Technical Report

Proper Parametrization of Real Algebraic Surfaces

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It is well-known that a real algebraic surface is real-rational if and only if it is complex-rational and connected. In this paper, we give an explicit construction for a large class of surfaces that satisfy this criterion.

1. Introduction

In this paper, we study the following problem: given an absolutely irreducible polynomial $F \in \mathbf{R}[x, y, z]$ (the implicit representation of a real algebraic surface S), we want to decide if S has a proper parametrization $(X, Y, Z) \in \mathbf{R}(s, t)$, and to compute one in the affirmative case. The analoguous problem for curves has been studied by several authors, for instance in (Alonso *et al.*, 1995; Recio and Sendra, 1997; Sendra and Winkler, 1997) (some of these authors start with a given complex parametrization). It has been observed that proper parametrizations have nicer properties than arbitrary ones; for instance, the image is dense in the usual (Euclidean) topology, whereas an improper parametrization may just parametrize a small portion of the given variety (see (Hong and Schicho, 1998)).

In the curve case, the theorem of Lüroth is very relevant, since it implies that unirationality is equivalent to rationality. The Lüroth theorem is still true for complex surfaces, but it is false for real surfaces (see (Shafarevich, 1974)). A different approach is necessary.

An important birational invariant is the number of connected components (see (Bochnak *et al.*, 1987)). It follows that this number being equal to 1 is necessary for the real-rationality. Conversely, it is known that the connectedness and real unirationality imply real-rationality ((Comesatti, 1912; Manin, 1967; Iskovskih, 1967), see also (Silhol, 1980)). Moreover, the unirationality condition can be weakened to complex-rationality (i.e. the existence of a parametrization with complex coefficients). Consequently, real-rationality can be decided: we can compute the number of connected components by (Canny *et al.*, 1992) and we can decide complex-rationality (see (Schicho, 1998b)). In contrast to this, we have examples of surfaces which are complex-rational but not real-rational, and for which real unirationality is still an open problem.

The first step in the proofs of the results of Comesatti/Manin is to start with a real surface which is complex-rational, and to reduce it birationally to a surface with a pencil of rational curves or to a Del Pezzo surface. This construction makes use of the Castelnuovo criterion $p_a = P_2 = 0$ (see (Hartshorne, 1977)). An algorithm, based on the concept of adjoints, can be found in (Schicho, 1998b) (there, it is formulated for the complex case, but it uses only field arithmetic and the existence of one nonsingular point).

The second step is to reduce a surface with given pencil of rational curves birationally to a "tubular surface", i.e. a surface with equation $A(t)x^2 + B(t)y^2 + C(t) = 0$, $A, B, C \in \mathbf{R}[t] - \{0\}$. This step amounts to the birational reduction of a genus zero curve defined over the field $\mathbf{R}(t)$ to a conic, which is also algorithmic. An algorithm for doing this is already described in (Nöether, 1870). A very explicit description can be found in (Sendra and Winkler, 1997).

The third step is to compute the number of connected components for a given tubular surface, and to construct a proper parametrization if this number is equal to one. However, the proofs in (Comesatti, 1912; Iskovskih, 1967; Silhol, 1980) are purely existential and do not give an idea for effective computation of such a proper parametrization.

In this paper, we solve the proper parametrization problem for tubular surfaces by applying the theory of quadratic forms. (The deep relation between the rationality problem and the theory of quadratic forms is also studied in (Ojanguren, 1988).) We make use of the fact that we have an efficient algorithm (Peternell, 1997; Schicho, 1998a) (the same algorithm was found independently by both authors) that computes an isotropic vector of the quadratic form $A(t)x^2 + B(t)y^2 + C(t)z^2$ over the field $\mathbf{R}(t)$, if exists. The costs of this algorithm are the costs of factoring the univariate polynomial ABC and of solving a system a linear system of equations of size $\mathcal{O}(\deg A + \deg B + \deg C)$. Our main result is that we can solve the proper parametrization problem within the same complexity.

It should be noted that the costs of factoring a univariate polynomial over the reals must not be underestimated: if the degree is larger than 4 and the polynomial is irreducible over $\mathbf{Q}$, then it is difficult to factor symbolically. We recommend to convert to floating point representation in such a situation.

