Optimal iterative Stokes solvers in the harmonic Velte subspace

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Abstract

We utilize the decomposition of the function space $(H_0^1)^n$ (where $n = 2, 3$) into three orthogonal subspaces introduced by Velte to propose Uzawa and Arrow-Hurwitz iterations which - after 2 auxiliary iterations - proceed fully in the third, smallest subspace. Corresponding to this situation, we also compute optimal iteration parameters. Here, for two-dimensional problems, the lower estimate of the inf-sup constant by Horgan and Payne proves useful and provides an inclusion of the spectrum under consideration.

Computational results show the effectiveness of our approach for discretizations which admit a discrete Velte decomposition.

Keywords:
Stokes equations, Velte decomposition, inf-sup constants, Uzawa and Arrow-Hurwitz iterations, optimal iterative parameters

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1 Introduction

The Velte decomposition introduced in [29] can be regarded as an $(H_0^1)^n$-equivalent of the well-known Helmholtz decomposition (see, e.g., [18]) for vector functions in $(L_2)^n$. The Velte decomposition contains, besides the

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rotation-free and divergence-free vector functions, a third orthogonal subspace consisting of $(H_0^1)^n$-functions which are neither rotation-free nor divergence-free; it is accompanied by a corresponding decomposition of the pressure space L_2 in which the third subspace consists of harmonic functions.

This paper considers the construction of iterative methods for the first kind Stokes problem, a topic of much research, see, e.g., [28], [20], [21], [5], [26], [7], and, for recent work, [10], [33].

Intimately connected with the Stokes problem and its iterative solution is the inf-sup condition (see [3], [12], [18], [4], [15]) in which a constant appears depending on the domain only.

By [29]-[31] and [19], the inf-sup condition is related to further important inequalities like that of [16] for analytic functions in the plane, to the Korn inequality, and to the Cosserat eigenvalue problem (for the latter, see [22]). Therefore inf-sup constants for specific domains Ω can be derived from these papers. Such constants have been analytically found in [1], [21], [6], [8], [9], [24], mostly for two-dimensional domains; some information on the three-dimensional case is contained in [29], [31]. Since the inf-sup constants are eigenvalues of the Schur complement operator of the Stokes problem, the papers [17], [9], [13], [27], [24] are also of interest here.

Numerically computed inf-sup constants for specific discretizations can be found in [2], [26], [27].

The aim of the paper is to propose iterative methods for the Stokes problem in two and three dimensions which - after two iterations - exactly reach the third Velte subspace in the sense that then the error of the iterative solution belongs to that subspace. The advantage of performing iterations in the third, harmonic Velte subspace lies in the fact that in this subspace the spectrum of the Schur complement operator is closer and can be bounded through estimates of the inf-sup constant. In this way, optimal iteration parameters can be determined.

In cases where the discretization admits a discrete Velte decomposition (see [27], [14]) the above ideas remain valid, only the constants change and then depend on h. One such discretization is the well-known difference approximation on the staggered grid in two (see [27]) and three space dimensions (see [14]).

The outline of the paper is as follows. In Section 2, the necessary notations as well as the Velte decomposition in the continuous case are introduced.

Section 3 serves to shortly survey the discrete Stokes problem and discrete Velte decompositions. In Section 4, Uzawa and Arrow-Hurwitz iterations are shown to reach after 2 steps the third Velte subspace. Further, optimal parameters for both methods are calculated. More specific formulae can be given for the optimal parameters in the two-dimensional case. Between others it follows that $\tau = 2$ is the optimal constant iteration parameter for the Uzawa iteration for all plane domains with nonzero inf-sup constant.

Finally we present computational results in Section 5.

2 The first-kind Stokes problem and the Velte decomposition

Let Ω be a bounded, simply connected open domain in $I\!R^n$, $n = 2, 3$, with Lipschitz-continuous boundary $\partial\Omega$. Denote by $V := (H_0^1(\Omega))^n$ the Sobolev space of vector functions $u(x) = (u_1(x), \ldots, u_n(x))^T$ defined for $x = (x_1, \ldots, x_n) \in \Omega$, with generalized derivatives in $(L_2(\Omega))^n$ and with zero boundary values in the sense of traces on $\partial\Omega$.

