Holonomic Functions in Computer Algebra

Slides of a Series of Talks

March 1999

Frédéric Chyzak

Frederic.Chyzak@inria.fr
http://algo.inria.fr/chyzak/

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Preface

The present document collects the slides from a series of talks given in Winter 1999 at the *Combinatorics Seminar* held by Peter Paule in RISC-Linz (Austria). The series is mainly devoted to a presentation of the doctoral thesis which I prepared in the *Algorithms Project*, INRIA-Rocquencourt (France), and with which the series shares its title. Its main goal is to present algorithms for symbolic summation and integration, and also for other algorithmic manipulations of special functions and combinatorial sequences. But it also provides the detailed background on holonomy required in algorithmic applications. The material consists in an assembled, more mature, and polished version of the talks I have given during the years 1996 to 1998. It is structured in the following way:

1. Motivation;
2. Implicit Representations of Special Functions and Sequences;
3. Gröbner Bases in Algebras of Linear Operators;
4. Holonomy and Holonomic Functions;
5. Holonomic Summation and Integration.

The majority of slides are printed on the left of a sheet, with comments on the right. These comments were primarily meant as a supporting skeleton for my discourse; some of them may be in a less polished form. However I left them since they also contain details, examples, questions, open problems, etc, which I hope will interest some of the readers.

I thank Peter Paule for accepting to devote roughly a third of a semester in his weekly seminar to the presentation of my work, and to give me the opportunity to think about these topics in great details again. I also thank the audience of the *Combinatorics Seminar* for their many questions and stimulating remarks during the talks.

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March 1999,
Frédéric Chyzak

Outline of the Series

This series splits into the following five topics:

1. Implicit Representations of Special Functions and Sequences;

   Chapters 1, 3, 5, and 6 in primer, Introduction and Chapters 1, 3, and 7 in thesis.

2. Gröbner Bases in Algebras of Linear Operators;

   Chapters 2, 7, 8, and 9 in primer, Chapters 2 and 3 in thesis.

3. Holonomy;

   Chapters 10 and 11 in primer, Chapters 4 and 8 in thesis.

4. Holonomic Summation and Integration;

   Chapter 20 in primer, Chapters 6 and 7 in thesis.

5. Kashiwara's Theorem and Applications.

   Chapters 4, 12 to 18, and 20 in primer, Chapter 8 in thesis.

Chapters refer to:

– F. Chyzak, *Fonctions holonomes en calcul formel*, PhD thesis (1998).

– S. C. Coutinho, *A Primer of Algebraic D-modules* (No. 33 in London Mathematical Society Student Texts, Cambridge University Press, 1995).

Holonomic Functions in Computer Algebra

Part 0

Motivation

Motivation

1. Motivation of the whole series through examples of algebraic manipulation of ∂-finite functions.

Automatic Calculation of Identities

$$\sum_{k=1}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right),$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3,$$

$$\sum_{k=0}^{n} \left(\sum_{j=0}^{k} \binom{n}{j} \right)^3 = n 2^{3n-1} + 2^{3n} - 3n 2^{n-2} \binom{2n}{n},$$

$$\sum_{n=0}^{+\infty} P_n(x) y^n = \frac{1}{\sqrt{1 - 2xy + y^2}},$$

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy - u(x^2+y^2))}{1-4u^2} \right)}{\sqrt{1-u^2}},$$

$$\frac{\pi}{2} J_0(z) = \int_0^1 \frac{\cos(zu)}{\sqrt{1-u^2}}\, du = \int_1^{+\infty} \frac{\sin(zu)}{\sqrt{u^2-1}}\, du,$$

$$\int_{-1}^{+1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}}\, dx = (-1)^n \pi I_n(p),$$

$$\int_0^{+\infty} x e^{-px^2} J_n(bx) I_n(cx)\, dx = \frac{1}{2p} \exp\left(\frac{c^2 - b^2}{4p} \right) J_n\left(\frac{bc}{2p} \right),$$

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2},$$

$$\oint_0 \frac{(1 + 2xy + 4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2} \right)}{y^{n+1} (1+4y^2)^{\frac{3}{2}}}\, dy = \frac{n! H_n(x)}{\lfloor n/2 \rfloor !},$$

$$\int_0^{\infty} \int_0^{\infty} e^{-x} e^{-y} J_1(x) J_1(y) J_2(c\sqrt{xy})\, dx\, dy = \frac{1}{2\pi c k} [(2-k^2) \boldsymbol{K}(k) - 2\boldsymbol{E}(k)],$$

$$\sum_{-\infty}^{+\infty} \frac{(a;q)_n x^n}{(b;q)_n} = \frac{(ax;q)_\infty \left(\frac{q}{ax}; q \right)_\infty (q;q)_\infty \left(\frac{b}{a}; q \right)_\infty}{(x;q)_\infty \left(\frac{b}{ax}; q \right)_\infty (b;q)_\infty \left(\frac{q}{a}; q \right)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{r_n(a,b;q) z^n}{(q;q)_n} = \frac{1}{(az;q)_\infty (bz;q)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{r_n(a,b;q) r_n(c,d;q) z^n}{(q;q)_n} = \frac{(acz;q)_\infty (adz;q)_\infty (bcz;q)_\infty (bdz;q)_\infty}{(abcdz^2;q)_\infty},$$

$$\sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{q^{(i+j)^2 + j^2}}{(q;q)_{n-i-j} (q;q)_i (q;q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2 + 1/2k}}{(q;q)_{n+k} (q;q)_{n-k}}.$$

Based on an effective theory of holonomy.

Automatic Calculation of Identities

1. Holonomy
   - large variety of objects, computable class
   - achieved goal: cover the class by my theory and the corresponding programmes
2. Decidability of the identities
   - simplifications of expressions in terms of special functions by computer algebra
   - parameterized integrations and summations
3. Sources for holonomic problems
   - combinatorics (problems of recursive nature and with ordering constraints)
   - analysis of algorithms
   - physical problems with cylindrical/spherical symmetry
   - statistical physics, q-calculus, divide and conquer
4. Thanks to implementation
   - each identity in a matter of minutes of calculation
   - recover recent research results

Holonomy for Special Functions and Sequences in Computer Algebra

Holonomic functions: derivatives live in a finite dimensional vector space.
Holonomic sequences: Taylor coefficients of holonomic functions.
$\Longrightarrow$ cancelled by linear PDE's and recurrences.

Class of ubiquitous functions and sequences:

Algebraic functions, exp, ln, sin, cos, sh, ch, Arcsin, Arccos, Arcsh, Arcch; $n!$, $\binom{n}{m}$, $\binom{n}{m_1,\dots,m_r}$, H_n, $H_n^{(p)}$, hypergeometric sequences; erf, erfc, Ci, Si; P_ν^μ, J_ν, Y_ν, I_ν, K_ν, $\boldsymbol{H}_\nu$, Ki; ${}_2F_1(a,b,c;z)$, ${}_pF_q((a_i),(b_i);z)$, $F_4(a,b,c,d;x,y)$, $M(a,b,z)$, $W_{\kappa,\mu}(z)$; $P_n^{(\alpha,\beta)}(x)$, $C_n^{(\alpha)}(x)$, $T_n(x)$, $U_n(x)$, $P_n(x)$, $L_n^{(\alpha)}(x)$, $L_n(x)$, $H_n(x)$, $r_n(q;a,b)$.

Algorithms to compute with **systems** of linear functional equations:

$+$, $\times$, ∂ ... ; simplification; definite and indefinite $\int$ and $\sum$.

New type of computation with implicit forms.

Unified theoretic **framework**, unified algorithms, unified implementation.

First full implementation (Mgfun in Maple).

Holonomy for Special Functions and Sequences in Computer Algebra

1. Holonomy = finitely specifiable $\Rightarrow$ computer algebra
   - description by
     * finite number of linear equations
     * finite numer of initial conditions
   - computations in finite dimensional vector spaces
2. Ubiquitous nature
   - 70% of sequences and functions from the handbook by Abramovitz and Stegun
   - half from Erdélyi's *Bateman Project*
   - 1/4 of sequences from the *Encyclopedia of Integer Sequences* by Sloane and Plouffe
3. Computation without closed forms (at least during intermediate calculations)
   - explicit forms (= closed f.)/implicits ones (systems)
   - CAS: failure of the methods that restrict to closed forms
   - my thesis: general method to compute even in the absence of closed forms
   - representation by systems
4. Unifying viewpoint
   - differential + recurrences
   - use of polynomial elimination
   - full implementation

Computations of Integrals and Sums

Wave equation: model that describes the vibration of a drum membrane:

$$\int_{-1}^{+1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}}\, dx \quad \longrightarrow \quad \pi(-1)^n I_n(p) \qquad (5\ sec).$$

Theory of special functions: Bessel functions integrals [M. L. Glasser and E. Montaldi (1994)]:

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx \quad \longrightarrow \quad -\frac{\ln(1-a^4)}{2\pi a^2} \qquad (75\ sec).$$

$$a^3(a^4-1)\phi^{(4)}(a) + 4a^2(4a^4-1)\phi^{(3)}(a) + a(73a^4+3)\phi''(a) \\ +(103a^4-3)\phi'(a) + 32a^3\phi(a) = 0.$$

Denumeration problems, combinatorial sums: generating functions of a product of Hermite functions:

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{u^n}{n!} \quad \longrightarrow \quad \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}} \qquad (30\ sec).$$

Probability computation: the expectance of the maximum of three Gaussian independent random variables is the limit of [N. J. Calkin (1994)]:

$$\sum_{k=0}^{n} \left(\sum_{j=0}^{k} \binom{n}{j} \right)^3 \quad \longrightarrow \quad \frac{n}{2} 8^n + 8^n - \frac{3n}{4} 2^n \binom{2n}{n} \qquad (195\ sec).$$

$$(n+1)u_{n+2} - (7n+12)u_{n+1} - 4(2n+1)u_n = (18n-20)8^n.$$

Computations of Integrals and Sums

1. Pre- and postprocessing pass from closed form to implicit form and conversely.
2. Variety of applications (maths, mathematical physics, ingineering sciences)
3. Computation of parameterized integrals and sums
   - ≥ 2 variables $\Rightarrow$ 1 linear functional equation $\Rightarrow$ 1 variable
   - my contribution: unified algorithmization and a palet of algorithms to get the linear functional equation
   - when no closed form available: possible following up of the calculations
4. Guaranteed general algorithms
   - recent research results due to specialists, experts $\Rightarrow$ short computer computations
   - first automatic calculations

Automatic Proof of Identities

Theory of special functions:

$$\int_0^\infty \int_0^\infty e^{-x} e^{-y} J_1(x) J_1(y) J_2(c\sqrt{xy})\, dx dy = \frac{1}{2\pi ck}[(2-k^2)\boldsymbol{K}(k) - 2\boldsymbol{E}(k)] \qquad (50\ sec),$$

where $k = 4c/(c^2+8)$ and $\boldsymbol{K}$ and $\boldsymbol{E}$ are elliptic functions.

Partition theory, hard hexagon models, statistical physics:

[Rogers–Ramanujan] $$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \qquad (80\ sec),$$

[Gordon] $$\sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i (q;q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \qquad (120\ sec),$$

where $(a;q)_n = (1-a)\cdots(1-aq^{n-1})$.

Number theory: Th. [Apéry (1978)]: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \notin \mathbb{Q}$.

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \qquad (60\ sec).$$

$$(n+2)^3 u_{n+2} - \left((n+2)^3 + (n+1)^3 + 4(2n+3)^3\right) u_{n+1} + (n+1)^3 u_n = 0.$$

Automatic Proof of Identites

1. Pre- and postprocessing pass from closed form to implicit form and conversely.
2. Variety of applications (maths, mathematical physics, ingineering sciences)
3. Proof of identities
   - absence of closed forms or resolution not possible
   - comparing/matching descriptions ⇒ zero test
4. Guaranteed general algorithms
   - Gordon: first automatic proof
     ⇒ solution in 2 min to a problem suggested by Knuth
   - Apéry: 1 min compared to several months of failure by Cohen + Van der Poorten

A Few Typical Basic Problems

PROBLEM: "Simplify"

$$
\begin{aligned}
y_{m,n}(x) &= 2T_m(x)T_n(x) - T_{n+m}(x) - T_{n-m}(x) \qquad (n \geq m),\\
z_n(a,b) &= p_{n+1} - p_{n-1} + \frac{2n}{a}\frac{\partial p_n}{\partial b} + \frac{2n}{b}\frac{\partial p_n}{\partial a},
\end{aligned}
$$

where $T_n(x)$ is the n-th Chebychev orthogonal polynomial and

$$p_n(a,b) = J_n(a)Y_n(b) - J_n(b)Y_n(a)$$

in terms of the Bessel functions of 1st and 2nd kinds and order n.
SOLUTION: $y_{m,n}(x) = z_n(a,b) = 0$.

PROBLEM: Rewrite the following expression which involves the Gauss hypergeometric function ${}_2F_1\left(\begin{smallmatrix}\alpha,\beta\\ \gamma\end{smallmatrix}\middle| u\right)$:

$$\frac{4^a}{\left(1+\sqrt{1-z}\right)^{2a}}\,{}_2F_1\left(\begin{matrix}2a, a-b+\frac{1}{2}\\ a+b+\frac{1}{2}\end{matrix}\middle|\frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}\right).$$

SOLUTION:

$${}_2F_1\left(\begin{matrix}2a, 2b\\ a+b+\frac{1}{2}\end{matrix}\middle|\frac{1-\sqrt{1-z}}{2}\right) = {}_2F_1\left(\begin{matrix}a, b\\ a+b+\frac{1}{2}\end{matrix}\middle| z\right).$$

A Few Typical Basic Problems

(= side-products of the method)

1. Maple: code for Wronskians in this particular case;
   algorithm for normal forms in a general class;
2. Maple: no way;
   algorithm for algebraic substitution in a certain class.

PROBLEM: Compute a simple particular solution of the LODE in F

$$F''(x) + \frac{F'(x)}{x} - q^2 F(x) = (1 - x^2) J_0(qx) - 2q^2 x J_1(qx).$$

SOLUTION:

$$F(x) = \frac{2q - 1 + x^2}{2q^2} J_0(qx) + \frac{(q^3 - 1)x}{q^3} J_1(qx).$$

PROBLEM: Compute the indefinite integral

$$\int e^{it} t^n J_n(t)\, dt.$$

SOLUTION: An expression in terms of the integrand is

$$\frac{e^{it} t^{n+1}}{2n+1} [J_n(t) - i J_{n+1}(t)].$$

PROBLEM: Compute the indefinite sum

$$\sum_n \frac{(-1)^n H_n}{\binom{m}{n}} \qquad \text{for} \qquad H_n = \sum_{k=1}^n \frac{1}{k}$$

(harmonic numbers).

SOLUTION: An expression in terms of the summand is

$$\frac{(-1)^n [(m+2) H_n - 1](n - 1 - m)}{\binom{m}{n} (m+2)^2}.$$

(strangely enough, same kind of problems)

1. Maple: ugly result that can no longer be manipulated;
   algorithm for searching for particular solutions in a general class;
2. Maple: fails even if n is given an integer value;
   algorithm for indefinite integration in a general class;
3. Maple: outside the scope of Gosper's algorithm;
   algorithm for indefinite summation in a general class.

PROBLEM: Compute the definite integral

$$\int_0^{+\infty} x e^{-px^2} J_n(bx) I_n(cx)\, dx$$

for the Bessel and modified Bessel functions of the 1st kind and order n.

SOLUTION:

$$\frac{1}{2p} \exp\left(\frac{c^2 - b^2}{4p}\right) J_n\left(\frac{bc}{2p}\right).$$

PROBLEM: Compute the definite sums

$$2\sum_{n=0}^{\infty} J_n(z)^2 \qquad \text{and} \qquad \sum_{-\infty}^{+\infty} \frac{(a;q)_n x^n}{(b;q)_n}$$

where $(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$ (q-Pochhammer symbols).

SOLUTION:

$$1 - J_0(z)^2 \qquad \text{and} \qquad \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty \left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty \left(\frac{b}{ax};q\right)_\infty (b;q)_\infty \left(\frac{q}{a};q\right)_\infty}.$$

1. Maple: fails because of parameters;

   algorithm for definite integration in a general class;

2. Maple: outside the scope of Gosper's algorithm;

   algorithm for definite summation in a general class; classical and q-calculus cases.

PROBLEM: Compute the generating series

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!}$$

where $H_n(x)$ is the n-th Hermite orthogonal polynomial.

SOLUTION:

$$\frac{\exp\left(\frac{4u[xy-u(x^2+y^2)]}{1-4u^2}\right)}{\sqrt{1-4u^2}}.$$

PROBLEM: Find a recurrence on the coefficients of the Chebychev series

$$x\exp x = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n T_n(x)$$

of $x\exp x$, where

$$c_n = \int_{-1}^{+1} \frac{x\exp x T_n(x)}{\sqrt{1-x^2}}\,dx.$$

SOLUTION:

$$c_{n+4} + 2(n+4)c_{n+3} + 2nc_{n+1} - c_n = 0.$$

1. Maple: fails because of parameters;
   application of summation algorithm;
2. Maple: fails because there is no closed form;
   application of integration algorithm.

PROBLEM: Compute a homogeneous LODE satisfied by

$$f(t) = \int_0^{+\infty} e^{-tz} \sin \frac{z}{1+z^2}\, dz.$$

SOLUTION:

$$\begin{aligned} &t^2 f^{(10)}(t) + 18t f^{(9)}(t) + 3(24+t^2) f^{(8)}(t) + 48t f^{(7)}(t) \\ &+(2t^2+169) f^{(6)}(t) + 36t f^{(5)}(t) + (117-2t^2) f^{(4)}(t) \\ &-3(t^2-9) f''(t) - 6t f'(t) - (t^2+1) f(t) + 1 = 0. \end{aligned}$$

PROBLEM: Prove that the double sum

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{n+r+s} \binom{n}{r} \binom{n}{s} \binom{n+r}{r} \binom{n+s}{s} \binom{2n-(r+s)}{n}$$

rewrites into the single summation

$$\sum_{k=0}^{\infty} \binom{n}{k}^4.$$

1. Maple: fails because there is no closed form; previous examples were cases of solvable LODE;
2. Maple: fails because there is no closed form; multivariate extension.

Apéry Numbers

Apéry numbers,

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

and Franel numbers,

$$\sum_{k=0}^{n} \binom{n}{k}^3,$$

satisfy the identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3.$$

First values are common:

$$1, 5, 73, 1445, 33001, 819005, 21460825, 584307365, 16367912425, \dots$$

Both side satisfy a common equation:

$$(n+2)^3 u_{n+2} - \left((n+2)^3 + (n+1)^3 + 4(2n+3)^3\right) u_{n+1} + (n+1)^3 u_n = 0.$$

Apéry Numbers

1. Declare $\binom{n}{k}^3$ as hypergeometric (two first-order equations).
2. Summation (creative telescoping) yields
$$(n+2)^2 F_{n+2} = (7n^2 + 21n + 16) F_{n+1} + 8(n+1)^2 F_n.$$
3. Declare $s_{n,k} = \binom{n}{k}\binom{n+k}{k}$ as hypergeometric:
$$\begin{aligned}(n+1-k) s_{n+1,k} &= (n+1+k) s_{n,k},\\ (k+1)^2 s_{n,k+1} &= (n(n+1) - k(k+1)) s_{n,k}.\end{aligned}$$
4. Product with F_n:
$$\begin{aligned}(n+1-k) p_{n+1,k} &= (n+1+k) p_{n,k},\\ (k+2)^4 p_{n,k+2} + 8(n+k+2)(n+k+1)(n-k) p_{n,k} &\\ = (7k^2 + 21k + 16)(n+k+2)(n-k-1) p_{n,k+1}.&\end{aligned}$$
5. Summation (creative telescoping) to get the right-hand side yields the announced equation.
6. Declare $t_{n,k} = \binom{n}{k}^2\binom{n+k}{k}^2$ as hypergeometric:
$$\begin{aligned}(n+1-k)^2 t_{n+1,k} &= (n+1+k)^2 t_{n,k},\\ (k+1)^4 t_{n,k+1} &= (n(n+1) - k(k+1))^2 t_{n,k}.\end{aligned}$$
7. Summation (creative telescoping) to get the left-hand side yields the announced equation.
8. Comparison of initial conditions:
$$A_0 = B_0 = 1, \qquad A_1 = B_1 = 5.$$
9. Faster comment on $\int J_1 I_1 Y_0 K_0$, with equation for Y_0 and resolution for the log.

Topics of the Series

1. manipulation of implicit representation of functions and sequences;
2. summation and integration of holonomic ∂-finite functions;
3. emphasis on $\mathcal{D}$-modules.

Sources

- F. Chyzak, *Fonctions holonomes en calcul formel*, PhD thesis (1998);
- S. C. Coutinho, *A Primer of Algebraic D-modules* (No. 33 in London Mathematical Society Student Texts, Cambridge University Press, 1995);
- the numerous references to be found in these books.

Content

1. ∂-Finiteness and Other Implicit Representations;
2. Gröbner Bases in Ore Algebras;
3. Holonomy;
4. Summation and Integration of Holonomic ∂-Finite Functions;
5. Kashiwara's Theorem and Applications.

Topics of the Series, Sources, Content

1. Method and tools
   - implicit representation by systems of linear functional equations
   - translation of linear operators in terms of "skew" (= non-commutative) polynomials
   - extension and application of methods of Gröbner bases
   - simplification, $+$, and $\times$: like "non-commutative algebraic numbers"
   - $\int$, $\sum$: non-commutative polynomial elimination
   - theory of holonomy
2. Last chapter of Coutinho's deals with *Automatic Proof of Identities*: his understanding is "strange" for combinatorialists, especially that of products
3. Applications of Kashiwara's theorem
   - multiplication of functions
   - enter $\mathcal{D}$-modules before further study

Holonomic Functions in Computer Algebra

Part I

Implicit Representations of Special Functions and Sequences

Implicit Representations of Special Functions and Sequences

1. Operator algebras; skew polynomial rings; "commutation algebras"; Ore algebras;
2. Annihilating ideals of a function with respect to an Ore algebra;
3. ∂-Finiteness;
4. ∂-Finite closures in a non-algorithmic way.

Linear Differential Operators

Functional notation: (Leibniz rule)

$$(xf(x))' = xf'(x) + f(x),$$
$$(P(x)f(x))' = P(x)f'(x) + P'(x)f(x).$$

Operator notation: (commutation rule)

$$D = \text{differentiation operator with respect to } x,$$
$$X = \text{operator of multiplication by } x,$$
$$DX = XD + 1,$$
$$DP = PD + P'.$$

Action on functions:

$$1 \cdot f = f, \qquad D \cdot f = f', \qquad D^2 \cdot f = f'', \qquad \dots$$
$$P(X)D^r \cdot f = P(x)f^{(r)}.$$

Multivariate case: here for $f(x_1, x_2)$,

$$1 \cdot f = f,$$
$$D_1 \cdot f = \frac{\partial f}{\partial x_1}, \qquad D_2 \cdot f = \frac{\partial f}{\partial x_2},$$
$$D_1^2 \cdot f = \frac{\partial^2 f}{\partial x_1^2}, \qquad D_1 D_2 \cdot f = \frac{\partial^2 f}{\partial x_1 \partial x_2}, \qquad D_2^2 \cdot f = \frac{\partial^2 f}{\partial x_2^2},$$
$$\vdots$$
$$P(X_1, X_2)D_1^r D_2^s \cdot f = P(x_1, x_2)\frac{\partial^{r+s} f}{\partial x_1^r \partial x_2^s}.$$

Linear Differential Operators

1. simple other description: $\mathbb{K}\langle X, D; DX = XD + 1\rangle$
2. possible extension
   - rational functions
   - formal power series
   - convergent series
   - Laurent series
   - analytic functions
   - meromorphic functions
   - any differential ring

Linear Difference Operators

Functional notation: (ring morphism)

$$[x \to x+1]\,(xf(x)) = (x+1)f(x+1),$$
$$[x \to x+1]\,(P(x)f(x)) = P(x+1)f(x+1).$$

Operator notation: (commutation rule)

$$S = \text{shift operator with respect to } x,$$
$$X = \text{operator of multiplication by } x,$$
$$SX = (X+1)S,$$
$$SP(X) = P(X+1)S.$$

Action on functions:

$$(1\cdot f)(x) = f(x), \quad (S\cdot f)(x) = f(x+1), \quad (S^2\cdot f)(x) = f(x+2), \quad \dots$$
$$(P(X)S^r\cdot f)(x) = P(x)f(x+r).$$

Multivariate case: here for $f(x_1,x_2)$,

$$(1\cdot f)(x_1,x_2) = f(x_1,x_2),$$
$$(S_1\cdot f)(x_1,x_2) = f(x_1+1,x_2), \quad (S_2\cdot f)(x_1,x_2) = f(x_1,x_2+1),$$
$$(S_1^2\cdot f)(x_1,x_2) = f(x_1+2,x_2), \quad (S_1S_2\cdot f)(x_1,x_2) = f(x_1+1,x_2+1),$$
$$(S_2^2\cdot f)(x_1,x_2) = f(x_1,x_2+2), \qquad \dots$$
$$(P(X_1,X_2)S_1^rS_2^s\cdot f)(x_1,x_2) = P(x_1,x_2)f(x_1+r,x_2+s).$$

Linear Difference Operators

1. difference vs shift
2. simple other description: $\mathbb{K}\langle X,S; SX = XS+S\rangle$
3. possible extension
   - rational functions
   - formal power series
   - Laurent series
   - analytic functions
   - meromorphic functions

Linear q-Difference Operators

Functional notation: (ring morphism)

$$[x \to qx]\,(xf(x)) = qxf(qx),$$
$$[x \to qx]\,(P(x)f(x)) = P(qx)f(qx).$$

Operator notation: (commutation rule)

$$H = \text{operator of } q\text{-dilation with respect to } x,$$
$$X = \text{operator of multiplication by } x,$$
$$HX = qXH,$$
$$HP(X) = P(qX)H.$$

Action on functions:

$$(1 \cdot f)(x) = f(x), \quad (H \cdot f)(x) = f(qx), \quad (H^2 \cdot f)(x) = f(q^2x), \quad \dots$$
$$(P(X)H^r \cdot f)(x) = P(x)f(q^r x).$$

Multivariate case: here for $f(x_1, x_2)$,

$$(1 \cdot f)(x_1, x_2) = f(x_1, x_2),$$
$$(H_1 \cdot f)(x_1, x_2) = f(qx_1, x_2), \quad (H_2 \cdot f)(x_1, x_2) = f(x_1, qx_2),$$
$$(H_1^2 \cdot f)(x_1, x_2) = f(q^2x_1, x_2), \quad (H_1H_2 \cdot f)(x_1, x_2) = f(qx_1, qx_2),$$
$$(H_2^2 \cdot f)(x_1, x_2) = f(x_1, q^2x_2), \qquad \dots$$
$$(P(X_1, X_2)H_1^r H_2^s \cdot f)(x_1, x_2) = P(x_1, x_2)f(q^r x_1, q^s x_2).$$

Linear q-Difference Operators

1. q-difference vs q-dilation
2. simple other description: $\mathbb{K}\langle X, H; HX = qXH\rangle$
3. possible extension
   - rational functions
   - formal power series
   - Laurent series
   - analytic functions
   - meromorphic functions

Mixed Cases as Well!