Another algorithm for computing isotropic vectors can be found in (Hillgarter, 1997). It is computationally more expensiv, but works for more general fields.

The paper is organized as follows. Section 2 contains a preliminary overview on the birational classification of complex-rational surfaces. The results are well-known, but they provide the context for the new results in the later sections. In section 3, we explain the relation of tubular surfaces, their spines, and their associated quadratic forms. We give a new proof for the fact that tubular surfaces can be classified by their spines, based on Witt's cancellation theorem. In section 4, we present an algorithm that performs Witt's cancellation. We apply it to the case of connected tubular surfaces and obtain an explicit construction for a proper parametrization and its inverse. Since this section contains some large formulas, we provide a Maple worksheet, downloadable by `http://www.risc.uni-linz.ac.at/people/jschicho/proper.html`, in order to facilitate the checks of the computations. This worksheet also contains an implementation of the Witt's cancellation algorithm.

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2. Generalities: the Birational Classification

Under a real algebraic surface, we understand the zero set of a prime ideal in affine or projective case, supposed that this zero set is two-dimensional. Given such a surface, we want to decide if it is rational, and to find a proper parametrization in the affirmative case.

A necessary condition for real-rationality is complex-rationality, i.e. the existence of a proper parametrization with complex coefficient. In this section, we give an overview on the classification of complex-rational surfaces up to birational equivalence.

A very important birational invariant is the number of connected components (see (Bochnak *et al.*, 1987)). Strictly speaking, this number is a birational invariant only for nonsingular projective varieties. In the general case, we have to count the components of the projectivization after the singularities have been resolved. As an immediate consequence of this invariance, rationality implies connectedness.

On the other hand, the following result is a cornerstone in the classification of complex-rational surfaces.

THEOREM 2.1. (ENRIQUES,MANIN) *A complex-rational surfaces has a pencil of rational curves or is birational equivalent to a Del Pezzo surface.*

Here, a pencil is a family of curves that arises as the family of fibers of a rational map $f : S \to \mathbf{P}^1_{\mathbf{R}}$. Del Pezzo surfaces are complex-rational surfaces with the property that the anticanonical divisor is base-point free (see (Manin, 1967)). (In the following, the precise knowledge of what a Del Pezzo surface is is inessential.) The paper (Schicho, 1998b) contains an algorithm with the following specification: given a surface in terms of its implicit representation, it decides whether the surface is rational. In the affirmative case, it computes either a rational pencil or a birational isomorphism to a Del Pezzo surface. The main costs of this algorithm is the problem of resolution of the singularities of the input surface (see (Bodnar and Schicho, 1999)).

Without proofs, we mention the following facts about Del Pezzo surfaces (for the proofs, we refer to (Manin, 1967; Manin, 1974)).

- A Del Pezzo surface has at most 5 connected components.
- Any connected Del Pezzo surface is real rational.
- Any Del Pezzo surface with 2 or 3 connected components has a pencil of rational curves.
- The Del Pezzo surfaces with 4 or 5 connected Components do not have a pencil of rational curves.
- Del Pezzo surfaces with 4 connected components are unirational.

It is not known whether Del Pezzo surfaces with 5 connected components are unirational.

The effective computation of a proper parametrization of a connected Del Pezzo surface is not easy. For instance, in order to parametrize a connected nonsingular cubic surface S properly, one has to find a real line on S. There are 27 complex lines on S, which cannot be distinguished algebraically. Thus, the computation of a line requires the computation of a real root of a degree 27 polynomial which is irreducible in general. See (Bajaj *et al.*, 1997) for details.

Let's now consider the case of surfaces with a pencil of rational curves (which is not disjoint from the Del Pezzo case, by the results above). Suppose that the pencil arises as the family of fibers of the rational map $f : S \to \mathbf{P}^1_{\mathbf{R}}$. Let $\mathbf{R}(t)$ be a simple transcendental extension of $\mathbf{R}$. Then $(t:1)$ is a generic point of $bfP^1_{\mathbf{R}}$. The generic fiber $f^{-1}((t:1))$ is a curve defined over the field $\mathbf{R}(t)$. By assumption, this curve is irreducible and rational over the algebraic closure $\mathbf{R}(t)$.