For a given $f \in (L_2(\Omega))^n$, a $g \in L_2(\Omega)$ satisfying $\int_\Omega g(x)\mathrm{d}x = 0$, and denoting by $L_{2,0}(\Omega)$ the subspace of $L_2(\Omega)$ of square integrable functions with zero integral over Ω, consider the following Stokes problem in variational formulation,

$$
\begin{aligned}
a(u, v) + b(v, p) &= (f, v) \quad \text{for all } v \in V, &(1)\\
b(u, q) &= (g, q) \quad \text{for all } q \in L_{2,0}(\Omega), &(2)
\end{aligned}
$$

where

$$
\begin{aligned}
(g, q) &= \int_\Omega g(x)q(x)\mathrm{d}x,\\
(f, v) &= \sum_{i=1}^n (f_i, v_i) = \int_\Omega \sum_{i=1}^n f_i(x)v_i(x)\mathrm{d}x,\\
a(u, v) &= \int_\Omega \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\mathrm{d}x,\\
b(u, p) &= -(\operatorname{div} u, p).
\end{aligned}
$$

The problem consists in finding a velocity vector $u \in V$ and a pressure $p \in L_{2,0}(\Omega)$. As is well known (see [23], [18]) this problem is solvable and its solution continously depends on the data since for Lipschitz domains the so-called inf-sup condition is satisfied, see [23], [15], [9]:

$$\sup_{u\neq 0} \frac{b^2(u,p)}{a(u,u)} \geq \beta_0^2 \|p\|^2 \quad \text{for all } 0 \neq p \in L_{2,0}; \quad \beta_0 = \beta_0(\Omega) > 0.$$

Closely connected both with the inf-sup condition and the Stokes problem is the following orthogonal decomposition of V which has been derived in [29]:

$$(H_0^1(\Omega))^n = V = V_0 \oplus V_1 \oplus V_\beta, \tag{3}$$

where

$$\begin{aligned} V_0 &:= \ker \operatorname{div} = \{w \in V,\ \operatorname{div} w = 0\}, \\ V_1 &:= \ker \operatorname{rot} = \{w \in V,\ \operatorname{rot} w = 0\}, \end{aligned}$$

and where in the two-dimensional case we mean by $\operatorname{rot} w$ the scalar $\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}$ often also denoted by $\operatorname{curl} w$.

The third orthogonal subspace V_β has been characterized in [29], see also Lemma 1 in [27]; it consists of biharmonic vector functions.

The space L_2 is decomposed similarly (see [27]) into three orthogonal subspaces:

$$L_2 = P_0 \oplus P_1 \oplus P_\beta, \tag{4}$$

where

$$P_0 := \ker \operatorname{grad}, \quad P_1 = \operatorname{div} \ker \operatorname{rot} = \operatorname{div} V_1, \quad P_\beta = \operatorname{div} V_\beta.$$

Here P_0 is the one-dimensional space of functions constant on Ω whereas P_β consists of harmonic functions [29].

We call both decompositions (3) and (4) *Velte decompositions* and remark (see [18] and [27]) that the operator $\Delta^{-1}\operatorname{grad}$ is an isomorphism from P_1 and P_β onto V_1 and V_β, respectively, whereas div is an isomorphism from V_1 and V_β onto P_1 and P_β; both mappings conserve orthogonalties.

[27] also contains the following result: in the two-dimensional case, there holds

$$P_1 = \operatorname{div} V_1 = \operatorname{rot} V_0 \ \perp\ \operatorname{rot} V_\beta = \operatorname{div} V_\beta = P_\beta. \tag{5}$$

Decisive for the solution of the Stokes problem and the inf-sup condition is the Schur complement operator $S = \operatorname{div} \Delta^{-1} \operatorname{grad}$ which has been investigated in [17], [13], and in [27]. The eigenvalues of the Schur complement operator S are known to lie in $[0, 1]$, the zero eigenvalue corresponds to the constant pressure (P_0) and to divergence-free velocities (V_0), the unit eigenvalue belongs to the rotation-free velocities (V_1) and to pressures (P_1) which are the divergence of such velocities, whereas the eigenvalues in $(0, 1)$ depend on the third Velte subspace only [27], and the inf-sup constant $\beta_0(\Omega)$ is the smallest between the latter eigenvalues. Due to (5), in two dimensions there holds for the restriction of S to P_β

$$\beta_0^2 \leq \lambda(S_{|P_\beta}) \leq 1 - \beta_0^2, \tag{6}$$

where both β_0^2 and $1 - \beta_0^2$ belong to the spectrum of S, see [13] and [27]; [30] contains the more general result that together with λ also $1 - \lambda$ is an eigenvalue of $S_{|P_\beta}$.

To formulate a practically useful estimate of $\beta_0(\Omega)$ for a class of domains Ω, we introduce the following concept.

Definition. Let Ω be a starlike domain with Lipschitz boundary $\partial\Omega$ and let $x_0 \in \Omega$. For all $x \in \partial\Omega$, consider the outer angle $\alpha(x)$ between a ray emitted from x_0 (hitting $\partial\Omega$ in x) and the outer normal vector to $\partial\Omega$ in x. Then we call the following angle ω a *Horgan-Payne angle* of Ω: $\omega = \frac{\pi}{2} - \max_{x\in\partial\Omega} \alpha(x)$. •

For domains for which there is a ball from all points of which all of $\partial\Omega$ is visible, the Horgan-Payne angle is positive.

