The construction of orthonormal wavelets using symbolic methods and a matrix analytical approach for wavelets on the interval

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Abstract

In this paper we discuss closed form representations of filter coefficients of wavelets on the real line, half real line and on compact intervals. We show that computer algebra can be applied to achieve this task. Moreover, we present a matrix analytical approach that unifies constructions of wavelets on the interval.

1 Introduction

Wavelets are one of the most popular tools in *signal–* and *image processing*. These functions are widely used in many practical applications such as *data compression* [1, 21, 14], or for the solution of *partial differential equations* (see e.g. [16]). Wavelets are special functions which often have a *fractal* character. This makes it relatively difficult to work with them explicitly; for example point evaluation of a wavelet function may already be a computational expensive task. To work with wavelets one uses the nice feature that they are defined by a small number of parameters, the so called *filter coefficients*. In general, any algorithm relying on wavelets only use the filter coefficients and not the wavelet function itself.

In this paper we review the basic equations for the filter coefficients. We show that these equations can be solved using computer algebra. In particular we can construct closed form representations of the wavelet coefficients; see section 3. The most popular (Daubechies [12]) wavelets form an orthonormal basis of the space of square integrable functions on $\mathbb{R}$. In many practical applications one requires a basis on the half-line or on a compact interval. In section 4

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we review several constructions of wavelets on the interval. We construct the filter coefficients of wavelets on the interval using a matrix analytical approach. This approach has the advantage to unify several constructions in the literature [18, 8, 9]. Moreover, our construction reveals that there exists a closed form representation of the filter coefficients of wavelets on the interval. Another advantage of using computer algebra is that one can avoid instabilities which occur in numerical calculations of filter coefficients.

In sections 4.5 and 5 we summarize the algorithms for calculating closed form coefficients of wavelets on the real line and on compact intervals and present some results.

2 Orthogonal Wavelets on $\mathbb{R}$

The construction of compactly supported orthogonal wavelet bases for $L^2(\mathbb{R})$ is well understood (see e.g. [11, 12, 5, 13]) but is summarized here for the readers convenience.

The construction of wavelets is related with the construction of a scaling function ϕ, which satisfies that for fixed $m \in \mathbb{Z}$ the functions $\phi_{m,k} := 2^{-m/2}\phi(2^{-m}x - k)$, $k \in \mathbb{Z}$ are orthonormal with respect to $L^2(\mathbb{R})$. Moreover, the spaces $V_m := \overline{\operatorname{span}\{\phi_{m,k}, k \in \mathbb{Z}\}}$ constitute a multiresolution analysis for $L^2(\mathbb{R})$, i.e.

$$V_m \subset V_{m-1}\,, \quad \text{for } m \in \mathbb{Z},$$

with

$$\bigcap_{m\in\mathbb{Z}} V_m = \{0\} \quad \text{and} \quad \overline{\bigcup_{m\in\mathbb{Z}} V_m} = L^2(\mathbb{R})\,.$$

The wavelet spaces W_m are the orthogonal complements of V_m in V_{m-1}, i.e.

$$W_m := V_m^{\perp} \cap V_{m-1}\,.$$

One defines the wavelet ψ such that the functions $\psi_{m,k} := 2^{-m/2}\psi(2^{-m}x - k)$, $k \in \mathbb{Z}$ form a an orthonormal basis for W_m. Since both, V_m and W_m are contained in V_{m-1} the scaling function ϕ must satisfy the *dilation equation*

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k \phi(2x-k)\,, \tag{1}$$

and the wavelet ψ satisfies

$$\psi(x) = \sum_{k\in\mathbb{Z}} g_k \phi(2x-k)\,, \tag{2}$$

where $g_k = (-1)^k h_{1-k}$.

Daubechies [11] established conditions on the filter sequence $\{h_k\}$ in order to ensure that the dilation equation (1) has a solution $\phi \in L^2(\mathbb{R})$, with $\operatorname{supp}\phi = [-N+1\,, N]$, that for fixed m the functions $\phi_{m,k}$ are orthogonal, and polynomials up to degree $N-1$ can be represented as linear combinations of $\phi_{m,k}$. A requirement for the existence of a solution of (1) is

$$\sum_k h_k = 2\,, \tag{3a}$$

which is equivalent to $\int \phi(x)\,dx = 1$. Compact support of ϕ in $[-N+1\,, N]$ is ensured by

$$h_k = 0 \qquad \text{if } k < -N+1 \text{ or } k > N. \tag{3b}$$

Orthonormality of the translates of ϕ, i.e. $\int \phi(x)\phi(x-k)\,dx = \delta_{0,l}$, can be translated into

$$\sum_k h_k\, h_{k-2l} = 2\,\delta_{0,l}\,, \tag{3c}$$

and the requirement that polynomials are representable by the $\phi_{m,k}$ leads to $\int x^l\, \psi(x)\,dx = 0$ for $l = 0, \ldots, N-1$, and thus

$$\sum_k (-1)^k h_{1-k}\, k^l = 0\,, \quad (l = 0, \ldots, N-1)\,. \tag{3d}$$

3 Closed Form Representation of Filter Coefficients

In this section we consider the calculation of the filter coefficients h_k from equations (3a)-(3d) using methods of computer algebra. In order to do so it is convenient to state explicitly the dependence of the filter coefficients h_k on N (the number of vanishing moments of the wavelet), i.e.

$$h_{N,k} := h_k\,.$$

Below we give a very brief and informal account on Gröbner bases, along with the calculation of the filter coefficients for the special case $N = 2$. Afterward we pass on to the cases $N > 2$ and present more computational details.