The Chebychev polynomials

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (n-k-1)!}{k!(n-2k)!} (2x)^{n-2k}$$

satisfy the equations

$$\begin{gathered}(1-x^2){T_n}''(x) - x{T_n}'(x) + n^2 T_n(x) = 0,\\ T_{n+2}(x) - 2xT_{n+1}(x) + T_n(x) = 0,\\ (1-x^2){T_{n+1}}'(x) - (n+1)xT_{n+1}(x) - (n+1)T_n(x) = 0.\end{gathered}$$

Equivalently, they are cancelled by the following operators:

$$\begin{gathered}(1-x^2)D_x^2 - xD_x + n^2,\\ S_n^2 - 2xS_n + 1,\\ (1-x^2)D_x S_n - (n+1)xS_n - (n+1).\end{gathered}$$

Mixed differential difference framework!

$$\begin{gathered}(1 \cdot f)(n,x) = f(n,x),\\ (S_n \cdot f)(n,x) = f(n+1,x), \quad (D_x \cdot f)(n,x) = \frac{\partial f}{\partial x}(n,x),\\ (S_n^2 \cdot f)(n,x) = f(n+2,x), \quad (S_n D_x \cdot f)(n,x) = \frac{\partial f}{\partial x}(n+1,x),\\ (D_x^2 \cdot f)(n,x) = \frac{\partial^2 f}{\partial x^2} f(n,x), \qquad \dots\\ (P(X_1,X_2)S_n^r D_x^s \cdot f)(n,x) = P(n,x)\frac{\partial^s f}{\partial x^s}(n+r,x).\end{gathered}$$

Mixed Cases as Well!

1. similar with any other family of orthogonal polynomials
2. Abandon closed forms
   - replaced by linear operators
   - possibility of mixed framework
3. Polynomial notation
   - in view of calculations by Gröbner bases
   - non-commutativity

Skew Polynomial Rings

General framework of skew polynomial rings $\mathbb{A}[\partial;\sigma,\delta]$:

$$\partial a = a^\sigma \partial + a^\delta \qquad \textit{[Ore]}.$$

$$a^\sigma = \sigma(a) = \text{conjugate of } a; \qquad a^\delta = \delta(a) = \sigma\text{-derivative of } a.$$

σ: ring endomorphism of $\mathbb{A}$; δ: σ-derivation of $\mathbb{A}$.

$$(fg)^\delta = f^\sigma g^\delta + f^\delta g \qquad \text{(skew Leibniz rule)}.$$

Special cases:

1. $\sigma = 1$ and $\delta = 0$: **commutative** polynomial rings;
2. $\sigma = 1$ and $\delta = '$: usual **derivation** rings; (usual) Leibniz rule;
3. $\delta = 0$ or $\delta = \sigma - 1$: usual **difference** rings; ring morphism.

Properties: ($\mathbb{A}$ a field)

- injective $\sigma \Rightarrow$ degree, right euclidean, left gcd;
- bijective $\sigma \Rightarrow$ degree, left and right euclidean, left and right gcd;
- left Ore condition, left fraction field.

Skew Polynomial Rings

1. indeterminate does not commute with coefficients
2. Ore's goals:
   - linear operators
   - fraction field with application to linear algebra
3. coefficients in a general ring of coefficients
4. general commutations
5. great deal of linear operators
6. no example of euclidean algorithm before two slides

Ore Algebras

Multivariate framework for several indeterminates ∂_i's $\mathbb{A}[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}]$:

$$\partial_j\partial_i = \partial_i\partial_j, \qquad \sigma_j\delta_i = \delta_i\sigma_j \qquad (i \neq j).$$

derivation	$f(x) \mapsto f'(x)$	$D_x x = x D_x + 1$
shift	$f(x) \mapsto f(x+1)$	$S_x x = (x+1) S_x$
difference	$f(x) \mapsto f(x+1) - f(x)$	$\Delta_x x = (x+1)\Delta_x + 1$
q-dilation	$f(x) \mapsto f(qx)$	$H_x^{(q)} x = qx H_x^{(q)}$
q-difference	$f(x) \mapsto f(qx) - f(x)$	$\Delta_x^{(q)} x = qx\Delta_x^{(q)} + (q-1)x$
q-derivation	$f(x) \mapsto \dfrac{f(qx)-f(x)}{x(q-1)}$	$D_x^{(q)} x = qx D_x^{(q)} + 1$
q-shift	$f(x) \mapsto f(x+1)$	$S_x q^x = q q^x S_x$
q-difference $(*)$	$f(x) \mapsto f(x+1) - f(x)$	$\Delta_x^{(q)} x = q q^x \Delta_x^{(q)} + (q-1) q^x$
Eulerian operator	$f(x) \mapsto x f'(x)$	$\Theta_x x = x\Theta_x + x$
derivation $(*)$	$f(x) \mapsto f'(x)$	$D_x e^x = e^x D_x + e^x$
Mahlerian operators	$f(x) \mapsto f(x^p)$	$M_x^{(p)} x = x^p M_x^{(p)}$
Frobenius operators	$f(x) \mapsto f(x)^p$	$F_x^{(p)} x = x^p F_x^{(p)}$
divided differences	$f(x) \mapsto \dfrac{f(x)-f(a)}{x-a}$	$\Delta_x^{(a)} x = a\Delta_x^{(a)} + 1$

q-shift $(*)$	$S_n^{(q)} n^\alpha (q^n)^\beta = q^\beta (n+1)^\alpha (q^n)^\beta S_n^{(q)}$
derivation $(**)$	$\mathbb{K}(x, e^x, e^{x^2}, e^{e^x}, \ln x, \tan x)[D_x; 1, D_x]$

Effective framework of Ore algebras $\mathbb{K}(\mathbf{x})[\mathbf{y}][\partial;\sigma,\delta]$:

$$\partial x = (qx+p)\partial + (sx+r) \qquad (q \neq 0).$$

Ore Algebras

1. Effective framework of algebras of linear operators
   - in view of calculations by Gröbner bases
   - description by commutations
   - independence from $\boldsymbol{\partial}$
   - the $\mathbf{x}$ and $\mathbf{y}$ and in fact the same variables, but viewed differently in different contexts
2. $\mathbb{K}$ commutative: algebras described by generators and relations

$$\mathbb{K}\langle x, \partial; \partial x = (qx+p)\partial + (sx+r)\rangle$$

Contiguity Relations

> Compute a contiguity relation for
>
> $$f_n(z) = \sum_{k=0}^{\infty} \binom{n}{k}^2 \binom{n+k}{k}^2 (-1)^k z^k,$$
>
> i.e. an equation of the form
>
> $$f_{n+1}(z) = \sum_{i=0}^{L} a_i(z) f_n^{(i)}(z).$$

¿From action of S_k on summand, $M(n,z,D_z) \cdot f = 0$ for

$$M = z^3(z+1)D_z^4 + 2z^2(4z+3)D_z^3 - z(-14z+2zn+2zn^2-7)D_z^2 \\ +(-4zn^2+4z-4zn+1)D_z + (n+1)^2n^2.$$

¿From action of S_n on summand, $Q(n,z,D_z)S_n \cdot f = P(n,z,D_z) \cdot f$ for

$$P = z^2D_z^2 + (2n+3)zD_z + (n+1)^2,$$
$$Q = z^2D_z^2 - (2n+1)zD_z + (n+1)^2.$$

By extended skew gcd algorithm, compute Bézout relation:

$$U(n,z,D_z)Q(n,z,D_z) + V(n,z,D_z)M(n+1,z,D_z) = 1.$$

Now, $S_n = UP \mod M$, whence the contiguity relation

$$S_n = 12\frac{z^3(z+1)}{(n+1)^3}D_z^3 + 4\frac{z^2(2n+2zn+11+14z)}{(n+1)^3}D_z^2 \\ -4\frac{z(5zn^2-n^2-4n-6+2zn-9z)}{(n+1)^3}D_z - \frac{16zn-n-1+4z}{n+1}.$$

Contiguity Relations

1. Ideas borrowed from Takayama (but avoiding Gröbner bases)
2. Action of S_k: either gfun or

$$\sum_{k=0}^{\infty} u_{n,k}z^k = \sum_{k=-1}^{\infty} u_{n,k+1}z^{k+1} = \\ u_{n,0} - z\sum_{k=0}^{\infty} \frac{(n-k)^2(n+k+1)^2}{(k+1)^4}u_{n,k}z^k.$$

Applying $(zD_z)^4$:

$$(zD_z)^4 \cdot f = -z\sum_{k=0}^{\infty}(n-k)^2(n+k+1)^2u_{n,k}z^k \\ = -z(zD_z-n)^2(zD_z+n+1)^2 \cdot f.$$

3. Action of S_n:

$$S_n \cdot f = \sum_{k=0}^{\infty} u_{n+1,k}z^k = \sum_{k=0}^{\infty}\left(\frac{n+k+1}{n-k+1}\right)^2 u_{n,k}z^k,$$

then same method (simpler: no constant).

4. Explain shift in Bézout relation, then perform the first few steps of gcd algorithm.
5. Iteration to express the S_n^i in terms of the D_z^i, next Gaussian elimination, yields pure recurrence in n (order 4, coeffs are polynomials of degree at most 10 in n and z).

Annihilating Ideals

"Function space": a $\mathbb{K}$-algebra $\mathcal{F}$ which is a left $\mathbb{O}$-module for the Ore algebra $\mathbb{O} = \mathbb{A}[\boldsymbol{\partial}; \boldsymbol{\sigma}, \boldsymbol{\delta}]$ over the $\mathbb{K}$-algebra $\mathbb{A}$:

$$\mathcal{F} + \mathcal{F} \subseteq \mathcal{F} \qquad \mathbb{O} \cdot \mathcal{F} \subseteq \mathcal{F}.$$

In applications, $\mathbb{K}$ is commutative and $\mathbb{A} = \mathbb{K}(\mathbf{x})[\mathbf{y}]$.

The annihilating set of $f \in \mathcal{F}$,

$$\operatorname{Ann} f = \big\{ P(\mathbf{x}, \mathbf{y}, \boldsymbol{\partial}) \in \mathbb{O} \mid P \cdot f = 0 \big\},$$

is a left ideal of $\mathbb{O}$:

$$\begin{cases} P \cdot f = 0 \\ Q \cdot f = 0 \end{cases} \Rightarrow (P + Q) \cdot f = 0,$$

$$P \cdot f = 0 \Rightarrow \begin{cases} x \cdot (P \cdot f) = x \times 0 = 0, \\ \partial \cdot (P \cdot f) = \partial \cdot 0 = 0. \end{cases}$$

Computations in the left module over $\mathbb{O}$

$$\mathbb{O} \cdot f \simeq \mathbb{O} / \operatorname{Ann} f.$$

Annihilating Ideals

1. Slide
   - Annihilator $\longleftrightarrow$ set of all linear functional equations
   - Difference between isomorphism of vector spaces, resp. modules (give product law in each module or quotient ring)
2. Examples
   - $\mathbb{O} = \mathbb{C}(x)[D_x; 1, D_x]$, $f(x) = J_0(x)$:

$$\begin{aligned} \operatorname{Ann} f &= \mathbb{O}(x D_x^2 + D_x + x), \\ \mathbb{O} \cdot f &= \mathbb{C}(x) f \oplus \mathbb{C}(x) f', \\ \mathbb{O} / \operatorname{Ann} f &= \mathbb{C}(x) \oplus \mathbb{C}(x) D_x. \end{aligned}$$

   - $\mathbb{O} = \mathbb{C}[x][D_x; 1, D_x]$, $f = \delta$ (Dirac distribution):

$$\begin{aligned} \operatorname{Ann} f &= \mathbb{O} x, \\ \mathbb{O} \cdot f &= \mathbb{C}[D_x] \cdot f = \bigoplus_{n \in \mathbb{N}} \mathbb{C} f^{(n)}, \\ \mathbb{O} / \operatorname{Ann} f &= \mathbb{C}[D_x], \qquad (x \text{ acts by } -D_\partial) \\ \mathbb{O} \cdot f &= \mathbb{O} \cdot f' \qquad (!), \\ \operatorname{Ann} f' &= \mathbb{O} x^2 + \mathbb{O}(x D_x + 2). \end{aligned}$$

(Introduce $P = A(D_x) + B(D_x) x + C(x, D_x) x^2 \in \operatorname{Ann} f'$ and write $P\partial \in \operatorname{Ann} f$.)

$$\begin{array}{ccc} \mathbb{O} / \operatorname{Ann} f & = & \mathbb{O} / \operatorname{Ann} f' \qquad (\simeq) \\ P & \longrightarrow & -Px \\ Q\partial & \longleftarrow & Q \end{array}$$

Hypergeometric Terms

$$\boxed{\begin{array}{c} t \text{ is hypergeometric} \\ \Updownarrow \\ \dfrac{t_{n+1}}{t_n} \in \mathbb{K}(n) \end{array}}$$

$\Rightarrow$ linear homogeneous recurrence of order 1 with coefficients in $\mathbb{K}(n)$

$\Rightarrow$ one-dimensional $\mathbb{K}(n)$-vector space.

Rational functions, products of binomial coefficients, more generally

$$t_n = R(n) \left(\prod_i \Gamma(a_i + b_i n)^{p_i} \right) z^n.$$

Closure under product.

No closure under addition: $n! + 1$ **is not** hypergeometric.

q-Hypergeometric Terms

$$\boxed{\begin{array}{c} t \text{ is } q\text{-hypergeometric} \\ \Updownarrow \\ \dfrac{t_{n+1}}{t_n} \in \mathbb{K}(q^n) \end{array}}$$

Hypergeometric Terms

1. Insist on 1-dimensional viewpoint = ∂-finite viewpoint
2. Suggest multivariate case
3. Term vs function
   - univariate case: meromorphic functions
   - multivariate case: ?
4. Intentionally excluded $\mathbb{K}(n, q^n)$

∂-Finite Terms

t is ∂-finite w.r.t. the operator algebra $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}]$

$$\Updownarrow$$

the $\boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot t$'s span a finite dimensional $\mathbb{K}(\mathbf{x})$-vector space:

$$\dim_{\mathbb{K}(\mathbf{x})} \mathbb{K}(\mathbf{x})[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}] \cdot t < +\infty.$$

- $\exp(z^2)$ for D_z over $\mathbb{Q}(z)$; $d = 1$
- $\exp\exp z$ for D_z over $\mathbb{Q}(\exp z)$; $d = 1$
- $n!$ for S_n over $\mathbb{Q}(n)$; $d = 1$
- $\binom{n}{k}$ for S_n and S_k over $\mathbb{Q}(n,k)$; $d = 1$
- $(x;q)_n$ for $S_{q,n}$ and $H_{q,x}$ over $\mathbb{Q}(q,q^n,x)$; $d = 1$
- $\mathrm{Ci}(z)$ for D_z over $\mathbb{Q}(z)$; $d = 2$
- F_n, H_n for S_n over $\mathbb{Q}(n)$; $d = 2$
- $J_n(z), T_n(z), P_n(z), \dots$ for S_n and D_z over $\mathbb{Q}(n,z)$; $d = 2$
- ${}_2F_1\left(\left.{a,b \atop c}\right| z\right)$ for S_a, S_b, S_c and D_z over $\mathbb{K}(a,b,c,z)$; $d = 2$
- $r_n(a,b;q), \dots$ for $S_{q,n}$ over $\mathbb{Q}(q,a,b,q^n)$; $d = 2$
- $J_\mu(x)J_\nu(y)$ for D_x, D_y, S_μ and S_ν over $\mathbb{Q}(\mu,\nu,x,y)$; $d = 4$
- $J_1(ax)I_1(ax)Y_0(x)K_0(x)$ for D_a and D_x over $\mathbb{Q}(a,x)$. $d = 16$

Special cases: (q)-hypergeometric terms, hyperexponential functions, Zeilberger's holonomic functions.

Algorithmic closure under $+$, $\times$ and the ∂_i's

$\Longrightarrow$ **simplification** and **zero test** of ∂-finite expressions.

∂-Finite Terms

1. ∂-finiteness vs (rational) holonomy; vs: D-finite f., P-finite f. and s., P-recursive s.
2. Term vs function
   - univariate case: meromorphic functions
   - multivariate case: ?
3. Describe:
   - algebra $\mathbb{O}$,
   - annihilator $\operatorname{Ann} f$,
   - vector space $\mathbb{O} \cdot f$ (choice of basis is irrelevant),
   - corresponding module structure
4. Bessel function: $z^2D_z^2 + zD_z + z^2 - \nu^2, \quad S_\nu + D_z - \dfrac{n}{z}$
5. Integral cosine: $\mathrm{Ci}(z) = \gamma + \ln z + \displaystyle\int_0^z \frac{\cos t - 1}{t}\,dt$

$$zD_z \cdot \mathrm{Ci}(z) = \cos(z),$$
$$(D_z^2 + 1)zD_z = zD_z^3 + 2D_z^2 + zD_z.$$

6. Gauss hyp. func.: ${}_2F_1\left(\left.\begin{matrix} a,b \\ c \end{matrix}\right| z\right) = \displaystyle\sum_{n\ge0} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}.$

   Classical results:

$$S_a = \frac{z}{a}D_z + 1, \quad S_b = \frac{z}{b}D_z + 1, \quad S_c^{-1} = \frac{z}{c-1}D_z + 1,$$
$$z(1-z)D_z^2 + (c - (a+b+1)z)D_z - ab.$$

   One computes:

$$(a-c)(b-c)S_c - c(1-z)D_z + c(a+b-c).$$

"Solving" Second Order Equations

Ore algebra $\mathbb{A} = \mathbb{F}[\partial; \sigma, \delta]$ over the $\mathbb{K}$-algebra $\mathbb{F}$.

Two-dimensional solution space of $J = \mathbb{A}(\partial^2 - a\partial - b)$:

$$\mathbb{K} \cdot f_1 \oplus \mathbb{K} \cdot f_2.$$

$$M = \mathbb{A}/J. \qquad M \simeq \mathbb{A} \cdot f_i? \qquad \operatorname{Ann} f_i?$$

1. Direct sum $M \simeq \mathbb{A} \cdot f_1 \oplus \mathbb{A} \cdot f_2$:

$$f_1 = 2^n, \qquad f_2 = \Gamma(n).$$

2. Isomorphic image of $\mathbb{A} \cdot f_2$ in $M \simeq \mathbb{A} \cdot f_1$:

$$f_1 = H_n = \sum_{k=1}^{n} \frac{1}{n}, \qquad f_2 = (n+1)(S-1) \cdot f_1 = 1.$$

3. No isomorphic image of $\mathbb{A} \cdot f_2$ in $M \simeq \mathbb{A} \cdot f_1$:

$$f_1 = (-3)^n \sum_{k=0}^{n-1} \left(-\frac{3}{4}\right)^k \frac{\Gamma(k+1)}{\Gamma(k+3/2)}, \qquad f_2 = (-3)^n.$$

4. Irreducible module $M \simeq \mathbb{A} \cdot f_1 \simeq \mathbb{A} \cdot f_2$:

$$f_1 = [x^n] \exp\left(-\arctan z\right), \qquad f_2 = [x^n] \exp\left(\arctan \frac{1}{z}\right).$$

Better interpretation by differential/difference/Ore Galois theory?

"Solving" Second Order Equations

1. Solution space loosely defined
2. Algorithm Hyper (to determine case)
3. Variation of constant for cases 2 and 3
   - related to indefinite summation problem
   - summand given by left factor
4. Classification
   - Direct sum:

$$\begin{aligned}(n-2)S^2 - (n^2+n-4)S + 2n(n-1) \\ = ((n-2)S - n(n-1))(S-2) \\ = ((n-2)S - 2(n-1))(S-n).\end{aligned}$$

   - Isomorphic image:

$$\begin{aligned}(n+2)S^2 - (2n+3)S + (n+1) \\ = ((n+2)S - (n+1))(S-1) \\ = (S-1)(n+1)(S-1).\end{aligned}$$

   $M \simeq \mathbb{F} \cdot f_1 \oplus \mathbb{F} \cdot f_2$ as $\mathbb{F}$-vector spaces only!
   $(\mathbb{O}/\operatorname{Ann} f_1)/(\mathbb{O}(S_n - 1)/\operatorname{Ann} f_1) \simeq \mathbb{F}H_n$,
   with action $S_n \cdot r(n)H_n = (n+2)^{-1}(n+1)r(n+1)H_n$.
   General pattern: $(S + \lambda/a - r)a(S-r)$.

   - No isomorphic image:

$$\begin{aligned}2(2n+3)S^2 + 3(5n+7)S + 9(n+1) \\ = (2(2n+3)S + 3(n+1))(S+3).\end{aligned}$$

   Related to the summation problem $\sum_{k\in\mathbb{Z}} (-1)^k \binom{n}{k}\binom{3k}{n}$.
   $(\mathbb{O}/\operatorname{Ann} f)/(\mathbb{O}(S_n + 3)/\operatorname{Ann} f) \simeq \mathbb{F}\left(-\frac{3}{4}\right)^n \frac{\Gamma(n+1)}{\Gamma(n+3/2)}$,
   with action $S_n \cdot r(n) f_1 = -3/2(2n+3)^{-1}(n+1)r(n+1)f_1$.

   - Irreducible module:

$$(n+2)S^2 + S + n.$$

   $u_0 + u_1 = 0$ and $u_0 = u_{-1}$ respectively.
   Generating function satisfies $(1+z^2)f + f' = 0$.

5. Algebraically closed field:

$$\cos \text{ and } \sin \qquad \text{vs} \qquad \exp(in) \text{ and } \exp(-in).$$

Closure Property under Addition

$$\boxed{\begin{array}{c} u \text{ and } v \text{ are } \partial\text{-finite w.r.t. } \mathbb{O} = \mathbb{K}(\mathbf{x})[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}] \\ \Downarrow \\ u+v \text{ is } \partial\text{-finite w.r.t. } \mathbb{O}. \end{array}}$$

$\mathcal{F}$: functional space over $\mathbb{O}$, $\quad u, v$: ∂-finite functions.

Each $\boldsymbol{\partial}^{\boldsymbol{\gamma}} \cdot (u+v)$ is a linear combination of the $\boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot u$'s and the $\boldsymbol{\partial}^{\boldsymbol{\beta}} \cdot v$'s.

Computational setting: direct sum $\mathbb{O} \cdot u \oplus \mathbb{O} \cdot v \subset \mathcal{F} \oplus \mathcal{F}$:

$$\begin{array}{ll} \sigma : \mathcal{F} \oplus \mathcal{F} \to \mathcal{F}, \quad \sigma(f \oplus g) = f + g & \text{(evaluation);} \\ \partial_i \cdot (f \oplus g) = \partial_i \cdot f \oplus \partial_i \cdot g & \text{(action);} \\ NF(f \oplus g) = NF(f) \oplus NF(g) & \text{(normal forms).} \end{array}$$

"Algorithm" = Normal Forms + Gaussian Elimination

$\to$ computes $\operatorname{Ann}(f \oplus g)$.

Closure Property under Addition

1. Suitable finite-dimensional vector space: direct sum
2. ∂-finiteness:

$$\mathbb{O} \cdot u = \bigoplus_{\boldsymbol{\alpha} \in I} \mathbb{K}(\mathbf{x}) \boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot u \subset \mathcal{F}, \qquad I \text{ a finite set;}$$

$$\mathbb{O} \cdot v = \bigoplus_{\boldsymbol{\beta} \in J} \mathbb{K}(\mathbf{x}) \boldsymbol{\partial}^{\boldsymbol{\beta}} \cdot v \subset \mathcal{F}, \qquad J \text{ a finite set.}$$

3. Finite basis of the $\boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot u \oplus \boldsymbol{\partial}^{\boldsymbol{\beta}} \cdot v$ for $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in I \times J$:

$$\begin{tikzcd} \mathbb{O} \cdot (u \oplus v) = \mathbb{O} \cdot u \oplus \mathbb{O} \cdot v \arrow[r, "\subset"] \arrow[d, "\sigma"] & \mathcal{F} \oplus \mathcal{F} \arrow[d, "\sigma"] \\ \mathbb{O} \cdot (u + v) = \mathbb{O} \cdot u + \mathbb{O} \cdot v \arrow[r, "\subset"] & \mathcal{F} \end{tikzcd}$$

4. Complete algorithm:
   - normal forms by Gröbner bases
   - efficiency by FGLM algorithm

Closure Property under Multiplication

$$\boxed{\begin{array}{c} u \text{ and } v \text{ are } \partial\text{-finite w.r.t. } \mathbb{O} = \mathbb{K}(\mathbf{x})[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}] \\ \Downarrow \\ uv \text{ is } \partial\text{-finite w.r.t. } \mathbb{O}. \end{array}}$$

$\mathcal{F}$: functional space over $\mathbb{O}$, u, v: ∂-finite functions.

Each $\boldsymbol{\partial}^{\boldsymbol{\gamma}} \cdot (uv)$ is a linear combination of the $(\boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot u)(\boldsymbol{\partial}^{\boldsymbol{\beta}} \cdot v)$'s,
if technical constraints satisfied!

$$\sigma_i = A_i(\partial_i) \quad \text{and} \quad \delta_i = B_i(\partial_i);$$
$$((A_i - 1) \cdot w)B_i = (B_i \cdot w)(A_i - 1) \quad \text{when } w \in \mathbb{K}(\mathbf{x}).$$

Computational setting: tensor product $\mathbb{O} \cdot u \otimes \mathbb{O} \cdot v \subset \mathcal{F} \otimes \mathcal{F}$:

$$\begin{array}{ll} \mu : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}, \qquad \mu(f \otimes g) = fg & \text{(evaluation);} \\ \partial_i \cdot (f \otimes g) = f^{\sigma_i} \otimes \partial_i \cdot g + \partial_i \cdot f \otimes g^{\delta_i} & \text{(action);} \\ NF(f \otimes g) = NF(f) \otimes NF(g) & \text{(normal forms).} \end{array}$$

"Algorithm" = Constraints on Commutations + NF's + Gaussian Elim

$\rightarrow$ computes $\operatorname{Ann}(f \otimes g)$.