Lemma 1. Let Ω be a starlike domain with Lipschitz boundary and positive Horgan-Payne angle ω. Then the inf-sup constant $\beta_0(\Omega)$ satisfies the following estimate:

$$\beta_0(\Omega) \geq \sin\frac{\omega}{2}. \tag{7}$$

Proof. This result is essentially contained in [19] (even though the angle introduced above does not explicitely appear there). We use the notations of the above definition and formula (6.29) in [19] according to which the constant Γ_R of an inequality considered by Friedrichs (see [16] and [19]) satisfies

$$\Gamma_R \le \max_{x\in\partial\Omega}\left\{\frac{1}{\cos\alpha(x)}+\left(\frac{1}{\cos^2\alpha(x)}-1\right)^{1/2}\right\}^2 .$$

Expressing the right-hand side as $((1+\cos\omega)/\sin\omega)^2$ and taking into account that by [19] and [29] there holds $\beta_0(\Omega) = 1/\sqrt{1+\Gamma_R}$, the result follows straightforwardly. $\bullet$

In cases where $\partial\Omega$ has inner angles different from π and no inner angles equal zero, an upper estimate of $\beta_0(\Omega)$ is also available, see [27] where on the basis of these bounds, for a number of computationally interesting domains Ω, an inclusion of $\beta_0(\Omega)$ has been obtained. For the square this inclusion reads

$$0.382683\ldots = \sin\frac{\pi}{8} \le \beta_0 \le \sqrt{\frac{1}{2}(1-\frac{2}{\pi})} = 0.426251\ldots, \qquad (8)$$

where the upper bound seems to be the exact value of the constant. For a rectangle R_L with sides 1 and $L \ge 1$, using the above approach, the upper bound remains the same as for the square, whereas from Lemma 1, with $\varepsilon = 1/L^2 \le 1$, it follows that

$$\beta_0(R_L) \ge \sin\frac{\omega(R_L)}{2} = \sqrt{\frac{1}{2}\frac{\sqrt{1+\varepsilon}-1}{\sqrt{1+\varepsilon}}}$$

which can elementary be further estimated by $\frac{1}{L}\sin\frac{\pi}{8}$. This improves the lower estimate obtained in [24], where for R_L the estimates

$$\frac{1}{L\sqrt{304}} \le \beta_0(R_L) \le \frac{\pi}{2\sqrt{3}L}$$

have been shown, so that $\beta_0(R_L) = \mathcal{O}(1/L)$.

3 The discrete Stokes problem, and discrete Velte decompositions

Using a suitable discretization by finite element or difference methods, from (1)-(2) we obtain the following discrete Stokes problem:

$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \qquad (9)$$

Here A_h corresponds to the vector Laplace operator and is a symmetric positive definite matrix; B_h corresponds to the negative divergence operator. Further, u, p and f, g now denote the nodal vectors of velocities, pressure, and of the projection of the force vector and the prescribed divergence, respectively.

As has been shown in [27] and [14], for the well-known difference approximation on a staggered grid, even for rather general two- and three-dimensional domains, a discrete Velte decomposition exists. Moreover, there are finite element families (of sufficiently high polynomial degree: with conforming velocities elementwise contained in the usual polynomial space $(P_k)^n$ and with discontinuous pressures elementwise contained in P_{k-1}, for $k \geq 4$) which also admit a discrete Velte decomposition whereas for the usual low-order finite elements inf-sup stably approximating the Stokes problem (first order Crouzeix-Raviart element: [11]; or the P0-P1 macro element: [15]; or the P1 element / P2 $\oplus$ bubble: [4]) a discrete Velte decomposition does not exist, see [14].

If a discrete Velte decomposition exists, then the eigenvalues of the discrete inf-sup problem are contained in $[0, 1]$, moreover, the eigenvalue 1 corresponds to discretely rotation-free velocities and the eigenvalue 0 to discretely divergence-free velocities [27].

For the efficiency of the iterative methods considered next it is important that the dimension of the third discrete Velte subspaces $P_{h\beta}$ and $V_{h\beta}$ is connected to the number of boundary points whereas the dimensions of the subspaces of discretely rotation-free (V_{h1}) resp. divergence-free functions (V_{h0}) correspond to the number of inner points. For illustration we mention that in [27] it has been shown that the dimensions in case of a square are $4N-9$ for $P_{h\beta}$ and $V_{h\beta}$, $(N-2)^2$ for V_{h0} and $(N-3)^2$ for V_{h1} and P_{h1} where N is the number of points along a side of the square, i.e. $(N-1)^2$ the number of cells of the staggered grid.

4 The algorithms of Uzawa and Arrow-Hurwitz in Velte subspaces

We describe and investigate these iterative methods to solve the Stokes problem (see [28]) for the differential operators. Let (p, u) be the exact solution of (1)-(2), its approximations $(p^{(i)}, u^{(i)})$ will be obtained in the i-th iteration and then $q^{(i)} := p - p^{(i)}$ is the pressure error and $v^{(i)} := u - u^{(i)}$ is the velocity error. By $w = (-\Delta)^{-1} v$ we denote the solution of the vector-Poisson equation $-\Delta w = v$ in $(H_0^1)^n$ for $v \in (H^{-1})^n$, where $n = 2, 3$.