3.1 Gröbner Bases

First of all note that due to the conditions imposed explicitly on the summation bounds in the equations (3a), (3c) and (3d), we can restrict our attention to the task of solving only those; the conditions (3b) can be satisfied separately by mere definition. But instead of solving, for fixed N, the equations (3a), (3c) and (3d) numerically, we try to find closed forms for the coefficients $h_{N,k}$, i.e., to approach the problem from the symbolic computation point-of-view. For solving systems of polynomial equations symbolically, the obvious tools to use are Gröbner bases; this is their most natural domain of application, and they were originally invented by B. Buchberger [3], [2] for that. For further introductory information see, e.g., [25] or [23]; an excellent source for additional references and for the state of art is [4].

The case $N = 1$ is trivial; $h_{1,0} = h_{1,1} = 1$ is the only solution. Hence we illustrate the Gröbner bases method for $N = 2$. In this case we are interested in all common roots of the polynomials:

$$
\begin{aligned}
& -2 + x_1 + x_2 + x_3 + x_4,\ -2 + x_1^2 + x_2^2 + x_3^2 + x_4^2, \\
& x_1 x_3 + x_2 x_4,\ x_1 - x_2 + x_3 - x_4,\ 2\,x_1 - x_2 + x_4;
\end{aligned} \tag{4}
$$

for the sake of simplicity we introduced the following renaming of variables:

$$x_1 = h_{2,-1},\ x_2 = h_{2,0},\ x_3 = h_{2,1},\ \text{and } x_4 = h_{2,2}.$$

Let I be the ideal in the polynomial ring $\mathbb{C}[x_1, x_2, x_3, x_4]$ generated by the polynomials from (4). Applying Buchberger's algorithm with respect to a certain order (here: "lexicographic") imposed on the monomials of $\mathbb{C}[x_1, x_2, x_3, x_4]$, delivers an alternative description of the ideal I, namely by the generators:

$$-1 - 4\,x_1 + 8\,x_1^2,\ -1 - 2\,x_1 + 2\,x_2,\ -1 + x_1 + x_3,\ -1 + 2\,x_1 + 2\,x_4. \tag{5}$$

The polynomials (5) again generate the ideal I, but additionally form a Gröbner basis of I. This means, besides having exactly the same variety of common roots as the generators from (4), they also possess the so-called "elimination property". Informally this means, the first polynomial in the Gröbner basis is a *univariate* polynomial (here in x_1), the second one a bivariate polynomial that involves only one further variable (here in x_1 and x_2), and so on. In other words, the role of the Gröbner basis algorithm in solving systems of algebraic equations is the same as that of Gaussian elimination in solving systems of linear equations, namely to triangularize the system or to carry out the elimination, respectively.

Remarkably, in our situation of solving filter coefficient equations an even nicer patter emerges. Namely, given the first univariate Gröbner basis polynomial $p_1(x_1)$ in x_1 only, the second Gröbner basis polynomial is the sum of a univariate polynomial in x_1 and a *linear* polynomial in x_2; the third Gröbner basis polynomial is the sum of a univariate polynomial in x_1 and a *linear* polynomial in x_3, and so on. This means, all other filter coefficients x_i, for $i > 1$ find a representation of the form

$$x_i = p_i(x_1) \tag{6}$$

where each $p_i(x_1)$ is a polynomial from $\mathbb{C}[x_1]$, i.e., depending on x_1 only. Consequently, there are as many different solutions of a system of filter coefficient equations as there are different roots of the first univariate Gröbner basis polynomial $p_1(x_1)$. So far we have strong computational evidence that this observation holds also for arbitrary N. For readers interested in ideal theory we state this in form of the following conjecture. (For more ideal theoretic background information see, for instance, the "shape lemma" in [25].)

Conjecture 1. *Polynomial ideals corresponding to Daubechies filter coefficient equations are* 0*-dimensional and radical.*

We conclude our informal discussion of the Gröbner bases approach by stating the solution of the case $N = 2$ explicitly. Since $(1+\sqrt{3})/4$ and $(1-\sqrt{3})/4$ are the roots of the first Gröbner basis polynomial $-1 - 4x_1 + 8x_1^2$, we obtain two solutions for the filter coefficients:

$$(x_1, x_2, x_3, x_4) = \left(\frac{1+\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\right) \tag{7}$$

and

$$(x_1, x_2, x_3, x_4) = \left(\frac{1-\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{1+\sqrt{3}}{4}\right) \tag{8}$$

3.2 Reduction and Transformation of Filter Coefficient Equations

For fixed N, the system (3a), (3c), (3d) consists of $2N+1$ equations in $2N$ unknowns, $2N$ also being the number of Gröbner basis polynomials we finally have to solve explicitly. In this section, as an important preprocessing step to Gröbner bases computation, we transform the system (3a), (3c), (3d) into a more economic form. More precisely, this system will consist of only N equations in N unknowns; the corresponding Gröbner bases will then consist of N polynomials for which one again observes the nice shape-pattern that was described above (see the discussion preceding Conjecture 1).

In a first step we introduce a normalization via multiplication by a binomial coefficient; namely, for any fixed positive integer N we define $a_{N,k}$ by the equation

$$h_{N,k} = \binom{2N-1}{N-k} \cdot a_{N,N-k}, \quad (k = -N+1, \ldots, N). \tag{9}$$

This implicitly installs conditions (3b) and thus enables to relax the explicit statement of the summation bounds in (3a), (3c), (3d).