Theorem 2.2. (Noether) *Any surface with a pencil of rational curves is birationally equivalent to a surface with equation $A(t)x^2 + B(t)y^2 + C(t) = 0$. Such surfaces are called* tubular surfaces*. In this birational model, the pencil of rational curves is the pencil to the family of conics where t is constant.*

An algorithm that computes such an equivalence, for a surface given implicitly, can be found in Noether's paper (Nöether, 1870). For a more modern and very explicit formulation, see (Sendra and Winkler, 1997). The second paper does not mention surfaces at all, but treats the more general case of curves defined over an arbitrary field (in our case, the field of definition is the function field $\mathbf{R}(t)$).

Remark 2.1. Even when the case is not given implicitly, it is sometimes possible to compute the equivalence above, see (Arrondo *et al.*, 1997) for generalized offsets and (Peternell and Pottmann, 1997) for pipe surfaces.

In order to compute the number of connected components of a tubular surface, we use the spine (which appears also in (Schicho, 1998a)). Let J be the set of all $t_0 \in \mathbf{R}$ such that the conic $A(t_0)x^2 + B(t_0)y^2 + C(t_0) = 0$ has a point in $\mathbf{R}^2$. It is a union of isolated points which are zeroes of A, B, C, and intervals which are limited by these zeroes. The *spine* is defined as the closure of the interior of J in $\mathbf{P}^1$, the one-point compactifaction of $\mathbf{R}$. (Passing to the closure of the interior is necessary to get rid of isolated points.) Any connected component of the spine is either a finite interval $[a, b]$, or an infinite interval $[-\infty, a]$ or $[a, +\infty]$, or a component $[-\infty, a] \cup [b, +\infty]$ containing $\pm\infty$ as an inner point.

The resolution of the singularities of a tubular surface is routine (see (Bodnar and Schicho, 1999)). The projection on the t-coordinate gives a one-to-one correspondence between the connected components of the desingularization and the connected components of the spine. This reveals two things.

For any positive integer n, there is a tubular surface with precisely n connected components (the surface with equation $x^2 + y^2 + t(t-1)\ldots(t-2n) = 0$).
A tubular surface is rational only if its spine is connected.

The converse of the second fact also holds: if the spine is connected, then the surface is rational. More general, if the spines of two tubular surfaces are projectively isomorphic (via some projective transformation of the projective line), then the two surfaces are birationally equivalent. In the next section, we give a new simple proof for this fact. As a consequence, it follows that for complex-rational surfaces, real-rationality is equivalent to connectedness.

Remark 2.2. For tubular surfaces with at least three connected components, the converse of the more general statement is also true: any birational equivalence induces a birational equivalence on spines (see (Comesatti, 1912; Silhol, 1980)). This is false for connected components: the surfaces $x^2+y^2-t^2-1=0$ and the surfaces $x^2+y^2+t^2-1=0$ are both real-rational, but their spines are not projectively equivalent.

(Silhol, 1980), theorem V.3.5 claims that any two surfaces with a pencil of rational curves with two connected components are isomorphic. However, his proof is wrong, and I have the counter-conjecture that any birational equivalence induces a birational equivalence on spines, also in this case. This is supported by the fact that the two rulings of a Del Pezzo surface with two connected components have the same cross ratio, which can be verified by explicit computation.

3. Tubular Surfaces and Quadratic Spaces

In this section, we apply the theory of quadratic forms (see (OḾeara, 1971)) to the classification problem for tubular surfaces. We obtain a new simple proof for the theorem that two tubular surfaces with projectively equivalent spines are birationally equivalent. Especially, it follows that any surface with connected spine is real-rational.

First, let us make a little simplification. The surface with equation $A(t)x^2+B(t)y^2+C(t)=0$ is birationally equivalent to the surface $A(t')C(t')x'^2+B(t')C(t')y'^2+1=0$ (via the birational transformation $x=C(t')x'$, $y=C(t')y'$, $t=t'$). This particular birational transformation also does not change the spine. Therefore, we may assume that $C=1$ without loss of generality, as far as birational questions or questions on the spine are concerned. A tubular surface with $C=1$ is called a normed tubular surface.