Starting from any $p^{(0)} \in L_2(\Omega)$, we proceed in case of the Uzawa algorithm as follows.

First we project $p^{(0)}$ into $L_{2,0}$ by replacing it with

$$p^{(0)} := p^{(0)} - \frac{1}{|\Omega|} \int_\Omega p^{(0)}(x) \mathrm{d}x.$$

Therefore now $p^{(0)} \in P_1 \oplus P_\beta$; for the error we have correspondingly $q^{(0)} = q_1^{(0)} + q_\beta^{(0)}$.

Secondly we perform one iteration with parameter $\tau = 1$:

$$u^{(1)} = (-\Delta)^{-1}(f - \operatorname{grad} p^{(0)}), \quad p^{(1)} = p^{(0)} - g - \operatorname{div} u^{(1)}.$$

For the errors $v^{(1)}$ and $q^{(1)}$, respectively, this means

$$\begin{aligned} v^{(1)} &= \Delta^{-1} \operatorname{grad} q^{(0)} = \Delta^{-1} \operatorname{grad} q_1^{(0)} + \Delta^{-1} \operatorname{grad} q_\beta^{(0)} \\ &=: v_1^{(1)} + v_\beta^{(1)} \in V_1 \oplus V_\beta, \\ q^{(1)} &= q^{(0)} - \operatorname{div} v^{(1)} = (q_1^{(0)} - \operatorname{div} v_1^{(1)}) + (q_\beta^{(0)} - \operatorname{div} v_\beta^{(1)}) \\ &= q_\beta^{(0)} - \operatorname{div} v_\beta^{(1)} = q_\beta^{(0)} - S q_\beta^{(0)} =: q_\beta^{(1)} \in P_\beta, \end{aligned}$$

since $q_1^{(0)} - \operatorname{div} v_1^{(1)} = q_1^{(0)} - \operatorname{div} \Delta^{-1} \operatorname{grad} q_1^{(0)} = q_1^{(0)} - S q_1^{(0)} = 0$.

Perform now a second iteration with parameter $\tau = 1$:

$$u^{(2)} = (-\Delta)^{-1}(f - \operatorname{grad} p^{(1)}), \quad p^{(2)} = p^{(1)} - g - \operatorname{div} u^{(2)}.$$

Thereafter the errors are

$$\begin{aligned} v^{(2)} &= \Delta^{-1} \operatorname{grad} q^{(1)} = \Delta^{-1} \operatorname{grad} q_\beta^{(1)} =: v_\beta^{(2)} \in V_\beta, \\ q^{(2)} &= q^{(1)} - \operatorname{div} v^{(2)} = q_\beta^{(1)} - S q_\beta^{(1)} =: q_\beta^{(2)} \in P_\beta. \end{aligned}$$

¿From here on the errors of all subsequent Uzawa iterations remain in the third Velte subspace independent of $\tau > 0$. In the usual way [25] we can calculate a constant optimal iteration parameter as

$$\tau = \tau_{\mathrm{opt}}^0 := 2/(\underline{\beta}^2 + \overline{\beta}^2) \tag{10}$$

when

$$0 < \underline{\beta}^2 \leq \lambda(S_{|_{P_\beta}}) \leq \overline{\beta}^2 < 1 \tag{11}$$

gives the inclusion of the spectrum of $S_{|_{P_\beta}}$. This holds in two and three dimensions, but in the latter case $\overline{\beta}^2 = 1 - \underline{\beta}^2$ is not true.

When this relation is satisfied, like in two dimensions according to (6), then the optimal iteration parameter is just $\tau = 2$. In case Ω is a circle we obtain the exact solution after the first such iteration (i.e. at the third iteration), because of $\beta_0^2 = \frac{1}{2}$, which follows via [19], [29] from [16] or directly from [12]. (We remark also that from [32] follows the existence of a whole class of domains in two dimensions for which there are only a few eigenvalues in the spectrum of $S_{|_{P_\beta}}$.) In proceeding in the Velte subspace V_β, for S restricted to P_β we have indeed from (6) that

$$\|I - 2S_{|_{P_\beta}}\|_0 = 1 - 2\beta_0^2(\Omega) < 1$$

for all two-dimensional domains Ω with nonzero inf-sup constant. (We remember that [28] shows convergence of the Uzawa algorithm for $0 < \tau < 2$.)

We can also calculate optimal sequences of parameters for a Chebyshev-type iteration, see [25] or [20]. For this aim it is important that in two dimensions, according to the estimates (6) and (7), when we know the Horgan-Payne angle ω of Ω, the eigenvalues of $S_{|_{P_\beta}}$ are bounded from below and above satisfying

$$\sin^2\frac{\omega}{2} \leq \lambda(S_{|_{P_\beta}}) \leq \cos^2\frac{\omega}{2}. \tag{12}$$

We turn now to the Arrow-Hurwitz algorithm (see [28], [20]), denoting the iteration parameters by τ and α and the intermediate velocities by $w^{(i)}$. Starting from a $p^{(0)} \in L_2(\Omega)$, we project it first into $L_{2,0}$ so that $p^{(0)} \in P_1 \oplus P_\beta$ and $q^{(0)} = q_1^{(0)} + q_\beta^{(0)}$.

