The Classification Problem for (Point) Symmetries of 2nd Order PDE's in one Dependent and two Independent Variables*

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1 Introduction

This report deals with the most important means for finding closed form solutions of differential equations (DE's for short), namely the so called non-trivial symmetries that a given DE may admit. Apart from being used in finding exact solutions of DE's, symmetries also reveal important information about the behavior of DE's; they can be used to verify and develop numerical schemes, to obtain conservation laws of a given DE, and even gauge theory is based on the

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continuous symmetries of certain relativistic fields. The study of symmetries has been initiated by Sophus Lie. Roughly speaking, a symmetry is a point- or contact-transformation that does not change the form of a DE. Obviously, the entirety of symmetry transformations of any given DE forms a group. The term *symmetry group* is applied to the largest group of transformations with this property. The symmetry problem for DE's comes in various versions. On the one hand, there is the classification problem. It aims to obtain a complete survey of all possible symmetry groups for a class of given DE's, e.g. DE's of a given order. We will be concerned with this version of the problem. The starting point of this approach is always a listing of groups, whose differential invariants determine the general form of a DE that may be invariant under the respective group. On the other hand, the symmetry group of any given particular DE needs to be determined *algorithmically* (and may then be applied for finding solutions of the equation). The key for solving the latter problem is the *Janet base* for the so-called determining system of the respective equation, since its coefficients determine the symmetry group. These two versions of the problem are closely related anyway.

In the second section, we introduce the theory necessary to treat the classification problem. In the third section, we give a listing of all continuous groups of the three dimensional space that is based on work by S. Lie and U. Amaldi completes this report. These are the groups that have to be processed in order to treat the classification problem for PDE's of order two in one dependent and two independent variables.

2 Basic Theoretical Concepts

In this section, we provide the basic notions and some theory in order to be able to tackle the above mentioned classification problem for PDE's of order two in one dependent and two independent variables. We present the theory for ordinary differential equations, since all the concepts and methods outlined there directly extend to the case of partial differential equations. For a more detailed treatment of the subject symmetry groups and differential equations consult any of the modern textbooks [1, 4, 5, 6].

We already mentioned that the starting point of this approach is always a listing of groups, whose differential invariants determine the general form of a DE that may be invariant under the respective group. Hence we first have to clarify what kind of groups we are talking about, and what it means for a DE to be invariant under such a group. This will be investigated in the subsections "Transformation Groups of Differential Equations" and "Infinitesimal Generators and Prolongations". Furthermore, we have to introduce the system of differential invariants for a given Lie group; this is a system of PDE's whose solutions are DE's invariant under the original group. This is done in the subsection "Differential Invariants of Point Transformations". Finally, two examples that demonstrate the whole procedure for two selected (space) groups follow in the subsection "Examples".

2.1 Transformation Groups of Differential Equations

Introducing new variables into a given (ordinary) DE is a widely used method in order to facilitate the solution procedure. Usually this is done in an ad hoc manner without guaranteed success. In particular, there is no criterion to decide whether a certain class of transformations will lead to an integrable equation or not. A critical examination of these methods was the starting point for Lie's symmetry analysis. We will now have a look on the behavior of DE's under special kind of transformations.

2.1.1 Properties of Point Transformations

Let an ODE of order n be given as

$$\omega(x, y, y', \ldots, y^{(n)}) = 0. \tag{1}$$

The general solution of such an equation is a set of curves in the x-y-plane depending on n parameters $C_1, \ldots, C_n$, given by

$$\Phi(x, y, C_1, \ldots, C_n) = 0, \tag{2}$$

compare any textbook on (ordinary) DE, e.g. [7]. Invertible analytic transformations between two planes with coordinates (x, y) and (u, v), respectively, of the form

$$u = \sigma(x, y),\ v = \rho(x, y), \tag{3}$$

are called *point transformations*. We will basically encounter them in the form of *one-parameter groups of point transformations*

$$u = \sigma(x, y, a), v = \rho(u, v, a). \tag{4}$$

Here the real parameter a ranges over an open interval including 0, such that for any fixed value of a equation (4) represents a point transformation. In addition, there exists a (real) group composition Φ such that

$$\begin{aligned}
&x' = \sigma(x, y, a),\ y' = \rho(u, v, a),\\
&x'' = \sigma(x', y', a'),\ y'' = \rho(u', v', a')\\
&\quad\Rightarrow\\
&x'' = \sigma(x, y, \Phi(a, a')),\ y'' = \rho(u, v, \Phi(a, a'))
\end{aligned}$$

Such group transformations can be reparametrized such that $\Phi(a, a') = a + a'$ and such that $a = 0$ represents the identity element.

An equation (1) is said to be *invariant* under the change of variables

$$x = \Phi(u, v),\ y = \Psi(u, v) \qquad (v \equiv v(u))$$

if it retains its form under this transformation., i.e. if the functional dependence of the transformed equation on u and v is the same as in the original equation

(1). Such a transformation is called a *symmetry* of the DE. The same transformation acts on the curves (2). If it is a symmetry, the functional dependence of the transformed curves of u and v must be the same as in (2). This is not necessarily true for the parameters $C_1, \ldots, C_n$ because they do not occur in the DE itself. This means, the entirety of curves described by the two equations is the same, to any fixed values for the constants however may correspond a different curve in either set, in other words the solution curves are permuted among themselves by a symmetry transformation. It is fairly obvious that all symmetry transformations of a given DE form a group, the *symmetry group* of that equation.

2.2 Infinitesimal Generators and Prolongations

Let a curve in the (x-y)-plane described by $y = f(x)$ be transformed under a point transformation of the form (3) into $v = g(u)$. Now the question arises how the derivative $y' = \frac{df}{dx}$ corresponds to $v' = \frac{dg}{du}$ under this transformation. A simple calculation leads to the *first prolongation*

$$v' = \frac{dv}{du} = \frac{\rho_x + \rho_y y'}{\sigma_x + \sigma_y y'} \equiv \chi(x, y, y').$$

Note that the knowledge of (x, y, y') and the equations of the point transformation (3) already determine v' uniquely, the knowledge of the equation of the curve is not required. This may be expressed by saying that the line element (x, y, y') is transformed into the line element (u, v, v') under the action of a point transformation. Similarly, the transformation law for derivatives of second order is obtained as

$$v'' = \frac{dv'}{du} = \frac{\chi_x + \chi_y y' + \chi_{y'} y''}{\sigma_x + \sigma_y y'}.$$

For later applications it would be useful to express the second derivative v'' explicitly in terms of σ and ρ. We do not give this more lengthy formula here, but turn now immediately to the extension of this prolongation concept to one-parameter groups of point transformations of the form

$$u = \sigma(x, y, a),\ v = \rho(u, v, a). \tag{5}$$

Here the transformation properties of the derivatives may be expressed in terms of the prolongation of the corresponding *infinitesimal generator*

$$U = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}, \tag{6}$$

where

$$\xi(x, y) = \frac{d}{da}\sigma(x, y, a)_{|a=0},\ \eta(x, y) = \frac{d}{da}\rho(x, y, a)_{|a=0}.$$

The n-th prolongation of (6) is now defined as (D is the operator of total differentiation w.r.t. x, i.e. $D = \frac{\partial}{\partial x} + \sum_{k=1}^{\infty} y^{(k)} \frac{\partial}{\partial y^{(k-1)}}$)

$$
\begin{aligned}
U^{(n)} &= U + \sum_{k=1}^{n} \zeta^{(k)} \frac{\partial}{\partial y^{(k)}}, \text{ where} \\
\zeta^{(1)} &= D(\eta) - y' D(\xi), \\
\zeta^{(k)} &= D(\zeta^{(k-1)}) - y^{(k)} D(\xi) \quad \text{for } k = 2, 3, \ldots
\end{aligned}
$$

Here we give the two lowest $\zeta' s$ explicitly:

$$
\begin{aligned}
\zeta^{(1)} &= \eta_x + (\eta_y - \xi_x) y' - \xi_y y'^2, \\
\zeta^{(2)} &= \eta_{xx} + (2\eta_{xy} - \xi_{xx}) y' + (\eta_{yy} - 2\xi_{xy}) y'^2 - \xi_{yy} y'^3 + (\eta_y - 2\xi_x) y'' - 3\xi_y y' y''.
\end{aligned}
$$

These two innocent looking expressions should not divert from the fact that the number of terms in $\zeta^{(k)}$ grows roughly as 2^k. But $\zeta^{(k)}$ is at least linear and homogeneous in $\xi(x,y)$ and $\eta(x,y)$ and its derivatives up to order k. In addition, $\zeta^{(k)}$ does not depend explicitly on x and y but only on $y', y'', \ldots, y^{(k)}$. For $k > 1$, $y^{(k)}$ occurs linearly and y' occurs with power $k+1$ in $\zeta^{(k)}$.