Let S be the normed tubular surface with equation $A(t)x^2+B(t)y^2+1=0$. Then the quadratic space $Q_S := \langle A, B, AB\rangle$ over the field $\mathbf{R}(t)$ is called the associated quadratic space to S.

Theorem 3.1. *Let S_1 and S_2 be normed tubular surfaces, such that their associated quadratic spaces are isometric. Then S_1 and S_2 are birationally equivalent. Moreover, the spines of S_1 and S_2 are the same.*

Proof. Let $T : Q_1 \to Q_2$ be an isometry between the two quadratic spaces, represented by a 3×3 matrix with values $T_{ij}\in\mathbf{R}(t)$. Then a birational equivalence of S_1 and S_2 is given by

$$(x,y,t)\mapsto \Big(\frac{T_{21}(t)B_1(t)y+T_{22}(t)A_1(t)x+T_{23}(t)}{B_2(t)(T_{31}(t)B_1(t)y+T_{32}(t)A_1(t)x+T_{33}(t))},$$

$$\frac{T_{11}(t)B_1(t)y+T_{12}(t)A_1(t)x+T_{13}(t)}{A_2(t)(T_{31}(t)B_1(t)y+T_{32}(t)A_1(t)x+T_{33}(t))},t\Big).$$

(Here, $A_1,\ldots,B_2$ are the polynomials in the equations of S_1 and S_2.) Since this birational map does not change the t-coordinate, it preserves the spine. □

Our goal is to show the following converse (corollary 3.1): if two normed tubular surfaces have the same spine, then their associated quadratic spaces are isometric. As a consequence, the surfaces are birationally equivalent.

The special case of tubular surfaces with full spine (i.e. the spine is $\mathbf{P}^1$) was solved in (Peternell, 1997; Schicho, 1998a). In these papers, one can find an algorithm that computes a *cross section*, i.e. a curve on S with parametrization $(X(t),Y(t),t)$, where

$X,Y\in\mathbf{R}(t)$. This cross section can be used to construct a proper parametrization of S, using the method of stereographic projection. For us, the following result will be useful.

THEOREM 3.2. *Let S be a normed tubular surface. The following are equivalent.*

1 *The tubular surface S has full spine.*
2 *The tubular surface S has a cross section.*
3 *The quadratic space $\langle 1\rangle\oplus Q_S$ is hyperbolic.*

PROOF. (1)→(2): see (Peternell, 1997; Schicho, 1998a).

(2)→(3): let $(X(t),Y(t),t)$ be the cross section. Then the vector $(1,X(t),Y(t),0)$ in $\langle 1\rangle Q_S$ is isotropic. Since $\langle 1\rangle\oplus Q_S$ is a Pfister form, it is hyperbolic (see (OMeara, 1971)).

(3)→(1): for all t_0 except for finitely many, the form $\langle 1,A(t_0),B(t_0),A(t_0)B(t_0)\rangle$ over $\mathbf{R}$ is hyperbolic. Thus, either $A(t_0)$ or $B(t_0)$ is negative, hence t_0 is in the spine. □

REMARK 3.1. An explicit construction for the isometry to the hyperbolic space, claimed in direction (2)→(3), is contained in the next section.

Let S be a normed tubular surface. A polynomial $F\in\mathbf{R}[t]$ is called a spine polynomial if and only if it is positive for all points in the interior of the spine and negative outside the spine and all its zeroes are real.

THEOREM 3.3. *Let S be a normed tubular surface. Let F be a spine polynomial for S. Then Q_S is isometric to $\langle 1,-F,-F\rangle$.*

PROOF. For those values t_0 where $A(t_0)$ and $B(t_0)$ have the same sign, the spine polynomial f has opposite sign. Thus, the spine of the tubular surface $A(t)x^2+B(t)y^2+F(t)=0$ is full. By theorem 3.2, the quadratic space $\langle 1,AF,BF,ABF^2\rangle$ is hyperbolic. By tensoring with $\langle F\rangle$, we get that

$$\langle F,AF^2,BF^2,ABF^3\rangle\cong\langle F,A,B,ABF\rangle =: Q_1$$

is hyperbolic.

