m-k} \, \|w\|_m \quad \forall\, w \in H_0^m, \quad 0 \le m \le k \le 2.$$

Then define the linear operator

$$T : H_0^2 \to V_h, \qquad T\widetilde{w} = (P_{V,l}^w P_l^{H^2} \widetilde{w}, P_{V,l}^\beta \nabla P_l^{H^2} \widetilde{w}) \tag{77}$$

We use the triangle inequality and $A \preceq D$ to estimate

$$\begin{aligned}\|(w,\beta)\|_{[A,D]_{1/2}} &\leq \|TP_{V,t}^w w\|_{[A,D]_{1/2}} + \|(w,\beta) - TP_{V,t}^w w\|_{[A,D]_{1/2}} \\ &\leq \|TP_{V,t}^w w\|_{[A,D]_{1/2}} + \|(w,\beta) - TP_{V,t}^w w\|_D.\end{aligned}$$

The prove of the theorem is complete after estimating the two terms in the following two Lemmata. □

Lemma 5. *The estimate*

$$\|TP_{V,t}^w w\|_{[A,D]_{1/2}} \preceq \|(w,\beta)\|_{V,l,\tilde{0}} \tag{78}$$

is valied.

Proof. First, we estimate

$$\|T\tilde{w}\|_D \preceq h^{-2}\|\tilde{w}\|_0 \qquad \forall\, \tilde{w} \in L_2 \tag{79}$$

Using the partition of unity we define

$$\varphi_i = \Psi_i P_l^{H^2} \tilde{w}$$

giving the splitting

$$P_l^{H^2} \tilde{w} = \sum_i \varphi_i,$$

what is L_2 stable

$$\sum_i \|\varphi_i\|_0^2 \preceq \|P_l^{H^2} \tilde{w}\|_0^2.$$

Due to the smoothness of Ψ_i the estimate

$$h^{-1}\|\varphi_i\|_1 + h^{-2}\|\varphi_i\|_2 \preceq \|\varphi_i\|_0$$

is valid. The domain decomposition and the interpolation operators have been design such that

$$(P_{V,l}^w \varphi_i, P_{V,l}^\beta \nabla \varphi_i) \in V_{l,i}.$$

So,

$$\begin{aligned}\|T\tilde{w}\|_D^2 &\preceq \sum_i \|(P_{V,l}^w \varphi_i, P_{V,l}^\beta \nabla \varphi_i\|_A^2 \\ &\preceq \sum \|P_{V,l}^\beta \nabla \varphi_i\|_1^2 + h^{-2}\|\nabla P_{V,l}^w \varphi_i - P_{V,l}^\beta \nabla \varphi_i\|_0^2 + h^{-2}\|\overline{\nabla P_{V,l}^w \varphi_i - P_{V,l}^\beta \nabla \varphi_i}\|_0^2 \\ &\preceq \sum h^{-4}\|\varphi_i\|_0^2 + 0 \\ &\preceq h^{-4}\|\tilde{w}\|_0^2.\end{aligned}$$

Then, we estimate

$$\|T\tilde{w}\|_A \preceq \|\tilde{w}\|_2 \qquad \forall\, \tilde{w} \in H_0^2 \tag{80}$$

by

$$\begin{aligned}
\|T\tilde{w}\|_A &\preceq \|P_{V,l}^{\beta}\nabla P_l^{H^2}\tilde{w}\|_1^2 + h^{-2}\|\nabla P_{V,l}^{w}P_l^{H^2}\tilde{w} - P_{V,l}^{\beta}\nabla P_l^{H^2}\tilde{w}\|_0 + 0 \\
&\preceq \|\tilde{w}\|_2^2 + h^{-2}\|\nabla(P_{V,l}^{w} - id)P_l^{H^2}\tilde{w}\|_0^2 + h^{-2}\|(P_{V,l}^{\beta} - id)\nabla P_l^{H^2}\tilde{w}\|_0^2 \\
&\preceq \|\tilde{w}\|_2^2.
\end{aligned}$$

Applying operator interpolation to (79) and (80) we conclude

$$\|T\tilde{w}\|_{[A,D]_{1/2}} \preceq h^{-1}\|\tilde{w}\|_1 \qquad \forall\, \tilde{w} \in H_0^1. \tag{81}$$

The necessary interpolation estimate $\|.\|_{H_0^1} \simeq \|.\|_{[H_0^2,L_2]_{1/2}}$ is found in [7].