2.3 Differential Invariants of Point Transformations

In order to obtain a classification of possible symmetry groups, the invariants of all finite groups of the plane have been determined by Lie. The starting point for this classification was a listing of all finite groups of point transformations of the plane.

Any r-parameter Lie group may be represented by r infinitesimal generators

$$
\xi_i \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial y} \quad (i = 1, \ldots, r). \tag{7}
$$

Any (ordinary) DE of order m that has such an r-parameter Lie group as symmetry group has to vanish under all m-th prolongations of the generators (7) and vice versa, i.e. this DE is a solution Φ of the following system of linear homogeneous partial differential equations:

$$
(\xi_i \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial y} + \zeta_i^{(1)} \frac{\partial}{\partial y'} + \ldots + \zeta_i^{(m)} \frac{\partial}{\partial y^{(m)}}) \Phi = 0 \quad (i = 1, \ldots, r). \tag{8}
$$

The system (8) is called *system of differential invariants*, its fundamental solutions are called the *differential invariants* of the respective Lie group.

2.3.1 Solvability of the System of Differential Invariants

The group property guarantees that (8) is a complete system for Φ with $m+r-2$ solutions. It may be brought into Jacobian normal form (a kind of analogon of the triangular form for matrices) before attempting to solve it. The dependencies of the fundamental solutions may then be chosen such that

$$\begin{aligned}
\Phi_1 &\equiv \Phi_1(x, y, y', \ldots, y^{(r-1)}),\\
\Phi_2 &\equiv \Phi_2(x, y, y', \ldots, y^{(r)}),\\
&\vdots\\
\Phi_{m-r+2} &\equiv \Phi_{m-r+2}(x, y, y', \ldots, y^{(m)}).
\end{aligned}$$

2.3.2 Example

In this subsubsection we give an example for the case of ODE's. Let $p = \frac{\partial}{\partial x}$, $q = \frac{\partial}{\partial y}$. We consider a group taken from Lie's listing of all finite groups of the plane, namely

$$g_{13} = \{p, 2xp + yq, x^2p + xyq\}.$$

Prolongation of its three generators up to the third order yields the following system of differential invariants:

$$\begin{aligned}
\Phi_x &= 0,\\
2x\Phi_x + y\Phi_y - y'\Phi_{y'} - 3y''\Phi_{y''} - 5y'''\Phi_{y'''} &= 0,\\
x^2\Phi_x + xy\Phi_y - (y'x - y)\Phi_{y'} - 3y''x\Phi_{y''} - (5y'''x + 3y'')\Phi_{y''} &= 0.
\end{aligned}$$

Its Jacobian normal form is

$$\begin{aligned}
\Phi_x &= 0,\\
\Phi_{y'} - \frac{3y''}{y}\Phi_{y'''} &= 0,\\
\Phi_y - \frac{3y''}{y}\Phi_{y''} - \frac{5y''y + 3y''y'}{y^2}\Phi_{y'''} &= 0.
\end{aligned}$$

Now, by using some strategies for solving systems of linear PDE's (which is a bit of an art anyway) we might arrive at the following two fundamental solutions:

$$\begin{aligned}
\Phi_1 &\equiv y''y^3,\\
\Phi_2 &\equiv y'''y^5 + 3y''y'y^4.
\end{aligned}$$

Note that they are DE's of order not higher than 3 that have the respective Lie group g_{13} as symmetry group.

2.4 Examples (PDE's)

The classification problem for partial differential equations has not yet been dealt with. So any contribution to the solution of the classification problem for the case of partial differential equations of order 2 in one dependent and two independent variables would be a natural extension of the existing theory. This is what I try to accomplish in the frame of an ongoing research project.

In this case, we have to start with a listing of the continuous space groups, acting on the variables x,y, and z. Finding the differential invariants may then be accomplished in analogy to the ordinary case. I did this already with a good part of the space groups listed by Lie (compare Section 3).

We give two examples of calculating the invariants from a given (space) Lie group. Both groups are taken from the listing in Section 3. Note that $p = \frac{\partial}{\partial x}$, $q = \frac{\partial}{\partial y}$, $r = \frac{\partial}{\partial z}$.

In our first example, we consider $\mathbf{ip}_{22} = \{p, q, xq, xp - yq, yp, xp + yq + r\}$. By prolongation up to order two we get the following system of differential invariants:

$$
\begin{aligned}
\frac{\partial}{\partial x}\Phi &= 0, \\
\frac{\partial}{\partial y}\Phi &= 0, \\
(x\frac{\partial}{\partial y} - z_y\frac{\partial}{\partial z_x} - z_{yy}\frac{\partial}{\partial z_{xy}} - 2z_{xy}\frac{\partial}{\partial z_{xx}})\Phi &= 0, \\
(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - z_x\frac{\partial}{\partial z_x} + z_y\frac{\partial}{\partial z_y} + 2z_{yy}\frac{\partial}{z_{yy}} - 2z_{xx}\frac{\partial}{\partial z_{xx}})\Phi &= 0, \\
(y\frac{\partial}{\partial x} - z_x\frac{\partial}{\partial z_y} - 2z_{xy}\frac{\partial}{z_{yy}} - z_{xx}\frac{\partial}{\partial z_{xy}})\Phi &= 0, \\
(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z} - z_x\frac{\partial}{\partial z_x} - z_y\frac{\partial}{\partial z_y} - 2z_{yy}\frac{\partial}{\partial z_{yy}} - 2z_{xy}\frac{\partial}{z_{xy}} - 2z_{xx}\frac{\partial}{\partial z_{xx}})\Phi &= 0.
\end{aligned}
$$

Its Jacobian normal form is

$$
\begin{aligned}
\frac{\partial}{\partial x}\Phi &= 0, \\
\frac{\partial}{\partial y}\Phi &= 0, \\
(\frac{\partial}{\partial z_x} + \frac{z_{yy}}{z_y}\frac{\partial}{\partial z_{xy}} + \frac{2z_{xy}}{z_y}\frac{\partial}{\partial z_{xx}})\Phi &= 0, \\
(\frac{\partial}{\partial z_y} + \frac{z_{xy}z_x z_{yy} - z_{xx}z_{yy}z_y}{z_{xy}z_y^2 - z_x z_{yy} z_y}\frac{\partial}{\partial z_{xy}} + \frac{2z_{xy}^2 z_x - 2z_{xy}z_{xx}z_y}{z_{xy}z_y^2 - z_x z_{yy}z_y}\frac{\partial}{\partial z_{xx}})\Phi &= 0, \\
(\frac{\partial}{\partial z_{yy}} + \frac{z_{xx}z_y^2 - z_x^2 z_{yy}}{2z_{xy}z_y^2 - 2z_x z_{yy}z_y}\frac{\partial}{\partial z_{xy}} - \frac{z_{xy}z_x^2 - z_{xx}z_x z_y}{z_{xy}z_y^2 - z_x z_{yy}z_y}\frac{\partial}{\partial z_{xx}})\Phi &= 0, \\
(\frac{\partial}{\partial z} - \frac{2z_{xy}z_y - 2z_x z_{yy}}{z_y}\frac{\partial}{\partial z_{xy}} + \frac{4z_{xy}z_x - 4z_{xx}z_y}{z_y}\frac{\partial}{\partial z_{xx}})\Phi &= 0.
\end{aligned}
$$

These six equations in eight variables lead to two fundamental solutions, namely

$$
\begin{aligned}
\Phi_1 &= (z_{xy}^2 - z_{xx}z_{yy})e^{-4z}, \\
\Phi_2 &= (z_{yy}z_x^2 - 2z_x z_y z_{xy} + z_{xx}z_y^2)e^{-4z}.
\end{aligned}
$$

Both constitute PDE's of order two in one dependent and two independent variables which have $\mathbf{ip}_{22}$ as their symmetry group.

We show for example that Φ_1 is invariant under p and yq. We have that p is the infinitesimal generator of the one-parameter transformation group

$$\begin{aligned} x' &= x + a, \\ y' &= y, \\ z' &= z. \end{aligned} \tag{9}$$

Now,

$$\Phi_1' = (z'^2_{x'y'} - z'_{x'x'}z'_{y'y'})e^{-4z'} = (z^2_{x'y} - z_{x'x'}z_{yy})e^{-4z} = (z^2_{xy} - z_{xx}z_{yy})e^{-4z} = \Phi_1,$$

i.e. Φ_1 is invariant under (9). Note that the third equality holds because of $z_{x'} = z_x \frac{dx'}{dx} = z_x$.