Next we perform one iteration with parameters $\tau = 1,\ \alpha = 1$ (i.e. an Uzawa iteration):

$$w^{(1)} = (-\Delta)^{-1}(f - \operatorname{grad} p^{(0)}), \quad u^{(1)} = w^{(1)}, \quad p^{(1)} = p^{(0)} - g - \operatorname{div} u^{(1)},$$

correspondingly obtaining the errors $v^{(1)}$ and $q^{(1)}$,

$$\begin{aligned} v^{(1)} = (-\Delta)^{-1}(f - \operatorname{grad} q^{(0)}) &= \Delta^{-1}\operatorname{grad} q_1^{(0)} + \Delta^{-1}\operatorname{grad} q_\beta^{(0)} \\ &=: v_1^{(1)} + v_\beta^{(1)} \in V_1 \oplus V_\beta, \\ q^{(1)} &= q_\beta^{(0)} - S q_\beta^{(0)} =: q_\beta^{(1)} \in P_\beta. \end{aligned}$$

Now follows an Arrow-Hurwitz iteration with parameters $\tau = 1, \;\; \alpha = \frac{1}{2}(\underline{\beta}^2 + \overline{\beta}^2) = 1/\tau^0_{\text{opt}}$, see (10):

$$w^{(2)} = (-\Delta)^{-1}(f - \operatorname{grad} p^{(1)}), \quad u^{(2)} = w^{(2)}, \quad p^{(2)} = p^{(1)} - \tau^0_{\text{opt}}(g + \operatorname{div} u^{(2)}).$$

Then the errors are

$$\begin{aligned} v^{(2)} &= \Delta^{-1} \operatorname{grad} q^{(1)} = \Delta^{-1} \operatorname{grad} q^{(1)}_\beta =: v^{(2)}_\beta \in V_\beta, \\ q^{(2)} &= q^{(1)} - \tau^0_{\text{opt}} \operatorname{div} v^{(2)} = q^{(1)}_\beta - \tau^0_{\text{opt}} S q^{(1)}_\beta =: q^{(2)}_\beta \in P_\beta. \end{aligned}$$

It is clear that here – to assure $v^{(2)}_\beta \in V_\beta$ and $q^{(2)}_\beta \in P_\beta$ – it is decisive that $\tau = 1$ but $\alpha > 0$ is free; therefore we already have chosen the optimal value $1/\alpha = \tau^0_{\text{opt}}$ (and when Ω is a circle we get the exact solution at this second iteration). Now the errors of the subsequent Arrow-Hurwitz iterations remain in the third Velte subspace independent of $\tau, \alpha > 0$.

Lemma 2. The Arrow-Hurwitz methods with initial steps $\tau = \alpha = 1$ and $\tau = 1, \;\; \alpha = 1/\tau^0_{\text{opt}}$ converges if, beginning from the third step, we select $\alpha > 0$ and τ according to

$$0 < \tau < \frac{\alpha}{\overline{\beta}^2} \left(\sqrt{1 + 4\frac{\overline{\beta}^2}{\alpha}} - 1 \right)$$

where $\overline{\beta}^2$ is the upper bound (11) of the spectrum of $S_{|P_\beta}$.

Proof. An analysis similar to what follows has been performed in [7] for $(P_0)^\perp$, i.e. not knowing the Velte decomposition. We shall use that $S_{|P_\beta}$, being selfadjoint (and positive definite) has a complete orthogonal system of eigenfunctions in P_β which we denote by $\{p^{(\beta)}_j\}^\infty_{j=1}$. The corresponding eigenvalues are denoted by $\{\beta^2_j\}^\infty_{j=1}$; they are contained in the interval (11). A complete orthogonal system in V_β is then given by $\{\Delta^{-1} \operatorname{grad} p^{(\beta)}_j\}^\infty_{j=1}$ (as follows from Section 2, see there above (5)). Let the errors of the i-th Arrow-Hurwitz iteration ($i \geq 2$) have the following expansions:

$$v^{(i)} = \sum_{j=1}^\infty \nu^{(i)}_j \Delta^{-1} \operatorname{grad} p^{(\beta)}_j, \quad q^{(i)} = \sum_{j=1}^\infty \mu^{(i)}_j p^{(\beta)}_j. \tag{13}$$