For those values t_0 where $A(t_0)$ and $B(t_0)$ have opposite sign, $F(t_0)$ is positive. Thus, the spine of the tubular surface $-F(t)x^2-A(t)B(t)y^2+1=0$ is full. By theorem 3.2, the quadratic space $Q_2:=\langle 1,-F,-AB,ABF\rangle$ is hyperbolic.

Now, we have

$$\begin{aligned} Q_2\oplus Q_S &= \langle 1,-F,-AB,ABF,A,B,AB\rangle \\ &\cong \langle 1,-F,A,B,ABF\rangle\oplus\langle -AB,AB\rangle \\ &\cong \langle 1,-F,A,B,ABF\rangle\oplus\langle -F,F\rangle \\ &\cong \langle 1,-F,-F,F,A,B,ABF\rangle = \langle 1,-F,-F\rangle\oplus Q_1. \end{aligned}$$

Since Q_1 and Q_2 are hyperbolic of the same dimension, they are isometric. By Witt's cancellation theorem, $\langle 1,-F,-F\rangle$ and Q_S are isometric. □

REMARK 3.2. In the next section, we will explicitly write down an isometry. The main idea for finding it is to apply a constructive version of Witt's cancellation theorem.

COROLLARY 3.1. *If two tubular surfaces have the same spine, then their associated quadratic forms are isometric.*

PROOF. If two tubular surfaces have the same spine, then the two spine polynomials differ at most by a positive factor λ. Therefore, the quadratic spaces $\langle 1, -F_1, -F_1\rangle$ and $\langle 1, -F_2, F_2\rangle$ are isometric by a scaling by $\sqrt{\lambda}$. □

COROLLARY 3.2. *If the spines of two tubular surfaces are projectively equivalent, then the surfaces are birationally equivalent.*

PROOF. Suppose that the spines are birationally equivalent. Then we apply a Möbius-transformation in the t-coordinate which transforms spine to spine. Thus, it suffices to consider the case where the spines are equal. But this case follows immediately by corollary 3.1 and theorem 3.1. □

4. A Construction using Witt's Theorem

For an effective construction of a proper parametrization, we need to do two things: first, we have to parametrize the *spine surface* with equation $x^2 + y^2 - F(t) = 0$ (where F is the spine polynomial) properly. This is possible if and only if $\deg(F) \leq 2$. If this is the case, then we have a cylinder ($\deg(F) = 0$), an elliptic paraboloid ($\deg(F) = 1$), an ellipsoid or a two-?? hyperboloid ($\deg F = 2$). All these surfaces can be easily properly parametrized (see, for instance, (Abhyankar and Bajaj, 1986)).

Second, we have to construct a birational map from the spine surface to the given surface, in order to transport the proper parametrization along this birational map. By theorem 3.1, such a map always exists. Since the proof uses Witt's cancellation theorem, we need to have a constructive version of that theorem.

The problem is the following. Given are an isometry of quadratic spaces $T : Q_1 \to Q_2$ and an isometry of two nondegenerate subspaces $S : P_1 \to P_2$, $P_1 \subset Q_1$, $P_2 \subset Q_2$, we have to find an isometry of the orthogonal complements. We choose orthogonal bases such that P_i is spanned by the first m basis vectors. The quadratic forms P_1, P_2, Q_1, Q_2 are given by their corresponding symmetric matrices (also denoted by P_1, P_2, Q_1, Q_2). By the choice of the bases, these matrices are diagonal, and P_i is the first $m \times m$ submatrix. By replacing P_1 by P_2 and T by $S^{-1} \oplus Id$, we may assume that S is the identity. The fact that T is an isometry is expressed by the matrix equation

$$P_1 = T^t P_2 T.$$

We need to find an isometry of the quadratic spaces represented by the last $(n-m) \times (n-m)$ submatrix (here, n is the dimension of the Q_i).