More important, it turns out that for fixed positive integer N we can restrict ourselves to consider $a_{N,k}$ as a polynomial in k of degree at most $N-1$. This means, we can write

$$a_{N,k} = \sum_{j=0}^{N-1} P_{N,j} \binom{k}{j} \tag{10}$$

where the $P_{N,j}$ are the new unknowns we have to solve for. Note that we have in total N of those — instead of $2N$ in the original setting (3a), (3c), (3d). In addition, we shall see below why it is convenient to work with basis elements $\binom{k}{j}$ instead of k^j.

With ansatz (9) and (10), respectively, in hand, we return to equations (3a) to (3d). It is not difficult to see that only two of those remain: (3b) is guaranteed due to the properties of the of the binomial coefficient we inserted in (9); also, equation (3d) is satisfied for arbitrary $l = 0, \ldots, N-1$ because of the following lemma which is taken from elementary combinatorics.

Lemma 1. *For any nonnegative integer n and complex numbers α_i:*

$$\sum_k (-1)^k \binom{n}{k} (\alpha_0 + \alpha_1 k + \cdots + \alpha_n k^n) = (-1)^n n!\, \alpha_n \,.$$

Proof. See, for instance, [15, (5.42)]. □

Now, with ansatz (9) and (10), respectively, equation (3d) is rewritten as

$$\sum_k (-1)^{k-N+1} \binom{2N-1}{k} a_{N,k} (k-N+1)^l = 0 \,,$$

and both, $a_{N,k}$ and $(k-N+1)^l$ are polynomials in k with degree less or equal to $N-1$. Hence by Lemma 1 equation (3d) is satisfied for all l in question.

Finally we state the new versions of the remaining equations (3a) and (3c) in form of propositions.

Proposition 1. *With ansatz* (9) *and* (10)*, equation* (3a) *can be rewritten as:*

$$\sum_{j=0}^{N-1} \binom{2N-1}{j} \frac{P_{N,j}}{2^j} = 2^{-2N+2}. \tag{11}$$

Proof. Substiting (9) and (10), and using the elementary fact $\binom{2N-1}{k}\binom{k}{j} = \binom{2N-1}{j}\binom{2N-j-1}{k-j}$, equation (3a) is rewritten as

$$2 = \sum_{j=0}^{N-1} \binom{2N-1}{j} P_{N,j} \sum_k \binom{2N-j-1}{k-j}.$$

But $\sum_k \binom{2N-j-1}{k-j} = 2^{2N-j-1}$ is a special instance of the binomial theorem, and the proposition follows. □

Proposition 2. *With ansatz* (9) *and* (10)*, equation* (3c) *can be rewritten as follows. For $l = 0, \ldots, N-1$:*

$$\sum_{i,j=0}^{N-1} \binom{2N-1}{i} \binom{2N-1}{j} \binom{4N-i-j-2}{2N+2l-i-1} P_{N,i}\; P_{N,j} = 2 \cdot \delta_{0,l}. \tag{12}$$

Proof. Substiting (9) and (10), and using the elementary facts $\binom{2N-1}{k}\binom{k}{j} = \binom{2N-1}{j}\binom{2N-j-1}{k-j}$ and $\binom{2N-1}{k+2l}\binom{k+2l}{i} = \binom{2N-1}{i}\binom{2N-i-1}{k+2l-i}$, the summation in (3c) is rewritten as

$$\sum_{j=0}^{N-1} \binom{2N-1}{i} \binom{2N-1}{j} P_{N,i}\; P_{N,j} \sum_k \binom{2N-j-1}{k-j} \binom{2N-i-1}{k+2l-i}.$$

The inner sum can be evaluated as follows: after applying binomial symmetry $\binom{n}{m} = \binom{n}{n-m}$, it becomes

$$\sum_k \binom{2N-j-1}{2N-k-1} \binom{2N-i-1}{k+2l-i} = \binom{4N-i-j-2}{2N+2l-i-1},$$

where the last line is a variant of standard Vandermonde summation (e.g, [15, (5.22)]). □

But there is still another reduction in the number of equations possible. For proving this, we need another elementary result.

Lemma 2. *For positive integers m and n:*

$$\sum_l \binom{m+n}{m+2l} = 2^{m+n-1}$$

Proof. Let $f(x) = \sum_l f_l x^l$ be a Laurent polynomial over the complex numbers. By "summing the even part" one gets $(f(x) + f(-x))/2 = \sum_l f_{2l} x^{2l}$. We apply this to

$$f(x) := \sum_l \binom{m+n}{m+l} x^l = \sum_l \binom{m+n}{l} x^{l-m} = \frac{(1+x)^{m+n}}{x^m}$$

where the last equality follows from the binomial theorem. Consequently,

$$\sum_l \binom{m+n}{m+2l} = \frac{f(1)+f(-1)}{2} = 2^{m+n-1}.$$

□

Remark 1. We would like to mention that due to Zeilberger's summation machinery [20], today proofs of binomial summations like Vandermonde's formula or Lemma 2 can be carried out in a purely automatic fashion; see, for instance, the Mathematica package [19].

Now we are ready to carry out the last reduction step.