On the other hand, yp is the infinitesimal generator of the one-parameter group of transformations

$$\begin{aligned} x' &= x + ay, \\ y' &= y, \\ z' &= z. \end{aligned} \tag{10}$$

Again, we have

$$\Phi_1' = (z'^2_{x'y'} - z'_{x'x'}z'_{y'y'})e^{-4z'} = (z^2_{x'y} - z_{x'x'}z_{yy})e^{-4z} = (z^2_{xy} - z_{xx}z_{yy})e^{-4z} = \Phi_1.$$

i.e. Φ_1 is invariant under (10). For the third equality, note that $z_{x'} = z_x \frac{dx'}{dx} + z_y \frac{dx'}{dy} = z_x + az_y$, and hence $z^2_{x'y} = (z_{xy} + az_{yy})^2 = z^2_{xy} + 2az_{xy}z_{yy} + a^2z^2_{yy}$. Furthermore, $z_{x'x'} = (z_x + az_y)_{x'} = z_{xx'} + az_{yx'} = (z_{xx}\frac{dx'}{dx} + z_{xy}\frac{dx'}{dy}) + a(z_{yx}\frac{dx'}{dx} + z_{yy}\frac{dx'}{dy}) = z_{xx} + az_{xy} + az_{xy} + a^2z_{yy} = z_{xx} + 2az_{xy} + a^2z_{yy}$. Hence we have $z^2_{x'y} - z_{x'x'}z_{yy} = (z^2_{xy} + 2az_{xy}z_{yy} + a^2z^2_{yy}) - (z_{xx}z_{yy} + 2az_{xy}z_{yy} + a^2z^2_{yy}) = z^2_{xy} - z_{xx}z_{yy}$.

As a second example, we consider the Lie group $\mathbf{ip}_{21} = \{p, q, xq, xp - yq, yp, xp + yq\}$. Again, by prolongation up to order two we get the system of differential invariants:

$$\frac{\partial}{\partial x}\Phi = 0,$$
$$\frac{\partial}{\partial y}\Phi = 0,$$
$$(x\frac{\partial}{\partial y} - z_y\frac{\partial}{\partial z_x} - z_{yy}\frac{\partial}{\partial z_{xy}} - 2z_{xy}\frac{\partial}{\partial z_{xx}})\Phi = 0,$$
$$(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - z_x\frac{\partial}{\partial z_x} + z_y\frac{\partial}{\partial z_y} + 2z_{yy}\frac{\partial}{z_{yy}} - 2z_{xx}\frac{\partial}{\partial z_{xx}})\Phi = 0,$$
$$(y\frac{\partial}{\partial x} - z_x\frac{\partial}{\partial z_y} - 2z_{xy}\frac{\partial}{z_{yy}} - z_{xx}\frac{\partial}{\partial z_{xy}})\Phi = 0,$$
$$(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z_x\frac{\partial}{\partial z_x} - z_y\frac{\partial}{\partial z_y} - 2z_{yy}\frac{\partial}{\partial z_{yy}} - 2z_{xy}\frac{\partial}{z_{xy}} - 2z_{xx}\frac{\partial}{\partial z_{xx}})\Phi = 0.$$

Its Jacobian normal form is

$$\frac{\partial}{\partial x}\Phi = 0,$$
$$\frac{\partial}{\partial y}\Phi = 0,$$
$$(\frac{\partial}{\partial z_x} + \frac{2z_{xy}^2 - 2z_{xx}z_{yy}}{z_{xy}z_y - z_x z_{yy}}\frac{\partial}{\partial z_{xx}})\Phi = 0,$$
$$(\frac{\partial}{\partial z_y} + \frac{2z_{xy}^3 z_x - 2z_{xy}^2 z_{xx} z_y - 2z_{xy}z_{xx}z_x z_{yy} + 2z_{xx}^2 z_{yy} z_y}{z_{xy}^2 z_y^2 - 2z_{xy}z_x z_{yy} z_y + z_x^2 z_{yy}^2}\frac{\partial}{\partial z_{xx}})\Phi = 0,$$
$$(\frac{\partial}{\partial z_{yy}} - \frac{z_{xy}^2 z_x^2 - 2z_{xy}z_{xx}z_x z_y + z_{xx}^2 z_y^2}{z_{xy}^2 z_y^2 - 2z_{xy}z_x z_{yy} z_y + z_x^2 z_{yy}^2}\frac{\partial}{\partial z_{xx}})\Phi = 0,$$
$$(\frac{\partial}{\partial z_{xy}} - \frac{2z_{xy}z_x - 2z_{xx}z_y}{z_{xy}z_y - z_x z_{yy}}\frac{\partial}{\partial z_{xx}})\Phi = 0.$$

Again we have six equations in eight variables leading to two fundamental solutions, namely

$$\Phi_1 \equiv z,$$
$$\Phi_2 \equiv \frac{z_{yy}z_x^2 - 2z_x z_y z_{xy} + z_{xx}z_y^2}{z_{xx}z_{yy} - z_{xy}^2}.$$

Here Φ_2 constitutes a PDE of order two in one dependent and two independent variables which has $\mathbf{ip}_{22}$ as its symmetry group.

3 The Finite Continuous Groups of the 3d Space

In this section we give a listing of all finite continuous space groups. We use some terminology by Lie, who divided them into primitive and imprimitive groups.

Lie gave a listing of all primitive groups and of some imprimitive groups. They can be extracted from [2, 3]. The remaining space groups are extracted from a paper by Ugo Amaldi, who completed Lie's listing.

As already laid out, this listing has to be the starting point for the classification problem for PDE's in one dependent and two independent variables. The next step will be to single out all those groups with less than eight generators, since those are the groups under which a second order PDE of the above type might be invariant.

3.1 All Primitive Groups

$\mathbf{p}_1$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial z}, y\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}+xz\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}+yz\frac{\partial}{\partial z}, xz\frac{\partial}{\partial x}+yz\frac{\partial}{\partial y}+z^2\frac{\partial}{\partial z}\}$

$\mathbf{p}_2$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial z}, y\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}\}$

$\mathbf{p}_3$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}-z\frac{\partial}{\partial z}, x\frac{\partial}{\partial z}, y\frac{\partial}{\partial z}\}$

$\mathbf{p}_4$: $\{\frac{\partial}{\partial x}-y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}+x\frac{\partial}{\partial z}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+2z\frac{\partial}{\partial z}, z\frac{\partial}{\partial x}-y(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}), z\frac{\partial}{\partial y}+x(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}), z(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})\}$

$\mathbf{p}_5$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}, x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x}\}$

$\mathbf{p}_6$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}, x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\}$

$\mathbf{p}_7$: $\{\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}, y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}, (xy-z)\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}+yz\frac{\partial}{\partial z}, \frac{\partial}{\partial x}+y\frac{\partial}{\partial z}, x\frac{\partial}{\partial x}+z\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x}+(xy-z)\frac{\partial}{\partial y}+xz\frac{\partial}{\partial z}\}$

$\mathbf{p}_8$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}, y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}, z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}, U, 2xU-S\frac{\partial}{\partial x}, 2yU-S\frac{\partial}{\partial y}, 2zU-S\frac{\partial}{\partial z}\}$, where $U := x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$, $S := x^2+y^2+z^2$.

3.2 The Imprimitive Groups

3.2.1 Some Imprimitive Groups by Lie

$\mathbf{ip}_1$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\}$

$\mathbf{ip}_2$: $\{x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, Z_1\frac{\partial}{\partial x}, \dots, Z_l\frac{\partial}{\partial x}, Z_1\frac{\partial}{\partial y}, \dots, Z_l\frac{\partial}{\partial y}\}$, wobei $l \geq 1$, $Z_i(z)$, o.b.d.A. $Z_1 = 1$.

$\mathbf{ip}_3$: $\{x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, Z_1\frac{\partial}{\partial x}, \dots, Z_l\frac{\partial}{\partial x}, Z_1\frac{\partial}{\partial y}, \dots, Z_l\frac{\partial}{\partial y}\}$, wobei $l \geq 1$, $Z_i(z)$, o.b.d.A. $Z_1 = 1$.

$\mathbf{ip}_4$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$

$\mathbf{ip}_5$: $\{x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, \frac{\partial}{\partial z},$
$e^{\lambda_k z}\frac{\partial}{\partial x}, ze^{\lambda_k z}\frac{\partial}{\partial x}, \dots, z^{m_k}e^{\lambda_k z}\frac{\partial}{\partial x}, e^{\lambda_k z}\frac{\partial}{\partial y}, ze^{\lambda_k z}\frac{\partial}{\partial y}, \dots, z^{m_k}e^{\lambda_k z}\frac{\partial}{\partial y}\}$,
wobei $k = 1, 2, \dots, h$ $(h \geq 1)$; $m_1, m_2, \dots, m_k \geq 1$, $\sum_{i=1}^{k} m_i = l - h$ $(l \geq 1)$; $\lambda_1, \lambda_2, \dots, \lambda_h \in \mathbb{Z}$, paarweise verschieden.