Closure Property under Multiplication

1. Suitable finite dimensional vector space: tensor product
2. Constraints on σ and δ (if $\sigma\delta = \delta\sigma$): first one because

$$\partial \cdot (fg) = f^{\sigma}(\partial \cdot g) + f^{\delta} g,$$
$$\partial^2 \cdot (fg) = f^{\sigma^2}(\partial^2 \cdot g) + 2f^{\sigma\delta}(\partial \cdot g) + f^{\delta^2} g,$$
$$\partial^3 \cdot (fg) = f^{\sigma^3}(\partial^3 \cdot g) + 3f^{\sigma^2\delta}(\partial^2 \cdot g) + 3f^{\sigma\delta^2}(\partial \cdot g) + f^{\delta^3} g,$$
$$\dots,$$

Secone one because of tensor product:

$$(w^{\sigma_i} - w)\delta_i = w^{\delta_i}(\sigma_i - 1).$$

Commonly: $\sigma_i = A_i = 1$ (derivation); or $\delta_i = B_i = 0$ (difference); or $\delta_i = \sigma_i - 1 = B_i = A_i - 1$ (difference). (Make the action explicit.)

3. ∂-finiteness:

$$\mathbb{O} \cdot u = \bigoplus_{\boldsymbol{\alpha} \in I} \mathbb{K}(\mathbf{x})\boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot u \subset \mathcal{F}, \qquad I \text{ a finite set;}$$
$$\mathbb{O} \cdot v = \bigoplus_{\boldsymbol{\beta} \in J} \mathbb{K}(\mathbf{x})\boldsymbol{\partial}^{\boldsymbol{\beta}} \cdot v \subset \mathcal{F}, \qquad J \text{ a finite set.}$$

4. Finite basis of the $\boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot u \otimes \boldsymbol{\partial}^{\boldsymbol{\beta}} \cdot v$ for $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in I \times J$:

$$\begin{array}{ccccc} \mathbb{O} \cdot (u \otimes v) & \xrightarrow{\subset} & \mathbb{O} \cdot u \otimes \mathbb{O} \cdot v & \xrightarrow{\subset} & \mathcal{F} \otimes \mathcal{F} \\ \downarrow \mu & & \downarrow \mu & & \downarrow \mu \\ \mathbb{O} \cdot (uv) & \xrightarrow{\subset} & (\mathbb{O} \cdot u)(\mathbb{O} \cdot v) & \xrightarrow{\subset} & \mathcal{F} \end{array}$$

5. Normal forms: $\mathbb{K}(\mathbf{x})$ is shared!
6. Complete algorithm:
   - normal forms by Gröbner bases
   - efficiency by FGLM algorithm

Single Variable Case

Functions f and g defined by:

$$f = \cos(x), \quad D^2 \cdot f + f = 0, \quad f(0) = 1, \quad f'(0) = 0;$$
$$g = \sin(x), \quad D^2 \cdot g + g = 0, \quad g(0) = 0, \quad g'(0) = 1.$$

Rewrite successive derivatives of $h = f^2 + g^2$:

$$\begin{aligned} h &= f^2 + g^2, & h(0) &= 1; \\ D \cdot h &= 2(fD \cdot f + gD \cdot g), & h'(0) &= 0; \\ D^2 \cdot h &= 2((D \cdot f)^2 - f^2 + (D \cdot g)^2 - g^2), & h''(0) &= 0; \\ D^3 \cdot h &= -8(fD \cdot f + gD \cdot g), & h'''(0) &= 0. \end{aligned}$$

Conclusion:

- $h = f^2 + g^2$ satisfies $D^3 \cdot h + 4D \cdot h = 0$;
- initial conditions $\Longrightarrow$ $h(x) = \boxed{\cos(x)^2 + \sin(x)^2 = 1}$.

Same algorithm applied to Fibonacci numbers F_n yields:

$$\boxed{F_{n+2}F_n - F_{n+1}^2 = (-1)^n}.$$

Single Variable Case

1. Direct calculation for a polynomial expression (also show calculation on $(f \otimes f) \oplus (g \otimes g)$).
2. Compare to calculation where $\sin = -D \cdot \cos$ (and $h' = 0$).
3. Compare to calculation with three different F's (and $S^3 - 2S^2 - 2S + 1$).
4. Allude to rectangular calculations, and announce example of the Mehler formula for Hermite polynomials.

The Mehler Formula for Hermite Polynomials

The sum

$$F(u) = \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{u^n}{n!}$$

equals the closed form

$$\frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}.$$

1. Computing product $H_n(x) \times H_n(y) \times \dfrac{u^n}{n!}$ yields:

$$\begin{aligned}(n+4)S_n^4 - 4xyuS_n^3 - 4(2n+5-2x^2-2y^2)u^2S_n^2 \\ -16xyu^3S_n + 16(n+1)u^4, \\ uD_u - n;\end{aligned}$$

2. **Derive** the following annihilator of the summand:

$$\begin{aligned}16u^3 + 16u^4D_u - 16u^2xyS_n + D_uS_n^4 - 8u^2S_n^2D_u \\ +4u(2x^2-1+2y^2)S_n^2 - 4xyS_n^3;\end{aligned}$$

3. Creative telescoping yields

$$((1-4u^2)^2D_u + 16u^3 + 4(2x^2+2y^2-1)u - 4xy(4u^2+1)) \cdot F = 0;$$

4. Solve to get the announced closed form.

The Mehler Formula for Hermite Polynomials

1. Multivariate product
   - perform first steps of the reduction by

     $$H_{n+2}(x) = 2xH_{n+1}(x) - 2(n+1)H_n(x)$$

   - describe lacks in the "algorithm":
     * procedure for normal forms
     * derivatives have to be ordered for more efficiency
2. Conclusion:
   - need of Gröbner bases:
     * normal forms for ∂-finite closures
     * elimination for holonomic closures
   - need of holonomy:
     * existence of non-trivial eliminated polynomial

Holonomic Functions in Computer Algebra

Part II

Gröbner Bases in Algebras of Linear Operators

Gröbner Bases in Algebras of Linear Operators

1. Gröbner bases for the skew case;
2. Graduations and filtrations of algebras and modules;
3. Variations on Gröbner bases:
   - FGLM algorithm;
   - Gröbner bases for modules;
   - computation of syzygies and their evaluation;
   - Gröbner walk.
4. ∂-Finite closures ($+$, $\times$, ∂^{-1}) in an algorithmic way.

Skew Gröbner Bases

Distinguished bases of an ideal satisfying a **divisibility property**.

Generalize the Euclidean division algorithm and Euclidean gcd calculations to multivariate (skew) polynomial rings.

Provide us with:

1. normal forms for ideals:

$$\{p_i\} \to \{g_i\} \text{ such that } \sum_i \mathcal{A} p_i = \sum_i \mathcal{A} g_i;$$

2. divisions with uniqueness of the remainder:

$$p \to \sum_i a_i g_i + r;$$

2'. normal forms in the quotient ring:

$$\mathcal{A} / \sum_i \mathcal{A} g_i;$$

3. polynomial elimination: $(\mathcal{A} = \mathcal{B}[x; \sigma, \delta])$

$$\{p_i\} \to \{g_i\} \text{ such that } \sum_i \mathcal{B} g_i = \sum_i \mathcal{A} p_i \cap \mathcal{B}.$$

Algorithmic procedures based on Buchberger's algorithm.

Skew Gröbner Bases

1. Effective calculations with multivariate ideals
   - originally developped in the commutative case
   - main difference to the Euclidean division: dependency in a term order
2. Same algorithms and program in Maple

Gröbner-Based Algorithms

Extension of Buchberger's algorithm to the skew case:
focus on left ideals; Gröbner bases still finite (noetherianity).

[Galligo, Takayama, Mora, Robbiano, Kandri-Rody et Weispfenning, Kredel]

Direct extension of:

- FGLM algorithm;
- Gröbner walk;
- Gröbner bases for (left-) modules;
- Syzygy calculations.

"Skew Varieties"

$\boxed{\text{Hilbert dimension}}$

For an Ore algebra $\mathbb{O}_{\text{rat}} = \mathbb{K}(\mathbf{x})[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}]$:

Dimension 0 $\longleftrightarrow$ ∂-finite function.

For an Ore algebra $\mathbb{O}_{\text{poly}} = \mathbb{K}[\mathbf{x}][\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}]$:

"Small" dimension $\longleftrightarrow$ holonomic modules and functions.

[Bernstein, Kashiwara, Sabbah, Zeilberger]

Gröbner-Based Algorithms and "Skew Varieties"

1. Generalization to Ore algebras:
   - as a subcase of polynomial rings of solvable type
   - possible restriction on term orders allowed
2. Skew Gröbner bases: different from
   - non-commutative Gröbner bases
   - differential Gröbner bases
3. Detail/define each application
4. Dimension theory
   - invariant = Hilbert dimension for ideals
     $\neq$ dimension as vector spaces
   - same as Gel'fand Kirillov dimension (and other ones)
   - link to (commutative) algebraic varieties
   - multiplicity
   - definition of holonomy originates
     in $\mathcal{D}$-module theory
5. Two frameworks
   - $+$, $\times$, simplifications: calculation with "non-commutative algebraic numbers"
   - $\int$, $\sum$: eliminations

Term Orderings

Set of terms = totally ordered monoid T (neutral e) with compatibility between $\preceq$ and product:

$$\begin{array}{ll} \forall t \in T & e \preceq t, \\ \forall (t,u,v) \in T^3 & u \preceq v \Rightarrow tu \preceq tv. \end{array}$$

Usually: $T = [x_1, \dots, x_n, \partial_1, \dots, \partial_n]$ = free commutative monoid.

Examples:

- lexicographical ordering $x \succ \partial$:

$$e \prec \partial \prec \partial^2 \prec \cdots \prec x \prec x\partial \prec x^2\partial \prec \cdots \prec x^3 \prec \dots$$

- total degree ordering $x \succ \partial$:

$$e \prec \partial \prec x \prec \partial^2 \prec x\partial \prec x^2 \prec \partial^3 \prec x\partial^2 \prec x^2\partial \prec x^3 \prec \dots$$

- intermediate orderings for elimination.

(Skew) polynomials = linear combinations of terms:

- leading term of a polynomial p;
- leading coefficient of a polynomial p;
- leading monomial of a polynomial p.

Term Orderings

1. Admissibility: positivity + continuity
   (positivity mainly for termination)
2. Compatibility for correctness
3. Term vs monomial
4. Two viewpoints on Gröbner bases
   - term rewriting
   - ring-theoretic
     * tools to be reused
     * non-conventional (?)
     * developed by Robbiano and Mora
5. Transition
   - euclidean ring = "principal ring with a degree", which makes algorithmic questions easier
   - similarly, filtered ring = "noetherian ring with a generalized degree", where Gröbner basis calculations are possible

Graduations and Filtrations of Algebras

$$\begin{array}{rcl} \text{Algebra } \mathbb{A} \text{ over a field } \mathbb{K} & \to & \text{algebra of skew polynomials;} \\ \text{Totally ordered monoid } \mathrm{M} & \to & \text{generalized degrees/set of terms.} \end{array}$$

M-graduation of $\mathbb{A}$ = family $(\mathcal{G}_\mu)_{\mu\in\mathrm{M}}$ of $\mathbb{K}$-vector spaces such that:

$$\begin{cases} \bigoplus_{\mu\in\mathrm{M}} \mathcal{G}_\mu = \mathbb{A}, \\ \mathcal{G}_\mu\mathcal{G}_{\mu'} \subseteq \mathcal{G}_{\mu\mu'}, \quad \text{whenever } (\mu,\mu') \in \mathrm{M}^2. \end{cases}$$

M-filtration of $\mathbb{A}$ = family $(\mathcal{F}_\mu)_{\mu\in\mathrm{M}}$ of vector spaces such that:

$$\begin{cases} \mathcal{F}_\mu \subseteq \mathcal{F}_{\mu'}, & \text{whenever } \mu \preceq \mu' \text{ in M}, \\ \bigcup_{\mu\in\mathrm{M}} \mathcal{F}_\mu = \mathbb{A}, & \\ \mathcal{F}_\mu\mathcal{F}_{\mu'} \subseteq \mathcal{F}_{\mu\mu'}, & \text{whenever } (\mu,\mu') \in \mathrm{M}^2. \end{cases}$$

Associated M-graded algebra $\mathrm{gr}_{\mathcal{F}}\,\mathbb{A}$ of the M-filtered algebra $\mathbb{A}$:

$$\mathrm{gr}_{\mathcal{F}}\,\mathbb{A} = \bigoplus_{\mu\in\mathrm{M}} \mathcal{G}_\mu \qquad \text{where} \qquad \begin{cases} \mathcal{G}_\mu = \mathcal{F}_\mu/\mathcal{V}_\mu, \\ \mathcal{V}_\mu = \bigcup_{\mu'\in\mathrm{M},\, \mu'\prec\mu} \mathcal{F}_{\mu'}. \end{cases}$$

Graduations and Filtrations of Algebras

1. More or less refined extension of univariate degree
   - $\mathrm{M} = \mathbb{N}$ for total degree
   - $\mathrm{M} = [\mathbf{x}]$ for term orderings
2. Graduations = for homogeneous decompositions
3. Filtration = for twisted products
4. Give definitions based on following examples
   - $\mathbb{K}[\mathbf{x},\mathbf{y}]$ with homogeneous components vs Weyl algebras
   - $\mathbb{K}\langle x, H; Hx - qxH\rangle$ vs $\mathbb{K}\langle x, D; Dx - qxD + q(x-1)\rangle$
5. Case of associative algebras:

$$\begin{array}{c} \mathbb{K}\langle x,\partial;\partial x = (qx+p)\partial + (sx+r)\rangle \\ \downarrow \mathrm{gr} \\ \mathbb{K}\langle x,\partial;\partial x = qx\partial\rangle \end{array}$$

6. Case of skew polynomial rings: $\mathbb{F}[u;\sigma,\delta] \to \mathbb{F}[u;\sigma,0]$
7. Link to term orderings
   - special filtration $\mathcal{F}_\mu = \bigoplus_{\mu'\in\mathrm{M},\,\mu'\preceq\mu} \mathbb{K}\mu'$
   - "positive filtrations" (consequence on associated graduations)

$$\mathcal{F}_\mu \simeq 0 \text{ if } \mu \prec e, \qquad \mathcal{F}_e \simeq \mathbb{K}$$

   - continuity is kept

Gröbner Bases in Filtered Algebras

M-filtered $\mathbb{K}$-algebra $\mathbb{A} = \bigcup_{\mu \in \mathrm{M}} \mathcal{F}_\mu$:

Degree: $\deg : \mathbb{A} \to \mathrm{M}, \quad \deg p = \min\{\mu \mid p \in \mathcal{F}_\mu\}$;

Principal symbol: $\theta : \mathbb{A} \to \mathrm{gr}_{\mathcal{F}} \mathbb{A}, \quad \theta\, p = p + \mathcal{V}_{\deg p}$.

Left-ideal $I \subseteq \mathbb{A}$, with basis G:

$\deg G$ = left-stable part of M spanned by the $\deg p$ for $p \in G$;

$\theta\, G$ = left-sided ideal of $\mathrm{gr}_{\mathcal{F}} \mathbb{A}$ spanned by the $\theta\, p$ for $p \in G$.

$$\boxed{\begin{array}{c} G \text{ is a } \textbf{Gröbner bases} \text{ for } I \text{ w.r.t. the filtration of } \mathbb{A} \\ \Updownarrow \\ \deg G = \deg I \\ \Updownarrow \\ \theta\, G = \theta\, I \end{array}}$$

Gröbner Bases in Filtered Algebras

1. Eschew discussing the degree of 0
2. $\theta\, \mathbb{A}$ only contains monomials: θ "=" lm!
3. Stable part of M $= \bigcup_{\mu \in M} \mathrm{M} \deg \mu$ = monoideal = stairs
4. Definition of Gröbner bases:
   - discuss it with stairs
   - $\deg I$ = monoideal of leading terms
   - $\theta\, I$ = ideal of leading terms
   - describe collisions in the commutative case
5. Relation to division, normal forms, etc (with stairs)
6. Two-sided theory by Mora

Reduction

$\boxed{\text{Left-division of } P \text{ by } Q\text{: find } L \text{ and } R \text{ such that } P = LQ + R \text{ and } \deg Q \text{ does not divide } \deg R.}$

Left-division of P by Q: find L and R such that

$$P = LQ + R$$

and $\deg Q$ does not divide $\deg R$.

Assume:

1. $\mathcal{G}_\mu \mathcal{G}_{\mu'} = \mathcal{G}_{\mu\mu'}$ and $\mathcal{F}_\mu \mathcal{F}_{\mu'} = \mathcal{F}_{\mu\mu'}$ for all $(\mu, \mu') \in \mathrm{M}^2$;
2. well-foundedness of $\preceq$.

Elementary division step:

1. set $\deg P = \mu$ and $\deg Q = \nu$;
2. if $\nu | \mu$, i.e. $\mu = \nu'\nu$, choose **any polynomial** $L \in \mathcal{F}_{\nu'}$ such that
$$\boxed{\theta\, P = \theta\, L\, \theta\, Q;}$$
3. then $R = P - LQ \notin \mathcal{F}_\mu$, i.e.
$$\boxed{\deg R \prec \deg P.}$$

Extension to the division of P by a family (Q_i).

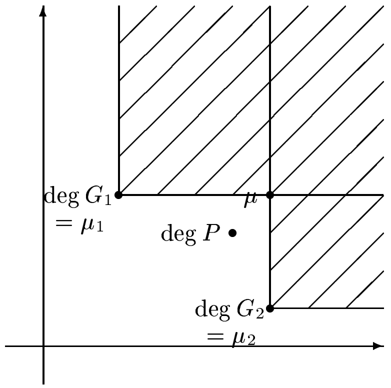
Reduction

1. Coefficient of $\theta\, P$ not necessarily product of coefficients of $\theta\, L$ and $\theta\, q$ although $\theta\, P = \theta\, L\, \theta\, Q$ (take σ's into account)
2. Iteration as long as $\operatorname{lt} Q \mid \operatorname{lt} R$

   $\rightarrow$ uniquely defines R in the univariate case
3. Assumptions
   (a) on products for solvability of θ equation
   (b) on order for termination
4. In implementation: L reduces to a single monomial:
$$L = \alpha\gamma^{-1} t Q'.$$
5. Traditional presentation with terms:

   For
$$\begin{aligned} P &= \operatorname{lm} P + P' = \alpha \operatorname{lt} P + P', \\ Q &= \operatorname{lm} Q + Q' = \beta \operatorname{lt} Q + Q', \end{aligned} \qquad \text{such that } \operatorname{lt} Q \mid \operatorname{lt} P,$$
   write $\operatorname{lt} P = t \operatorname{lt} Q$. Then $tQ = \gamma \operatorname{lt} P + tQ'$ with
$$\boxed{\operatorname{lm}(tQ) = \gamma \operatorname{lt} P.}$$
   Choose any polynomial L (traditionally, a monomial) such that
$$\operatorname{lm} L = \alpha\gamma^{-1} t$$
   (α uniquely defined) and write $L = \alpha\gamma^{-1} t + L'$. Then
$$R = P - LQ = P' + \alpha\gamma^{-1} tQ' + L'Q$$
   with
$$\begin{aligned} \operatorname{lt} P' &\prec \operatorname{lt} P && \text{by definition,} \\ \operatorname{lt}(\alpha\gamma^{-1} tQ') &\prec \operatorname{lt} P && \text{by continuity,} \\ \operatorname{lt}(L'Q) &\prec \operatorname{lt} P && \text{by continuity.} \end{aligned}$$
   Thus
$$\boxed{\operatorname{lt}(P - LQ) \prec \operatorname{lt} P.}$$

S-Polynomials



$$\begin{cases} P = \sum_i L_i G_i \\ \deg P \notin \bigcup_i \mathrm{M} \deg G_i \end{cases} \implies \text{cancellation of leading monomials!}$$

Restrict to finitely many S-polynomials:

$$\deg P_1 = \mu_1, \qquad \deg P_2 = \mu_2, \qquad \mu = \operatorname{lcm}(\mu_1, \mu_2) = \nu_i \mu_i,$$
$$\operatorname{Spoly}(P_1, P_2) = L_1 P_1 + L_2 P_2 \qquad \text{for } \textbf{any polynomials } L_i \text{ such that}$$
$$\deg L_i = \nu_i \qquad \text{and} \qquad \theta\, L_1\, \theta\, G_1 + \theta\, L_2\, \theta\, G_2 = 0,$$
$$\text{i.e.} \qquad \operatorname{Spoly}(P_1, P_2) \notin \mathcal{F}_\mu.$$

Theorem:

$$\boxed{\text{All } \operatorname{Spoly}(P_i, P_j)\text{'s reduce to 0 by } G \implies G \text{ is a Gröbner basis.}}$$

S-Polynomials

1. Proof of theorem: by considering the evaluation map
$$\phi : \mathbb{A}^n \longrightarrow \mathbb{A}, \qquad \phi(L_1, \ldots, L_n) = \sum_{i=1}^n L_i G_i,$$
and using standard representations in the module $\mathbb{A}^n$ (i.e. divisions in $\mathbb{A}^n$, not in $\mathbb{A}$!), makes link between S-polynomials and syzygies ($= \ker \phi$).
2. Adjust coefficients by taking σ's into account
3. In implementation
   - L_i's reduce to single monomials
   - calculations in $\mathbb{A}[x]\langle\partial\rangle$, with $\mathbb{A}$
     * a commutative ring
     * a left Ore ring ($ua = vb$)
4. Example:
$$\sum_{-\infty}^{+\infty} \frac{(a;q)_n x^n}{(b;q)_n} = \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty \left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty \left(\frac{b}{ax};q\right)_\infty (b;q)_\infty \left(\frac{q}{a};q\right)_\infty},$$
$$\mathbb{A} = \mathbb{Q}(q,a,b,x)[q^n][H, S; \ldots], \quad Hx = qxH, \quad Sq^n = qq^n S,$$
$$\mathbb{A}(H - q^n) + \mathbb{A}((1 - q^n b)S - q(1 - q^n a)) \subseteq \operatorname{Ann} f$$
$$(f \text{ is the summand}),$$
$$P = q^n - H, \qquad Q = bq^n S - aqq^n - S + q,$$
$$\operatorname{Spoly}(P, Q) = (bS + \cdots)P + (-q + \cdots)Q = qq^2 q^n + \cdots$$

Spoly(P, Q) is $bSP - qQ$ and not $bSP - Q$ (!)

Buchberger Algorithm

INPUT: a finite set P of polynomials p_i, a term order $\preceq$.
OUTPUT: a Gröbner basis G for the same ideal.

1. initialize G to P;
2. initialize S to the set of unordered pairs of elements of G;
3. while S is not empty,
   (a) select a pair $p = \{g, g'\}$ and remove it from S;
   (b) if p is useless, forget it;
   (c) if p is triviale, forget it;
   (d) compute and reduce $\mathrm{Spoly}(g, g')$;
   (e) if the remainder r is non-zero then
      i. adjoin all new pairs $\{g, r\}$ for $g \in G$ to S;
      ii. adjoin r to G;
4. return G.

A lot of freedom in pair selection and reductions $\to$ several strategies.

Avoiding reductions:

- useless pair = redundant calculation;
- trivial pair = usefull, but straightforwardly reduces to 0.

Buchberger Algorithm

1. Ore algebras = special case of skew polynomial rings of solvable type
2. Reduction:
   - order on S-polynomials,
   - order on reducers,
   - modular arithmetic,
   - trace lifting.
3. Strategies:
   - normal strategy
   - sugar strategy
4. Example: $J_n(x)$ for the algebra $\mathbb{Q}(n, x)[S_n, D_x; \ldots]$ with lexicographical order $S_n \prec D_x$:

$$p = x^2 D_x^2 + x D_x + x^2 - n^2, \quad q = x D_x S_n + (n+1) S_n - x,$$
$$\theta\, p = x^2 D_x^2, \qquad \theta\, q = x D_x S_n.$$

$$\begin{aligned}\mathrm{Spoly}(p, q) &= S_n p - x D_x q \\ &= -x(n+1) D_x S_n + x^2 D_x \\ &\qquad + (x^2 - (n+1)^2) S_n + x \\ &\to x^2 D_x + x^2 S_n - nx = r\end{aligned}$$

This new r is already a contiguity relation!

$$\mathrm{Spoly}(q, r) = xq - S_n r = -x^2 S_n^2 + 2x(n+1) S_n - x^2$$

(Stop calculation here.)

The FGLM Algorithm

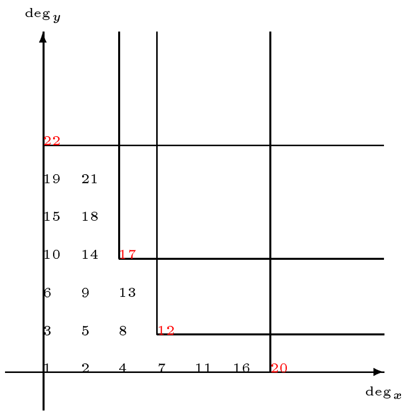
INPUT: $\begin{cases} \mathcal{A} = \mathbb{K}\langle x_1, \dots, x_n; \dots \rangle, \mathfrak{I} \text{ a left ideal;} \\ \text{an algorithm } NF \text{ for normal forms in } \mathcal{A}/\mathfrak{I}; \\ \text{a term order } T \text{ on } \mathcal{A}; \end{cases}$

OUTPUT: a Gröbner basis of $\mathfrak{I}$ with respect to T.

ALGORITHM: consider terms in increasing order and find linear dependencies between their normal forms.

$\boxed{\text{Termination for zero-dimensional ideals only.}}$

Example: $\mathcal{A} = \mathbb{K}[x, y]$, T is the total degree order with $x < y$.