Proposition 3. *The case $l = 0$ of Proposition 2 is a consequence of Proposition 1 together with the cases $l = 1, \dots, N-1$ of Proposition 2.*

Proof. By $L(i,j;l)$ we denote the double sum on the left hand side of (12). First we note that due to symmetry, $L(i,j;l) = L(i,j;-l)$ for $l = 0, \dots, N-1$. Next we sum $L(i,j;l)$ over all $l = -N+1, \dots, N-1$. Because of Lemma 2 this gives

$$\sum_{l=-N+1}^{N-1} L(i,j;l) = 2^{4N-3} \left(\sum_{j=0}^{N-1} \binom{2N-1}{j} \frac{P_{N,j}}{2^j} \right)^2 = 2$$

where the last equality is by Proposition 1. On the other hand, because of the symmetry property $L(i,j;l) = L(i,j;-l)$, we have

$$\sum_{l=-N+1}^{N-1} L(i,j;l) = L(i,j;0) + 2 \sum_{l=1}^{N-1} L(i,j;l).$$

Combining things and applying Proposition 2 for the cases $l = 1, \dots, N-1$, results in the desired $L(i,j;0) = 2$. □

Finally we summarize what we need in order to solve the $2N+1$ Daubechies equations in $2N$ unknows $h_{N,k}$.

Theorem 1. *Any solution of the N algebraic equations*

$$\sum_{j=0}^{N-1} \frac{Q_{N,j}}{2^j} = \frac{1}{2^{2N-2}}, \tag{13}$$

and

$$\sum_{i,j=0}^{N-1} \binom{4N-i-j-2}{2N+2l-i-1} Q_{N,i}\, Q_{N,j} = 0, \quad (l = 1, \ldots, N-1) \tag{14}$$

gives rise to a solution of the Daubechies filter coefficient equations (3a) *to* (3d) *via*

$$h_{N,k} = \binom{2N-1}{N-k} \sum_{j=0}^{N-1} Q_{N,j} \frac{\binom{N-k}{j}}{\binom{2N-1}{j}} \tag{15}$$

where $k = -N+1, \ldots, N$.

We conclude this section with a few comments on various computational aspects of our approach.

First of all the reduction of the original system to N equations in N unknowns enables the computation of the corresponding Gröbner bases up to $N = 6$. We used the computer algebra system MATHEMATICA and the built-in procedure `GroebnerBasis`. The case $N = 6$ takes about 20 seconds on a computing platform equipped with a Pentium II processor and with 125.42 MB memory.

Another important aspect concerns the observation that the Gröbner bases computed with respect to the N equations from Theorem 1 have the same nice triangulation property as those computed with respect to (3a), (3c), (3d). But even more seems to be true; namely, in all instances the first univariate Gröbner basis polynomial turns out to be the same in both cases. This leads us to the following conjecture.

Conjecture 2. *Both systems of algebraic equations,* (3a), (3c), (3d) *and that one from Theorem 1, have only finitely many solutions, and the total number of different solutions in both cases is the same.*

Since the Gröbner bases for both systems seem to be of the same "triangular" shape with a common univariate polynomial p_1, the degree of this polynomial is a bound on the number of solutions. In all the cases $N = 1, \ldots, 6$ it turns . out to be of degree 2^{N-1}.

Conjecture 3. *Both systems of algebraic equations,* (3a), (3c), (3d) *and that one from Theorem 1, have at most* 2^{N-1} *different solutions.*

Also here more seems to be true. For instance, up to $N = 6$ the common univariate Gröbner basis polynomial always has 2^{N-1} *different* solutions. In particular, we have two real solutions in the cases $N = 2, 3$; four real solutions in the cases $N = 4, 5$; and eight real solutions if $N = 6$. In the Appendix we give the MATHEMATICA procedure we have used together with same Gröbner bases output.

Finally, at the end of this section, we display two of the four solutions corresponding to case $N = 3$. In order to obtain those, first one has to find all solutions of the univariate Gröbner basis polynomial, which is:

$$p_1(x_1) = 9 - 96x_1 - 1536x_1^2 - 4096x_1^3 + 16384x_1^4. \tag{16}$$

From this, one computes the two real solutions as described above:

$$(h_{3,-2}, h_{3,-1}, h_{3,0}, h_{3,1}, h_{3,2}, h_{3,3}) = \\ \left(\tfrac{1+\sqrt{10}+\sqrt{5+2\sqrt{10}}}{16}, \tfrac{5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}}{16}, \tfrac{5-\sqrt{10}+\sqrt{5+2\sqrt{10}}}{8}, \right. \\ \left. \tfrac{5-\sqrt{10}-\sqrt{5+2\sqrt{10}}}{8}, \tfrac{5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}}{16}, \tfrac{1+\sqrt{10}-\sqrt{5+2\sqrt{10}}}{16}\right)$$

and

$$(h_{3,-2}, h_{3,-1}, h_{3,0}, h_{3,1}, h_{3,2}, h_{3,3}) = \\ \left(\tfrac{1+\sqrt{10}-\sqrt{5+2\sqrt{10}}}{16}, \tfrac{5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}}{16}, \tfrac{5-\sqrt{10}-\sqrt{5+2\sqrt{10}}}{8}, \right. \\ \left. \tfrac{5-\sqrt{10}+\sqrt{5+2\sqrt{10}}}{8}, \tfrac{5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}}{16}, \tfrac{1+\sqrt{10}+\sqrt{5+2\sqrt{10}}}{16}\right). \tag{17}$$

4 Wavelets on the interval

4.1 Meyer's Construction

To our knowledge the first construction of orthogonal wavelets on the interval was proposed by Yves Meyer [18]. His construction restricts compactly supported orthonormal wavelets on $\mathbb{R}$ (as considered in §2) to the interval $I := [0\,,1]$ and manipulates the restricted functions in such a way that they form an orthonormal basis on I.