Obviously, we may "cancel" the subspaces spanned by the first m basis vector one after the other. The idea for constructing the isometry in the case $m = 1$ is to reflect along the vector $A := TE - E$, where E is a spanning vector for P_1. Reflection along A is an isometry that takes A into $-A$. It can be computed by the formula

$$R_A(V) = V - \frac{2\langle A, V\rangle}{\langle A, A\rangle} A.$$

The formula does not work in the case when A is an isotropic vector. In this case, one takes $A := TE + E$. The two vectors cannot both be isotropic, because

$$\langle TE - E, TE - E\rangle + \langle TE + E, TE + E\rangle = 4\langle E, E\rangle \neq 0.$$

Here is the algorithm for solving the problem.

repeat m times
    $E :=$ the first unit vector;
    $A := TE - E$;
    if $A^t Q_2 A = 0$ then
        $A := TE + E$;
    $R = Id - \frac{2}{A^t Q_2 A} A A^t Q_2$;
    $T := RT$;
    remove the first row and first column in Q_1, Q_2, T;

In order to construct the input isometry T, we have to use the fact that the tubular surfaces $A(t)u^2 + B(t)v^2 + F(t) = 0$ and $z^2 - F(t)w^2 - A(t)B(t)$ have full spine. By theorem 3.2, there are cross sections $(U(t), V(t), t)$ and $(Z(t), W(t), t)$, which can be constructed by the efficient algorithm (Peternell, 1997; Schicho, 1998a). Then, the isometries of the associated quadratic forms to the hyperbolic space of dimension 4 can be constructed as follows.

$$T_1 = \begin{bmatrix} \frac{-AU}{2F} & \frac{-BV}{2F} & \frac{1}{2} & 0 \\ AU & BV & F & 0 \\ \frac{-V}{2F} & \frac{U}{2F} & 0 & \frac{1}{2} \\ ABV & -ABU & 0 & ABF \end{bmatrix},$$

$$T_2 = \begin{bmatrix} \frac{Z}{2AB} & \frac{-FW}{2AB} & \frac{1}{2} & 0 \\ Z & -FW & -AB & 0 \\ \frac{W}{2AB} & \frac{-Z}{2AB} & 0 & \frac{1}{2} \\ -FW & FZ & 0 & ABF \end{bmatrix},$$

$$T_1^t \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} T_1 = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & ABF \end{bmatrix},$$

$$T_2^t \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -F & 0 & 0 \\ 0 & 0 & -AB & 0 \\ 0 & 0 & 0 & ABF \end{bmatrix}.$$

After the proper arrangment by composing these isometries, scaling the hyperbolic line by the isometry

$$\begin{bmatrix} C & 0 \\ 0 & 1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix},$$

and rearranging the variables by permutation matrices, this leads to the following situation.

$$T^t \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ABF & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & BA \end{bmatrix} T = \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ABF & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -F & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -F \end{bmatrix},$$

$$T = \begin{bmatrix} 0 & 0 & 1/4\,\frac{\left(B^2A^2-F^2\right)U}{BAF} & 0 & T_{15} & T_{16} & 1/4\,\frac{(BA+F)^2U}{BAF} \\ 0 & 0 & 1/4\,\frac{\left(B^2A^2-F^2\right)V}{BAF} & 0 & T_{25} & T_{26} & 1/4\,\frac{(BA+F)^2V}{BAF} \\ 0 & 0 & -1/4\,\frac{(-F+BA)^2}{BAF} & 0 & T_{35} & -1/2\,\frac{(BA+F)W}{BA} & -1/4\,\frac{B^2A^2-F^2}{BAF} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2\,\frac{BA+F}{BA} & 0 & 0 & 0 & 1/2\,\frac{-F+BA}{BA} \end{bmatrix},$$

$$T_{15} = 1/2\,\frac{UZF-UZAB+2\,VWBF}{BAF}, T_{16} = -1/2\,\frac{-WUBA+2\,VZB+WUF}{BA},$$

$$T_{25} = -1/2\,\frac{VZAB-VZF+2\,WUAF}{BAF}, T_{26} = -1/2\,\frac{-VWBA-2\,UZA+VWF}{BA},$$