We split $w = w_0 + w_r$ such that $\|w_0\|_1 + t^{-1}\|w_r\|_0 \preceq \|w\|_{V^{0,w}}$ and finish the lemma by

$$\begin{aligned}
&\|TP_{V,t}^{w}w\|_{[A,D]_{1/2}} \preceq h^{-1}\|P_{V,t}^{w}w\|_1 \\
&\quad\preceq\; h^{-1}\|P_{V,t}^{w}w_0\|_1 + h^{-1}\|P_{V,t}^{w}w_r\|_1 \\
&\quad\preceq\; h^{-1}\|w_0\|_1 + h^{-1}t^{-1}\|w_r\|_0 \\
&\quad\preceq\; h^{-1}\|w\|_{V^{0,w}} \preceq \|(w,\beta)\|_{V,l,\tilde{0}}.
\end{aligned}$$

Lemma 6. *The estimate*

$$\|TP_{V,t}^{w}w - (w,\beta)\|_D \preceq \|(w,\beta)\|_{V,l,\tilde{0}} \tag{82}$$

is valid.

First, we prove for all $(v,\eta) \in V_l$ the splitting estimate

$$\|(v,\eta)\|_D \preceq h^{-1}\|\eta\|_0 + h^{-1}(h+t)^{-1}\|v\|_0 + t^{-1}\|\overline{\nabla v - \eta}\|_0. \tag{83}$$

The case $h \preceq t$ is trivial, and we may assume $t \preceq h$. We set

$$\begin{aligned}
v_i &= P_{V,l}^{w}(\Psi_i v), \\
\eta_i &= P_{V,l}^{\beta}(\Psi_i \eta) + P_{V,l}^{\beta,2}\left[\nabla(\Psi_i v) - (\Psi_i \eta)\right],
\end{aligned}$$

such that we can represent

$$\begin{aligned}
v &= \sum v_i, \\
\eta &= \sum \eta_i + P_{V,l}^{\beta,2}(\eta - \nabla v).
\end{aligned}$$

Because $P_{V,l}^{\beta,2}(\eta - \nabla v)$ contains only A-orthogonal bubble functions, which can be split locally, we get

$$\|(0, P_{V,l}^{\beta,2}(\eta - \nabla v)\|_D \preceq \|(0, P_{V,l}^{\beta,2}(\eta - \nabla v))\|_A \preceq h^{-2}\|v\|_0 + h^{-1}\|\eta\|_0 + t^{-1}\|\overline{\nabla v - \eta}\|_0.$$

We estimate

$$\begin{aligned}\|(v_i,\eta_i)\|_A^2 &\preceq \|\eta_i\|_1 + h^{-2}\|\nabla v_i - \eta_i\|_0^2 + \|\overline{\nabla v_i - \eta_i}\|_0 \\ &\preceq h^{-2}\|(\Psi_i v)\|_0 + h^{-1}\|(\Psi_i \eta)\|_0 + 0.\end{aligned}$$

Thus, we obtain estimate (83) by

$$\begin{aligned}\|(v,\eta)\|_D^2 &\preceq \sum_i \|(v_i,\eta_i)\|_A^2 + \|(0, P_{V,l}^{\beta,2}(\eta - \nabla v)\|_D^2 \\ &\preceq \sum_i h^{-4}\|(\Psi_i v)\|_0^2 + h^{-2}\|(\Psi_i \eta)\|_0^2 + \|(0, P_{V,l}^{\beta,2}(\eta - \nabla v)\|_D^2 \\ &\preceq h^{-4}\|v\|_0^2 + h^{-2}\|\eta\|_0^2 + t^{-2}\|\overline{\nabla v - \eta}\|_0^2.\end{aligned}$$

We split $w = w_0 + w_r$ such that $\|w_0\|_1 + t^{-1}\|w_r\|_0 \preceq \|w\|_{V^{0,w}}$ and estimate

$$\begin{aligned}\|(TP_{V,t}^w w)_w - w\|_0 &\preceq \|P_{V,l}^w P_l^{H^2} P_{V,t}^w w_0 - w_0\|_0 + \|P_{V,l}^w P_l^{H^2} P_{V,t}^w w_r - w_r\|_0 \\ &\preceq (h+t)\|w_0\|_1 + t\,t^{-1}\|w_r\|_0 \preceq (h+t)\|w\|_{V^{0,w}},\end{aligned}$$

and, using (72),

$$\|(TP_{V,t}^w)_\beta\|_0 \preceq \|P_{V,l}^\beta \nabla P_l^{H^2} P_{V,t}^w w\|_0 \simeq \|P_{V,t}^w w\|_1 \preceq \|w\|_{V^{0,w}}.$$

Combining (83) with the last two estimates, and using $\overline{\nabla(T)_w - (T)_\beta} = 0$, we obtain estimate (82), and the proof is complete □

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