It follows then that the errors of the next iteration are given by

$$
\begin{aligned}
v^{(i+1)} &= \sum_{j=1}^{\infty}((1-\tau)\nu_j^{(i)} + \tau\mu_j^{(i)})\Delta^{-1}\,\mathrm{grad}\,p_j^{(\beta)},\\
q^{(i+1)} &= \sum_{j=1}^{\infty}\left(\frac{\tau(\tau-1)}{\alpha}\beta_j^2\nu_j^{(i)} + \left(1-\frac{\tau^2}{\alpha}\beta_j^2\right)\mu_j^{(i)}\right)p_j^{(\beta)}.
\end{aligned}
$$

These expansions obey the form

$$
v^{(i+1)} = \sum_{j=1}^{\infty}\lambda_j\nu_j^{(i)}\Delta^{-1}\,\mathrm{grad}\,p_j^{(\beta)} \quad =: \quad \sum_{j=1}^{\infty}\nu_j^{(i+1)}\Delta^{-1}\,\mathrm{grad}\,p_j^{(\beta)}, \tag{14}
$$

$$
q^{(i+1)} = \sum_{j=1}^{\infty}\lambda_j\mu_j^{(i)}p_j^{(\beta)} \quad =: \quad \sum_{j=1}^{\infty}\mu_j^{(i+1)}p_j^{(\beta)}, \tag{15}
$$

if the λ_j are eigenvalues of the following problems:

$$
A_j\begin{pmatrix}\nu\\ \mu\end{pmatrix} = \lambda\begin{pmatrix}\nu\\ \mu\end{pmatrix}, \quad A_j := \begin{pmatrix} 1-\tau & \tau \\ \frac{\tau(\tau-1)}{\kappa_j^2} & 1-\frac{\tau^2}{\kappa_j^2}\end{pmatrix} \tag{16}
$$

where we have introduced $\kappa_j := \frac{\sqrt{\alpha}}{\beta_j}$. The characteristic polynomial of the matrix A_j here is

$$
\lambda^2 - \left(2-\tau-\frac{\tau^2}{\kappa_j^2}\right)\lambda + 1 - \tau,
$$

its roots are

$$
\lambda_{1,2}^{(j)} = 1-\gamma_j \pm \sqrt{\gamma_j^2 - \frac{\tau^2}{\kappa_j^2}}, \quad \gamma_j := \frac{\tau}{2} + \frac{\tau^2}{2\kappa_j^2}. \tag{17}
$$

We use now the solutions $(\lambda_1^{(j)}, (\nu_{(j,1)}, \mu_{(j,1)})^T)$ and $(\lambda_2^{(j)}, (\nu_{(j,2)}, \mu_{(j,2)})^T)$ of (16) to rewrite the coefficients of the j-th terms of (13) for fixed i $(=2)$ as

$$
\begin{aligned}
\nu_j^{(i)} &= \rho_{j,1}^{(i)}\nu_{(j,1)} + \rho_{j,2}^{(i)}\nu_{(j,2)},\\
\mu_j^{(i)} &= \rho_{j,1}^{(i)}\mu_{(j,1)} + \rho_{j,2}^{(i)}\mu_{(j,2)}.
\end{aligned} \tag{18}
$$

Since here $\nu_j^{(i)}$ and $\mu_j^{(i)}$ can be considered to be given, this amounts to solving the 2×2 systems (18) for $\rho_{j,1}^{(i)}$ and $\rho_{j,2}^{(i)}$. As long as $\lambda_1^{(j)} \neq \lambda_2^{(j)}$, this is possible since then the eigenvectors of (16) are independent. However, for $\tau > 0$ and $\alpha < 4\beta_j^2$ it can happen that

$$\gamma_j^2 = \frac{\tau^2}{\kappa_j^2}. \tag{19}$$

In fact, (19) means in detail

$$\alpha^2 - 4\alpha\beta_j^2 + 2\alpha\beta_j^2\tau + \beta_j^4\tau^2 = 0 \tag{20}$$

and this equation has the positive root

$$\tau_1 = \tau_1(\alpha, \beta_j) = \kappa_j(2 - \kappa_j) \qquad \left(, \ \kappa_j = \frac{\sqrt{\alpha}}{\beta_j} < 2\right). \tag{21}$$

When (19) holds, by (17) we have a double root

$$\lambda_1^{(j)} = \lambda_2^{(j)} = \lambda^{(j)} = 1 - \gamma_j = 1 - \gamma_j(\tau_1(\alpha, \beta_j)),$$

and since the matrix in (16) is then similar to a nontrivial 2×2-Jordan block, there is only one eigenvector $y^{(j)} := (\nu_{(j,1)}, \mu_{(j,1)})^T$. Instead of $(\nu_{(j,2)}, \mu_{(j,2)})^T$, we take a root vector $z^{(j)}$ satisfying $A_j z^{(j)} = y^{(j)}$, e.g. $z^{(j)} = (0, -1)^T$ when $y^{(j)} = (-\tau, 1 - \tau - \lambda^{(j)})^T$. Then (18) again is solvable with respect to $\rho_{j,1}^{(i)}$ and $\rho_{j,2}^{(i)}$. Proceeding from (13) to (14) and further, continuing the iteration, we find for $k \geq 1$:

$$\rho_{j,1}^{(i+k)} = \left(\lambda_1^{(j)}\right)^k \rho_{j,1}^{(i)}, \quad \rho_{j,2}^{(i+k)} = \left(\lambda_2^{(j)}\right)^k \rho_{j,2}^{(i)}$$

in case of different roots $\lambda_{1,2}^{(j)}$, and

$$\rho_{j,1}^{(i+k)} = \left(\lambda^{(j)}\right)^k \rho_{j,1}^{(i)} + k\left(\lambda^{(j)}\right)^{k-1} \rho_{j,2}^{(i)}, \quad \rho_{j,2}^{(i+k)} = \left(\lambda^{(j)}\right)^k \rho_{j,2}^{(i)}$$

for a double root. In both cases, clearly, the iteration converges if and only if $|\lambda_{1,2}^{(j)}| < 1$ for all j. We finish the proof of the lemma by straightforwardly analysing (17) (which, for $4\beta_j^2 > \alpha$, includes the consideration of complex and double roots). The result is that, with $\kappa_j^2 = \frac{\alpha}{\beta_j^2}$,

$$0 < \tau < \kappa_j^2\left(\sqrt{1 + \frac{4}{\kappa_j^2}} - 1\right) \tag{22}$$

is necessary and sufficient for convergence. Then, taking into account (11) and remarking that the right-hand side of (22) is a monotonously decreasing function of β_j^2, the lemma follows. $\bullet$

Theorem. For the Arrow-Hurwitz methods with initial steps $\alpha = \tau = 1$ and $\alpha = 1/\tau^0_{\text{opt}}$, $\tau = 1$, the following values of $\alpha, \tau > 0$ (as iteration parameters from the third step on) are optimal with respect to the asymptotic speed of convergence and for a given inclusion (11) of the spectrum of $S_{|P_\beta}$:

$$\alpha = \alpha_{\text{opt}} := \left(\frac{2\underline{\beta}\overline{\beta}}{\underline{\beta} + \overline{\beta}} \right)^2, \tag{23}$$

$$\tau = \tau_1(\alpha_{\text{opt}}) := \frac{4\underline{\beta}\overline{\beta}}{(\underline{\beta} + \overline{\beta})^2}. \tag{24}$$

Proof. According to the proof of Lemma 2, optimizing the asymptotic speed of the Arrow-Hurwitz iteration means minimizing $\lambda^{(j)}_{\max} := \max(|\lambda^{(j)}_1|, |\lambda^{(j)}_2|)$, see (17). We have $\lambda^{(j)}_1 \lambda^{(j)}_2 = 1 - \tau$, hence, if $0 < \tau \leq 1$, $\lambda^{(j)}_{\max} > \sqrt{1-\tau}$ for $\lambda^{(j)}_{1,2}$ being real and not double roots, and

$$\lambda^{(j)}_{\max} = \sqrt{1-\tau} \tag{25}$$

for real double or for complex roots. Therefore this latter case is advantageous; it happens for all j if

$$\gamma_j^2 \leq \frac{\tau^2}{\kappa_j^2} \tag{26}$$

i.e. if the left-hand expression in (20) is non-positive for all j. For this to be possible, $\alpha \leq 4\underline{\beta}^2$ must hold. According to (25), a maximal $\tau \leq 1$ is optimal. We therefore take the positive root (21) of (20); then $\tau = \tau_1 \leq 1$ indeed will hold. As usual, we perform the optimal choice of α $(\leq 4\underline{\beta}^2)$ not for the set of eigenvalues β_j^2 of $S_{|P_\beta}$ but for the interval (11).

The optimal choice then consists in selecting $\alpha = \alpha_{\text{opt}}$ such that

$$\tau_1(\alpha_{\text{opt}}, \beta) > \tau_1(\alpha_{\text{opt}}, \underline{\beta}) = \tau_1(\alpha_{\text{opt}}, \overline{\beta}) =: \tau_{\text{opt}} \tag{27}$$

for $\underline{\beta}^2 < \beta^2 < \overline{\beta}^2$. The τ_1-values belonging to $\underline{\kappa} := \frac{\sqrt{\alpha}}{\overline{\beta}}$ and to $\overline{\kappa} := \frac{\sqrt{\alpha}}{\underline{\beta}}$ will be equal for $2 = \underline{\kappa} + \overline{\kappa}$, i.e. if

$$2 = \sqrt{\alpha}\frac{\underline{\beta} + \overline{\beta}}{\underline{\beta}\overline{\beta}},$$

from where (23) follows, and $\alpha \leq 4\underline{\beta}^2$ holds indeed. Further we find from (27) that the optimal $\tau_1 \leq 1$ is (24). •

Remarks. 1. One checks straightforwardly that for the values (23), (24) of α and τ and for $\underline{\beta}^2 < \beta_j^2 < \overline{\beta}^2$, relation (26) holds with strong inequality, i.e. then $\lambda_{1,2}^{(j)}$ are complex whereas for $\beta_j = \underline{\beta}$ and $\beta_j = \overline{\beta}$, $\lambda_{1,2}^{(j)}$ is a double root. The case $\underline{\beta}^2 = \overline{\beta}^2 =: \beta^2$ seems to happen only for the circle; at any rate, instead of (23), (24) one better takes $\tau = 1, \alpha = \beta^2$ then – which means application of Uzawa iterations with $\tau = 1/\beta^2$.