Terms reduced: 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4, x^5, x^2y^3, xy^4, y^5, x^6, xy^5, y^6.

Leading terms of the computed Gröbner basis: x^3y, x^2y^3, x^6, y^6.

The FGLM algorithm

1. Original problem (commutative case):
   - compute a Gröbner basis by a change of term ordering
   - reduction by original Gröbner basis = taking a normal form
   - linear dependencies obtained by Gaussian elimination
2. Non zero-dimensional case: Gröbner walk
3. Example:
   - original Gröbner basis is

$$y^{18} + 16y^9 - 128y^5 + 64,$$
$$64x + y^{14} + 8y^5 - 128y$$

   - equivalently, normal form obtained by:
     (a) substituting x by $2y - y^5/8 - y^{14}/128$;
     (b) reducing modulo $y^{18} + 16y^9 - 128y^5 + 64$.
   - final Gröbner basis is

$$4x^3y - 2x^4 - y^3, \qquad x^2y^3 - 2,$$
$$x^6 + 2y^5 + xy^4 - 16x + 1,$$
$$y^6 - 8xy + 4x^2$$

Algorithmic ∂-Finite Closures

Compute an annihilating system for $f+g$, $f\times g$, $\partial\cdot f$ for ∂-finite functions f and g.

Example: the product $h = T_n(x) \times \dfrac{e^{-ux}}{\sqrt{1-x^2}}$.

Ore algebra: $\mathbb{O} = \mathbb{Q}(x,n,u)\langle D_x, S_n, D_u\rangle$.

Term order: total degree in $D_x \succ S_n \succ D_u$.

Gröbner basis for $f = T_n(x)$:

$$D_u, \qquad (x^2-1)D_x - nS_n + nx, \qquad S_n^2 - 2xS_n + 1.$$

Gröbner basis for $g = \frac{e^{-ux}}{\sqrt{1-x^2}}$:

$$D_u + x, \qquad S_n - 1, \qquad (x^2-1)D_x + ux^2 + x - u.$$

$\longrightarrow$ **Normal forms** in

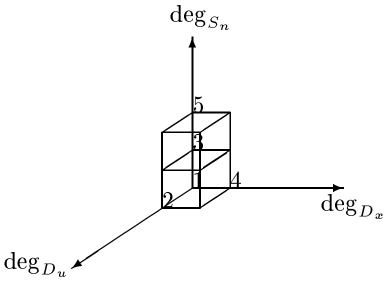
$$\mathbb{O}\cdot f = \mathbb{Q}(x,n,u)f \oplus \mathbb{Q}(x,n,u)(S_n\cdot f) \quad \text{and} \quad \mathbb{O}\cdot g = \mathbb{Q}(x,n,u)g.$$

Algorithmic ∂-Finite Closures

1. Goal: $(\mathrm{sys}_i)_{i\in I} \longrightarrow \mathrm{sys}$
2. Algorithm:
   - normalize the action of the successive $\boldsymbol{\partial}$ at the input expression
   - look for linear combinations
   - order of calculations: à la FGLM
   - termination due to 0-dimensionality
3. Example of $\times$:
   - action on $\otimes$
   - normal forms

In $\mathbb{O}$:

$$1 \prec \boxed{D_u} \prec S_n \prec \boxed{D_x} \prec D_u^2 \prec D_u S_n \prec D_u D_x \prec \boxed{S_n^2} \prec S_n D_x \prec D_x^2 \dots$$



Normal forms in $\mathbb{O} \cdot f \otimes \mathbb{O} \cdot g$:

$$\begin{aligned}
1 \cdot (f \otimes g) &= f \otimes g, \\
\boxed{D_u} \cdot (f \otimes g) &= (D_u \cdot f) \otimes g + f \otimes (D_u \cdot g) = -x(f \otimes g), \\
S_n \cdot (f \otimes g) &= (S_n \cdot f) \otimes (S_n \cdot g) = (S_n \cdot f) \otimes g, \\
\boxed{D_x} \cdot (f \otimes g) &= (D_x \cdot f) \otimes g + f \otimes (D_x \cdot g) \\
&= (x^2 - 1)^{-1}[n(S_n \cdot f) \otimes g - (ux^2 + (n+1)x - u)(f \otimes g)], \\
\boxed{S_n^2} \cdot (f \otimes g) &= (S_n^2 \cdot f) \otimes (S_n^2 \cdot g) = 2x(S_n \cdot f) \otimes g - f \otimes g.
\end{aligned}$$

Gröbner basis **computed** for $f \otimes g$:

$$\boxed{D_u} + x, \quad (x^2 - 1)\boxed{D_x} - nS_n + ux^2 + (n+1)x - u, \quad \boxed{S_n^2} - 2xS_n + 1.$$

Example of $\otimes$

1. Emphasize
   - colored terms: leading terms in input GB's and rewritten to obtain NF
   - boxed terms: leading terms in output GB that allow to avoid considering their multiples
2. Simpler action on $\oplus$

Graduations and Filtrations of Modules

Filtered $\mathbb{K}$-algebra $\mathbb{A}$	$\to$	algebra of skew polynomials;
Module $\mathfrak{m}$ over $\mathbb{A}$	$\to$	module of functions;
Totally ordered monoid M	$\to$	generalized degrees/set of terms.

M-graduation of $\mathfrak{m}$ = family $(\Gamma_\mu)_{\mu\in\mathrm{M}}$ of $\mathbb{K}$-vector spaces such that:

$$\begin{cases} \bigoplus_{\mu\in\mathrm{M}} \Gamma_\mu = \mathfrak{m}, \\ \mathcal{G}_\mu \Gamma_{\mu'} \subseteq \Gamma_{\mu\mu'}, \quad \text{whenever } (\mu,\mu') \in \mathrm{M}^2. \end{cases}$$

M-filtration of $\mathfrak{m}$ = family $(\Phi_\mu)_{\mu\in\mathrm{M}}$ of vector spaces such that:

$$\begin{cases} \Phi_\mu \subseteq \Phi_{\mu'}, & \text{whenever } \mu \preceq \mu' \text{ in M}, \\ \bigcup_{\mu\in\mathrm{M}} \Phi_\mu = \mathfrak{m}, \\ \mathcal{F}_\mu \Phi_{\mu'} \subseteq \Phi_{\mu\mu'}, & \text{whenever } (\mu,\mu') \in \mathrm{M}^2. \end{cases}$$

Associated M-graded module $\mathrm{gr}_\Phi\, \mathfrak{m}$ of the M-filtered module $\mathfrak{m}$:

$$\mathrm{gr}_\Phi\, \mathfrak{m} = \bigoplus_{\mu\in\mathrm{M}} \Gamma_\mu \qquad \text{where} \qquad \begin{cases} \Gamma_\mu = \Phi_\mu / \Upsilon_\mu, \\ \Upsilon_\mu = \bigcup_{\mu'\in\mathrm{M},\, \mu'\prec\mu} \Phi_{\mu'}. \end{cases}$$

$\boxed{\textbf{gr}_\Phi\, \mathfrak{m} \textbf{ is a } \mathrm{gr}_\mathcal{F}\, \mathbb{A}\textbf{-module!}}$

Graduations and Filtrations of Modules

1. Intuition: in order to apply results of modules over a graded algebra, make:
   - the (only) filtered algebra into a graded algebra
   - the (only) filtered module into a graded module
2. More or less refined extension of univariate degree
   - $\mathrm{M} = \mathbb{N}$ for total degree and dimension theory
   - $\mathrm{M} = [\mathbf{x}]$ for term orderings and Gröbner bases
3. Generalization to a sorted set X endowed with a compatible action of M?
   - compatibility:

$$\mu \preceq \mu' \implies \mu x \preceq \mu' x$$
$$x \preceq x' \implies \mu x \preceq \mu x'$$

   - X-graduation of a module $\mathfrak{m}$ would be

$$\bigoplus_{x\in X} \Gamma_x = \mathfrak{m}, \qquad \mathcal{G}_\mu \Gamma_x \subseteq \Gamma_{\mu x}.$$

   - X-filtration of a module $\mathfrak{m}$ would be

$$x \preceq x' \implies \Phi_x \subseteq \Phi_{x'},$$
$$\bigcup_{x\in X} \Phi_x = \mathfrak{m}, \qquad \mathcal{F}_\mu \Phi_x \subseteq \Phi_{\mu x}.$$

Never seen any (complete) theory yet!

Gröbner Bases in Filtered Free Modules

$(\mathrm{M} \times I)$-filtered free module $\mathfrak{f} = \bigoplus_{i\in I} \mathbb{A} \cdot e_i = \bigcup_{(\mu,i)\in \mathrm{M}\times I} \Phi_{\mu,i}$
over the M-filtered $\mathbb{K}$-algebra $\mathbb{A} = \bigcup_{\mu\in\mathrm{M}} \mathcal{F}_\mu$:

Degree: $\deg : \mathfrak{m} \to \mathrm{M} \times I, \quad \deg f = \min\{(\mu, i) \mid f \in \Phi_{\mu,i}\};$

Principal symbol: $\theta : \mathfrak{m} \to \mathrm{gr}_\Phi\, \mathfrak{m}, \qquad \theta\, f = f + \Upsilon_{\deg f}.$

Left-submodule $\mathfrak{m} \subseteq \mathfrak{f}$, with **finite** basis G:

$\deg G$ = left-stable part of $M \times I$ spanned by the $\deg f$ for $f \in G$;

$\theta\, G$ = left-sided module of $\mathrm{gr}_\Phi\, \mathfrak{m}$ spanned by the $\theta\, f$ for $f \in G$.

G is a **Gröbner bases** for $\mathfrak{m}$ w.r.t. its filtration

$$\Updownarrow$$

$$\deg G = \deg \mathfrak{m}$$

$$\Updownarrow$$

$$\theta\, G = \theta\, \mathfrak{m}$$

Gröbner Bases in Filtered Free Modules

1. Never seen any (complete) theory yet!
2. Termination:
   - same conditions on M as previously
   - well-foundedness of $\mathrm{M} \times I$ (I need not be finite!)
3. Stable part of $\mathrm{M} \times I$ = family of stairs indexed by I
4. Examples
   - triangularization of matrices of linear operators
     * $(N \times M)$-matrices $(a_{i,j})$ with coefficients in algebra $\mathbb{A} = \mathbb{K}\langle \mathbf{x}, \boldsymbol{\partial} \rangle$
     * $I = \{1, \dots, M\}$, where M is the number of columns
     * encode each line as element of $\mathbb{A}^M$: $\sum_i a_i \cdot e_i$
     * term order: first sort the e_i's lexicographically, refine by any order on the x_i's and ∂_i's
   - $\mathbb{K}[\mathbf{x}, y]$ viewed as a $\mathbb{K}[\mathbf{x}]$-module with (infinite) free basis $I = \{y^n,\ n \in \mathbb{N}\}$

Syzygies

For $\mathfrak{m}$ a finitely generated $\mathbb{A}$-module, $\mathfrak{m} = \sum_{i=1}^{n} \mathbb{A} \cdot g_i$, define:

$$\mathbb{A}^n \times \mathfrak{m}^n \to \mathfrak{m}, \qquad (s_1, \dots, s_n, p_1, \dots, p_n) \mapsto \sum_{i=1}^{n} s_i \cdot p_i;$$

$$\phi : \mathbb{A}^n \to \mathfrak{m}, \qquad (s_1, \dots, s_n) \mapsto \sum_{i=1}^{n} s_i \cdot g_i.$$

(First) Module of syzygies is the finitely generated $\mathbb{A}$-module:

$$\mathrm{Syz}(g_1, \dots, g_n) = \ker \phi = \sum_{i=1}^{n'} \mathbb{A} \cdot g_i'.$$

Iteration yields an exact sequence:

$$\cdots \longrightarrow \mathbb{A}^{n''} \longrightarrow \mathbb{A}^{n'} \longrightarrow \mathbb{A}^{n} \longrightarrow \mathfrak{m} \xrightarrow{\pi} 0.$$
$$\mathrm{im}\, \phi'' = \ker \phi', \qquad \mathrm{im}\, \phi' = \ker \phi, \qquad \mathrm{im}\, \phi = \ker \pi.$$

Calculation:

1. introduce the free module $\mathfrak{f} = \bigoplus_{i=0}^{n} \mathbb{A} \cdot e_i$;
2. eliminate e_0 from the submodule generated by the $g_i \cdot e_0 - e_i$.

Syzygies

1. Finiteness of exact sequence in the case $\mathbb{A}$ is an Ore algebra, because of noetherianity of associated graded ring of $\mathbb{A}$ (Hilbert).
2. Result of calculation
   - polynomials free of e_0 = basis of the syzygy module
   - coefficients of e_0 in computed GB = GB for the original module
   - computed GB expressed in terms of original polynomials (!)

Syzygetic Calculations

Indefinite ∂^{-1} of ∂-finite functions.

Goal: compute all ∂-finite F such that $\partial' \cdot F = f$ for a ∂-finite f w.r.t. $\mathbb{O} = \mathbb{K}[\mathbf{x}, x'][\boldsymbol{\partial}, \partial'; \boldsymbol{\sigma}, \sigma', \boldsymbol{\delta}, \delta']$.

Idea: assuming natural boundaries, and $\operatorname{Ann} f = \sum_{i=1}^{n} \mathbb{A} g_i$,

1. eliminate x' to obtain $N_i = P_i(\mathbf{x}, \boldsymbol{\partial}) + Q_i(\mathbf{x}, \boldsymbol{\partial})\partial' \in \operatorname{Ann} f$;
2. for each i, determine an ideal $\mathfrak{I}_i$ such that $\mathfrak{I}_i Q_i \cdot f = 0$:
   (a) compute $\mathrm{Syz}(Q_i, g_1, \dots, g_n) \subseteq \mathbb{O} \oplus \mathbb{O}^n$;
   (b) project onto the first component $\mathbb{O}$ to obtain $\mathfrak{I}_i$;
   (c) $N_i \cdot F$ is a constant w.r.t. x', assumed to be zero, so that:
   $$0 = \mathfrak{I}_i N_i \cdot F = \mathfrak{I}_i P_i \cdot F + \mathfrak{I}_i Q_i \cdot f = \mathfrak{I}_i P_i \cdot F;$$
3. combine $(\operatorname{Ann} f)\partial'$ and the $\mathfrak{I}_i Q_i$ into a description of F.

Addition of ∂-finite functions.

$$\operatorname{Ann} f = \sum_{i=1}^{r} \mathbb{A} p_i, \qquad \operatorname{Ann} g = \sum_{i=1}^{s} \mathbb{A} q_i, \qquad \mathbb{O} = \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}; \boldsymbol{\sigma}, \boldsymbol{\delta}].$$

Idea: in view of $\operatorname{Ann} f \cap \operatorname{Ann} g \subseteq \operatorname{Ann}(f+g)$,

1. compute $\mathrm{Syz}(p_1, \dots, p_r, q_1, \dots, q_s) \subseteq \mathbb{O}^r \oplus \mathbb{O}^s$;
2. project onto the first component $\mathbb{O}^r$ to obtain $\operatorname{Ann} f \cap \operatorname{Ann} g$.

Syzygetic Calculations

1. Indefinite ∂^{-1}:
   - ∂-finite analogue of Gosper's problem (is F element of $\mathbb{O} \cdot f$?)
   - no ∂-finite description of F by $(\operatorname{Ann} f)\partial'$
   - eliminations assumed to yield non-trivial results
   - $\partial' \cdot (N_i \cdot F) = N_i \cdot (\partial' \cdot F) = N_i \cdot f = 0$: $N_i \cdot F$ is constant w.r.t. x'
   - only heuristic (and restricted) method
   - example:
   $$F(x,y) = -\int_y^{+\infty} f(x,t)\,dt \quad \text{for} \quad f(x,y) = \frac{1}{(1+xy+y^2)^2}.$$
   $$\mathbb{O}((1+xy+y^2)D_x + 2y) + \mathbb{O}((1+xy+y^2)D_y + 2x + 4y) \subseteq \operatorname{Ann} f.$$
   $$N = P + QD_y, \quad \begin{cases} P = -x(x-2)(x+2)D_x^2 - 4(x^2+1)D_x, \\ Q = x(x-2)(x+2)D_x + xD_y + 2(x^2+1). \end{cases}$$
   $\mathfrak{I}$ too large to be displayed; final GB is
   $$x(x^2-4)(1+xy+y^2)D_x^2 + 4(x^2+1)(1+xy+y^2)D_x \\ -(2x^2y^2 + 2y^2 + 6xy + 2)D_y.$$

2. Addition:
   - note:
   $$\phi_{p,q}(a,b) = \sum_{i=1}^{r} a_i p_i + \sum_{i=1}^{s} b_i q_i = 0 \\ \Longrightarrow \phi_p(a) = \sum_{i=1}^{r} a_i p_i \in \operatorname{Ann} f \cap \operatorname{Ann} g.$$
   - for $\mathbb{O}^r \oplus \mathbb{O}^s \xrightarrow{\pi} \mathbb{O}^r$ and $\mathbb{O}^r \oplus \mathbb{O}^s \xrightarrow{\phi_{p,q}} \mathbb{O}$:

$$\begin{tikzcd}
\mathbb{O}^r \oplus \mathbb{O}^s \arrow[r, "\delta=\phi_p\oplus\phi_q"] & \operatorname{Ann} f \oplus \operatorname{Ann} g \arrow[r, "\alpha"] & \mathbb{O} \\
\mathrm{Syz} = \ker \phi_{p,q} \arrow[u, "\iota"] \arrow[r, "\delta"] \arrow[d, "\pi"] & \delta(\mathrm{Syz}) \subseteq \{(u,-u)\} \arrow[u, "\iota"] \arrow[r, "\alpha"] \arrow[d, "\pi"] & 0 \arrow[u, "\iota"] \\
\pi(\mathrm{Syz}) \arrow[r, "\phi_p"] & \operatorname{Ann} f \cap \operatorname{Ann} g &
\end{tikzcd}$$

Holonomic Functions in Computer Algebra

Part III

Holonomy and Holonomic Functions

Holonomy and Holonomic Functions

1. Three fundamental types of operator algebras; their ideal and module structures;
2. Graduations and filtrations of algebras and modules; Hilbert dimensions;
3. Bernstein-like inequalities;
4. Complete definitions of holonomy in each of the three fundamental cases;
5. Dimension and elimination; fundamental lemma;
6. Variant definition for the recurrence case in terms of the differential case;
7. Holonomic closures;
8. Rational holonomy vs holonomy.

Analytic Continuation Problem

Gel'fand's Problem (ICM, 1954):

For $P \in \mathbb{R}[x_1, \dots, x_r]$ and $f(\mathbf{x}) \in C_0^\infty(\mathbb{R}^r)$, the integral

$$\Gamma_f(\lambda) = \int_{\mathbb{R}^r} P^\lambda(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x}$$

is analytic in $\{\lambda \mid \operatorname{Re} \lambda > 0\}$.

Is Γ_f continuable as a meromorphic function of λ?

Two Difficult Proofs by Atiyah (1968), and by Bernstein and Gel'fand (1968):

Hironaka's resolution of singularities.

Elementary Solution by Bernstein (1972):

There exist a polynomial $b \in \mathbb{C}[\lambda]$ and a **linear differential operator** $L \in \mathbb{C}[\lambda]\langle x_1, \dots, x_r, \partial_1, \dots, \partial_r\rangle$ such that

$$b(\lambda)\Gamma_f(\lambda) = \int_{\mathbb{R}^r} P^{\lambda+1}(\mathbf{x})(L \cdot f)(\mathbf{x})\, d\mathbf{x} = \Gamma_{L \cdot f}(\lambda + 1).$$

Using the theory of holonomy (Bernstein, 1971).

Analytic Continuation Problem

1. Holonomy: studied started by Joseph Bernstein in the 1970's, further developed by Masaki Kashiwara (and others)
2. Solution to problem:
   - meromorphic continuation by recurrence
   - poles are on arithmetical progressions
   - further developments of the theory: functional equations obtained for formal series or analytical function instead of polynomial P
3. Bernstein-Sato polynomial $b(\lambda)$:
   - existence by holonomy (Bernstein)
     * $\mathcal{W}_r(\mathbb{K}(s))$-module $\mathbb{K}(s)[\mathbf{x}, p^{-1}] \cdot p^s$ is holonomic
     * so is its $\mathcal{W}_r(\mathbb{K}(s))$-submodule generated by p^s
     * finite length: for sufficiently large k,

       $$p^k p^s \in \mathcal{W}_r(\mathbb{K}(s))p^{k+1} \cdot p^s$$

     * clearing denominators, shifting by S_λ^{-k},

       $$b(\lambda)p^s \in \mathcal{W}_r(\mathbb{K}[s])p \cdot p^s$$

   - zeroes are negative rational numbers (Kashiwara)
4. Example: $r = 1$, $P = x^2$,

$$(2\lambda + 1)(2\lambda + 2) \int_{\mathbb{R}} x^{2\lambda} f(x)\, dx = \int_{\mathbb{R}} x^{2(\lambda+1)} f(x)\, dx.$$

Holonomic Identities

Zeilberger's Observation (1990):

Many special functions and sequences are holonomic.
The holonomic class is closed under:

$$+,\ \times,\ \partial,\ \int,\ \sum,\ \int_a^b,\ \sum_a^b,\ \dots$$

$$2T_m(x)\times T_n(x) - T_{n+m}(x) - T_{n-m}(x) = 0.$$

$T_n(x) =$ Chebychev orthogonal polynomials.

$$\int_0^{+\infty} xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx = -\frac{\ln(1-a^4)}{2\pi a^2}.$$

$$a^3(a^4-1)\phi^{(4)}(a) + 4a^2(4a^4-1)\phi^{(3)}(a) + a(73a^4+3)\phi''(a)$$
$$+(103a^4-3)\phi'(a) + 32a^3\phi(a) = 0.$$

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3 = \frac{n}{2}8^n + 8^n - \frac{3n}{4}2^n\binom{2n}{n}.$$

$$(n+1)u_{n+2} - (7n+12)u_{n+1} - 4(2n+1)u_n = (18n-20)8^n.$$

Holonomic Identities

1. Holonomic class
   - large variety of objects, computable class
   - ubiquitous nature
   - finitely specifiable $\Rightarrow$ computer algebra
2. Decidability of the identities
   - simplifications of expressions in terms of special functions by computer algebra
   - parameterized integrations and summations

Experimental Facts

$$f = \exp(x^2).$$

$$h_n = \text{ dimension of the } \mathbb{Q}\text{-vector space generated by the } x^i \frac{d^j f}{dx^j}$$
for $i \geq 0$, $j \geq 0$, and $i + j \leq n$.

n	0	1	2	3	4	5	6	7	8	9	10
h_n	1	2	3	4	5	6	7	8	9	10	11
$\Delta_n h_n$		1	1	1	1	1	1	1	1	1	1

$h_n = n^1 + 1$, the exponent 1 is the number of variables!

$$f = \frac{1}{x^2 - x + y}.$$

$$h_n = \text{ dimension of the } \mathbb{Q}\text{-vector space generated by the } x^i y^j \frac{d^{k+l} f}{dx^k dy^l}$$
for $i \geq 0$, $j \geq 0$, $k \geq 0$, $l \geq 0$, and $i + j + k + l \leq n$.

n	0	1	2	3	4	5
h_n	1	5	13	25	41	61
$\Delta_n h_n$		4	8	12	16	20
$\Delta_n^2 h_n$			4	4	4	4

$h_n = 2n^2 + 2n + 1$, the exponent 2 is the number of variables!

$$f = \frac{1}{\sqrt{y^3 - y^2 + xy}}.$$

$$h_n = \text{ dimension of the } \mathbb{Q}\text{-vector space generated by the } x^i y^j \frac{d^{k+l} f}{dx^k dy^l}$$
for $i \geq 0$, $j \geq 0$, $k \geq 0$, $l \geq 0$, and $i + j + k + l \leq n$.

n	0	1	2	3	4	5
h_n	1	5	14	29	50	77
$\Delta_n h_n$		4	9	15	21	27
$\Delta_n^2 h_n$			5	6	6	6

$h_n = 3n^2 + 2$ for large enough n, 2 is the number of variables!

$$f = J_\mu(x) J_\nu(y).$$

$$h_n = \text{ dimension of the } \mathbb{Q}\text{-vector space generated by the } x^i y^j \frac{d^{k+l} f}{dx^k dy^l}$$
for $i \geq 0$, $j \geq 0$, $k \geq 0$, $l \geq 0$, and $i + j + k + l \leq n$.

n	0	1	2	3	4	5
h_n	1	5	15	35	68	116
$\Delta_n h_n$		4	10	20	33	48
$\Delta_n^2 h_n$			6	10	13	15

6	7	8	9	10	11	12
180	260	356	468	596	740	900
64	80	96	112	128	144	160
16	16	16	16	16	16	16

$h_n = 8n^2 - 24n + 36$ for large enough n, 2 is the number of variables!

$f = \cos(xy - z).$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by

$$\text{the } x^i y^j z^k \frac{d^{t+u+v} f}{dx^t dy^u dz^v} \text{ for } \begin{cases} i \geq 0,\ j \geq 0,\ k \geq 0,\ t \geq 0,\ u \geq 0,\ v \geq 0, \\ \text{and } i + j + k + t + u + b \leq n. \end{cases}$$

n	0	1	2	3	4	5
h_n	1	7	19	39	69	111
$\Delta_n h_n$		6	12	20	30	42
$\Delta_n^2 h_n$			6	8	10	12
$\Delta_n^3 h_n$				2	2	2

$h_n = \frac{n^3}{3} + 2n^2 + \frac{11n}{3} + 1,$ the exponent 3 is the number of variables!

$f = \binom{x}{y}.$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by the $x^i y^j \binom{x+k}{y+l}$

for $i \geq 0$, $j \geq 0$, $k \geq 0$, $l \geq 0$, and $i + j + k + l \leq n$.

n	0	1	2	3	4	5	6
h_n	1	5	12	22	35	51	70
$\Delta_n h_n$		4	7	10	13	16	19
$\Delta_n^2 h_n$			3	3	3	3	3

$h_n = \frac{3n^2}{2} + \frac{5n}{2} + 1,$ the exponent 2 is the number of variables!

$f = 1.$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by

$$\text{the } x_1^{\alpha_1} \dots x_r^{\alpha_r} \frac{d^{\beta_1 + \dots + \beta_r} f}{dx_1^{\beta_1} \dots dx_r^{\beta_r}} \text{for } \alpha_k \geq 0,\ \beta_k \geq 0, \text{ and } \sum_{k=1}^r \alpha_k + \sum_{k=1}^r \beta_k \leq n.$$

$h_n = \binom{n+r}{r} = \Theta(n^r),$ the exponent r is the number of variables!