$\mathbf{ip}_6$: $\{x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \frac{\partial}{\partial z},$
$e^{\lambda_k z}\frac{\partial}{\partial x}, ze^{\lambda_k z}\frac{\partial}{\partial x}, \dots, z^{m_k}e^{\lambda_k z}\frac{\partial}{\partial x}, e^{\lambda_k z}\frac{\partial}{\partial y}, ze^{\lambda_k z}\frac{\partial}{\partial y}, \dots, z^{m_k}e^{\lambda_k z}\frac{\partial}{\partial y}\}$,
wobei $k = 1, 2, \dots, h$ $(h \geq 1)$; $m_1, m_2, \dots, m_k \geq 1$, $\sum_{i=1}^{k} m_i = l - h$ $(l \geq 1)$; $\lambda_1, \lambda_2, \dots, \lambda_h \in \mathbb{Z}$, paarweise verschieden.

$\mathbf{ip}_7$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, z\frac{\partial}{\partial z}\}$

$\mathbf{ip}_8$: $\{x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, z\frac{\partial}{\partial z} + a(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), \frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, z^2\frac{\partial}{\partial x}, \dots, z^m\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial y}, \dots, z^m\frac{\partial}{\partial y}\}$, $a \in \mathbb{R}$

$\mathbf{ip}_9$: $\{x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, z^2\frac{\partial}{\partial x}, \dots, z^m\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial y}, \dots, z^m\frac{\partial}{\partial y}\}$

$\mathbf{ip}_{10}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{11}$: $\{x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, 2z\frac{\partial}{\partial z} + m(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), z^2\frac{\partial}{\partial z} + mz(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}),$
$\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, z^2\frac{\partial}{\partial x}, \dots, z^m\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial y}, \dots, z^m\frac{\partial}{\partial y}\}$

$\mathbf{ip}_{12}$: $\{x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z} + mz(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}),$
$\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, z^2\frac{\partial}{\partial x}, \dots, z^m\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial y}, \dots, z^m\frac{\partial}{\partial y}\}$

$\mathbf{ip}_{13}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z}, y\frac{\partial}{\partial x} - z^2\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{14}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\}$ $[= \mathbf{ip}_2[l = 1]]$

$\mathbf{ip}_{15}$: $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\}$ $[= \mathbf{ip}_5[h = 1, m_1 = \lambda_1 = 0]]$

$\mathbf{ip}_{16}$: $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} + \frac{1}{2}x^2\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x} + \frac{1}{2}y^2\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{17}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x^\rho y^\sigma \frac{\partial}{\partial z}\}, \sigma + \rho = 0, 1, 2, \ldots, h$ $[h = 0 \to \mathbf{ip}_{15}]$

$\mathbf{ip}_{18}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial z}, x^\rho y^\sigma \frac{\partial}{\partial z}\}$, $\sigma + \rho = 0, 1, 2, \ldots, h$ $[h = 0 \to \mathbf{ip}_8[a = m = 0]]$

$\mathbf{ip}_{19}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}\}$ $[= \mathbf{ip}_{11}[m = 0]]$

$\mathbf{ip}_{20}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z}, y\frac{\partial}{\partial x} - z^2\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\}$

$\mathbf{ip}_{21}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\}$ $[= \mathbf{ip}_3[l = 1]]$

$\mathbf{ip}_{22}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\}$

$\mathbf{ip}_{23}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + az\frac{\partial}{\partial z}, x^\rho y^\sigma \frac{\partial}{\partial z}\}, \sigma + \rho = 0, 1, 2, \ldots, h;$ $[a = h = 0 \to \mathbf{ip}_6[h = 1, m_1 = \lambda_1 = 0]; a \neq 0, h = 0 \to \mathbf{ip}_8[m = 0]]$

$\mathbf{ip}_{24}: \{\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} + \frac{1}{2}x^2\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x} + \frac{1}{2}y^2\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{25}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}, x^\rho y^\sigma \frac{\partial}{\partial z}\}, \sigma + \rho = 0, 1, 2, \ldots, h;$ $[h = 0 \to \mathbf{ip}_9[m = 0]]$

$\mathbf{ip}_{26}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}\}$ $[= \mathbf{ip}_{12}[m = 0]??]$

$\mathbf{ip}_{27}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z}, y\frac{\partial}{\partial x} - z^2\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (y - xz)\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z(y - xz)\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{28}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\}$ $[= \mathbf{ip}_1?]$

$\mathbf{ip}_{29}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + \frac{3}{2}x\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + \frac{3}{2}y\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{30}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{2h}{3}z\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + hxz\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + hyz\frac{\partial}{\partial z}, x^\rho y^\sigma \frac{\partial}{\partial z}\}, \sigma + \rho = 0, 1, 2, \ldots, h;$ $[h = 0 \to \mathbf{ip}_4?]$

$\mathbf{ip}_{31}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{32}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + hxz\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + hyz\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, x^\rho y^\sigma \frac{\partial}{\partial z}\}$, $\sigma + \rho = 0, 1, 2, \ldots, h;$ $[h = 0 \to \mathbf{ip}_7?]$

$\mathbf{ip}_{33}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}\}$ $[= \mathbf{ip}_{10}?]$

3.2.2 The Remaining Imprimitive Groups by Amaldi

The Groups of Type A

$\mathbf{ip}^1_{1,A}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^{t-1}\frac{\partial}{\partial y}, x^t\frac{\partial}{\partial y} + \binom{t}{t}x^0\frac{\partial}{\partial z}, x^{t+1}\frac{\partial}{\partial y} + \binom{t+1}{t}x^1\frac{\partial}{\partial z}, \ldots, x^s\frac{\partial}{\partial y} + \binom{s}{t}x^{s-t}\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y} + (c-t)z\frac{\partial}{\partial z}\}$ $(1 \le t \le s+1)$

$\mathbf{ip}^2_{1,A}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^{t-1}\frac{\partial}{\partial y}, x^t\frac{\partial}{\partial y} + \binom{t}{t}x^0\frac{\partial}{\partial z}, x^{t+1}\frac{\partial}{\partial y} + \binom{t+1}{t}x^1\frac{\partial}{\partial z}, \ldots, x^s\frac{\partial}{\partial y} + \binom{s}{t}x^{s-t}\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\}$ $(1 \le t)$

$\mathbf{ip}^1_{2,A}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, x^2\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial z}, \ldots, x^s\frac{\partial}{\partial y} + sx^{s-1}\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, 2x\frac{\partial}{\partial x} + sy\frac{\partial}{\partial y} + (s-2)z\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} + sxy\frac{\partial}{\partial y} + [(s-2)zx + sy]\frac{\partial}{\partial z}\}$ $(s \ge 1)$

$\mathbf{ip}^2_{2,A}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial y}, \ldots, x^s\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, 2x\frac{\partial}{\partial x} + sy\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} + sxy\frac{\partial}{\partial y} + z(x + cz^2)\frac{\partial}{\partial z}\}$ $(s \ge 1)$

$\mathbf{ip}^3_{2,A}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, x^2\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y} + 2(cx + y)\frac{\partial}{\partial z}\}$

$\mathbf{ip}^1_{3,A}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^{t-1}\frac{\partial}{\partial y}, x^t\frac{\partial}{\partial y} + \binom{t}{t}x^0\frac{\partial}{\partial z}, x^{t+1}\frac{\partial}{\partial y} + \binom{t+1}{t}x^1\frac{\partial}{\partial z}, \ldots, x^{s-1}\frac{\partial}{\partial y} + \binom{s-1}{t}x^{s-1-t}\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + (sy + x^s)\frac{\partial}{\partial y} + [(s-t)z + \binom{s}{t}x^{s-t}]\frac{\partial}{\partial z}\}$ $(t < s, s \ge 1)$

$\mathbf{ip}^2_{3,A}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^{s-1}\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + (s\frac{\partial}{\partial y} + x^s)\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\}$ $(s \ge 1)$

$\mathbf{ip}_{4,A}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^{t-1}\frac{\partial}{\partial y}, x^t\frac{\partial}{\partial y} + \binom{t}{t}x^0\frac{\partial}{\partial z}, x^{t+1}\frac{\partial}{\partial y} + \binom{t+1}{t}x^1\frac{\partial}{\partial z}, \ldots, x^s\frac{\partial}{\partial y} + \binom{s}{t}x^{s-t}\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} - tz\frac{\partial}{\partial z}\}$ $(s \ge 1)$

$\mathbf{ip}_{5,A}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^{t-1}\frac{\partial}{\partial y}, x^t\frac{\partial}{\partial y} + \binom{t}{t}x^0\frac{\partial}{\partial z}, x^{t+1}\frac{\partial}{\partial y} + \binom{t+1}{t}x^1\frac{\partial}{\partial z}, \ldots, x^s\frac{\partial}{\partial y} + \binom{s}{t}x^{s-t}\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} - tz\frac{\partial}{\partial z}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} + sxy\frac{\partial}{\partial y} + (s-2t)xz\frac{\partial}{\partial z}\}$ $(s \ge 1)$

$\mathbf{ip}_{6,A}$: $\{\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - az\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - (2axz + c)\frac{\partial}{\partial z}\}$, $a \in \{0, 1\}$