To avoid notational difficulties we restrict our attention to the construction of wavelets on $\mathbb{R}^+ := [0\,,\infty)$. From our presentation it becomes evident how the construction can be generalized to obtain a wavelet basis for $L^2(I)$.

We introduce the family of scaling functions restricted to $\mathbb{R}^+$

$$\phi^{\text{half}}_{m,k}(x) := \begin{cases} 0 & \text{if } x < 0 \\ \phi_{m,k}(x) & \text{if } x \geq 0 \end{cases}$$

and the according spaces

$$V^{\text{half}}_m := \overline{\operatorname{span}\left\{\phi^{\text{half}}_{m,k},\ k \in \mathbb{Z}\right\}}.$$

The spaces V^{half}_m form a multiresolution analysis for $L^2(\mathbb{R}^+)$. The according wavelets spaces W^{half}_m are given by

$$W^{\text{half}}_m := \left(V^{\text{half}}_m\right)^{\perp} \cap V^{\text{half}}_{m-1}\,. \tag{18}$$

We denote by $\mathbb{P}_{W_m^{\text{half}}}$ and $\mathbb{P}_{V_m^{\text{half}}}$ the orthogonal projection operators onto the spaces W_m^{half} and V_m^{half}, respectively.

Since the scaling function ϕ has support in $[-N+1\,,N]$, $\phi_{m,k}^{\text{half}} = 0$ for $k \leq -N$ and $\phi_{m,k}^{\text{half}} = \phi_{m,k}$ for $k \geq N-1$.

Wavelets on $\mathbb{R}^+$ can be constructed in the following way:

1. Orthonormalize the set of functions $\left\{\phi_{m,k}^{\text{half}}, k \geq -N+1\right\}$. The orthonormal basis of V_m^{half} is denoted by $\left\{\phi_{m,k}^{\text{edge}}\,, k \geq -N+1\right\}$.
2. Compute $\mathbb{P}_{W_m^{\text{half}}}\,\phi_{m-1,k}^{\text{edge}}$ and orthonormalize them to obtain an orthonormal basis $\psi_{m,k}^{\text{edge}}$ of W_m^{half}.

The functions $\phi_{m,k}^{\text{half}}$ can be orthonormalized by making a basis transformation

$$\phi_m^{\text{edge}} = A\,\phi_m^{\text{half}} \tag{19}$$

where

$$\phi_m^{\text{edge}} := \begin{pmatrix} \phi_{m,-N+1}^{\text{edge}} \\ \phi_{m,-N+2}^{\text{edge}} \\ \vdots \end{pmatrix} \quad \text{and} \quad \phi_m^{\text{half}} := \begin{pmatrix} \phi_{m,-N+1}^{\text{half}} \\ \phi_{m,-N+2}^{\text{half}} \\ \vdots \end{pmatrix}.$$

Using the notation

$$\phi_m^{\text{edge}} {\phi_m^{\text{edge}}}^t := \left(\left\langle \phi_{m,k}^{\text{edge}}\,,\,\phi_{m,l}^{\text{edge}} \right\rangle\right)_{k,l \geq -N+1},$$

we see that the orthonormality of the functions ϕ_m^{edge} is equivalent to the matrix equation

$$\phi_m^{\text{edge}} {\phi_m^{\text{edge}}}^t = I\,.$$

From (19) it follows that

$$I = A\,\phi_m^{\text{half}} {\phi_m^{\text{half}}}^t\,A^t\,.$$

If

$$\Lambda := \phi_m^{\text{half}} {\phi_m^{\text{half}}}^t\,, \tag{20}$$

the matrix of inner products of the truncated scaling functions $\phi_{m,k}^{\text{half}}$, is known, then the matrix A in (19) can be obtained by the Cholesky factorization

$$\Lambda = \left(A^{-1}\right)\left(A^{-1}\right)^t\,, \tag{21}$$

where A^{-1} is regular and of lower triangular form. Therefore the matrix A is also lower triangular. This in particular ensures staggered support of the functions $\phi_{m,k}^{\text{edge}}$, i.e. $\operatorname{supp}\phi_{m,k}^{\text{edge}} \subseteq [0\,,2^m(N+k)]$.

In the following we derive the refinement equations (similar to (1) and (2)) for ϕ_m^{edge} and the according wavelets ψ_m^{edge}. These equations are the basis for the implementation of multiresolution cascade algorithms [17], as they are used e.g. in data compression (see e.g. [24]).

The truncated scaling functions $\phi^{\mathrm{half}}_{m,k}$ satisfy the dilation equation

$$\phi^{\mathrm{half}}_{m,k} = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}} h_{r-2k} \phi^{\mathrm{half}}_{m-1,r} \,.$$

Writing this equation in matrix form yields

$$\phi^{\mathrm{half}}_{m} = H\, \phi^{\mathrm{half}}_{m-1} \,, \tag{22}$$

where the dilation matrix H is a 1×2 block Toeplitz matrix:

$$H_{k,l} = \begin{cases} \frac{h_{l-2k}}{\sqrt{2}} & \text{if } -N+1 \le l-2k \le N, \\ 0 & \text{otherwise.} \end{cases}$$

From (19) and (22) it follows that

$$\phi^{\mathrm{edge}}_{m} = A\, \phi^{\mathrm{half}}_{m} = AH\, \phi^{\mathrm{half}}_{m-1} = AHA^{-1} \phi^{\mathrm{edge}}_{m-1} \,. \tag{23}$$

Thus the refinement matrix H^{edge} for the dilation equation of the edge scaling functions ϕ^{edge}_{m} is

$$H^{\mathrm{edge}} = AHA^{-1} \,. \tag{24}$$

Note that H^{edge} is no longer a 1×2 block Toeplitz matrix (which is the case for wavelets on $\mathbb{R}$). This reflects the fact that the edge scaling functions *cannot* be obtained as shifts of a single function.