$f = \exp(\sin(x)).$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by the $x^i \frac{d^j f}{dx^j}$

for $i \geq 0$, $j \geq 0$, and $i + j \leq n$.

n	0	1	2	3	4	5	6
h_n	1	3	6	10	15	21	28
$\Delta_n h_n$		2	3	4	5	6	7
$\Delta_n^2 h_n$			1	1	1	1	1

$\rightarrow$ $h_n = \frac{n^2}{2} + \frac{3n}{2} + 1,$

the exponent 2 is **not**

the number of variables!?

(But $2 \geq 1$.)

$f = \frac{1}{x^2 + y^2}.$

$h_n =$ dimension of the $\mathbb{Q}$-vector space generated by

$$\text{the } \frac{x^i y^j}{(x+k)^2 + (y+l)^2} \text{ for } \begin{cases} i \geq 0,\ j \geq 0,\ k \geq 0,\ l \geq 0, \\ \text{and } i + j + k + l \leq n. \end{cases}$$

$h_n = \binom{n+4}{4} = \Theta(n^4),$ the exponent 4 is **not** the number of variables!?

(But $4 \geq 2$.)

Fundamental Types of Operator Algebras

Three types of associative algebras over the ring $\mathbb{A}$:

$$\begin{aligned}
\mathcal{W}_1(\mathbb{A}) &= \mathbb{A}\langle x, \partial; \partial x - (x\partial + 1)\rangle, \\
\mathcal{R}_1(\mathbb{A}) &= \mathbb{A}\langle x, \sigma; \sigma x - (x + 1)\sigma\rangle, \\
\mathcal{Q}_1(\mathbb{A}) &= \mathbb{A}\langle x, \tau; \tau x - qx\tau\rangle \qquad (q \in \mathbb{A} \setminus \{0\});
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_r(\mathbb{A}) &= \mathbb{A}\langle x_1, \dots, x_r, \partial_1, \dots, \partial_r; \partial_i x_j - (x_j \partial_i + \delta_{i,j})\rangle, \\
\mathcal{R}_r(\mathbb{A}) &= \mathbb{A}\langle x_1, \dots, x_r, \sigma_1, \dots, \sigma_r; \sigma_i x_j - (x_j + \delta_{i,j})\sigma_i\rangle, \\
\mathcal{Q}_r(\mathbb{A}) &= \mathbb{A}\langle x_1, \dots, x_r, \tau_1, \dots, \tau_r; \tau_i x_j - q^{\delta_{i,j}} x_j \tau_i\rangle.
\end{aligned}$$

The $\mathcal{W}_r(\mathbb{A})$'s are known as the Weyl algebras.

For the field $\mathcal{M}$ of meromorphic functions, $\operatorname{End}_{\mathbb{C}} \mathcal{M}$ contains:

$$D : f \mapsto (z \mapsto f'(z)), \quad S : f \mapsto (z \mapsto f(z+1)), \quad T : f \mapsto (z \mapsto f(qz)),$$
$$Z : f \mapsto (z \mapsto zf(z)), \qquad (q \in \mathbb{C} \setminus \{0\}).$$

In the algebra $\operatorname{End}_{\mathbb{C}} \mathcal{M}$:

$$DZ = ZD + 1, \qquad SZ = (Z+1)S, \qquad TZ = qZT.$$

Therefore:

- $\mathcal{W}_1(\mathbb{C}) \simeq$ subalgebra of $\operatorname{End}_{\mathbb{C}} \mathcal{M}$ generated by Z and D;
- $\mathcal{R}_1(\mathbb{C}) \simeq$ subalgebra of $\operatorname{End}_{\mathbb{C}} \mathcal{M}$ generated by Z and S;
- $\mathcal{Q}_1(\mathbb{C}) \simeq$ subalgebra of $\operatorname{End}_{\mathbb{C}} \mathcal{M}$ generated by Z and T.

Fundamental Types of Operator Algebras

1. Multivariate case $\mathcal{A}_r$ = external tensor product of r copies of $\mathcal{A}_1$ (where $\mathcal{A} = \mathcal{W}, \mathcal{R}, \mathcal{Q}$)
2. Mixed case = external tensor product $\mathcal{W}_r \otimes \mathcal{R}_s \otimes \mathcal{Q}_t$ (over the same ring $\mathbb{A}$)
3. Several possible extensions to algebras with inverse indeterminates
$$x^{-1}, \partial^{-1}, \sigma^{-1}, \tau^{-1};$$
stick to these in view of Gröbner bases
4. As an abuse of notation, single notation for the associative algebras and any of their concrete representations
5. Total degree: two notions
   - in the $2r$ generators
   - in the r non-x generators
   - well-defined in view of the commutations
   - usual properties with respect to $+$ and $\times$

Ideal and Module Structures

Ideal $\mathfrak{I}$ of a ring $\mathbb{A}$ = non-empty subset of $\mathbb{A}$ such that:

- left ideal: $\mathfrak{I}+\mathfrak{I}\subseteq\mathfrak{I}$, $\mathbb{A}\mathfrak{I}\subseteq\mathfrak{I}$;
- two-sided ideal: $\mathfrak{I}+\mathfrak{I}\subseteq\mathfrak{I}$, $\mathbb{A}\mathfrak{I}\mathbb{A}\subseteq\mathfrak{I}$.

$\longrightarrow$ Describes the linear functional equations satisfied by a function.

Module $\mathfrak{m}$ over a ring $\mathbb{A}$ = non-empty set such that:

- left module: $\mathfrak{m}+\mathfrak{m}\subseteq\mathfrak{m}$, $\mathbb{A}\cdot\mathfrak{m}\subseteq\mathfrak{m}$;
- two-sided module: $\mathfrak{m}+\mathfrak{m}\subseteq\mathfrak{m}$, $\mathbb{A}\cdot\mathfrak{m}\cdot\mathbb{A}\subseteq\mathfrak{m}$.

$\longrightarrow$ Suitable to describe the action of operators on a function.

Simple ring $\mathbb{A}$ = ring whose only two-sided ideals are 0 and $\mathbb{A}$.

$\mathcal{W}_r(\mathbb{K})$ and variants of $\mathcal{R}_r(\mathbb{K})$ and $\mathcal{Q}_r(\mathbb{K})$ are simple.

(char $\mathbb{K}=0$, q not a root of unity.)

Simplicity + action of operators on the left

$\longrightarrow$ study retricted to left ideals and modules.

Noetherian ring $\mathbb{A}$ = such that ideals and modules are finitely generated.

$\longrightarrow$ required for a theory of Gröbner bases.

Ideal and Module Structures

1. Examples of modules
   - $\mathcal{M}$ is a left module over each of the algebras $\mathcal{W}_1(\mathbb{C})$, $\mathcal{R}_1(\mathbb{C})$, and $\mathcal{Q}_1(\mathbb{C})$;
   - for each $f\in\mathcal{M}$, $\mathcal{W}_1(\mathbb{C})\cdot f$, $\mathcal{R}_1(\mathbb{C})\cdot f$, and $\mathcal{Q}_1(\mathbb{C})\cdot f$ are left modules.
2. Simplicity: examples of simple ring are, for $\mathbb{K}$ a field,

   $\mathbb{K}\langle x_1,\dots,x_r,\partial_1,\dots,\partial_r;\dots\rangle$

   $\mathbb{K}\langle x_1,\dots,x_r,\sigma_1,\dots,\sigma_r,\sigma_1^{-1},\dots,\sigma_r^{-1};\dots\rangle$

   $\mathbb{K}\langle x_1,\dots,x_r,x_1^{-1},\dots,x_r^{-1},\tau_1,\dots,\tau_r,\tau_1^{-1},\dots\tau_r^{-1};\dots\rangle$

   any variant with the x_i^{-1}'s

   Proofs of simplicity use that
   - $P\mapsto[\partial,P]=\partial P-P\partial=dP/dx$ decreases degree in x
   - $P\mapsto[x,P]=-dP/d\partial$ decreases degree in ∂
   - $P\mapsto[\sigma,P]$ decreases degree in x
   - $P\mapsto[x,P]\sigma^{-1}$ decreases degree in σ
   - $P\mapsto\tau P\tau^{-1}-q^{\deg_x P}P$ decreases degree in x
   - $P\mapsto[x,P]\tau^{-1}$ decreases degree in τ

   Non-simplicity of $\mathcal{R}_r(\mathbb{K})$ and $\mathcal{Q}_r(\mathbb{K})$:

   The left ideals $\mathcal{R}_1(\mathbb{K})\sigma_i$ and $\mathcal{Q}_1(\mathbb{K})\tau_i$ are in fact non-trivial two-sided ideals of $\mathcal{R}_1(\mathbb{K})$ and $\mathcal{Q}_1(\mathbb{K})$ respectively $\to$ but are degenerate cases $\sigma_i=0$ and $\tau_i=0$
3. Noetherianity: examples of noetherian rings are PID's and all the above algebras (over a field)

   Link to:
   - theory of standard bases
   - Gröbner bases
   - division theorems
   - Stafford's theorem (any left ideal in $\mathcal{W}_r(\mathbb{K})$ can be generated by two of its elements)

Graduations of Algebras and Modules

$\mathcal{A}_r$ one of $\mathcal{S}_r$ or $\mathcal{Q}_r$; $\mathfrak{m}$ a left $\mathcal{A}_r$-module.

$$\mathcal{S}_r(\mathbb{A}) = \mathbb{A}[x_1, \dots, x_r].$$

Graduation of $\mathcal{A}_r$ = family $(\mathcal{G}_n)_{n\in\mathbb{Z}}$ of vector spaces such that:

$$\bigoplus_{n\in\mathbb{Z}} \mathcal{G}_n = \mathcal{A}_r, \qquad \mathcal{G}_n\mathcal{G}_m \subseteq \mathcal{G}_{n+m}.$$

Example:

$$\mathcal{G}_n = \{\text{homogeneous } P \in \mathcal{A}_r(\mathbb{K}) \text{ with total degree } n\},$$

which additionally satisfy

$$\mathcal{G}_n = 0 \text{ when } n < 0, \qquad \mathcal{G}_0 = \mathbb{K}, \qquad \mathcal{G}_n \text{ is finite-dimensional.}$$

Graduation of $\mathfrak{m}$ = family $(\Gamma_n)_{n\in\mathbb{Z}}$ of vector spaces such that:

$$\bigoplus_{n\in\mathbb{Z}} \Gamma_n = \mathfrak{m}, \qquad \mathcal{G}_n\Gamma_m \subseteq \Gamma_{n+m}.$$

Example:

$$\Gamma_n = \sum_{i\in I} \mathcal{G}_{n-d_i} \cdot \mu_i \qquad \text{for fixed } d_i \text{ when} \qquad \mathfrak{m} = \sum_{i\in I} \mathcal{A}_r \cdot \mu_i.$$

If I is finite, the Γ_n's are finite-dimensional.

Graduations of Algebras and Modules

1. Total degree: in the r or $2r$ generators
2. Product of two homogeneous polynomials is homogeneous
3. Special case of graduations with respect to the additive monoid $\mathbb{Z}$ (or $\mathbb{N}$); Gröbner bases by considering refined graduations
4. Graduations of a module defined with respect to a given graduation of the algebra
5. Generator μ_i is assigned a formal degree d_i
6. Example: looking at the stairs for the graded module $\mathcal{S}_2/(x^2y - xy^2)$: (as vector spaces)

$$\Gamma_0 = \mathbb{K}, \qquad \Gamma_1 = \mathbb{K}x \oplus \mathbb{K}y,$$
$$\Gamma_2 = \mathbb{K}x^2 \oplus \mathbb{K}xy \oplus \mathbb{K}y^2,$$
$$\Gamma_p = \mathbb{K}x^p \oplus \mathbb{K}x^{p-1}y \oplus \mathbb{K}y^p \qquad (p \geq 3).$$

The action of $\mathcal{S}_2$ can be viewed as:

$$x \cdot x^p = x^{p+1}, \quad x \cdot x^{p-1}y = x^py, \quad x \cdot y^p = x^py,$$
$$y \cdot x^p = x^py, \quad y \cdot x^{p-1}y = x^py, \quad y \cdot y^p = y^{p+1}.$$

Filtrations of Algebras and Modules

$\mathcal{A}_r$ one of $\mathcal{S}_r$, $\mathcal{W}_r$, $\mathcal{R}_r$, or $\mathcal{Q}_r$; $\mathfrak{m}$ a left $\mathcal{A}_r$-module.

Filtration of $\mathcal{A}$ = family $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ of vector spaces such that:

$$\mathcal{F}_n \subseteq \mathcal{F}_{n+1}, \qquad \bigcup_{n\in\mathbb{Z}} \mathcal{F}_n = \mathcal{A}_r, \qquad \mathcal{F}_n\mathcal{F}_m \subseteq \mathcal{F}_{n+m}.$$

Example:

$$\mathcal{F}_n = \{P \in \mathcal{A}_r(\mathbb{K}) \text{ with total degree less than or equal to } n\},$$

which additionally satisfy

$$\mathcal{F}_n = 0 \text{ when } n < 0, \qquad \mathcal{F}_0 = \mathbb{K}, \qquad \mathcal{F}_n \text{ is finite-dimensional}.$$

Filtration of $\mathfrak{m}$ = family $(\Phi_n)_{n\in\mathbb{Z}}$ of vector spaces such that:

$$\Phi_n \subseteq \Phi_{n+1}, \qquad \bigcup_{n\in\mathbb{Z}} \Phi_n = \mathfrak{m}, \qquad \mathcal{F}_n\Phi_m \subseteq \Phi_{n+m}.$$

Example:

$$\Phi_n = \sum_{i\in I} \mathcal{F}_{n-d_i} \cdot \mu_i \qquad \text{for fixed } d_i \text{ when} \qquad \mathfrak{m} = \sum_{i\in I} \mathcal{A}_r \cdot \mu_i.$$

If I is finite, the Φ_n's are finite-dimensional.

Filtrations of Algebras and Modules

1. Total degree: in the (r or) $2r$ generators
2. Product of two "homogeneous" polynomials need not be "homogeneous"
3. Special case of filtrations with respect to the additive monoid $\mathbb{Z}$ (or $\mathbb{N}$); Gröbner bases by considering refined filtrations
4. Filtration of a module defined with respect to a given filtration of the algebra
5. Generator μ_i is assigned a formal degree d_i
6. Example: the $\mathcal{W}_2$-module spanned by $f = J_0(x+y)$ (notes for this given in appendix)

Associated Graded Algebras and Modules

$\mathcal{A}_r$ one of $\mathcal{S}_r$, $\mathcal{W}_r$, $\mathcal{R}_r$, or $\mathcal{Q}_r$; $\mathfrak{m}$ a left $\mathcal{A}_r$-module.

Associated graded algebra $\operatorname{gr}_{\mathcal{F}} \mathcal{A}_r$ of the filtered algebra $\mathcal{A}_r$:

$$\operatorname{gr}_{\mathcal{F}} \mathcal{A}_r = \bigoplus_{n \in \mathbb{Z}} \mathcal{G}_n \qquad \text{where} \qquad \mathcal{G}_n = \mathcal{F}_n / \mathcal{F}_{n-1}.$$

Product: defined from $\mathcal{G}_n \times \mathcal{G}_m$ to $\mathcal{G}_{n+m}$; extended by bilinearity.

Associated graded module $\operatorname{gr}_\Phi \mathfrak{m}$ of the filtered module $\mathfrak{m}$:

$$\operatorname{gr}_\Phi \mathfrak{m} = \bigoplus_{n \in \mathbb{Z}} \Gamma_n \qquad \text{where} \qquad \Gamma_n = \Phi_n / \Phi_{n-1}.$$

Action: defined from $\mathcal{G}_n \times \Gamma_m$ to Γ_{n+m}; extended by bilinearity.

$\boxed{\textbf{gr}_\Phi\, \mathfrak{m} \textbf{ is a } \textbf{gr}_{\mathcal{F}}\, \mathcal{A}_r\textbf{-module!}}$

Special cases:

$$\operatorname{gr}_{\mathcal{F}} \mathcal{S}_r = \mathcal{S}_r, \qquad \operatorname{gr}_{\mathcal{F}} \mathcal{W}_r \simeq \operatorname{gr}_{\mathcal{F}} \mathcal{R}_r \simeq \mathcal{S}_{2r}, \qquad \operatorname{gr}_{\mathcal{F}} \mathcal{Q}_r = \mathcal{Q}_r,$$

$\longrightarrow$ (classical and q-) algebraic geometry.

Associated Graded Algebras and Modules

1. Intuition: in order to apply results of modules over a graded algebra, make:
   - the (only) filtered algebra into a graded algebra
   - the (only) filtered module into a graded module
2. Product in $\operatorname{gr}_{\mathcal{F}} \mathcal{A}_r$: if $\alpha = P + \mathcal{F}_{n-1} = P' + \mathcal{F}_{n-1} \in \mathcal{G}_n$ and $\beta = Q + \mathcal{F}_{m-1} = Q' + \mathcal{F}_{m-1} \in \mathcal{G}_m$,
$$\begin{aligned} PQ - P'Q' &= (P - P')Q + P'(Q - Q') \\ &\in \mathcal{F}_{n-1}\mathcal{F}_m + \mathcal{F}_n\mathcal{F}_{m-1} \subseteq \mathcal{F}_{n+m-1} \end{aligned}$$
$$\longrightarrow \alpha\beta = PQ + \mathcal{F}_{n+m-1} = P'Q' + \mathcal{F}_{n+m-1} \in \mathcal{G}_{n+m}$$
3. Action on $\operatorname{gr}_\Phi \mathfrak{m}$: if $\alpha = P + \mathcal{F}_{n-1} = P' + \mathcal{F}_{n-1} \in \mathcal{G}_n$ and $\mu = f + \Phi_{m-1} = f' + \Phi_{m-1} \in \Gamma_m$,
$$\begin{aligned} P \cdot f - P' \cdot f' &= (P - P') \cdot f + P' \cdot (f - f') \\ &\in \mathcal{F}_{n-1} \cdot \Phi_m + \mathcal{F}_n \cdot \Phi_{m-1} \subseteq \Phi_{n+m-1} \end{aligned}$$
$$\longrightarrow \alpha \cdot \mu = P \cdot f + \Phi_{n+m-1} = P' \cdot f' + \Phi_{n+m-1} \in \Gamma_{n+m}$$
4. Dimensions: for the module $\mathfrak{m} = \sum_{i \in I} \mathcal{A}_r \cdot \mu_i$ (with I a finite set), and its filtration $\Phi_n = \sum_{i \in I} \mathcal{F}_{n-d_i} \cdot \mu_i$,
$$\dim_{\mathbb{K}} \Phi_n \mathfrak{m} = \sum_{j=0}^{n} \dim_{\mathbb{K}} \Gamma_j \operatorname{gr}_\Phi \mathfrak{m}$$
5. Link to algebraic geometry = link to dimension theories for modules over $\mathcal{S}_{2r}$ and $\mathcal{Q}_r$
6. Example: the $\mathcal{S}_4$-module $\operatorname{gr}(\mathcal{W}_2 \cdot J_0(x+y))$ (notes for this given in appendix)

Hilbert Dimension

Th. [Hilbert]: for a finitely generated graded $\mathcal{S}_r(\mathbb{K})$-module $\mathfrak{m}$, let

$$h(n) = \dim_{\mathbb{K}} \mathcal{G}_n \mathfrak{m} \qquad \text{(Hilbert function)}.$$

There exists positive integers d (dimension) and e (multiplicity), and a polynomial

$$p(n) = \frac{en^d}{d!} + \text{lower order terms} \qquad \text{(Hilbert polynomial)}$$

such that $h(n) = p(n)$ for all sufficiently large n.

$$0 \le d(\mathfrak{m}) \le r \quad (\mathfrak{m} \ne 0), \qquad d(0) = -\infty, \qquad d(\mathcal{S}_r(\mathbb{K})) = r.$$

Cor.: for a finitely generated filtered module $\mathfrak{m}$ over $\mathcal{W}_r(\mathbb{K})$ or $\mathcal{R}_r(\mathbb{K})$, let

$$h(n) = \dim_{\mathbb{K}} \Phi_n \mathfrak{m} = \sum_{j=0}^{n} \dim_{\mathbb{K}} \Gamma_j \operatorname{gr}_\Phi \mathfrak{m} \qquad \text{(Hilbert function)}.$$

There exists positive integers d (dimension) and e (multiplicity), and a polynomial

$$p(n) = \frac{en^d}{d!} + \text{lower order terms} \qquad \text{(Hilbert polynomial)}$$

such that $h(n) = p(n)$ for all sufficiently large n.

$$0 \le d(\mathfrak{m}) \le 2r \quad (\mathfrak{m} \ne 0),$$

$$d(0) = -\infty, \qquad d(\mathcal{S}_r(\mathbb{K})) = r, \qquad d(\mathcal{A}_r(\mathbb{K})) = 2r.$$

Hilbert Dimension

1. Dimension for ideals and modules: to refine between infinite-dimensional vector spaces
2. Dimensions: equivalent/related definitions by
   - analytic dimension of an algebraic variety (two concepts for two different filtrations) in the commutations case
   - Hilbert dimension in the commutative case
   - extends to q-case by standard bases (= "combinatorial" dimension)
   - Krull dimension of a quotient ring in the commutative case
   - Gel'fand-Kirillov dimension (all contexts)
   - homological dimension (all contexts)

   $\longrightarrow$ coincide if $\mathbb{K}$ is algebraically closed in the commutative case

   (link to algebraic variety detailed in next slide)
3. Hilbert dimension and multiplicity: independent from the graduation
4. Proof that $d(\mathcal{S}_r(\mathbb{K})) = r$ as an $\mathcal{A}_r(\mathbb{K})$-module: consider

$$\mathcal{A}_r / \sum_{i=1}^{r} \mathcal{A}_r \partial_i, \quad \mathcal{A}_r / \sum_{i=1}^{r} \mathcal{A}_r \sigma_i, \quad \text{and} \quad \mathcal{A}_r / \sum_{i=1}^{r} \mathcal{A}_r \tau_i$$

5. Example of $\mathcal{W}_2 \cdot J_0(x+y)$ (notes given in appendix)

Characteristic Variety

$\mathfrak{m}$: finitely generated filtered left module over $\mathcal{W}_r$ or $\mathcal{R}_r$

$$\Downarrow$$

$\mathrm{gr}_\Phi\,\mathfrak{m}$: graded $\mathcal{S}_{2r}$-module

$$\Downarrow$$

$\mathrm{Ann}\,\mathrm{gr}_\Phi\,\mathfrak{m} = \{P \in \mathcal{S}_{2r}, P \cdot \mu = 0 \text{ for } \mu \in \mathrm{gr}_\Phi\,\mathfrak{m}\}$: ideal in $\mathcal{S}_{2r}$

$$\Downarrow$$

$\mathrm{rad}\,\mathrm{Ann}\,\mathrm{gr}_\Phi\,\mathfrak{m}$: radical ideal, independent from Φ

$$\Downarrow$$

$\mathrm{Ch}(\mathfrak{m}) = \mathcal{Z}(\mathrm{rad}\,\mathrm{Ann}\,\mathrm{gr}_\Phi\,\mathfrak{m})$: algebraic variety of $\mathbb{K}^{2r}$,

the characteristic variety of $\mathfrak{m}$

dimension of the $\mathcal{W}_r$- or $\mathcal{R}_r$-module $\mathfrak{m}$

$=$

Hilbert dimension of $\mathrm{gr}_\Phi\,\mathfrak{m}$

$=$

dimension of the algebraic variety $\mathrm{Ch}(\mathfrak{m})$

Characteristic Variety

1. Extension to the q-case: is it possible to define a characteristic variety in the quantum planes (non-commutative geometry)?
2. Link to symplectic geometry:
   - orthogonality with respect to the matrix
   $$\begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}$$
   - simple result: ideal $\mathrm{Ann}\,\mathrm{gr}_\Phi\,\mathfrak{m}$ of
   $$\mathcal{S}_{2r}(\mathbb{K}) = \mathbb{K}[x_1, \dots, x_r, \xi_1, \dots, \xi_r]$$
   is closed under Poisson bracket:
   $$\{P, Q\} = \sum_{i=1}^{r} \frac{dP}{dx_i}\frac{dQ}{d\xi_i} - \frac{dP}{d\xi_i}\frac{dQ}{dx_i},$$
   which follows from
   $$\sigma([P, Q]) = \{\sigma(P), \sigma(Q)\}$$
   - deep result: radical ideal $\mathrm{rad}\,\mathrm{Ann}\,\mathrm{gr}_\Phi\,\mathfrak{m}$ is closed under Poisson bracket [Kashiwara, Kawai, Sato]
   - therefore, characteristic variety $\mathrm{Ch}(\mathfrak{m})$ has dimension $\geq r$: involutive variety
   - special case of lagrangian varieties for which
   $$V = V^\perp,$$
   thus of dimension r
3. Link between various definitions of dimensions (previous slide) can be better understood now
4. Example of $\mathcal{W}_2 \cdot J_0(x + y)$ (notes given in appendix)

Bernstein's Inequality

For any finitely generated left module $\mathfrak{m}$ over $\mathcal{A}_r = \mathcal{W}_r$, $\mathcal{R}_r$, or $\mathcal{Q}_r$,

$$d(\mathfrak{m}) \geq r \qquad \text{[Bernstein, Sabbah, \ldots]},$$

unless it is zero, and, assuming the injectivity of the σ_i's in the case of $\mathcal{R}_r$, or the injectivity of the x_i's and the τ_i's in the case of $\mathcal{Q}_r$.

Lemma: we have the injection

$$\begin{array}{ccc} \mathcal{F}_p\mathcal{A}_r & \to & \mathrm{Hom}_{\mathbb{K}}(\Gamma_p\mathfrak{m}, \Gamma_{2p}\mathfrak{m}). \\ P & \mapsto & (f \mapsto P \cdot f) \end{array}$$

Proof of lemma: induction on p.

For any non-zero $P \in \mathcal{F}_{p+1}$ such that $P \cdot \Gamma_{p+1}\mathfrak{m} = 0$,

derive a non-zero $Q \in \mathcal{F}_p$ such that $Q \cdot \Gamma_p\mathfrak{m} = 0$.