$\mathbf{ip}_{7,A}$: $\{\frac{\partial}{\partial x}, 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + y^2\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{8,A}$: $\{\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + c(y - x)\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{9,A}$: does not exist

$\mathbf{ip}_{10,A}$: does not exist

$\mathbf{ip}_{11,A}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$

$\mathbf{ip}_{12,A}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{13,A}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\}$

$\mathbf{ip}^1_{14,A}$: $\{\frac{\partial}{\partial y}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + (2zy + xz^2)\frac{\partial}{\partial z}\}$

$\mathbf{ip}^2_{14,A}$: $\{\frac{\partial}{\partial y}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + (2zy + cz^2)\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{15,A}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + (2y + az)z\frac{\partial}{\partial z}\}$

$\mathbf{ip}^1_{16,A}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + (2y + az)z\frac{\partial}{\partial z}\}$

$\mathbf{ip}^2_{16,A}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + az\frac{\partial}{\partial z}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + 2zy\frac{\partial}{\partial z}\}$

$\mathbf{ip}^3_{16,A}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}, y\frac{\partial}{\partial y}, y^2\frac{\partial}{\partial y}\}$

$\mathbf{ip}^1_{17,A}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial y} + (2y + az)z\frac{\partial}{\partial z}\}$

$\mathbf{ip}^2_{17,A}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + az\frac{\partial}{\partial z}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} + 2azx\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + 2zy\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{18,A}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \Psi_1(x)\frac{\partial}{\partial y}, \ldots, \Psi_{i-1}(x)\frac{\partial}{\partial y}, \Psi_i(x)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \Psi_{i+1}(x)\frac{\partial}{\partial y} + \varphi_{i+1}\frac{\partial}{\partial z}, \ldots, \Psi_s(x)\frac{\partial}{\partial y} + \varphi_s\frac{\partial}{\partial z}\}$? $(s \geq 0)$

$\mathbf{ip}_{19,A}$: $\{\frac{\partial}{\partial y}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, x\frac{\partial}{\partial y}, \Psi_1(x)\frac{\partial}{\partial y}, \ldots, \Psi_i(x)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \Psi_{i+1}(x)\frac{\partial}{\partial y} + \varphi_{i+1}\frac{\partial}{\partial z}, \ldots, \Psi_s(x)\frac{\partial}{\partial y} + \varphi_s\frac{\partial}{\partial z}\}$ $(s \geq 0)$

$\mathbf{ip}_{20,A}$: $\{\frac{\partial}{\partial x}, e^{a_1x}\frac{\partial}{\partial y}, xe^{a_1x}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, x^2e^{a_1x}\frac{\partial}{\partial y} + \pi_{12}\frac{\partial}{\partial z}, \ldots, x^{s_1}e^{a_1x}\frac{\partial}{\partial y} + \pi_{1s_1}\frac{\partial}{\partial z}, e^{a_ix}\frac{\partial}{\partial y} + \pi_{i0}\frac{\partial}{\partial z}, xe^{a_ix}\frac{\partial}{\partial y} + \pi_{i1}\frac{\partial}{\partial z}, x^2e^{a_ix}\frac{\partial}{\partial y} + \pi_{i2}\frac{\partial}{\partial z}, \ldots, x^{s_i}e^{a_ix}\frac{\partial}{\partial y} + \pi_{is_i}\frac{\partial}{\partial z}\}$, $a_1(a_1 - 1) = 0$, $i = 2, \ldots, q$, size: $1 + q + \sum_{i=1}^q s_i$

$\mathbf{ip}_{21,A}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} + \pi_i\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial y} + \pi_2\frac{\partial}{\partial z}, \ldots, x^s\frac{\partial}{\partial y} + \pi_s\frac{\partial}{\partial z}, e^{a_ix}(\frac{\partial}{\partial y} + \pi_{i0}\frac{\partial}{\partial z}), e^{a_ix}(x\frac{\partial}{\partial y} + \pi_{i1}\frac{\partial}{\partial z}), \ldots, e^{a_ix}(x^{s_i}\frac{\partial}{\partial y} + \pi_{is_i}\frac{\partial}{\partial z})\}$, $i = 1, 2, \ldots, h$, $a_1 = 1$, size: $3 + s + h + \sum_{i=1}^h s_i$

The Groups of Type B

$\mathbf{ip}^1_{1,B}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \dots, x^{l-1}\frac{\partial}{\partial y}, x^l\frac{\partial}{\partial y} + ax^{m_i+1}y^{n-i}\frac{\partial}{\partial z}, \dots,$
$x^s\frac{\partial}{\partial y} + \frac{(t+1)(t+2)\dots(s-1)sa}{(m_i+2)\dots(m_i+s-t+1)}x^{m_i+s-t+1}y^{n-i}\frac{\partial}{\partial z},$
$x\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y} + [m_i + 1 + c(n-i+1) - t]z\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$, $j = 0, 1, \dots, n$; $1 \le t \le s+1$; $m \le m_j$; $m_{j+1} \ge m_j + s$ for $j \ne i$; $m_{i+1} \ge m_i + 2s - t$

$\mathbf{ip}^2_{1,B}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \dots, x^{l-1}\frac{\partial}{\partial y}, x^l\frac{\partial}{\partial y} + ax^{m_n+1}\frac{\partial}{\partial z}, \dots,$
$x^s\frac{\partial}{\partial y} + \frac{(t+1)(t+2)\dots(s-1)sa}{(m_n+2)\dots(m_n+s-t+1)}x^{m_n+s-t+1}\frac{\partial}{\partial z},$
$x\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y} + [m_n + c - t + 1]z\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$, $j = 0, 1, \dots, n$; $m \le m_j$; $m_{j+1} \ge m_j + s$

$\mathbf{ip}^3_{1,B}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \dots, x^s\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y} + c_1 z\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$, $j = 0, 1, \dots, n$; $m \le m_j$; $m_{j+1} \ge m_j + s$; $c_1 \in \{0, 1\}$

$\mathbf{ip}^1_{2,B}$: $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial y} + cx\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + (cy + c_1 x)\frac{\partial}{\partial z}\}$, $c, c_1 \in \{0, 1\}$

$\mathbf{ip}^2_{2,B}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \dots, x^s\frac{\partial}{\partial y}, 2x\frac{\partial}{\partial x} + sy\frac{\partial}{\partial y} + sz\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} + sxy\frac{\partial}{\partial y} + sxz\frac{\partial}{\partial z}, \frac{\partial}{\partial z},$
$x\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial z}, \dots, x^s\frac{\partial}{\partial z}\}$

$\mathbf{ip}^3_{2,B}$: $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \dots, x^s\frac{\partial}{\partial y}, 2x\frac{\partial}{\partial x} + sy\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + sxy\frac{\partial}{\partial y} + cx\frac{\partial}{\partial z}\}$, $c \in \{0, 1\}$, $s \ne 2$

$\mathbf{ip}^4_{2,B}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \dots, x^s\frac{\partial}{\partial y}, 2x\frac{\partial}{\partial x} + sy\frac{\partial}{\partial y} + (m_0 + sn)z\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} + sxy\frac{\partial}{\partial y} +$
$(m_0 + sn)xz\frac{\partial}{\partial z}, x^m y^{n-j}\}$; $j = 0, 1, \dots, n$; $m \le m_0 + sj$

$\mathbf{ip}^1_{3,B}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \dots, x^{t-1}\frac{\partial}{\partial y}, x^t\frac{\partial}{\partial y} + ax^{m_i+1}y^{n-i}\frac{\partial}{\partial z}, \dots,$
$x^{s-1}\frac{\partial}{\partial y} + \frac{(t+1)\dots(s-1)a}{(m_i+2)\dots(m_i+s-t)}x^{m_i+s-t}y^{n-i}\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} + (sy + x^s)\frac{\partial}{\partial y} + ([m_i +$
$1 + s(n-i+1) - t]z + \frac{(t+1)\dots(s-1)sa}{(m_i+2)\dots(m_i+s-t)} \cdot x^{m_i+s-t+1}y^{n-i})\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\},$
$j = 0, 1, \dots, n$; $1 \le t \le s$; $m \le m_j$; $m_{j+1} \ge m_j + s$ for $j \ne i$;
$m_{i+1} \ge m_i + 2s - t$

$\mathbf{ip}^2_{3,B}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \dots, x^{s-1}\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + (sy + x^s)\frac{\partial}{\partial y} + cz\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$; $j =$
$0, 1, \dots, n$; $m \le m_j$; $m_{j+1} \ge m_j + s$; $c \in \{0, 1\}$