Now we construct the edge wavelets and derive their refinement matrix. The projections of $\phi^{\mathrm{edge}}_{m-1,k}$ onto W^{half}_{m} as defined by (18) are given by

$$\psi^{\mathrm{half}}_{m,k} := \mathbb{P}_{W^{\mathrm{half}}_{m}} \phi^{\mathrm{edge}}_{m-1,k} = \phi^{\mathrm{edge}}_{m-1,k} - \sum_{l} \left\langle \phi^{\mathrm{edge}}_{m-1,k} \,, \phi^{\mathrm{edge}}_{m,l} \right\rangle \phi^{\mathrm{edge}}_{m,l} \,,$$

or equivalently

$$\psi^{\mathrm{half}}_{m} = \phi^{\mathrm{edge}}_{m-1} - \left(\phi^{\mathrm{edge}}_{m-1} {\phi^{\mathrm{edge}}_{m}}^{t} \right) \phi^{\mathrm{edge}}_{m} \,. \tag{25}$$

From (23) and (24) it follows that

$$\psi^{\mathrm{half}}_{m} = \left(I - {H^{\mathrm{edge}}}^{t} H^{\mathrm{edge}} \right) \phi^{\mathrm{edge}}_{m-1} =: G^{\mathrm{half}}\, \phi^{\mathrm{edge}}_{m-1} \,. \tag{26}$$

The matrix G^{half} does not have 1×2 lower block triangular form, i.e. it does not fulfill $G^{\mathrm{half}}_{k,l} = 0$ for $l > N + 2k$. Consequently the functions $\psi^{\mathrm{half}}_{m,k}$ do not have staggered support. In [8] it is established that there exists a basis transformation U such that

$$\psi^{\mathrm{stag}}_{m} := U\, \psi^{\mathrm{half}}_{m} \tag{27}$$

has staggered support. In §4.5 we give a simple constructive algorithm for calculating U. The functions $\psi^{\mathrm{stag}}_{m,k}$ are orthonormalized to get the edge wavelets

$$\psi^{\mathrm{edge}}_{m} = B\, \psi^{\mathrm{stag}}_{m} \,. \tag{28}$$

In the following we outline the orthonormalization procedure, i.e. the calculation of the matrix B. Let Λ_w be the matrix of inner products of the functions $\tilde{\tilde{\psi}}^{\text{half}}_{m,k}$. Then from (27) and (26) it follows that

$$\Lambda_w := \psi^{\text{stag}}_m {\psi^{\text{stag}}_m}^t = U\, G^{\text{half}} {G^{\text{half}}}^t U^t \,. \tag{29}$$

From (28) and the orthonormality of the functions ψ^{edge}_m we get

$$\Lambda_w = \left(B^{-1}\right)\left(B^{-1}\right)^t \,.$$

Thus the matrix B can be calculated from Λ_w by a Cholesky factorization and inversion. From (28), (27) and (26) it follows that

$$\psi^{\text{edge}}_m = B\, U\, G^{\text{half}}\, \phi^{\text{edge}}_{m-1} \,.$$

Thus the matrix G^{edge} for the refinement equation of the edge wavelets is given by

$$G^{\text{edge}} = BUG^{\text{half}} \,. \tag{30}$$

To make the calculations complete we have to determine the matrix Λ in (20). The matrix Λ is independent of the scale m as one can see from the following argument. Since

$$\phi^{\text{half}}_{m,k}(2x) = \sqrt{2}\phi^{\text{half}}_{m-1,k}(x)$$

it follows that

$$\left\langle \phi^{\text{half}}_{m-1,k} \,,\, \phi^{\text{half}}_{m-1,l} \right\rangle = 2 \left\langle \phi^{\text{half}}_{m,k}(2\,\cdot) \,,\, \phi^{\text{half}}_{m,l}(2\,\cdot) \right\rangle = \left\langle \phi^{\text{half}}_{m,k} \,,\, \phi^{\text{half}}_{m,l} \right\rangle \,. \tag{31}$$

From (31) and (22) it follows that

$$\Lambda = \phi^{\text{half}}_m {\phi^{\text{half}}_m}^t = H\,\Lambda\, H^t \,. \tag{32}$$

Since $\Lambda_{k,l} = \left\langle \phi^{\text{half}}_{m,k} \,,\, \phi^{\text{half}}_{m,l} \right\rangle = \delta_{k,l}$ if k or $l \geq N-1$, equation (32) above can be reduced to

$$\boldsymbol{\Lambda} = \boldsymbol{H} \boldsymbol{\Lambda}^{\text{ext}} \boldsymbol{H}^t \,, \tag{33}$$

where $\boldsymbol{\Lambda} \in \mathbb{R}^{(2N-2)\times(2N-2)}$, $\boldsymbol{H} \in \mathbb{R}^{(2N-2)\times(4N-4)}$ with $\boldsymbol{H}_{k,l} = h_{l-2k}/\sqrt{2}$, for $-N+1 \leq k \leq N-2$ and $-N+1 \leq l \leq 3N-4$, and

$$\boldsymbol{\Lambda}^{\text{ext}} = \begin{pmatrix} \boldsymbol{\Lambda} & 0 \\ 0 & I \end{pmatrix} \in \mathbb{R}^{(4N-4)\times(4N-4)} \,.$$