Proof of theorems: when p goes to infinity,

$$\begin{aligned} \dim_{\mathbb{K}} \mathcal{F}_p\mathcal{A}_r = \Theta\left(p^{2r}\right) &\leq \dim_{\mathbb{K}} \mathrm{Hom}_{\mathbb{K}}(\Gamma_p\mathfrak{m}, \Gamma_{2p}\mathfrak{m}) \\ &= \Theta\left(p^{d(\mathfrak{m})}\right)\Theta\left((2p)^{d(\mathfrak{m})}\right) = \Theta\left(p^{2d(\mathfrak{m})}\right). \end{aligned}$$

Bernstein's Inequality

1. A priori: dimensions are $-\infty$ or between 0 and $2r$
   $\longrightarrow$ reduce to $-\infty$ or between r and $2r$
2. Injectivity assumption:
   - $\mathfrak{m} \to \mathbb{K}\langle x, x^{-1}, \tau, \tau^{-1}\rangle \otimes \mathfrak{m}$ is an injection
   - both modules describe the same information
   - sequences identified when they match for $n \geq n_0$ and $m \geq m_0$ (n_0 and m_0 sufficiently large)
3. Proof of lemma: induction on p using:
   - Weyl algebras:
     * if $P \in \mathcal{F}_p$ involves x_i, $Q = [\partial_i, P] = dP/dx_i \in \mathcal{F}_{p-1}$
     * if $P \in \mathcal{F}_p$ involves ∂_i, $Q = [x_i, P] = -dP/d\partial_i \in \mathcal{F}_{p-1}$
   - recurrences (is this proof, mine, new?):
     * if $P \in \mathcal{F}_p$ involves x_i, $[\sigma_i, P] = \sigma_i Q$ for $Q \in \mathcal{F}_{p-1}$, and one needs σ_i^{-1}
     * if $P \in \mathcal{F}_p$ involves σ_i, $[x_i, P] = \sigma_i Q$ for $Q \in \mathcal{F}_{p-1}$, and one needs σ_i^{-1}
   - q-calculus (either gap, or inexplicit use of a different filtration and homological algebra in Sabbah's proof):
     * if $P \in \mathcal{F}_p$ involves x_i with degree d, $\tau_i P - q^d P\tau_i = \tau_i Q$ where $\deg_{x_i} Q \leq d-1$, and one needs τ_i^{-1}
       $\longrightarrow$ but Q need not be in $\mathcal{F}_{p-1}$
     * if $P \in \mathcal{F}_p$ involves τ_i with degree d, $q^d x_i P - P x_i = x_i Q$ where $\deg_{\tau_i} Q \leq d-1$, and one needs x_i^{-1}
       $\longrightarrow$ but Q need not be in $\mathcal{F}_{p-1}$

Holonomy

A finitely generated left module $\mathfrak{m}$ over $\mathcal{A}_r = \mathcal{W}_r$, $\mathcal{R}_r$, or $\mathcal{Q}_r$, is holonomic when

$$\mathfrak{m} = 0 \qquad \text{or} \qquad d(\mathfrak{m}) = r,$$

and when the σ_i's are injective in the case of $\mathcal{R}_r$, or the x_i's and the τ_i's are injective in the case of $\mathcal{Q}_r$. [Bernstein, Sabbah, …].

$$f \in \mathfrak{n} \text{ is holonomic in the left } \mathcal{A}_r\text{-module } \mathfrak{n}$$

$$\Updownarrow$$

$$f \in \mathcal{A}_r \cdot f \subseteq \mathfrak{m} \subseteq \mathfrak{n} \text{ for a holonomic submodule } \mathfrak{m}.$$

$\mathfrak{m} = \mathcal{S}_r$ (polynomial ring); $f = 1$ (constant function or sequence).

For holonomic $\mathcal{A}_r$-modules $\mathfrak{m}$ and $\mathfrak{n}$, $\mathfrak{m} \oplus \mathfrak{n}$ is a holonomic $\mathcal{A}_r$-module.

For holonomic f and g, $f + g$ is holonomic.

For holonomic $\mathcal{A}_r$-modules $\mathfrak{m}$ and $\mathfrak{n}$, $\mathfrak{m} \otimes \mathfrak{n}$ is a holonomic module over $\mathcal{A}_r \otimes \mathcal{A}_r \simeq \mathcal{A}_{2r}$ (tensoring over $\mathbb{K}$).

For holonomic f and g, $f(x)g(y)$ (in separated variables) is holonomic.

Holonomy

1. Simplified definition: to avoid
   - algebras with inverse indeterminates
   - need of homological algebra
2. Holonomic elements:
   - equivalently, $\mathcal{A}_r \cdot f$ is a holonomic module
   - holonomic module has all its elements holonomic
   - holonomic elements span a holonomic module
3. $f(x)g(y)$ is represented by $f \otimes g \in \mathfrak{m} \otimes \mathfrak{n}$

Dimension and Elimination

$\mathfrak{I}$ = left ideal of $\mathcal{A}_r = \mathcal{W}_r$, $\mathcal{R}_r$, or $\mathcal{Q}_r$.

$\mathcal{B}$ = subalgebra generated by u generators chosen from the $2r$ ones of $\mathcal{A}_r$.

$$d(\mathcal{A}_r/\mathfrak{I}) < u \implies \mathcal{A}_r \cap \mathfrak{I} \text{ is not empty.}$$

Any set of at most $2r - d - 1$ generators can be eliminated simultaneously.

Fundamental Lemma

f a holonomic element in a left $\mathcal{A}_r$-module

$\Downarrow$

$x_2, \dots, x_r$ can be eliminated simultaneously from $\operatorname{Ann} f$.

f a holonomic element in a left $\mathcal{A}_r$-module

$\Downarrow$

$\partial_2, \dots, \partial_r$ (resp. $\sigma_2, \dots, \sigma_r$, resp. $\tau_2, \dots, \tau_r$) can be eliminated simultaneously from $\operatorname{Ann} f$.

Dimension and Elimination, and Fundamental Lemma

1. More strongly: dimension is $d \Leftrightarrow$ any set of $2r - d - 1$ indeterminates can be eliminated
2. Proof: consider the linear map

$$\begin{array}{ccc} \mathcal{F}_p\mathcal{B} = \mathcal{F}_p\mathcal{A}_r \cap \mathcal{B} & \to & \Gamma_p(\mathcal{A}_r/\mathfrak{I}) = \mathcal{F}_p\mathcal{A}_r \cdot f \\ P & \mapsto & P \cdot f \end{array}$$

   for $f = 1 + \mathfrak{I} \in \mathcal{A}_r/\mathfrak{I}$, and note that for p going to infinity

$$\dim_{\mathbb{K}} \mathcal{F}_p\mathcal{B} = \Theta\left(p^u\right) > \dim_{\mathbb{K}} \Gamma_p(\mathcal{A}_r/\mathfrak{I}) = \Theta\left(p^d\right).$$

3. Proof: f is holonomic $\implies d = r$.

Differential Action on Sequences

Weyl algebras act on sequences! $(\mathcal{W}_1 = \mathbb{K}\langle x, \partial\rangle,\ \mathcal{R}_1 = \mathbb{K}\langle n, \sigma\rangle)$

$$
\begin{aligned}
(\partial \cdot u)(n) &= (n+1)u_{n+1} && = ((n+1)\sigma \cdot u)(n),\\
(x \cdot u)(n) &= u_{n-1} && = (\sigma^{-1} \cdot u)(n).
\end{aligned}
$$

For compatible actions of $\mathcal{W}_r$ and $\mathcal{R}_r$ and invertible actions of x_i and σ_i such that

$$\partial_i = (n_i+1)\sigma_i, \qquad x_i = \sigma_i^{-1}, \qquad n_i = x_i\partial_i, \qquad \sigma_i = x_i^{-1}:$$

$$
\begin{gathered}
(u(n_1,\dots,n_r)) \text{ is holonomic with respect to } \mathcal{R}_r\\
\Updownarrow\\
(u(n_1,\dots,n_r)) \text{ is holonomic with respect to } \mathcal{W}_r\\
\Updownarrow\\
\sum_{n_1,\dots,n_r} u(n_1,\dots,n_r)x_1^{n_1}\dots x_r^{n_r} \text{ is holonomic with respect to } \mathcal{W}_r
\end{gathered}
$$

Holonomy with respect to $\mathcal{W}_r$ suffices for holonomic summation:

$$P(x_1,\dots,x_r,\partial_1)\cdot u \quad \longrightarrow \quad Q(n_1,\sigma_1,\dots,\sigma_r)\cdot u = 0.$$

Differential Action on Sequences

1. $\mathcal{W}_r \Leftrightarrow \mathcal{W}_r$:
   - this is Zeilberger's definition of holonomy for sequences
   - relate to the discussion about D-finite functions vs P-recursive sequences
2. $\mathcal{R}_r \Leftrightarrow \mathcal{W}_r$:
   - any set of $2r-d-1$ indeterminates can be eliminated $\Leftrightarrow$ dimension is d
   - view the same module as a module over

     $$\mathbb{K}\langle x_1,\dots,x_r,\theta_1,\dots,\theta_r\rangle$$

     for $\theta_i = x_i\partial_i$; equivalently, consider both filtrations

     $$
     \begin{aligned}
     \mathcal{F}_n &= \{P \in \mathcal{A}_r(\mathbb{K}) \mid \deg_{\mathbf{x},\boldsymbol{\partial}} P \le n\},\\
     \mathcal{F}'_n &= \{P \in \mathcal{A}_r(\mathbb{K}) \mid \deg_{\mathbf{x}} P \le n, \deg_{\boldsymbol{\partial}} P \le n\},
     \end{aligned}
     $$

     and use $\mathcal{F}_n \subseteq \mathcal{F}'_n \subseteq \mathcal{F}_{2n}$
3. For a sequence: $\operatorname{Ann} u$ with respect to $\mathcal{W}_r$ seems computationally extremely difficult to obtain

Holonomic Closures

Differential, difference and q-difference case:

$$f \text{ and } g \text{ are holonomic} \implies f+g \text{ is holonomic;}$$

$$f \text{ is holonomic and } P \in \mathcal{A}_r \implies P \cdot f \text{ is holonomic.}$$

Differential case:

$$f \text{ is holonomic and } P \in \mathcal{S}_r \implies P^{-1} f \text{ is holonomic}$$

(known counter examples in the difference and q-difference cases);

$$f \text{ is holonomic} \implies \text{the Fourier transform of } f \text{ is holonomic.}$$

Differential and difference case:

$$f \text{ is holonomic for } \mathcal{A}_r \implies f(x_1, \dots, x_{r-1}, 0) \text{ is holonomic for } \mathcal{A}_{r-1};$$

$$f \text{ and } g \text{ are holonomic} \implies fg \text{ and } f * g \text{ are holonomic.}$$

Differential case:

$$f \text{ is holonomic for } \mathcal{W}_r \implies \int_{\mathbb{R}} f \, dx_r \text{ is holonomic for } \mathcal{W}_{r-1}.$$

Difference case:

$$f \text{ is holonomic for } \mathcal{R}_r \implies \sum_{x_r \in \mathbb{Z}} f \text{ is holonomic for } \mathcal{R}_{r-1}.$$

Holonomic Closures

1. $f+g$: we have a surjection
$$((\mathcal{F}_p \mathcal{A}_r) \cdot f) \oplus ((\mathcal{F}_p \mathcal{A}_r) \cdot g) \to (\mathcal{F}_p \mathcal{A}_r) \cdot (f+g),$$
whence $\Theta(p^r) + \Theta(p^r) \ge \Theta(p^d)$ for $d = d(\mathcal{A}_r \cdot (f+g))$
2. $P \cdot f$: $(\mathcal{F}_p \mathcal{A}_r) \cdot (P \cdot f) \subseteq (\mathcal{F}_{p+\deg P} \mathcal{A}_r) \cdot f$,
thus $d(\mathcal{A}_r \cdot (P \cdot f)) \le r$
3. $P^{-1} f$:
For $f = 1$: there exists an m such that
$$(\mathcal{F}_p \mathcal{W}_r) \cdot P^{-1} \subseteq \Phi_p = P^{-p} \mathcal{F}_{mp} \mathcal{S}_r$$
and (Φ_p) is a filtration of $\mathfrak{m} = \mathcal{W}_r \cdot P^{-1}$ thus $d(\mathfrak{m}) \le r$
When $f \ne 1$, use the closure under product below.
4. Fourier transform of f: the algebra automorphism $x_i \mapsto -\partial_i$, $\partial_i \mapsto x_i$ preserves the filtration
5. $x_r = 0$:
   - direct proof [Kashiwara]:
   for a holonomic $\mathcal{A}_r$-module $\mathfrak{m}$,
   $$\mathfrak{m}/x_r\mathfrak{m}$$
   is a holonomic $\mathcal{A}_{r-1}$-module
   - using products [Bernstein] and Fourier transform
   (specialization = convolution with Dirac distribution)
6. fg: direct proof [Bernstein], or
using specialization [Kashiwara]: $f(\mathbf{x})$ and $g(\mathbf{x})$ are holonomic with respect to $\mathcal{W}_r$
$\implies$ $f(\mathbf{x})g(\mathbf{y})$ is holonomic with respect to $\mathcal{W}_{2r}$
$\implies$ $f(\mathbf{x})g(\mathbf{x}+\mathbf{y})$ is holonomic with respect to $\mathcal{W}_{2r}$
$\implies$ $f(\mathbf{x})g(\mathbf{x})$ is holonomic with respect to $\mathcal{W}_r$
7. $f * g$: use holonomy of fg and Fourier transform
8. $\int f$: use $\mathfrak{m}/\partial_r\mathfrak{m}$ instead of $\mathfrak{m}/x_r\mathfrak{m}$ (by Fourier transform)
9. $\sum f$ (and also specialization, fg and $f*g$ in the difference case): I actually have never understood the direct proofs; I believe in indirect proofs (taking generating functions)
10. q-case: I don't understand any result concerning closure

Rational Holonomy

V is a rational $\mathcal{A}_r(\mathbb{K})$-module

$\Updownarrow$

V is a $\mathbb{K}(x_1, \ldots, x_r)$-vector space and an $\mathcal{A}_r(\mathbb{K})$-module

(with compatible actions)

$\Updownarrow$

V is a module over $\mathbb{K}(x_1, \ldots, x_r) \otimes \mathcal{A}_r(\mathbb{K})$

(tensoring over $\mathbb{K}[x_1, \ldots, x_r]$)

H is a rationally holonomic $\mathcal{A}_r(\mathbb{K})$-module

$\Updownarrow$

the rational $\mathcal{A}_r(\mathbb{K})$-module H is finite-dimensional over $\mathbb{K}(x_1, \ldots, x_r)$

$f \in V$ is rationally holonomic in the rational $\mathcal{A}_r$-module V

$\Updownarrow$

$f \in (\mathbb{K}(x_1, \ldots, x_r) \otimes \mathcal{A}_r) \cdot f \subseteq H \subseteq V$

for a rationally holonomic submodule H.

$\Updownarrow$

f is ∂-finite (with respect to $\mathbb{K}(x_1, \ldots, x_r) \otimes \mathcal{A}_r$)

Rational Holonomy

1. Goal: give a large class of examples of holonomic functions
2. Idea: extend operators to rational function coefficients
3. Rational holonomy is what we encounter in combinatorics

Holonomy vs Rational Holonomy

$\mathfrak{m}$ is a holonomic $\mathcal{A}_r$-module

$\Downarrow$ (simple)

$\mathbb{K}(x_1, \dots, x_r) \otimes \mathfrak{m}$ is a rationally holonomic $\mathcal{A}_r$-module

(tensoring over $\mathbb{K}[\mathbf{x}]$; for a suitable action of $\mathcal{A}_r$ on the tensor product)

V is a rationally holonomic $\mathcal{W}_r$-module

$\Downarrow$ [Kashiwara]

any finitely generated $\mathcal{W}_r$-submodule $\mathfrak{m}$ of V is holonomic

Example: rational functions are holonomic (with usual derivations).

V is a rationally holonomic $\mathcal{Q}_r$-module

$\Downarrow$ [Sabbah]

for any non-zero finitely generated $\mathcal{Q}_r$-submodule $\mathfrak{n}$ of V,

there exists a non-zero holonomic $\mathcal{Q}_r$-submodule $\mathfrak{m}$ of $\mathfrak{n}$

Example: $\dfrac{1}{q^{2n} + q^{2k}}$ is rationally holonomic but not holonomic!

Holonomy vs Rational Holonomy

1. Give counterparts with holonomic elements and rationally holonomic elements

2. Tensor product: extended to an $\mathcal{A}_r$-module by

$$\partial_i \cdot (r \otimes m) = (\partial_i \cdot r) \otimes m + r \otimes (\partial_i \cdot m),$$
$$\sigma_i \cdot (r \otimes m) = (\sigma_i \cdot r) \otimes (\sigma_i \cdot m),$$
$$\tau_i \cdot (r \otimes m) = (\tau_i \cdot r) \otimes (\tau_i \cdot m)$$

3. Case of recurrences: presumably similar to the case of q-recurrences

4. Differential case [Kashiwara+Takayama]: for

$$\mathfrak{m} = \sum_k \mathcal{W}_r \cdot b_k$$

a finitely generated $\mathcal{W}_r$-submodule of V, let $L \in \mathbb{K}[\mathbf{x}]$ be such that for a fixed d but each i and k

$$\partial_i \cdot b_k \in L^{-1} \sum_k \mathcal{F}_d \mathcal{W}_r \cdot b_k \subseteq L^{-1}\mathfrak{m} \subseteq \mathfrak{n} = \bigcup_{p \geq 0} L^{-p}\mathfrak{m}.$$

Set $m = \max(1 + \deg P, d)$. Then $\mathfrak{n}$, therefore $\mathfrak{m}$ is holonomic

$\to \dim_{\mathbb{K}} \Phi_p \mathfrak{n} = \Theta\left(p^r\right)$ for $\Phi_p \mathfrak{n} = \sum_k P^{-p} \mathcal{F}_{mp} \mathcal{Q}_r \cdot b_k$

5. Counter-example: prove that it is not holonomic by starting from a recurrence free from q^k and obtain a contradiction

6. Suggest expected result in the (q-)recurrence case (with “nice” singularities)

Holonomic Functions in Computer Algebra

Part IV

Holonomic Summation and Integration

Holonomic Summation and Integration

1. Creative telescoping;
2. Naive, but most intuitive, Gröbner basis approach;
3. Resultant, skew gcd, and Sylvester's dialytic method;
4. Takayama's algorithm for the case of natural boundaries;
5. Extended Zeilberger's algorithm.

Creative Telescoping

Let $\mathcal{A} = \mathbb{Q}(n,k)\langle S_n, S_k\rangle$ and $u_{n,k}$; we want to determine

$$U_n = \sum_{k=a}^{b} u_{n,k}.$$

$$P(n,S_n)\cdot u = (S_k - 1)Q(n,k,S_n,S_k)\cdot u$$

$$\Downarrow$$

$$P(n,S_n)\cdot \sum_{k=a}^{b} u_{n,k} = [Q(n,k,S_n,S_k)\cdot u]_a^{b+1}.$$

Let $\mathcal{A} = \mathbb{Q}(z,t)\langle D_z, D_t\rangle$ and $u(z,t)$; we want to determine

$$U(z) = \int_a^b u(z,t)\,dt.$$

$$P(z,D_z)\cdot u = D_t Q(z,t,D_z,D_t)\cdot u$$

$$\Downarrow$$

$$P(z,D_z)\cdot \int_a^b u(z,t)\,dt = [Q(z,t,D_z,D_t)\cdot u]_a^{b}.$$

P is known $\Rightarrow$ indefinite $\sum$ or $\int$ finds Q;

Existence of a non-trivial (P,Q) guaranteed when u is holonomic.

Creative Telescoping

1. Paradigm:

   linear functional system which defines u_n, resp. $u(z)$, up to initial conditions

   $\longrightarrow$ equation that defines U_n, resp. $U(z)$, up to initial conditions

2. First simple examples:

   – $\sum_{k\in\mathbb{Z}} \binom{n}{k}$ is 2^n: from Pascal's rule

   $$(S_nS_k - S_k - 1)\cdot\binom{n}{k} = 0$$

   one obtains $P = S_n - 2$ and $Q = 1 - S_n$

   – $\sum_{k\in\mathbb{Z}} \binom{n}{k}^2$ is $\binom{2n}{n}$: by hand, one rather painfully gets

   $$\begin{aligned}(n+2)S_n^2S_k^2 - (2n+3)S_nS_k^2 + (n+1)S_k^2 \\ -(2n+3)S_nS_k - 2(n+1)S_k + n + 1,\end{aligned}$$

   and derives

   $$\begin{aligned}P &= (n+2)S_n^2 - 2(2n+3)S_n,\\ Q &= (n+2)S_n^2S_k + (n+2)S_n^2 - (2n+3)S_nS_k\\ &\quad -2(2n+3)S_n + (n+1)S_k - (n+1)\end{aligned}$$

   – $\int_{-\infty}^{+\infty} \exp(-xy^2)\,dy$ is $\sqrt{\frac{\pi}{x}}$: from

   $$(D_y^2 + 4x^2D_x + 2x)\cdot\exp(-xy^2) = 0,$$

   one obtains $P = 4x^2D_x + 2x$ and $Q = -D_y$

3. Multivariate case, mixed case and other operators possible

4. Possibility of linear dependency of the bounds of summation in n, and of algebraic dependency of the bounds of integration in z

Hypergeometric Terms

Determined by first-order linear recurrence equations:

$$\frac{t(x_1,\dots,x_{i-1},x_i+1,x_{i+1},\dots,x_r)}{t(x_1,\dots,x_r)} \in \mathbb{K}(x_1,\dots,x_r).$$

As a consequence:

$$\dim_{\mathbb{K}(\mathbf{x})} \mathbb{K}(\mathbf{x})[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}]\cdot t = 1.$$

∂-Finite Terms

Determined by higher-order linear functional equations:

$$\dim_{\mathbb{K}(\mathbf{x})} \mathbb{K}(\mathbf{x})[\boldsymbol{\partial};\boldsymbol{\sigma},\boldsymbol{\delta}]\cdot t \ < +\infty.$$

Holonomic Terms

Smallest possible Hilbert dimension (unless $t=0$):

$$d(\mathbb{K}[x_1,\dots,x_r][\partial_1;\sigma_1,\delta_1]\dots[\partial_1;\sigma_1,\delta_1]\cdot t) = r.$$

Fundamental property:

There exist non-trivial annihilating operators
free from $r-1$ generators.

Hypergeometric, ∂-Finite, and Holonomic Terms

1. Hypergeometric terms:
   - closure under product, but no closure under addition ($n!+1$ is not hypergeometric)
   - concept of similarity: $t \simeq u$ iff $t/u \in \mathbb{K}(\mathbf{x})$; $t+u$ is again hypergeometric
   - special case of proper hypergeometric terms

     $$t_{n,k} = P(n,k)\prod_{i=1}^{\ell}\Gamma(a_in+b_ik+c_i)^{d_i}z^k$$

     (for $a_i,b_i,d_i \in \mathbb{Z}$, $c_i, z \in \mathbb{K}$, and $P \in \mathbb{K}[n,k]$)
   - similar definitions for q-hypergeometric terms and hyperexponential terms
2. Holonomic terms
   - class of operator algebras restricted to differential, difference and q-difference operators
   - proper hypergeometric terms are holonomic
   - differentially ∂-finite terms are holonomic
   - no satisfying result in other ∂-finite cases
3. Two frameworks
   - $+$, $\times$, simplifications: calculation with “non-commutative algebraic numbers”
   - $\int$, $\sum$: eliminations
4. Visual representation of the various classes in the appendix

Definite Summation and Integration Algorithms

1. Resultants, gcd's: (full elimination)
   - Sylvester's dialytic elimination and Zeilberger's slow algorithm;
   - recursive fraction-free gcd algorithm.
2. Gröbner bases:
   - naive elimination; (full elimination)
   - Takayama's algorithm (partial result). (partial elimination)
3. Undetermined coefficients:
   - Sister Celine's method; (full elimination)
   - the Verbaeten-Wegschaider approach. (partial elimination)
4. Solving linear functional equations: (partial elimination)
   - Zeilberger's fast algorithm for hypergeometric definite summation;
   - Chyzak's ∂-finite extension of it (uses Gröbner bases).

Indefinite Summation and Integration Algorithms

1. Gröbner bases: (full elimination)
   - syzygetic method.
2. Solving linear functional equations: (partial elimination)
   - Gosper's algorithm for hypergeometric indefinite summation;
   - Chyzak's ∂-finite extension of it (uses Gröbner bases).
3. Differential and difference field extensions: (-)
   - Risch's and Karr's algorithms.

Definite/Indefinite Summation and Integration Algorithms

1. Zeilberger: efficient algorithmic framework for the hypergeometric holonomic case only
2. Chyzak: extension to the ∂-finite holonomic case
3. Partial elimination: faster and smaller orders
4. Takayama-like algorithms: return partial results, which can then be extended by an indefinite algorithm
5. Risch and Karr: not based on holonomy
6. Research program: integrate the approach followed by Verbaeten and Wegschaider into the ∂-finite fast algorithm

Summation and Integration by Creative Telescoping: Gröbner Bases for Elimination

Compute an annihilating system for the **definite** sum or integral of a ∂-finite function.

Idea:

1. Case $U(z) = \int_a^b u(z,t)\,dt$ w.r.t. $\mathbb{Q}\langle z, t, D_z, D_t\rangle$: the elements Z from
$$\mathbb{Q}\langle z, D_z, D_t\rangle \cap \operatorname{Ann} u$$
are given by a Gröbner basis for $\operatorname{Ann} u$ for any order such that
$$t \ggg D_z, \qquad t \ggg D_t, \qquad \text{and} \qquad t \ggg z.$$
Any such Z rewrites into the form $Z = P - D_t Q$.

2. Case $U_n = \sum_{k=a}^{b} u_{n,k}$ w.r.t. $\mathbb{Q}(n,k)\langle S_n, S_k\rangle$: the elements Z from
$$\mathbb{Q}\langle n, S_n, S_k\rangle \cap \operatorname{Ann} u.$$
are given by a Gröbner basis for $\operatorname{Ann} u$ for any order such that
$$k \ggg S_n, \qquad k \ggg S_k, \qquad \text{and} \qquad k \ggg n.$$
Any such Z rewrites into the form $Z = P - (S_k - 1)Q$.