$\mathbf{ip}^1_{4,B}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \dots, x^{t-1}\frac{\partial}{\partial y}, x^t\frac{\partial}{\partial y} + ax^{m_i+1}y^{n-i}\frac{\partial}{\partial z},$
$x^s\frac{\partial}{\partial y} + \frac{(t+1)(t+2)\dots sa}{(s-t)!}x^{m_i+s-t+1}y^{n-i}\frac{\partial}{\partial z}, y\frac{\partial}{\partial y} + (n-i+1)(z + cx^{m_n+1})\frac{\partial}{\partial z}, \frac{\partial}{\partial x},$
$x\frac{\partial}{\partial x} + [(m_i + 1 - t)z + c(m_i - m_n - t)x^{m_n+1}]\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$; $j = 0, 1, \dots, n$;
$m \le m_j$; $m_{j+1} \ge m_j + s$ for $j \ne i$; $m_{i+1} \ge m_i + 2s - t$

$\mathbf{ip}^2_{4,B}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \dots, x^{s-1}\frac{\partial}{\partial y}, x^s\frac{\partial}{\partial y} + a(n-i+1)x^{m_i+1}y^{n-i}\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} + [bz + c(b_i -$
$m_n - 1)x^{m_n+1} + a(m_i + 1 - s - b)x^{m_i-s+1}y^{n-i+1}]\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y} + (n - i +$
$1)(z + cx^{m_n+1})\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$, $j = 0, 1, \dots, n$; $m \le m_j$; $m_{j+1} \ge m_j + s$

$\mathbf{ip}^3_{4,B}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^s\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}+(bz+\sum_{j=0}^n a_j(m_j+1-b)x^{m_j+1}y^{n-j}+aby^{n+1})\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}+(cz+\sum_{j=0}^n a_j(n-j-c)x^{m_j+1}y^{n-j}+a(c-n-1)y^{n+1})\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$; $j=0,1,\ldots,n$; $m \le m_j$; $m_0 \ge s$; $m_{j+1} > m_j + s$

$\mathbf{ip}^1_{5,B}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^{s-1}\frac{\partial}{\partial y}, x^s\frac{\partial}{\partial y}+ax^{s-1}y^n\frac{\partial}{\partial z}, x\frac{\partial}{\partial x}-(z+c(n+1)sx^{s(n+1)-1})\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}+(n+1)[z+cx^{s(n+1)-1}]\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x}+sxy\frac{\partial}{\partial y}+[(s(n+1)-2)xz+\frac{ay^{n+1}}{n+1}-cx^{s(n+1)}]\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$, $j=0,1,\ldots,n$; $m \le (j+1)s-2$; $s \ge 2$

$\mathbf{ip}^2_{5,B}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^s\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}+(bz+\sum_{j=0}^n a_j(m_0+sj+1-b)x^{m_0+sj+1}y^{n-j}+aby^{n+1})\frac{\partial}{\partial z}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y}+\frac{1}{s}((m_0+sn-2b)z-\sum_{j=0}^n a_j(m_0+sj-2b)x^{m_0+sj+1}y^{n-j}+a(m_0-s-2b)y^{n+1})\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x}+sxy\frac{\partial}{\partial y}+[(m_0+sn)xz+\sum_{j=0}^n a_jx^{m_0+sj+2}y^{n-j}+a(m_0-s)y^{n-1}x]\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$; $j=0,1,\ldots,n$; $m \le m_0+sj$

$\mathbf{ip}^3_{5,B}$: $\{x^2\frac{\partial}{\partial x}+sxy\frac{\partial}{\partial y}+[(s(n+1)-2)xz+\sum_{j=0}^n a_jx^{s(j+1)}y^{n-j}-2axy^{n+1}+b'y^{n+1}]\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$; $j=0,1,\ldots,n$; $m \le (j+1)s-2$

$\mathbf{ip}^4_{5,B}$: $\{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^s\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}+\sum a_j(m_6+sj+1)x^{m_0+sj+1}y^{n-j}\frac{\partial}{\partial z}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y}+\frac{1}{s}((m_0+sn)z-\sum_{j=0}^n a_j(m_0+s_j)x^{m_0+sj+1}y^{n-j}+a(m_0-s)y^{n+1})\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x}+sxy\frac{\partial}{\partial y}+[(m_0+s_n)xz+\sum_{j=0}^n a_jx^{m_0+s_j+2}y^{n-j}+\sum_{j=0}^n b_jx^{m_0+s_j+1}y^{n-j}+a(m_0-s)xy^{n+1}]\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$; $j=0,1,\ldots,n$; $m \le m_0+sj$

$\mathbf{ip}^1_{6,B}$: $\{\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}+cx^{m_i+1}y^{m_i+2}\frac{\partial}{\partial z}, x\frac{\partial}{\partial x}+(m_i+1)z\frac{\partial}{\partial z}, x^2+\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}[2(m_i+1)xz+\frac{c}{m_i+2}x^{m_i+2}y^{m_i+2}]\frac{\partial}{\partial z}, x^m y^{2m_i+2-m_j}\frac{\partial}{\partial z}\}$; $j=0,1,\ldots,h$; $m \le m_j$; $m_{j+1} > m_j$

$\mathbf{ip}^2_{6,B}$: $\{\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}+bz\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}+2bxz\frac{\partial}{\partial z}, x^m y^{2b-m_j}\}$; $j=0,1,\ldots,h$; $m \le m_j$; $m_{j+1} > m_j$

$\mathbf{ip}_{7,B}$: $\{\frac{\partial}{\partial x}, 2x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}+(cx+c_ly^2)\frac{\partial}{\partial z}, x^l y^{-n}\frac{\partial}{\partial z}\}$; $l=0,1,\ldots,m$; $n=l,l+1,\ldots,m$

$\mathbf{ip}_{8,B}$: $\{\frac{\partial}{\partial x}+\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{m}{2}z\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}+[mxz+c(y-x)^{\frac{m}{2}+1}]\frac{\partial}{\partial z}, x^l\frac{\partial}{\partial z}\}$; $l=0,1,\ldots,m$

$\mathbf{ip}_{9,B}$: $\{\frac{\partial}{\partial y}, \Psi_j^1(x)y^{n_j}e^{c_jy}\frac{\partial}{\partial z}, \Psi_j^2(x)y^{n_j}e^{c_jy}, \ldots, \Psi_j^{l_0}(x)y^{n_j}e^{c_jy}, \Psi_j^1(x)y^{n_j-1}e^{c_jy}\frac{\partial}{\partial z}, \Psi_j^2(x)y^{n_j-1}e^{c_jy}, \ldots, \Psi_j^{l_0}(x)y^{n_j-1}e^{c_jy}, \Psi_j^{l_0+1}(x)y^{n_j-1}e^{c_jy}, \ldots, \Psi_j^{l_0+l_1}(x)y^{c_jy}, \Psi_j^1(x)e^{c_jy}\frac{\partial}{\partial z}, \ldots, \Psi_j^2(x)e^{c_jy}, \ldots, \Psi_j^{l_0}(x)e^{c_jy}, \ldots, \Psi_j^{l_0+l_1+\ldots+l_{nj}}(x)e^{c_jy}\}$, $j=1,2,\ldots,s$

$\mathbf{ip}^1_{10,B}$: $\{\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}+cz\frac{\partial}{\partial z}, \Psi_j(x)y^i\frac{\partial}{\partial z}\}$; $j=1,2,\ldots,l_i$; $i=0,1,\ldots,n$; $l_n \le l_{n-1} \le \ldots \le l_0$

$\mathbf{ip}^2_{10,B}$: $\{\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}+c\frac{\partial}{\partial z}, \Psi_j(x)y^i\frac{\partial}{\partial z}\}$; $j=1,2,\ldots,l_i$; $i=0,1,\ldots,n$; $l_n \le l_{n-1} \le \ldots \le l_0$

$\mathbf{ip}_{11,B} : \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + (\sum_{ij} h_{ij} x^{p_{ij}+1} y^{q_{ij}+1} e^{d_j y} + \sum_i k_i x^{t_i+1} y^{u_i}) \frac{\partial}{\partial z},$
$x^m y^n e^{a_j x + b_j y} \frac{\partial}{\partial z}{}_{[j=1,2,\ldots,h_1;\ m \leq m_{ij},\ n \leq n_{ij};\ i=1,2,\ldots,k_j;\ a_j=|=0,\ b_j=|=0]},$
$x^p y^q e^{d_j y} \frac{\partial}{\partial z}{}_{[j=1,2,\ldots,h_2;\ p \leq p_{ij},\ q \leq q_{ij};\ i=1,2,\ldots,l_j;\ d_j=|=0]},$
$x^r y^s e^{c_j} \frac{\partial}{\partial z}{}_{[j=1,2,\ldots,h_3;\ r \leq r_{ij},\ s \leq s_{ij};\ i=1,2,\ldots,v_j;\ c_j=|=0]}, x^t y^u \frac{\partial}{\partial z}\},$
$t \leq t_j, u \leq u_j : j = 1, 2, \ldots, h$