Equation (33) is a non-homogeneous linear system for as many unknowns as equations which can be solved symbolically. We have strong evidence that there exists a solution $\boldsymbol{\Lambda}$ of (33) but so far we have no proof: the existence of a solution is closely related to the eigenvalues of the matrix $H_1 \in \mathbb{R}^{(2N-2)\times(2N-2)}$ which is the restriction of $\boldsymbol{H}$ to the first $2N-2$ columns. If the absolute values of the eigenvalues of H_1 are less than 1, then from (33) it follows that

$$\Lambda = \sum_{n=0}^{\infty} H_1^n H_2 H_2^t \left(H_1^t\right)^n \,,$$

where H_2 are the last $2N-2$ columns of $\boldsymbol{H}$.

We mention a result in [22], which shows that N eigenvalues of H_1 are given by

$$2^{-k-1/2}, \quad \text{with } k = 0, \ldots, N-1.$$

Since there is no estimate for the other $N-1$ eigenvalues available, this result does not give existence of a solution. However, in all our considered examples the largest eigenvalue turned out to be $1/\sqrt{2}$.

4.2 The construction of Cohen, Daubechies and Vial

The starting point is again a compactly supported orthogonal wavelet family on $\mathbb{R}$. As in Meyer's approach the construction of Cohen, Daubechies and Vial [8] retains the interior scaling functions and adds adapted edge scaling functions.

In [8, 7] the family of transformed scaling functions restricted to $\mathbb{R}^+$ is introduced as follows:

$$\phi_{m,k}^{\text{mod}} = \begin{cases} \sum_l \binom{N-1-l}{N-1-k} \phi_{m,l}^{\text{half}} & \text{if } 0 \le k \le N-1, \\ \phi_{m,k}^{\text{half}} & \text{if } k \ge N. \end{cases}$$

The functions $\phi_{m,k}^{\text{mod}}$ can generate all polynomials up to degree $N-1$ [8, Proposition 4.1.]. In contrast to Meyer's construction this approach requires less edge scaling functions to fulfill this task. While in Meyer's construction the spaces V_m^{half} are just the projections of V_m onto $L^2(\mathbb{R}^+)$, here the space

$$V_m^{\text{half}} := \overline{\operatorname{span}\left\{\phi_{m,k}^{\text{mod}}, k \in \mathbb{N}_0\right\}} = T(V_m),$$

where $T = (T_{k,l})$ is a matrix with indices $0 \le k$ and $-N+1 \le l$ which satisfies

$$T_{k,l} = \begin{cases} \binom{N-1-l}{N-1-k} & \text{if } 0 \le k \le N-1, \\ \delta_{k,l} & \text{if } k \ge N. \end{cases}$$

Since $T_{k,l} = 0$ if $l > k$ the family

$$\phi_m^{\text{mod}} = T\phi_m^{\text{half}} \tag{34}$$

has staggered support. The spaces V_m^{half} define a multiresolution analysis on $L^2(\mathbb{R}^+)$ and the according wavelet spaces are given by

$$W_m^{\text{half}} := \left(V_m^{\text{half}}\right)^{\perp} \cap V_{m-1}^{\text{half}}.$$

The functions $\phi_{m,k}^{\text{mod}}$ can be orthonormalized by a basis transformation

$$\phi_m^{\text{edge}} = A\,\phi_m^{\text{mod}}. \tag{35}$$

Again the orthonormalization matrix A is determined by the Cholesky decomposition of

$$\tilde{\Lambda} := \phi_m^{\text{mod}} {\phi_m^{\text{mod}}}^t = T\,\Lambda\,T^t, \tag{36a}$$

where Λ is as in (32), i.e.

$$\tilde{\Lambda} = \left(A^{-1}\right)\left(A^{-1}\right)^t . \tag{36b}$$

In the following we determine the filter matrix H^{edge}; once the filter matrix H^{edge} is constructed, the refinement matrix G^{edge} of the edge wavelets can be calculated analogously to the construction presented in §4.1.

The filter matrix for the dilation equation satisfies

$$\phi_m^{\text{edge}} = H^{\text{edge}}\, \phi_{m-1}^{\text{edge}} . \tag{37}$$

From (34) and (22) we get

$$\phi_m^{\text{mod}} = T\, \phi_m^{\text{half}} = TH\phi_{m-1}^{\text{half}} . \tag{38}$$

Suppose that there exists a dilation equation for ϕ_m^{mod}, i.e.

$$\phi_m^{\text{mod}} = H^{\text{mod}}\, \phi_{m-1}^{\text{mod}} , \tag{39}$$

then from (38) and (39) it follows that

$$TH = H^{\text{mod}}\, T .$$

Multiplication of this equation by a right inverse $T^\dagger$ of T from the right gives

$$H^{\text{mod}} = TH\,T^\dagger . \tag{40}$$

This yields the following condition on T and $H\ T^\dagger$:

$$TH\,T^\dagger T = TH ,$$

which is equivalent to

$$\mathcal{N}(T) \subset \mathcal{N}(TH) , \tag{41}$$

where $\mathcal{N}$ denotes the nullspace. In particular this shows that the condition (41) is independent on the choice of the right inverse $T^\dagger$.