$$T_{35} = -1/2\,\frac{VZAB-VZF+2\,WUAF}{BAF}.$$

We apply the algorithm above and obtain the identity

$$\begin{bmatrix} \frac{VW}{A} & \frac{-VZ}{A} & U \\ \frac{-UW}{B} & \frac{UZ}{B} & V \\ \frac{Z}{AB} & \frac{-FW}{AB} & 0 \end{bmatrix}^t \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & AB \end{bmatrix} \begin{bmatrix} \frac{VW}{A} & \frac{-VZ}{A} & U \\ \frac{-UW}{B} & \frac{UZ}{B} & V \\ \frac{Z}{AB} & \frac{-FW}{AB} & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -F & 0 \\ 0 & 0 & -F \end{bmatrix}.$$

This computations have been done by Maple.

Remark 4.1. Actually, the first result computed by Maple was quite complicated. I then tried several times to permute (in the isometry T) basis vectors of the same length, and, to my pleasure, one of the possible ways lead to the surprisingly simple formula above!

Now, the birational map from the spine surface $x'^2 + y'^2 - F(t) = 0$ and the given surface $A(t)x^2 + B(t)y^2 + 1 = 0$ can be written down easily.

$$x = \frac{U(t)W(t)F(t) - U(t)Z(t)x' - V(t)y'B(t)}{F(t)(-Z(t) + W(t)x')},$$

$$y = \frac{-V(t)W(t)F(t) + V(t)Z(t)x' - U(t)y'A(t)}{F(t)(-Z(t) + W(t)x')}, t = t.$$

The inverse is

$$x' = \frac{A(t)U(t)Z(t)x - B(t)V(t)Z(t)y - F(t)W(t)}{A(t)U(t)W(t)x - B(t)V(t)W(t)y - Z(t)},$$

$$y' = \frac{-A(t)B(t)V(t)x - A(t)B(t)U(t)y}{A(t)U(t)W(t)x - B(t)V(t)W(t)y - Z(t)}, t = t.$$

To obtain a proper parametrization, we just plug the proper parametrization of the spine surface in the birational map above. For the reverse of the proper parametrization, we just plug the inverse map above into the inverse of the proper parametrization of the spine surface. The dominant costs in this method are the construction of the two cross sections $(U(t), V(t), t)$ and $(Z(t), W(t), t)$. Using the algorithm (Peternell, 1997; Schicho, 1998a), this can be done by decomposing the polynomials A, B, F into irreducible factors and solving a linear equation system in $\mathcal{O}(\deg A + \deg B)$ unknowns.

Example 4.1. Let S be the normed tubular surface with equation

$$(t-2)x^2 + (t^3 - t^2 + 2t + 1)y^2 + 1 = 0.$$

Then the spine is $[-\infty, 2]$, and the spine polynomial is $-t+2$. The spine surface $x'^2 + y'^2 + t - 2 = 0$ has the obvious proper parametrization

$$(x', y', t) = (p, q, -p^2 - q^2 + 2),$$

with inverse

$$(p, q) = (x', y').$$

We have to find a cross section of the surface

$$(t-2)u^2 + (t^3 - t^2 + 2t + 1)v^2 + (-t + 2) = 0.$$

In this example, we can easily guess the solution $U := 1$, $V := 0$. Next, we have to find a cross section of the surface

$$z^2 - (-t+2)w^2 - (t-2)(t^3 - t^2 + 2t + 1) = 0.$$

The algorithm in (Schicho, 1998a) computes the solution

$$Z := t^2 - 2t, W := t + 1.$$

Now, we just plug the proper parametrization of the spine surface into the formula above and get the proper parametrization

$$(x, y, t) = (\frac{p^3 + p^2 + pq^2 - 2p - 3 + q^2}{p^4 + 2p^2q^2 - 2p^2 + q^4 - 2q^2 + p^3 + pq^2 - 3p},$$

$$\frac{q}{p^4 + 2p^2q^2 - 2p^2 + q^4 - 2q^2 + p^3 + pq^2 - 3p}, -p^2 - q^2 + 2),$$

with inverse

$$(p, q) = (\frac{(t^3 - t^2 + 2t + 1)y}{xt - t + x}, \frac{xt^2 - 2xt + t + 1}{xt - t + x}).$$

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