Summation and Integration by Creative Telescoping: Gröbner Bases for Elimination

1. Existence of a non-zero (P,Q) ensured by holonomy:
   - related to the Hilbert dimension of $\operatorname{Ann} h$
   - this dimension can be computed from a Gröbner basis for $\operatorname{Ann} h$
2. More requirements for the success of creative telescoping
   - $P \neq 0$: if $P = 0$, another pair (P', Q') with $P' \neq 0$ exists
     * Zeilberger's proof (requires natural boundaries)
     * Wegschaider's general proof
   - Q non-singular: I know no result, but was said that Schorn has partial results

Definite Summation and Integration: ∂_Ω^{-1}

$\mathcal{A}_r = \mathbb{K}[x_1, \dots, x_r][\partial_1; \sigma_1, \delta_1] \dots [\partial_r; \sigma_r, \delta_r]$;

$\mathcal{F}_r$ = left $\mathcal{A}_r$-module = "space of functions in r variables".

$\partial_{r,\Omega}^{-1}$ = "definite anti-∂" from $\mathcal{F}_r$ to $\mathcal{F}_{r-1}$ = generalization of

$$f \mapsto \int_\Omega f(x_1, \dots, x_r)\, dx_r \qquad \text{and} \qquad f \mapsto \sum_{x_r \in \Omega} f(x_1, \dots, x_r).$$

Requirements: derivation/shift under $\int / \sum$, and $\int / \sum$ by parts:

$$\text{if } i \neq r, \qquad \partial_i \partial_{r,\Omega}^{-1} = \partial_{r,\Omega}^{-1} \partial_i;$$

$$\partial_{r,\Omega}^{-1} \cdot ((\partial_r \cdot f) g) = (\partial_{r,\Omega}^{-1} \partial_r) \cdot (fg) - \partial_{r,\Omega}^{-1} \cdot (\sigma_r(f)(\partial_r \cdot g)).$$

Natural Boundaries: $\partial_\Omega^{-1}\partial = 0$.

Analytical contexts:

- $\sum$ and $\int$ over compact support;
- $\int$ over closed contours (residues);
- $\sum$ and $\int$ of exponentially decreasing functions.

Definite Summation and Integration and Natural Boundaries

∂_Ω^{-1} of ∂-Finite Functions by Creative Telescoping: Gröbner Bases for Elimination

$\boxed{\text{Compute an annihilating system for the } \textbf{definite } \partial_\Omega^{-1} \text{ of a } \partial\text{-finite function.}}$

$\mathcal{A}_r = \mathbb{K}[x_1, \dots, x_r][\partial_1; \sigma_1, \delta_1] \dots [\partial_r; \sigma_r, \delta_r]$;

$\mathcal{F}_r$ = left $\mathcal{A}_r$-module = "space of functions in r variables".

Idea: $U = \partial_{r,\Omega}^{-1} \cdot u$ w.r.t. $\mathcal{A}_r$: the elements from

$$\mathcal{A}_{r-1}[\partial_r] \cap \operatorname{Ann} u$$

are given by a Gröbner basis for $\operatorname{Ann} u$ for any order such that

$$x_r \gg \partial_i, \qquad x_r \gg x_j \qquad (j \neq r).$$

Any such Z rewrites into the form $Z = P - \partial_r Q$.

∂_Ω^{-1} of ∂-Finite Functions by Creative Telescoping: Gröbner Bases for Elimination

1. Existence of a non-zero (P, Q) ensured by holonomy:
   - related to the Hilbert dimension of $\operatorname{Ann} h$
   - this dimension can be computed from a Gröbner basis for $\operatorname{Ann} h$
2. More requirements for the success of creative telescoping
   - $P \neq 0$: if $P = 0$, another pair (P', Q') with $P' \neq 0$ exists
     * Zeilberger's proof
       (requires natural boundaries)
     * Wegschaider's general proof
   - Q non-singular: I know no result, but was said that Schorn has partial results

$$\int_{-1}^{+1} T_n(x) \frac{e^{-ux}}{\sqrt{1-x^2}}\, dx = \pi(-1)^n I_n(u) \qquad \textbf{(I)}$$

The integrand $h = T_n(x)\dfrac{e^{-ux}}{\sqrt{1-x^2}}$ is annihilated by the Gröbner basis

$$\begin{cases} (x^2-1)D_x^2 + (2px^2+3x-2p)D_x + p^2x^2 + 3px - n^2 - p^2 + 1, \\ (x^2-1)D_xS_n + (px^2 - nx - p)S_n + n + 1, \\ S_n^2 - 2xS_n + 1, \qquad D_u + x, \end{cases}$$

in the Ore algebra $\mathbb{O} = \mathbb{Q}[n,u,x]\langle D_x, S_n, D_u\rangle$.

Eliminating x by the term order

$$x \ggg D_x \succ S_n \succ D_u \succ n \succ u$$

yields another Gröbner basis:

$$\begin{aligned} &S_n^2 + 2S_nD_u + 1, \\ &D_x(D_u^2-1) - nS_n + uD_u^2 - (n-1)D_u - u, \qquad x + D_u. \end{aligned}$$

After integration, the integral $H = \displaystyle\int_{-1}^{+1} h\, dx$ satisfies

$$\begin{aligned} (S_n^2 + 2S_nD_u + 1) \cdot H &= 0, \\ [nS_n - uD_u^2 + (n-1)D_u + u] \cdot H &= [(D_u^2-1)\cdot h]_{x=-1}^{x=+1} = 0. \end{aligned}$$

$$\int_{-1}^{+1} T_n(x) \frac{e^{-ux}}{\sqrt{1-x^2}}\, dx = \pi(-1)^n I_n(u) \qquad \textbf{(I)}$$

1. used to model the vibration of a drum membrane

Generating Function of the Jacobi Polynomials

The Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}$$

are determined up to initial conditions by

$$\begin{aligned}
&(1-x^2)P_n^{(\alpha,\beta)\prime\prime}(x)\\
&\quad + (\beta-\alpha-(\alpha+\beta+2)x)P_n^{(\alpha,\beta)\prime}(x)\\
&\quad + n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) = 0,\\
&2(n+2)(n+\alpha+\beta+2)(2n+\alpha+\beta+2)P_{n+2}^{(\alpha,\beta)}(x)\\
&\quad - (2n+\alpha+\beta+3)((\alpha^2-\beta^2)\\
&\qquad + (2n+\alpha+\beta+2)(2n+\alpha+\beta+4)x)P_{n+1}^{(\alpha,\beta)}(x)\\
&\quad + 2(n+\alpha+1)(n+\beta+1)(2n+\alpha+\beta+4)P_n^{(\alpha,\beta)}(x) = 0,\\
&(2n+\alpha+\beta+2)(1-x^2)P_{n+1}^{(\alpha,\beta)\prime}(x)\\
&\quad + (n+1)(\alpha-\beta-(2n+\alpha+\beta+2)x)P_{n+1}^{(\alpha,\beta)}(x)\\
&\quad - 2(n+\alpha+1)(n+\beta+1)P_n(x)^{(\alpha,\beta)} = 0.
\end{aligned}$$

We **prove** that their generating function is

$$\sum_{n\geq 0} P_n^{(\alpha,\beta)}(x)y^n = \Big(R(x,y)\,(1-y+R(x,y))^\alpha\,(1+y+R(x,y))^\beta\Big)^{-1},$$

for $R(x,y) = \sqrt{1-2yx+y^2}$.

Initialization and calculation of equations satisfied by $P_n^{(\alpha,\beta)}(x)y^n$ in $\mathbb{K}(x,y,n)\langle D_x, D_y, S_n\rangle$:

```
G:=[...,...,...]:
with(Ore_algebra):
A:=skew_algebra(diff=[Dx,x],diff=[Dy,y],
    shift=[Sn,n],comm=[alpha,beta],polynom=n):
G:=map(primpart,map(numer,[op(subs(Sn=Sn/y,G)),
    y*Dy-n]),[Sn,Dx,Dy]):
```

Compute the sum by creative telescoping (elimination of n):

```
with(Groebner):
T:=termorder(A,lexdeg([n],[Sn,Dx,Dy])):
GB:=gbasis(G,T):
CT:=map(collect,subs(Sn=1,remove(has,GB,n)),
    [Dx,Dy],distributed,factor);
```

(Maximum degree 3 in D_x and D_y, too large to be displayed.)

Proving the identity:

```
A:=diff_algebra([Dx,x],[Dy,y],comm=[alpha,beta]):
T:=termorder(A,tdeg(Dx,Dy)):
GB:=gbasis(CT,T):
collect(GB,[Dx,Dy],factor);
```

$$(\dots)D_x + (\dots)D_y + (\dots), \qquad (\dots)D_y^2 + (\dots)D_y + (\dots).$$

(Two initial conditions needed.)

Orthogonality of the Legendre Polynomials

The Legendre polynomials are:

$$P_n(x) = P_n^{(0,0)}(x) = \frac{1}{2^n} \sum_{k\in\mathbb{Z}} \binom{n}{k}^2 (x-1)^k (x+1)^{n-k}.$$

Their generating function is:

$$\phi(x,y) = \frac{1}{\sqrt{1-2xy+y^2}}.$$

We want to **prove** that:

$$\begin{aligned}\eta(u,v) &= \int_{-1}^{+1} \phi(x,u)\phi(x,v)dx \\ &= \sum_{(n,m)\in\mathbb{N}^2} \langle P_n(x) \mid P_m(x)\rangle u^n v^m\end{aligned}$$

is

$$\frac{2\,\mathrm{atanh}(\sqrt{uv})}{\sqrt{uv}} = \sum_{n\in\mathbb{N}} \frac{2}{2n+1}(uv)^n.$$

After computing the product, the elimination of x yields:

$$\begin{cases} v-u+(-2vu+v^2+1)D_v+(-u^2+2vu-1)D_u \\ \qquad +2(vu-1)(v-u)D_uD_v, \\ (u+v)(vu-1)+2v^2(u-1)(u+1)D_v \\ \qquad +2u^2(v-1)(v+1)D_u \\ \qquad +(u-1)(u+1)(v-1)(v+1)D_x. \end{cases}$$

Non-homogeneous creative telescoping then yields:

$$\begin{aligned}0 = P \cdot \int_{-1}^{+1} &\phi(x,u)\phi(x,v)dx \\ &+ [(u-1)(u+1)(v-1)(v+1)\phi(x,u)\phi(x,v)]_{-1}^{+1}.\end{aligned}$$

The non-homogeneous term equals $2(u+v)$.

Three equations in $\eta(u,v)$. Solving yields the closed form.

A last creative telescoping yields the coefficients by the Cauchy formula:

$$\int_{\text{contour}} \frac{2\,\mathrm{atanh}\left(\sqrt{z}\right)}{z^{n+1}\sqrt{z}}\,dz$$

gives the orthogonality coefficients, $\dfrac{2}{2n+1}$.

Sylvester's Dialytic Elimination and Skew Gaussian Elimination

For polynomials P and Q of degree n and m in x and coefficients in $\mathbb{A}$, let

$$\left(P, \dots, x^{m-1} P, Q, \dots, x^{n-1} Q\right)^T = M \left(x^d, \dots, x, 1\right)^T.$$

where $d = n + m - 1$ and M is a square matrix (with entries in $\mathbb{A}$).

If, after recombining rows by $\mathbb{A}$-linearity,

$$\Lambda \left(P, \dots, x^{m-1} P, Q, \dots, x^{n-1} Q\right)^T = U \left(x^d, \dots, x, 1\right)^T,$$

for an upper-triangular matrix $U = \Lambda M$, then

$$(\Lambda_{d,1}, \dots, \Lambda_{d,d}) \left(P, \dots, x^{m-1} P, Q, \dots, x^{n-1} Q\right)^T = U_{d,d}$$

is in

$$\mathbb{A} \cap (\mathbb{A}[x] P + \mathbb{A}[x] Q).$$

[Ore, 1931]: same result for polynomials in $\mathbb{A}[x; \sigma, \delta]$ (for an invertible σ), if coefficients are in a left Ore ring $\mathbb{A}$:

$$\mathbb{A}a \cap \mathbb{A}b \neq \{0\} \qquad \text{even for } (a, b) \neq (0, 0).$$

$\longrightarrow$ recombine rows by left-linearity.

Sylvester's Dialytic Elimination and Skew Gaussian Elimination

1. Zeilberger's slow algorithm:
   - does not realize that $\mathbb{A}$ is a left Ore ring
   - consider more rows instead
   - reduce to commutative determinant theory
2. Computing annihilators (u, v) for $ua = vb \in \mathbb{A}a \cap \mathbb{A}b$: by an extended skew gcd calculation (which in fact also performs the elimination)
3. Difference to the commutative case: rows are no longer merely shifted when multiplied by x

Neumann's Addition Theorem

$J_0(z)^2 + 2\sum_{k=1}^{\infty} J_k(z)^2 = 1$:

an Example of Summation by Sylvester's Dialytic Elimination

For the algebra $\mathbb{Q}(z,k)\langle D_z, S_k\rangle$, $J_k(z)$ satisfies:

$$\begin{cases} z^2D_z^2 + zD_z + z^2 - k^2, \\ zS_k + zD_z - k. \end{cases}$$

By algorithmic closure under product, $J_k(z)^2$ satisfies:

$$\begin{cases} zD_z^2 + (-2k+1)D_z - 2zS_k + 2z, \\ zD_zS_k + zD_z + (2k+2)S_k - 2k, \\ z^2S_k^2 - 4(k+1)^2S_k - 2z(k+1)D_z + 4k(k+1) - z^2, \end{cases}$$

and, by rewriting with k on the right, also

$$\begin{cases} (-4S_k+4)k^2 + (4-2zD_z)k + (z^2(S_k^2-1) - 2zD_z), \\ -2D_zk^2 + (zD_z^2 + D_z + 2z(1-S_k))k + 2zS_k, \\ -2D_zk + (zD_z^2 + D_z + 2z(1-S_k)). \end{cases}$$

This rewrites into the matrix notation:

$$\begin{pmatrix} -4S_k+4 & 4-2zD_z & z^2(S_k^2-1) - 2zD_z \\ -2D_z & zD_z^2 + D_z + 2z(1-S_k) & 2zS_k \\ 0 & -2D_z & zD_z^2 + D_z + 2z(1-S_k) \end{pmatrix}$$

Using $2D_z(-4S_k+4) + (-4S_k+4)(-2D_z) = 0$:

$$\begin{pmatrix} -2D_z & zD_z^2 + D_z + 2z(1-S_k) & 2zS_k \\ 0 & -2D_z & zD_z^2 + D_z + 2z(1-S_k) \\ 0 & \begin{array}{r} -4zS_kD_z^2 - 4D_zS_k + 8zS_k^2 \\ +8D_z + 8z - 16zS_k \end{array} & \begin{array}{r} -4z + 2D_zz^2S_k^2 + (-2z^2-4)D_z \\ +8zS_k - 4zD_z^2 - 4zS_k^2 \end{array} \end{pmatrix}$$

Using $u(-2D_z) + (1-zD_z)b = 0$ for

$$b = -4zS_kD_z^2 - 4D_zS_k + 8zS_k^2 + 8D_z + 8z - 16zS_k,$$

$$u = -4z^2S_k^2 + 8z^2S_k - 4z^2 - 4zD_z + 2z^2D_z^2S_k + 4 - 2S_k + 2zD_zS_k :$$

$$\begin{pmatrix} -2D_z & zD_z^2 + D_z + 2z(1-S_k) & 2zS_k \\ 0 & -2D_z & zD_z^2 + D_z + 2z(1-S_k) \\ 0 & 0 & \begin{array}{r} -8z^3 + 2z(6z^2+1)D_z^2S_k - 14D_zz^2S_k^2 \\ -6D_zz^2 + 24z^3S_k - 2z^3D_z^2 \\ -24z^3S_k^2 + (20z^2-2)D_zS_k + 8z^2D_z^3S_k \\ -10z^3D_z^2S_k^2 + 2z^3S_kD_z^4 + 8z^3S_k^3 \end{array} \end{pmatrix}$$

Finally:

1. rewrite into the form $P(z,D_z) - (S_k-1)Q(z,D_z,S_k)$;
2. sum for k between 0 and ∞;
3. solve non-homogeneous differential equation to obtain the announced identity.

Summation and Integration over Natural Boundaries: Takayama's Algorithm and Gröbner Bases for Modules

Compute an annihilating system for the **definite** sum or integral of a ∂-finite function in the case of **natural boundaries**.

Goal: compute the P's such that there exists a Q such that

$$[P(n,u,D_x,S_n,D_u) - D_x Q(x,n,u,D_x,S_n,D_u)] \cdot h = 0,$$

or equivalently,

$$P(n,u,S_n,D_u) \cdot H = [Q(x,n,u,D_x,S_n,D_u) \cdot h]_{x=-1}^{x=+1},$$

without computing the Q's.

Idea:

if $\qquad \operatorname{Ann} h = \sum_{i=1}^{\ell} \mathbb{Q}[n,u,x]\langle D_x,S_n,D_u\rangle g_i,$

then $\qquad \operatorname{Ann} h = \lim_{N\to\infty} \sum_{i=1}^{\ell}\sum_{j=0}^{N} \mathbb{Q}[n,u]\langle D_x,S_n,D_u\rangle x^j g_i.$

Summation and Integration over Natural Boundaries: Takayama's Algorithm and Gröbner Bases for Modules

1. Algorithm:
   - by troncation of the module according the the degree in x
   - using Gröbner bases for modules: multiplication by x forbidden!
   - from finitely-generated $\mathcal{A}_r$-modules (ideals) to non-finitely-generated $\mathcal{A}_{r-1}$-modules
   - calculations modulo D_x on the left

∂_Ω^{-1} over Natural Boundaries: Takayama's Algorithm and Gröbner Bases for Modules

Compute an annihilating system for the **definite** ∂_Ω^{-1} of a ∂-finite function in the case of **natural boundaries**.

Recall: the elimination of x_r by a lexicographical Gröbner basis computes

$$\mathcal{A}_{r-1}[\partial_r] \cap \operatorname{Ann} u.$$

Better: determine

$$(\mathcal{A}_{r-1} \oplus \partial_r \mathcal{A}_r) \cap \operatorname{Ann} u = (\mathcal{A}_{r-1} \oplus \partial_r \mathcal{A}_{r-1}[x_r][\partial_r; \sigma_r, \delta_r]) \cap \operatorname{Ann} u.$$

Even better: determine

$$\mathcal{A}_{r-1} \cap (\partial_r \mathcal{A}_r + \operatorname{Ann} u).$$

Idea:

if $$\operatorname{Ann} u = \sum_{i=1}^{\ell} \mathcal{A}_{r-1}[x_r][\partial_r; \sigma_r, \delta_r] g_i,$$

then $$\operatorname{Ann} u = \lim_{N\to\infty} \sum_{i=1}^{\ell} \sum_{j=0}^{N} \mathcal{A}_{r-1}[\partial_r; \sigma_r, \delta_r] x^j g_i.$$

$\longrightarrow$ straighforward extension to **simultaneous** $\partial_{i,\Omega}^{-1}$'s.

∂_Ω^{-1} over Natural Boundaries: Takayama's Algorithm and Gröbner Bases for Modules

1. Algorithm:
   - by troncation of the module according the degree in x_r
   - from finitely-generated modules over $\mathcal{A}_r$ (= ideals in $\mathcal{A}_r$) to non-finitely-generated modules over $\mathcal{A}_{r-1}$
   - calculations modulo ∂_r on the left, resp. using

$$P(x_r) D_r^d = (-1)^d P^{(d)}(x_r) + D_r Q(x_r, D_r),$$
$$P(x_r) S_r^d = S_r^d P(x_r - d), \quad P(x_r) H_r^d = H_r^d P(q^{-d} x_r).$$

2. Computing $\displaystyle\int_0^1 \frac{\cos(zu)}{\sqrt{1-u^2}}\,du$ so as to prove that it is $\dfrac{\pi}{2} J_0(z)$
   - naive Gröbner elimination: order 6 in 5 sec

$$z^2 D_z^6 + 10 z D_z^5 + (20 + 2z^2) D_z^4 + 14 z D_z^3 + (19 + z^2) D_z^2 + 4 z D_z + 2$$

   - Takayama-like algorithm: order 4 in 1.3 sec

$$z D_z^4 + 3 z D_z^3 + z D_z^2 + 2 D_z$$

3. Rogers-Ramanujan identity (partitions theory):

$$\sum_{k\in\mathbb{Z}} \frac{q^{k^2}}{(q)_k (q)_{n-k}} = \sum_{k\in\mathbb{Z}} (-1)^k \frac{q^{\frac{k(5k-1)}{2}}}{(q)_{n-k}(q)_{n+k}}$$

   - 2-order operator for the left-hand side

$$(1 - q^2 q^n) S_n^2 + (-q^3 q^{n\,2} - 1 - q + q^2 q^n) S_n + q$$

   - 5-order operator for the right-hand side
   - naive Gröbner elimination: 1.2+53 sec
   - Takayama-like algorithm: 1.2+23.5 sec

4. Simultaneous multivariate ∂_Ω^{-1}: first automatic proofs of Gordon's identities in a matter of minutes

$$\int_{-1}^{+1} T_n(x)\frac{e^{-ux}}{\sqrt{1-x^2}}\,dx = \pi(-1)^n I_n(u) \qquad \textbf{(II)}$$

For $N = 3$:

$$1D_u + x, \qquad xD_u + x^2, \qquad x^2D_u + x^3,$$
$$1(-D_x - nS_n - u) + x(n+1) + x^2(u + D_x),$$
$$x(-D_x - nS_n - u) + x^2(n+1) + x^3(u + D_x),$$
$$1(S_n^2+1) - x2S_n, \qquad x(S_n^2+1) - x^22S_n, \qquad x^2(S_n^2+1) - x^32S_n.$$

Optimisation: Taking remainder modulo D_x on the left by $p(x)D_x = D_xp(x) - p'(x)$ yields

$$D_u1 + x, \qquad D_ux + x^2, \qquad D_ux^2 + x^3,$$
$$(-nS_n - u)1 + (n-1)x + ux^2,$$
$$1 + (-nS_n - u)x + (n-2)x^2 + ux^3,$$
$$(S_n^2+1)1 - 2S_nx, \qquad (S_n^2+1)x - 2S_nx^2, \qquad (S_n^2+1)x^2 - 2S_nx^3.$$

Gröbner basis for $x \ggcurly S_n \succ D_u$:

$$\boxed{(-u - nS_n - (n-1)D_u + uD_u^2)1}, \qquad \boxed{(S_n^2 + 1 + 2D_uS_n)1},$$
$$D_u1 + x, \qquad (-u - nS_n - (n-1)D_u)1 + ux^2,$$
$$(unS_nD_u + n(n-2)S_n - u + (u^2 + n^2 - 3n + 2)D_u + nu)1 + u^2x^3.$$

$\longrightarrow$ same two eliminated polynomials.

$$\int_{-1}^{+1} T_n(x)\frac{e^{-ux}}{\sqrt{1-x^2}}\,dx = \pi(-1)^n I_n(u) \qquad \textbf{(II)}$$

1. Reminiscent to the resultant method
2. Contribution: optimisation

Indefinite Hypergeometric Summation: Gosper's Algorithm

For hypergeometric terms t_k and T_k such that $T_{k+1} - T_k = t_k$:

$$T_k = \theta(k) t_k \qquad \text{for a rational function } \theta.$$

Writing $\frac{t_{k+1}}{t_k} = R(k)$ yields $R(k)\theta(k+1) - \theta(k) = 1$, which can be solved for rational solutions by Abramov's algorithm.

Writing $R(k) = \frac{p_{k+1}}{p_k}\frac{q_k}{r_{k+1}}$ (+ technical conditions) and $s_k = \frac{p_k}{r_k}\theta(k)$:

$$q_k s_{k+1} - r_k s_k = p_k \qquad \text{where } s_k \text{ is a polynomial.}$$

INPUT: a hypergeometric term t_k.
OUTPUT: a hypergeometric term T_k from $\mathbb{K}(k)t_k$ such that $T_{k+1} - T_k = t_k$, or $\perp$.

1. write $\frac{t_{k+1}}{t_k} = \frac{p_{k+1}}{p_k}\frac{q_k}{r_{k+1}}$ (+ technical conditions);
2. solve $q_k s_{k+1} - r_k s_k = p_k$ for a polynomial solution s_k;
3. if solvable, return $T_k = \frac{r_k s_k}{p_k} t_k$, otherwise return $\perp$.

Indefinite Hypergeometric Summation: Gosper's Algorithm

1. Decision algorithm: $\perp$ means "does not terminate"
2. Key idea:
   - work in the 1-dimensional vector space of hypergeometric terms similar to t_k
   - each hypergeometric term u_k under consideration can be represented by a single rational function r such that $u_k = r(k)t_k$
3. Summation problem: $\sum_{j=0}^{k-1} t_j = T_k - T_0$
4. Generalizes to q-calculus

Example of Indefinite Hypergeometric Summation by Gosper's Algorithm

$$\boxed{\text{Evaluate } \sum_{j=0}^{k} \frac{4^j}{\binom{2j}{j}}.}$$

For $t_k = \dfrac{4^k}{\binom{2k}{k}}$,

$$R(k) = \frac{2(k+1)}{2k+1},$$
$$p_k = 1, \quad q_k = 2(k+1), \quad r_k = 2k-1.$$

Solving for polynomial solutions yields $s_k = 1/3$, so that

$$T_k = \frac{2k-1}{3} t_k = \frac{2k-1}{3} \boxed{\frac{4^k}{\binom{2k}{k}}}.$$

Finally,

$$\sum_{j=0}^{k} \boxed{\frac{4^j}{\binom{2j}{j}}} = (T_k - T_0) + t_k = \frac{2(k+1)}{3} \boxed{\frac{4^k}{\binom{2k}{k}}} + \frac{1}{3}.$$

Example of Indefinite Hypergeometric Summation by Gosper's Algorithm

Indefinite ∂-Finite ∂^{-1}

ALGORITHM: $T = \textbf{Indefinite}(t, (b_i)_{i=1,\dots,N}, \mathcal{A})$.

INPUT: $\begin{cases} \text{a } \partial\text{-finite term } t \in V = \bigoplus_{i=1}^{N} \mathbb{F}\, b_i, \\ \text{an operator algebra } \mathcal{A} = \mathbb{F}\langle \partial_0, \partial_1, \dots, \partial_r \rangle, \\ \text{the action of } \mathcal{A} \text{ on } V. \end{cases}$

OUTPUT: $T \in V$ such that $\partial_0 \cdot T = t$; or $\perp$.