$\mathbf{ip}_{12,B} : \{\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, y^l x^{a_i} \log^u x \cdot \frac{\partial}{\partial z}\},\ l = 0, 1, 2, \ldots, n;\ i = 1, 2, \ldots, i_l;$
$u = 0, 1, \ldots, u_{i_l};\ i_l \leq i_{l-1} \leq \ldots \leq i_0;\ u_{i_l} \leq u_{i_l - 1} \leq \ldots \leq u_{i_0}$

$\mathbf{ip}^1_{13,B} : \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y} + (cz + c' y^{u_1+1})\frac{\partial}{\partial z}, x^m y^n e^{a_j x} \frac{\partial}{\partial z}{}_{[j=1,2,\ldots,h_1;\ m \leq m_{ij},\ldots,n \leq n_{ij};\ i=1,2,\ldots,k_j]},$
$x^t y^u \frac{\partial}{\partial z}{}_{[t \leq t_j, u \leq u_j;\ j=1,2,\ldots,h;\ u_1 \geq u_2 \geq \ldots \geq u_h]}\}$

$\mathbf{ip}^2_{13,B} : \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + a x^{t_i+1} y^{u_i} \frac{\partial}{\partial z}, y\frac{\partial}{\partial y} + [(u_i + 1)z + c' y^{u_1+1}]\frac{\partial}{\partial z},$
$x^m y^n e^{a_j x} \frac{\partial}{\partial z}{}_{[j=1,2,\ldots,h_1;\ m \leq m_{ij}; n \leq n_{ij};\ i=1,2,\ldots,\frac{\partial}{\partial z}{}_j]}, x^t y^u \frac{\partial}{\partial z}\},\ t = t_j;\ u \leq u_j;$
$j = 1, 2, \ldots, h;\ u_1 \geq u_2 \geq \ldots \geq u_h$

$\mathbf{ip}^1_{14,B} : \{\frac{\partial}{\partial y}, y\frac{\partial}{\partial y} + (\frac{n}{2}z + cx)\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + (nz + 2cx)y\frac{\partial}{\partial z}, \Psi_j(x) y^i \frac{\partial}{\partial z}\}, j = 1, 2, \ldots, l_i;$
$i = 0, 1, \ldots, n;\ l_n \leq l_{n-1} \leq \ldots \leq l_0;\ c \in \{0, 1\}$

$\mathbf{ip}^2_{14,B} : \{\frac{\partial}{\partial y}, y\frac{\partial}{\partial y} + (cx + c')\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + 2(cx + c')y\frac{\partial}{\partial z}, \Psi_j(x)\frac{\partial}{\partial z}\},\ j = 1, 2, \ldots, h;$
$cc' = 0$

$\mathbf{ip}^1_{15,B} : \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y} + (\frac{n}{2}z + a y^{n+1})\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + (nyz + \frac{2a}{n+2}y^{n+2})\frac{\partial}{\partial z},$
$x^m y^l e^{a_i x} \frac{\partial}{\partial z}$
${}_{[l=0,1,\ldots,n;\ i=1,\ldots,i_l;\ m=0,1,\ldots,m_{i_l};\ i_n \leq i_{n-1} \leq \ldots \leq i_0;\ m_{i_n} \leq m_{i_{n-1}} \leq \ldots \leq m_{i_0};\ a_i=|=0]},$
$x^m y^l \frac{\partial}{\partial z}{}_{[m \leq m_0;\ l \leq n]}, x^m \frac{\partial}{\partial z}{}_{[m \leq m_1 > m_0]}\}$

$\mathbf{ip}^2_{15,B} : \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} - (2yz + a x^{m_0-1})\frac{\partial}{\partial z}, x^m e^{a_j x} \frac{\partial}{\partial z}{}_{[j=1,\ldots,h;\ m \leq m^j;\ a_j \neq 0]},$
$x^m \frac{\partial}{\partial z}{}_{[m \leq m_0]}\}$

$\mathbf{ip}^3_{15,B} : \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, y^2\frac{\partial}{\partial y} + ay\frac{\partial}{\partial z}, x^m e^{a_j x}\frac{\partial}{\partial z}, x^m\frac{\partial}{\partial z}\}$

$\mathbf{ip}^1_{16,B} : \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y} + (\frac{n}{2}z + a y^{n+1})\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} + b(z - \frac{2a}{n+2}y^{n+1})\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + (nyz +$
$\frac{2a}{n+2}y^{n+2})\frac{\partial}{\partial z}, x^m y^l \frac{\partial}{\partial z}{}_{[m \leq m_0,\ l \leq n]};\ x^m \frac{\partial}{\partial z}{}_{[m \leq m_1 \geq m_0]}\}$

$\mathbf{ip}^2_{16,B} : \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + (m_0 + 1)z\frac{\partial}{\partial z}, y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} - (2yz + a x^{m_0+1})\frac{\partial}{\partial z}, x^m\frac{\partial}{\partial z}\},$
$m \leq m_0$

$\mathbf{ip}^3_{16,B} : \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial y} + ay\frac{\partial}{\partial z}, x^m\frac{\partial}{\partial z}\},\ m \leq m_0$

$\mathbf{ip}^4_{16,B} : \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + bz\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y}, x^m\frac{\partial}{\partial z}\},\ m \leq m_0$

$\mathbf{ip}^1_{17,B} : \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y} + (\frac{n}{2}z + a y^{n+1})\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} + (\frac{m}{2}z - \frac{ma}{n+2}y^{n+1})\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + (nyz +$
$\frac{2a}{n+2}y^{n+2})\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} + (myz - \frac{2ma}{n+2}x y^{n+1})\frac{\partial}{\partial z}, x^p y^q \frac{\partial}{\partial z}{}_{[p \leq m;\ q \leq n]}\}$

$\mathbf{ip}^2_{17,B}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + (m_0+1)z\frac{\partial}{\partial z}, y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}, x^2\frac{\partial}{\partial x} + 2(m_0+1)xz\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} - (2yz + ax^{m_0+1})\frac{\partial}{\partial z}, x^m\frac{\partial}{\partial z}_{[m\leq m_0]}\}$

$\mathbf{ip}^3_{17,B}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + ax\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + by\frac{\partial}{\partial z}, x^m\frac{\partial}{\partial z}_{[m\leq m_0]}\}$

$\mathbf{ip}^4_{17,B}$: $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}+m_0z\frac{\partial}{\partial z}, y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x}+(2m_0zx+ax^{m_0+1})\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y}, x^m z_{[m\leq m_0]}\}$

$\mathbf{ip}_{18,B}$: $\{\frac{\partial}{\partial y}+cz\frac{\partial}{\partial z}, x\frac{\partial}{\partial y}+bz\frac{\partial}{\partial z}, \Psi_l(x)\frac{\partial}{\partial y}+b_lz\frac{\partial}{\partial z}, y^{u-m_0-m_1-\ldots-m_s-i}\theta_{uj}(x)x^{m_0}\cdot \prod_{l=1}^s \Psi_l(x)^{m_l}\frac{\partial}{\partial z}\}$, $u = n, n-1, \ldots, 1, 0$; $i + m_0 + m_1 + \ldots + m_s \leq u$; $j = 1, 2, \ldots, p_m$

$\mathbf{ip}_{19,B}$: $\{\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}+cz\frac{\partial}{\partial z}, x\frac{\partial}{\partial y}, \Psi_1(x)\frac{\partial}{\partial y}, \ldots, \Psi_s(x)\frac{\partial}{\partial y}, y^{u-m_0-m_1-\ldots-m_s-i}\theta_{uj}(x)x^{m_0}\cdot \prod_{l=1}^s \Psi_l(x)^{m_l}\frac{\partial}{\partial z}\}$

$\mathbf{ip}_{20,B}$: $\{[\frac{\partial}{\partial x}, e^{a_ix}(\frac{\partial}{\partial y} + \pi_{i0}\frac{\partial}{\partial z}), e^{a_ix}(x\frac{\partial}{\partial y} + \pi_{i1}\frac{\partial}{\partial z}), \ldots, e^{a_ix}(x^{s_i}\frac{\partial}{\partial y} + \pi_{is_i}\frac{\partial}{\partial z})]_{[i=1,2,\ldots,q;\ \pi_{10}=0]},$
$x^m y^{n-t} e^{c_{j_{n-t}}x}\frac{\partial}{\partial z}_{[t=0,1,\ldots,n;\ j=1,2,\ldots,q_{n-t};\ m=0,1,\ldots,m_{j_{n-t}}]},$
$x^m y^{n-t-\sum h_l} e^{(c_{j_{n-t}}+\sum h_l a_{i_l})x}\frac{\partial}{\partial z}_{[t=0,1,\ldots,n;\ j=1,2,\ldots,q_{n-t};\ m\geq m_{j_{n-t}}+\sum h_l s_{i_l}]}\}$