2. For the optimal values of α and τ, the error reduction rate of one Arrow-Hurwitz iteration is given by

$$q_{\text{opt}} = \lambda_{\max}(\tau_{\text{opt}}) = \sqrt{1 - \tau_{\text{opt}}} = \frac{\overline{\beta} - \underline{\beta}}{\overline{\beta} + \underline{\beta}}.$$

This is obtained for the constant optimal parameters (23), (24) but else is typical for the convergence speed reachable by Chebyshev iteration. •

Specifying the formulae for the purpose of an illustration to the case of Ω being a square, we have from (8)

$$\begin{aligned}
\underline{\beta} &= \sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2 - \sqrt{2}}, \\
\overline{\beta} &= \cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2 + \sqrt{2}}, \\
\tau_{\text{opt}} &= \frac{2\sin\frac{\pi}{4}}{1 + \sin\frac{\pi}{4}} = 2(\sqrt{2} - 1), && (28) \\
\alpha_{\text{opt}} &= \frac{1}{2(1 + \sin\frac{\pi}{4})} = 1 - \frac{\sqrt{2}}{2}, && (29) \\
q_{\text{opt}} &= \sqrt{2} - 1 \approx 0.4142. && (30)
\end{aligned}$$

5 Numerical results

All numerical results refer to the staggered grid approximation of the Stokes problem in the unit square with $h = 1/(N-1)$, where $(N-1)^2$ is the number of grid cells.

First we list values of $\tau_{\text{opt},N}$ as computed from the smallest nonzero and the largest nonunit eigenvalue of the discrete Schur complement operator. These values have to be compared with $\tau = 2$.

Table 1

N	4	8	16	32	64	128
$\tau_{\text{opt},N}$	1.37	1.53	1.65	1.73	1.78	1.81

Next we show the final errors e of the iteratively computed pressure $p_{it,h}$ as reached after maximally 61 Uzawa iterations or earlier if the criterion $e := \|p_{ex,h} - p_{it,h}\|_{(2)} \leq 10^{-10}$ was satisfied, along with the corresponding iteration numbers. The exact discrete solution $p_{ex,h}$ was known, a constant parameter $\tau = 2$ was taken (after the initial steps with $\tau = 1$ which are also counted).

Table 2

N	4	8	16	32	64	128	256
iter.	58	55	58	61	61	61	61
e	5.04-11	8.31-11	6.20-11	3.91-11	6.37-11	4.18-10	1.14-9

Table 3 below shows results of a Chebyshev iteration with 3 parameters used cyclically and computed on the basis of the smallest nonzero and the largest nonunit eigenvalue of the discrete Schur complement operator. The notations and the stopping criterion are the same as above, however, maximally 30 iterations have been performed.

Table 3

N	4	8	16	32	64	128	256
iter.	12	17	20	23	26	30	30
e	9.46-11	1.85-11	5.07-11	8.26-11	9.64-11	1.21-10	1.14-9

Now we show the initial errors e_0 and the final errors e of the iteratively computed pressure $p_{it,h}$ after maximally 46 Uzawa iterations or earlier if $e := \|p_{ex,h} - p_{it,h}\|_{(2)} \leq 10^{-9}$ was satisfied, and the corresponding iteration numbers. The constant iteration parameter $\tau = 2$ was taken (after the initial steps with $\tau = 1$) but every 18 iterations one step with $\tau = 1/0.855 \approx 1/\overline{\beta}$ was performed.

Table 4

N	8	16	32	64	128	256	361
iter.	37	37	37	37	37	46	46
e_0	2.03	4.78	10.36	21.55	43.94	88.73	125.48
e	1.29-11	2.11-11	3.48-11	6.04-11	9.68-11	1.23-9	3.62-9

Finally we give the results for an application of an Arrow-Hurwitz iteration where, according to Section 4, $\tau = \alpha = 1$ and $\tau = 1,\ \alpha = 1/2$ were taken initially, then the values (28) and (29) were used. The maximal iteration number was 30, the stopping criterion 10^{-9}. The initial errors are the same as above. From these data, an average error reduction rate q has been calculated, too, which is to be compared with (30).

Table 5

N	8	16	32	64	128	256	512
iter.	24	26	27	28	29	30	30
e	3.87-10	8.64-10	8.94-10	4.80-10	7.00-10	1.30-9	2.64-8
q	0.3936	0.4220	0.4239	0.4165	0.4243	0.4354	0.4703

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