1. let $T = \sum_{i=1}^{N} \phi_i b_i$ for undetermined coefficients $\phi_i \in \mathbb{F}$;
2. extract the coefficients of $\partial_0 \cdot T - t$ in the b_i's to obtain a first order functional system in the ϕ_i's;
3. solve for rational solutions [Abramov et al.];
4. if solvable return T; otherwise return $\perp$.

Special case: (a variant of) Gosper's algorithm when $r = 0$, $\partial_0 = S_n$, $\mathcal{A} = \mathbb{Q}(n)\langle S_n \rangle$, $N = 1$.

Indefinite ∂-Finite ∂^{-1}

1. Gosper's algorithm
   - indefinite hypergeometric summation
   - crucial step in Zeilberger's "fast" algorithm for definite hypergeometric summation
   - notion of similarity in the hypergeometric case
2. Algorithm
   - any finite-dimensional vector space
   - use undetermined coefficients
3. Two methods to solve for rational solutions:
   - by uncoupling the first-order functional system to get a higher-order equation in a single ϕ_i
   - recent algorithms to work at the level of a system [Barkatou et al.]
4. Extension to a linearly parametrized input
   $$t = \sum_i \eta_i t_i$$
   (for parameters η_i's, $t_i \in V$)
5. Extended algorithms:
   - replacing $\partial_0 \cdot T - t$ by $H \cdot T - t$ yields a method to obtain particular solutions of non-homogeneous functional equations
   - possibility of an extension to algebras where ∂a would be a polynomial in ∂ (not necessarily linear)?

Examples of Indefinite ∂-Finite ∂^{-1}

Example of indefinite ∂-finite summation:

$$\sum_{j=1}^{k} t_j \text{ for } t_k = \binom{k}{p} H_k,$$

$$(\dots)t_{k+2} + (\dots)t_{k+1} + (\dots)t_k = 0,$$

$$\mathcal{A} = \mathbb{Q}(p,k)\langle S_k \rangle, \qquad T = \phi_0 t_k + \phi_1 t_{k+1},$$

$$\text{Output:} \quad \phi_0(k) = \frac{(k+1)^2}{(p+1)^2} - 1, \quad \phi_1(k) = -\frac{(k-p)(k-p+1)}{(p+1)^2},$$

$$T = \sum t_k \,\delta k = \binom{k}{p} \frac{k-p}{(p+1)^2} \left((p+1)H_k - 1\right).$$

Example of indefinite ∂-finite integration:

$$\int \operatorname{Ci}(z)\, dz \text{ for } t(z) = \operatorname{Ci}(z) = \int_0^z \frac{\cos(t)}{t}\, dt,$$

$$(\dots)t'''(z) + (\dots)t''(z) + (\dots)t'(z) = 0,$$

$$\mathcal{A} = \mathbb{Q}(z)\langle D_z \rangle, \qquad T = \phi_0 t(z) + \phi_1 t'(z) + \phi_2 t''(z),$$

$$\text{Output:} \qquad \phi_0(z) = z, \qquad \phi_1(z) = 1, \qquad \phi_2(z) = z,$$

$$T = \int \operatorname{Ci}(z)\, dz = z \operatorname{Ci}(z) - \sin(z).$$

Examples of Indefinite ∂-Finite ∂^{-1}

1. Example of the indefinite integral

$$F = \int f\, dx = \int \left(P \cdot T_n(x) \frac{e^{-ux}}{\sqrt{1-x^2}} \right) dx,$$

where

$$P = u^2 D_u^4 + 5u D_u^3 + (4 - 2u^2 - n^2) D_u^2 \\ -5u D_u + n^2 + u^2 - 2$$

shows the necessity of Gröbner bases

Definite Hypergeometric Summation: Zeilberger's Fast Algorithm

For a holonomic hypergeometric term $h_{n,k}$:

$$P(n, S_n) \cdot h + [Q(n,k) \cdot h]_k^{k+1} = 0.$$

If P is known, Gosper's algorithm applies to find $Q \in \mathbb{K}(n,k)$.

ALGORITHM: $T = \mathbf{Zeilberger}(h_{n,k})$. INPUT: a holonomic hypergeometric term $h_{n,k}$. OUTPUT: a recurrence satisfied by the definite sum over k.

1. for $L = 0, 1, 2, \dots$, while unsuccessful:
   (a) write $P = \sum_{i=0}^{L} \eta_i(n) S_n^i$ with undetermined coefficients;
   (b) write $t_k = P \cdot h$ and apply a variant of Gosper's algorithm to simultaneously solve for $T_k = Q \cdot h$ and rational η_i's;
2. output (P, Q).

Generalizes to integration and q-summation.

Example of Definite Hypergeometric Summation

$$\boxed{\text{Evaluate } f_n = \sum_{k \in \mathbb{Z}} \binom{n}{2k} \binom{2k}{k} 4^{-k}.}$$

For $L = 1$, write $P = \eta_0 + \eta_1 S_n$:

$$t_k = P \cdot h = \left(\eta_0 + \frac{(n+1)\eta_1}{n+1-2k} \right) h_{n,k}.$$

Gosper's modified algorithm returns:

$$\eta_0 = 2n+1, \quad \eta_1 = -(n+1), \quad \theta_k = \frac{T_k}{t_k} = n - 2k.$$

The sum f_n satisfies $(n+1) f_{n+1} = (2n+1) f_n$, from which follows

$$f_n = \frac{\binom{2n}{n}}{2^n}.$$

Definite ∂-Finite ∂_Ω^{-1}

ALGORITHM: $(P, Q) = \mathbf{Definite}(u, (b_i)_{i=1,\dots,N}, \mathcal{A})$.
INPUT: $\begin{cases} \text{a } \partial\text{-finite term } u \text{ w.r.t. } \mathcal{A} = \mathbb{F}\langle \partial_0, \partial_1 \rangle, \\ \text{a finite } \mathbb{F}\text{-basis } (b_i)_{i=1,\dots,N} \text{ of } \mathcal{A} \cdot u. \end{cases}$
OUTPUT: $\begin{cases} P \in \mathbb{F}\langle \partial_1 \rangle \text{ that commutes with } \partial_0, \\ Q \in \mathcal{A} \text{ such that } P \cdot u = \partial_0 Q \cdot u. \end{cases}$

For increasing values of r:

1. let $P = \sum_{i=0}^{r} \eta_i \partial_1^i$ and $t = P \cdot u$ for undetermined coefficients η_i that commute with ∂_0;
2. let $(T, (\eta_i)) = \mathbf{ParamIndefinite}(t, (b_i), \mathcal{A}, \{\eta_0, \dots, \eta_r\})$.
3. if $T \neq \perp$ return (P, Q) for $Q \cdot u = T$.

Termination: when u is holonomic.

Special case: Zeilberger's algorithm when $N = 1$.

Definite ∂-Finite ∂_Ω^{-1}

1. Advantages:
   - few calculations with Gröbner bases
   - Gröbner bases for term orders in two times less indeterminates
   - does not require a holonomic representation of the input
2. For a ∂-finite holonomic term $h_{n,k}$:

   $$[P(n, S_n) + (S_k - 1)Q(n, k, S_n, S_k)] \cdot h = 0$$

   If P is known such that such a Q exists, the previous algorithm finds it
3. $t = P \cdot h$ depends linearly on the η_i's:

   ParamIndefinite computes them together with the coefficients of Q

Neumann's Addition Theorem

$$J_0(z)^2 + 2\sum_{k=1}^{\infty} J_k(z)^2 = 1$$

$\mathcal{A} = \mathbb{Q}(z,k)\langle D_z, S_k\rangle, \quad J_k(z)$ satisfies :

$$\begin{cases} z^2D_z^2 + zD_z + z^2 - k^2, \\ zS_k + zD_z - k. \end{cases}$$

$J_k(z)^2$ satisfies :

$$\begin{cases} zD_z^2 + (-2k+1)D_z - 2zS_k + 2z, \\ zD_zS_k + zD_z + (2k+2)S_k - 2k, \\ z^2S_k^2 - 4(k+1)^2S_k - 2z(k+1)D_z + 4k(k+1) - z^2. \end{cases}$$

Basis: $\left(J_k(z)^2, S_k \cdot J_k(z)^2, D_z \cdot J_k(z)^2\right)$.

Output: $P = D_z, \qquad Q = \frac{k}{z} + \frac{1}{2}D_z,$

$$P \cdot \left(\sum_{k=0}^{\infty} J_k(z)^2\right) + \left[Q \cdot J_k(z)^2\right]_{k=0}^{k=\infty} = 0, \qquad \text{i.e.}$$

$$D_z \cdot \left(2\sum_{k=1}^{\infty} J_k(z)^2 + J_0(z)^2 - 1\right) = 0.$$

Initial condition: $2\sum_{k=1}^{\infty} J_k(0)^2 + J_0(0)^2 - 1 = 0.$

$$\frac{2}{\pi}\int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}}\,dt = J_0(z)$$

$\mathcal{A} = \mathbb{Q}(z,t)\langle D_z, D_t\rangle, \quad \frac{\cos(zt)}{\sqrt{1-t^2}}$ satisfies :

$$\begin{cases} D_z^2 + t^2, \\ t(1-t^2)D_t - (1-t^2)zD_z - t^2. \end{cases}$$

Basis: $\left(\frac{\cos(zt)}{\sqrt{1-t^2}}, D_z \cdot \frac{\cos(zt)}{\sqrt{1-t^2}} = -t\frac{\sin(zt)}{\sqrt{1-t^2}}\right)$.

Output: $P = zD_z^2 + D_z + z, \qquad Q = \frac{1-t^2}{t}D_z,$

$$P \cdot \left(\int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}}\right) + \left[Q \cdot \frac{\cos(zt)}{\sqrt{1-t^2}}\right]_0^1 = 0, \qquad \text{i.e.}$$

$$(zD_z^2 + D_z + z) \cdot \frac{2}{\pi}\int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}}\,dt = 0.$$

Correct initial conditions.

Generating Function of the Rogers-Szegő Polynomials

Rogers–Szegő: $$r_n(a,b;q)=\sum_{k=0}^{n}\binom{n}{k}_q a^k b^{n-k},$$

Generating function: $$R(a,b,z;q)=\sum_{n=0}^{\infty}\frac{r_n(a,b;q)z^n}{(q;q)_n}$$

We want to prove: $$R(a,b,z;q)=\frac{1}{(az;q)_\infty(bz;q)_\infty}.$$

$\mathcal{A}=\mathbb{Q}(a,b,q,z,q^n)\langle H_{q,z},S_{q,n}\rangle$, $\dfrac{r_n(a,b;q)z^n}{(q;q)_n}$ satisfies :

$$\begin{cases}(1-q^n)(1-qq^n)S_{q,n}^2-(a+b)z(1-q^n)S_{q,n}+abz^2(1-qq^n),\\ \hfill H_{q,z}-q^n.\end{cases}$$

Basis: $$\left(\frac{r_n(a,b;q)z^n}{(q;q)_n},S_{q,n}\cdot\frac{r_n(a,b;q)z^n}{(q;q)_n}\right).$$

Output: $$\begin{cases}P=H_{q,z}-(1-az)(1-bz),\\ Q=(1-qq^n)S_{q,n}+[1-q^n-(a+b)z].\end{cases}$$

Non-homogeneous term: $$\left[Q\cdot\frac{r_n(a,b;q)z^n}{(q;q)_n}\right]_{n=0}^{n=+\infty}=0;$$

$$\boxed{R(a,b,qz;q)=(1-az)(1-bz)R(a,b,z;q).}$$

Correct initial conditions.

Mehler Formula for the Rogers-Szegő Polynomials

G.F. for products: $$R(a,b,c,d,z;q)=\sum_{n=0}^{\infty}\frac{r_n(a,b;q)r_n(c,d;q)z^n}{(q;q)_n}$$

Goal: $$R(a,b,c,d,z;q)=\frac{(acz;q)_\infty(adz;q)_\infty(bcz;q)_\infty(bdz;q)_\infty}{(abcdz^2;q)_\infty}.$$

$\mathcal{A}=\mathbb{Q}(a,b,q,z,q^n)\langle H_{q,z},S_{q,n}\rangle$, $\dfrac{r_n(a,b;q)r_n(c,d;q)z^n}{(q;q)_n}$ satisfies :

$$\begin{cases}(1-q^4q^n)S_{q,n}^4-(a+b)(c+d)zS_{q,n}^3\\ +[ab(c^2+d^2)+(a^2+b^2)cd+2abcd+abcdq^2(1+q)q^n]z^2S_{q,n}^2\\ \hfill -abcd(a+b)(c+d)z^3S_{q,n}+a^2b^2c^2d^2(1-qq^n)z^4,\\ \hfill H_{q,z}-q^n.\end{cases}$$

Basis: $$\left(S_{q,n}^i\cdot\frac{r_n(a,b;q)r_n(c,d;q)z^n}{(q;q)_n}\right)\text{ for } i=0,\dots,3.$$

Output: $$\begin{cases}P=(1-abcdz^2)(1-qabcdz^2)H_{q,z}\\ \qquad -(1-acz)(1-adz)(1-bcz)(1-bdz),\\ Q=(1-q^3q^n)S_{q,n}^3+\dots.\end{cases}$$

Non-homogeneous term: $$\left[Q\cdot\frac{r_n(a,b;q)r_n(c,d;q)z^n}{(q;q)_n}\right]_{n=0}^{n=+\infty}=0;$$

$$\boxed{\frac{R(qz)}{R(z)}=\frac{(1-acz)(1-adz)(1-bcz)(1-bdz)}{(1-abcdz^2)(1-qabcdz^2)}.}$$

Correct initial conditions.

Calkin's Curious Identity

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{k}\right)^3 = n2^{3n-1} + 2^{3n} - 3n2^{n-2}\binom{2n}{n}$$

Only 3 minutes!

Starting from a system for $\binom{n}{k}$, summation and cubation yield a system of second and third order recurrence equations for

$$u_{n,k} = \left(\sum_{j=0}^{k}\binom{n}{j}\right)^3.$$

Reduce $P + (S_k - 1)Q$ where

$$P = \sum_{i=0}^{2}\eta_i(n)S_n^i,$$
$$Q = \phi_0(n,k) + \phi_1(n,k)S_k + \phi_2(n,k)S_k^2 + \psi(n,k)S_nS_k.$$

Uncoupling yields a 4-th order recurrence in $\phi(k)$.

After non-homogeneous summation, the sum U_n satisfies

$$(n+1)U_{n+2} - (7n+12)U_{n+1} - 4(2n+1)U_n = 2(9n-10)8^n,$$

from which the result follows by Petkovšek's algorithm.

Calkin's Curious Identity

Multivariate Definite ∂-Finite ∂_Ω^{-1}

ALGORITHM: $(P, Q) = \textbf{MultiDefinite}(u, (b_i)_{i=1,\dots,N}, \mathcal{A})$.

INPUT: $\begin{cases} \text{a } \partial\text{-finite term } u \text{ w.r.t. } \mathcal{A} = \mathbb{F}\langle \partial_0, \dots, \partial_r \rangle, \\ \text{a finite } \mathbb{F}\text{-basis } (b_i)_{i=1,\dots,N} \text{ of } \mathcal{A} \cdot u. \end{cases}$

OUTPUT: $\begin{cases} P \in \mathbb{F}\langle \partial_1, \dots, \partial_r \rangle \text{ that commutes with } \partial_0, \\ Q \in \mathcal{A} \text{ such that } P \cdot u = \partial_0 Q \cdot u. \end{cases}$

For increasing supports of the polynomials P:

1. let $P = \sum_{\boldsymbol{\alpha}} \eta_{\boldsymbol{\alpha}} \partial_1^{\alpha_1} \dots \partial_r^{\alpha_r}$ and $t = P \cdot u$ for undetermined coefficients $\eta_{\boldsymbol{\alpha}}$ that commute with ∂_0;
2. let $(T, (\eta_{\boldsymbol{\alpha}})) = \textbf{ParamIndefinite}(t, (b_i), \mathcal{A}, \{\eta_{\boldsymbol{\alpha}}\})$.
3. if $T \neq \perp$ return (P, Q) for $Q \cdot u = T$.

Termination: when u is holonomic.

Possible strategies:

- any sequence of terms;
- filtration by total degree;
- filtration by a term order (FGLM-like).

Multivariate Definite ∂-Finite ∂_Ω^{-1}

1. Idea: replace candidate of a given degree by candidate of a given support (= set of terms with a non-zero coefficient)
2. Extension already possible in the hypergeometric case (Zeilberger's algorithm)
3. Iterated summation: possible if each inner sum is holonomic ∂-finite
4. Merging with the approach suggested by Verbaeten and Wegschaider could provide a 4th strategy in the future

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+r}{r}\binom{n+s}{s}\binom{2n-(r+s)}{n}=\sum_{k=0}^{\infty}\binom{n}{k}^4$$

The summand u is cancelled by:

$$\begin{cases}(n+1)(n+1-r)(n+1-s)(n+1-r-s)S_n\\ \quad +(n+s+1)(n+r+1)(2n+1-r-s)(2n+2-r-s),\\ (r+1)^2(2n-r-s)S_r+(n+r+1)(n-r)(n-r-s),\\ (s+1)^2(2n-r-s)S_s+(n+s+1)(n-s)(n-r-s),\end{cases}$$

with respect to the algebra $\mathbb{Q}(n,r,s)[S_n;S_n,0][S_r;S_r,0][S_s;S_s,0]$.

1. Inner simple sum over s: for $L=2$, introducing

$$\begin{cases}P=\eta_{0,0}+\eta_{1,0}S_n+\eta_{0,1}S_r+\eta_{2,0}S_n^2+\eta_{1,1}S_nS_r+\eta_{0,2}S_r^2,\\ Q=\phi(s).\end{cases}$$

one obtains:

$$P_1=(\dots)S_n^2+(\dots)S_r^2+(\dots)S_n+(\dots)S_r+(\dots),$$
$$P_2=(\dots)S_r^2+(\dots)S_n+(\dots)S_r+(\dots),$$
$$Q_1=\frac{(\dots)}{(\dots)},\qquad Q_2=\frac{(\dots)}{(\dots)}.$$

Homogeneous summation: $[Q_i\cdot u]_{s=-\infty}^{s=+\infty}=0$.

$\longrightarrow$ the inner sum is ∂-finite w.r.t. $\mathbb{Q}(n,r)[S_n;S_n,0][S_r;S_r,0]$.

2. Simple case of the algorithm for the summation over r yields

$$(n+2)^3S_n^2-2(2n+3)(3n^2+9n+7)S_n-4(4n+5)(4n+3)(n+1).$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+r}{r}\binom{n+s}{s}\binom{2n-(r+s)}{n}$$

1. Sum to be found in $A=B$; a better example, with non-hypergeometric summand, has to be found
2. Difficult part is left-hand side (Wegschaider gave another derivation)
3. Iteration: because P_1 and P_2 span a ∂-finite ideal
4. End of the proof:
   - same second-order operator for the right-hand side
   - both sides agree at $n=0$ and $n=1$, where they are 1 and 2,
   - leading coefficient $(n+2)^3$ does not vanish for non-negative integers n

Holonomic Functions in Computer Algebra

Part V

Kashiwara's Theorem and Applications

Kashiwara's Theorem and Applications

I had announced this part in a tentative way. The talk actually did not take place for lack of time. Kashiwara's theorem is a rather technical result. The main application from my point of view is a better understanding of the product of two holonomic functions, and for the specialization of a holonomic function. The proof (to be found in Coutinho's book) introduces many tools used in the theory of $\mathcal{D}$-module, which makes it a nice starting point for those who want to understand more.

Holonomic Functions in Computer Algebra

Appendices

Appendices

1. Blackboard example: holonomy of the module spanned by

$$f = J_0(x+y)$$

   over the Weyl algebra;

2. Hierarchy of classes (general vs differential cases).

$\mathcal{W}_2$-module spanned by $f = J_0(x+y)$

A Gröbner basis of the annihilating ideal of f for the total degree order $\partial_x > \partial_y > x > y$ is:

$$\partial_x - \partial_y, \quad x\partial_y^2 + y\partial_y^2 + \partial_y + x + y$$

(leading terms are ∂_x and $x\partial_y^2$.) In $\mathcal{W}_2 \cdot f$: (as vector spaces)

$$\Phi_0 \simeq \mathbb{K}, \qquad \Phi_1 \simeq \mathbb{K}x \oplus \mathbb{K}y \oplus \mathbb{K}\partial_y \oplus \Phi_0,$$
$$\Phi_2 \simeq \mathbb{K}x^2 \oplus \mathbb{K}xy \oplus \mathbb{K}y^2 \oplus \mathbb{K}x\partial_y \oplus \mathbb{K}y\partial_y \oplus \mathbb{K}\partial_y^2 \oplus \Phi_1,$$
$$\Phi_p \simeq \left(\bigoplus_{i=0}^{p} \mathbb{K}x^i y^{p-i}\right) \oplus \left(\bigoplus_{i=0}^{p-1} \mathbb{K}x^i y^{p-1-i}\partial_y\right) \oplus \left(\bigoplus_{i=0}^{p-2} \mathbb{K}y^i \partial_y^{p-i}\right) \oplus \Phi_{p-1} \quad (p \geq 2)$$

$$\dim_{\mathbb{K}} \Phi_p = 3p + \dim_{\mathbb{K}} \Phi_{p-1} \quad (p \geq 2).$$

Action of $\mathcal{W}_2$ is described by the Gröbner basis above; in particular:

$$\begin{aligned}
x \cdot y^i\partial_y^{p-i} &= y^{i+1}\partial_y^{p-i} + \text{ lower degree terms},\\
\partial_y \cdot x^i y^{p-1-i}\partial_y &= y^{p-1}\partial_y^2 + \text{ lower degree terms},\\
\partial_x \cdot x^i y^{p-i} &= x^i y^{p-i}\partial_y + \text{ lower degree terms},\\
\partial_x \cdot x^i y^{p-1-i}\partial_y &= y^{p-1}\partial_y^2 + \text{ lower degree terms},\\
\partial_x \cdot y^i\partial_y^{p-i} &= y^i\partial_y^{p+1-i} + \text{ lower degree terms}
\end{aligned}$$

$\mathcal{S}_4$-module $\mathrm{gr}(\mathcal{W}_2 \cdot J_0(x+y))$

A Gröbner basis of the annihilating ideal of $\mathrm{gr}(\mathcal{W}_2 \cdot J_0(x+y))$ for the total degree order $\partial_x > \partial_y > x > y$ is

$$\partial_x, \quad x\partial_y^2.$$

In $\mathrm{gr}(\mathcal{W}_2 \cdot f)$: (as vector spaces)

$$\Gamma_0 \simeq \mathbb{K}, \qquad \Gamma_1 \simeq \mathbb{K}x \oplus \mathbb{K}y \oplus \mathbb{K}\partial_y,$$
$$\Gamma_2 \simeq \mathbb{K}x^2 \oplus \mathbb{K}xy \oplus \mathbb{K}y^2 \oplus \mathbb{K}x\partial_y \oplus \mathbb{K}y\partial_y \oplus \mathbb{K}\partial_y^2,$$
$$\Gamma_p \simeq \left(\bigoplus_{i=0}^{p} \mathbb{K}x^i y^{p-i}\right) \oplus \left(\bigoplus_{i=0}^{p-1} \mathbb{K}x^i y^{p-1-i}\partial_y\right) \oplus \left(\bigoplus_{i=0}^{p-2} \mathbb{K}y^i \partial_y^{p-i}\right) \quad (p \geq 2)$$

$$\dim_{\mathbb{K}} \Gamma_p = 3p \quad (p \geq 2).$$

Action of $\mathcal{S}_4$ is described by the Gröbner basis above; in particular:

$$\begin{aligned}
x \cdot y^i\partial_y^{p-i} &= y^{i+1}\partial_y^{p-i},\\
\partial_y \cdot x^i y^{p-1-i}\partial_y &= y^{p-1}\partial_y^2,\\
\partial_x \cdot x^i y^{p-i} &= x^i y^{p-i}\partial_y,\\
\partial_x \cdot x^i y^{p-1-i}\partial_y &= y^{p-1}\partial_y^2,\\
\partial_x \cdot y^i\partial_y^{p-i} &= y^i\partial_y^{p+1-i}
\end{aligned}$$

Dimension and Characteristic Variety for $J_0(x+y)$

1. Hilbert function = Hilbert polynomial:

$$h_p = \frac{3p^2}{2} + \frac{3p}{2} + 1$$

2. Dimension = 2
3. Multiplicity = 3: two Bessel functions

$$J_0(x+y) = \sum_{k \geq 0} \frac{(-1)^k}{4^k} \frac{z^{2k}}{(k!)^2}, \qquad Y_0(x+y)$$

and the distribution $\mathcal{D}$ defined by

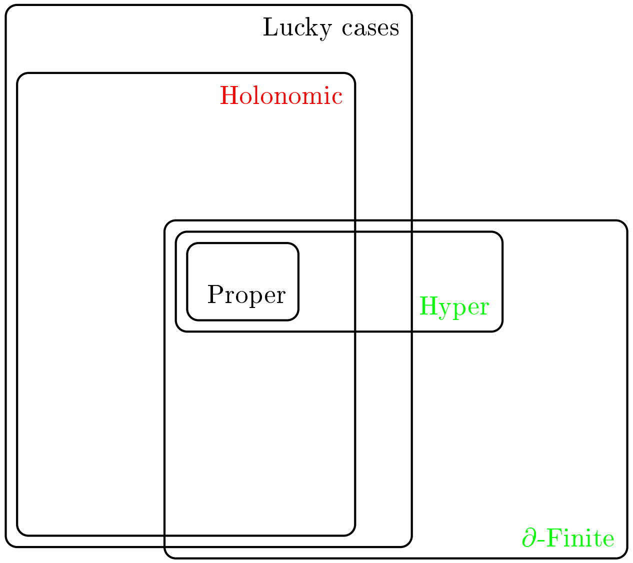
$$\langle \mathcal{D}, f \rangle = \sum_{k \geq 0} \frac{(-1)^k}{4^k} \binom{2k}{k} f^{(2k)}(0)$$

4. Characteristic variety:

$$\{x = \partial_x = 0\} \cup \{\partial_x = \partial_y = 0\}$$

(both subvarieties are of dimension 2)

Hierarchy of Classes: General Case



Hierarchy of Classes: Differential Case