$\mathbf{ip}_{21,B}$: $\{\frac{\partial}{\partial x}, y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} + \pi_1\frac{\partial}{\partial z}, \ldots, x^s\frac{\partial}{\partial y} + \pi_s\frac{\partial}{\partial z},$
$e^{a_ix}(\frac{\partial}{\partial y} + \pi_{i0}\frac{\partial}{\partial z}), e^{a_ix}(x\frac{\partial}{\partial y} + \pi_{i1}\frac{\partial}{\partial z}), \ldots, e^{a_ix}(x^{s_i}\frac{\partial}{\partial y} + \pi_{is_i}\frac{\partial}{\partial z}),$
$x^m y^{n-t} e^{c_{j_{n-t}}x}\frac{\partial}{\partial z}_{[t=0,1,\ldots,n;\ j=1,2,\ldots,q_{n-t};\ m=0,1,\ldots,m_{j_{n-t}}]},$
$x^m y^{n-t-\sum h_l-h} e^{(c_{j_{n-t}}+\sum h_l a_{i_l})x}\frac{\partial}{\partial z}$
$_{[t=0,1,\ldots,n;\ j=1,2,\ldots,q_{n-t};\ \text{mass. di } m\geq m_{j_{n-t}}+\sum h_l s_{i_l}+hs]}\}$

The Groups of Type C

$\mathbf{ip}_{1,C}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^s\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$, $j = 0, 1, \ldots, n$; $m \le m_j$; $m_{j+1} \ge m_j + s$

$\mathbf{ip}_{2,C}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^s\frac{\partial}{\partial y}, 2x\frac{\partial}{\partial x} + sy\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + sx\frac{\partial}{\partial y} + (m_0 + sn)xz\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$, $j = 0, 1, \ldots, n$; $m \le m_0 + sj$

$\mathbf{ip}_{3,C}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^{s-1}, x\frac{\partial}{\partial x} + (sy + x^s\frac{\partial}{\partial y}), z\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$, $j = 0, 1, \ldots, n$; $m \le m_j$; $m_{j+1} \ge m_j + s$

$\mathbf{ip}_{4,C}: \{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^s\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, z\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$, $j = 0, 1, \ldots, n$; $m \le m_j$; $m_{j+1} \ge m_j + s$

$\mathbf{ip}_{5,C}: \{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \ldots, x^s\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x} + sxy\frac{\partial}{\partial y} + (m_0 + sn)xz\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, x^m y^{n-j}\frac{\partial}{\partial z}\}$, $j = 0, 1, \ldots, n$; $m \le m_2 + s_j$

$\mathbf{ip}_{6,C}: \{\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + cxz\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, x^m y^{c-m_j}\frac{\partial}{\partial z}\}$, $m \le m_j$; $j = 0, \beta, \ldots, h$; $m_{j+1} > m_j$

$\mathbf{ip}_{7,C}: \{\frac{\partial}{\partial x}, 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}, x^l y^{-n}\frac{\partial}{\partial z}\}$, $l = 0, 1, \ldots, m$; $n = l, l+1, \ldots, m$

$\mathbf{ip}_{8,C}: \{\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + mxz\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, x^l\frac{\partial}{\partial z}\}$, $l = 0, 1, \ldots, m$

$\mathbf{ip}_{9,C}: \{\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}, \Psi_{j_l}(x)y^{n_j - n}e^{c_j y}\frac{\partial}{\partial z}\}$, $j = 1, 2, \ldots, s$; $n = 0, 1, \ldots, n_j$; $l = 0, 1, \ldots, l_n$; $l_0 \le l_1 \le \ldots \le l_{n_j}$

$\mathbf{ip}_{11,C}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, z\frac{\partial}{\partial z}, x^m y^n e^{a_j x + b_j y}\frac{\partial}{\partial z}\}$, $j = 1, 2, \ldots, h$; $m \le m_{ij}$, $n \le n_{ij}$; $i = 1, 2, \ldots, k_j$

$\mathbf{ip}_{12,C}: \{\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}, y^l x^{a_i}(\log^u x)\frac{\partial}{\partial z}\}$, parameters as in $\mathbf{ip}_{12,B}$

$\mathbf{ip}_{13,C}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}, x^m y^n e^{a_i x}\frac{\partial}{\partial z}\}$, $j = 1, 2, \ldots, s$; $m \le m_{ij}$, $n \le n_{ij}$; $j = 1, 2, \ldots, h_i$

$\mathbf{ip}_{14,C}: \{\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, y^2\frac{\partial}{\partial y} + nyz\frac{\partial}{\partial z}, \Psi_j(x)y^i\frac{\partial}{\partial z}\}$, $j = 1, 2, \ldots, l_i$; $i = 0, 1, \ldots, n$; $l_n \le l_{n-1} \le \ldots \le l_0$

$\mathbf{ip}_{15,C}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, y^2\frac{\partial}{\partial y} + nyz\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, y^l x^m e^{a_i x}\}$, $l = 0, 1, \ldots, n$; $i = 1, 2, \ldots, i_l$; $m = 0, 1, \ldots, m_{il}$; $i_n \le i_{n-1} \le \ldots \le i_0$; $m_{in} \le m_{in-1} \le \ldots \le m_{io}$

$\mathbf{ip}_{16,C}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, y^2\frac{\partial}{\partial y} + nyz\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, x^m y^l\frac{\partial}{\partial z}_{[m \le m_0,\, l \le n]}, x^m\frac{\partial}{\partial z}_{[m \le m_1 \ge m_0]}\}$

$\mathbf{ip}_{17,C}: \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + m_0 xz\frac{\partial}{\partial z}, y^2\frac{\partial}{\partial y} + n_0 yz\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, x^m y^n\frac{\partial}{\partial z}_{[m \le m_0,\, n \le n_0]}\}$

$\mathbf{ip}_{18,C}: \{\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, \Psi_1(x)\frac{\partial}{\partial y}, \ldots, \Psi_s(x)\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}, y^{u - m_0 - m_1 - \ldots - m_s - i}\theta_{uj}(x)x^{m_0} \cdot \prod_{l=1}^{s}\Psi_l(x)^{m_l}\frac{\partial}{\partial z}\}$, $u = n, n-1, \ldots, 1, 0$; $i + m_0 + m_1 + \ldots + m_s \le u$; $j = 1, 2, \ldots, q_u$

$$
\begin{aligned}
\mathbf{ip}_{19,C} :\ & \{ \tfrac{\partial}{\partial y}, y\tfrac{\partial}{\partial y}, x\tfrac{\partial}{\partial y}, \Psi_1(x)\tfrac{\partial}{\partial y}, \ldots, \Psi_s(x)\tfrac{\partial}{\partial y}, z\tfrac{\partial}{\partial z}, \\
& y^{u-m_0-m_1-\ldots-m_s-i}\theta_{uj}(x)x^{m_0} \cdot \textstyle\prod_{l=1}^{s} \Psi_l(x)^{m_l}\tfrac{\partial}{\partial z} \}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{ip}_{20,C} :\ & \{ [\tfrac{\partial}{\partial x}, e^{a_i x}\tfrac{\partial}{\partial y}, xe^{a_i x}\tfrac{\partial}{\partial y}, \ldots, x^{s_i}e^{a_i x}\tfrac{\partial}{\partial y}, z\tfrac{\partial}{\partial z}]_{[i=1,2,\ldots,q]}, \\
& x^m y^{n-t} e^{c_{j_{n-t}} x}\tfrac{\partial}{\partial z}{}_{[t=0,1,\ldots,n;\ j=1,2,\ldots,q_{n-t};\ m=0,1,\ldots,m_{j_{n-t}}]}, \\
& x^m y^{n-t-\sum h_l} e^{(c_{j_{n-t}}+\sum h_l a_{i_l})x}\tfrac{\partial}{\partial z}{}_{[t=0,1,\ldots,n;\ j=1,2,\ldots,q_{n-t};\ m\geq m_{j_{n-t}}+\sum h_l s_{i_l}]} \}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{ip}_{21,C} :\ & \{ \tfrac{\partial}{\partial x}, y\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}, x\tfrac{\partial}{\partial y}, \ldots, x^s\tfrac{\partial}{\partial y}, \\
& e^{a_i x}\tfrac{\partial}{\partial y}, xe^{a_i x}\tfrac{\partial}{\partial y}, \ldots, x^{s_i}e^{a_i x}\tfrac{\partial}{\partial y}, z\tfrac{\partial}{\partial z}, x^m y^{n-t} e^{c_{j_{n-t}} x}\tfrac{\partial}{\partial z} \\
& {}_{[t=0,1,\ldots,n;\ j=1,2,\ldots,q_{n-t};\ m=0,1,\ldots,m_{j_{n-t}}]}, \\
& x^m y^{n-t-\sum h_l-h} e^{(c_{j_{n-t}}+\sum h_l a_{i_l})x}\tfrac{\partial}{\partial z} \\
& {}_{[t=0,1,\ldots,n;\ j=1,2,\ldots,q_{n-t};\ \text{mass. di}\ m\geq m_{j_{n-t}}+\sum h_l s_{i_l} h+?s]} \}
\end{aligned}
$$

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