From (40) and (37) it follows that

$$H^{\text{edge}} = ATH\,T^\dagger A^{-1} . \tag{42}$$

Now the further procedure to construct G^{edge} is analogous as in §4.1. For the readers convenience we have summarized the calculation of the refinement matrices in §4.5.

This matrix analytical approach clearly reveals the similarity between the constructions proposed by Meyer and Cohen, Daubechies & Vial. In fact the only difference in both constructions is that the construction of the filter matrix H^{edge} incorporates the matrix T. Any right-invertible matrix T satisfying (41) can be used to construct wavelets on $\mathbb{R}^+$ with different properties. The special form of the matrix T proposed in [8] guarantees that the scaling functions have staggered support and that any polynomial up to degree $N-1$ can be represented as a linear combination of the scaling functions. Setting $T = I$ gives the construction proposed by Meyer.

4.3 The biorthogonal case – the constructions of Dahmen et al.

In this section we show that the our matrix approach for the construction of wavelets on the interval can be generalized in a natural way to the construction of biorthogonal wavelets on the interval. This outlines the constructions proposed by Dahmen et al. [9, 10].

In the biorthogonal case one requires two scaling functions ϕ and $\tilde{\phi}$ satisfying dilation equations

$$(43) \qquad \phi(x) = \sum_{k=-N+1}^{N} h_k \phi(2x-k) \qquad \text{and} \qquad \tilde{\phi}(x) = \sum_{k=-\tilde{N}+1}^{\tilde{N}} \tilde{h}_k \tilde{\phi}(2x-k)\,.$$

Both scaling functions satisfy (3a), (3d) and are biorthogonal, i.e.

$$(44) \qquad \sum_k h_k \tilde{h}_{k+2l} = \delta_{0,l}\,.$$

The corresponding multiresolution analyses are given by

$$V_m := \overline{\operatorname{span}\{\phi_{m,k}, k \in \mathbb{Z}\}}\,, \qquad \tilde{V}_m := \overline{\operatorname{span}\{\tilde{\phi}_{m,k}, k \in \mathbb{Z}\}}\,.$$

The wavelet spaces W_m and $\tilde{W}_m$ are then defined by

$$W_m = V_{m-1} \cap \tilde{V}_m^{\perp}\,, \quad \text{and} \quad \tilde{W}_m = \tilde{V}_{m-1} \cap V_m^{\perp}\,.$$

For more background on biorthogonal wavelets we refer to [6].

Following the notation of the previous chapters we define the modified scaling functions on $\mathbb{R}^+$ by

$$(45) \qquad \phi_m^{\text{mod}} = T\,\phi_m^{\text{half}} \quad \text{and} \quad \tilde{\phi}_m^{\text{mod}} = \tilde{T}\,\tilde{\phi}_m^{\text{half}}\,,$$

where again ϕ_m^{half} and $\tilde{\phi}_m^{\text{half}}$ are the restrictions to the positive real line.

The two families ϕ_m^{mod} and $\tilde{\phi}_m^{\text{mod}}$ are *biorthogonalized* by two basis transforms A and $\tilde{A}$, i.e.

$$(46) \qquad \phi_m^{\text{edge}} := A\,\phi_m^{\text{mod}} \quad \text{and} \quad \tilde{\phi}_m^{\text{edge}} := \tilde{A}\,\tilde{\phi}_m^{\text{mod}}$$

satisfy

$$\phi_m^{\text{edge}}\,{\tilde{\phi}_m^{\text{edge}}}^{t} = I\,.$$

Analogously to (36) the last equation is equivalent to

$$(47) \qquad \left(A^{-1}\right)\left(\tilde{A}^{-1}\right)^t = T\Lambda\tilde{T}^t\,,$$

where $\Lambda := \phi_m^{\text{half}}\,{\tilde{\phi}_m^{\text{half}}}^{t}$.

For a given matrix $T\Lambda\tilde{T}^t$ the factorization into the matrices A^{-1} and $\tilde{A}^{-1}$ can be computed in several ways: one could use e.g. a factorization by means of a SVD, as suggested by Dahmen et al. [9], a LU-decomposition, or simply set $A = I$

and $\tilde{A} = (T\Lambda\tilde{T}^t)^{-1}$. Each possible factorization results in different biorthogonal bases for the same multiresolution spaces V_m^{half} and $\tilde{V}_m^{\text{half}}$. For orthogonal wavelets we calculated the factorization by a Cholesky decomposition.

The matrix Λ can be calculated similarly to the orthogonal case (cf. (32)) as the solution of the following linear inhomogeneous system:

$$\Lambda = H\,\Lambda\,\tilde{H}^t\,.$$

The dilation matrices H^{edge} and $\tilde{H}^{\text{edge}}$ are given by

$$H^{\text{edge}} = ATHT^{\dagger}A^{-1} \quad \text{and} \quad \tilde{H}^{\text{edge}} = \tilde{A}\tilde{T}\tilde{H}\tilde{T}^{\dagger}\tilde{A}^{-1}\,,$$

where $T^{\dagger}$ and $\tilde{T}^{\dagger}$ denote the right inverses of T and $\tilde{T}$ satisfying

$$\mathcal{N}(T) \subset \mathcal{N}(TH) \quad \text{and} \quad \mathcal{N}(\tilde{T}) \subset \mathcal{N}(\tilde{T}\tilde{H})\,. \tag{48}$$

Note the similarity of the constructions of H^{edge} in the orthogonal and biorthogonal case!

The construction of the biorthogonal wavelet bases can be carried over from the orthogonal case. Since

$$V_m \oplus W_m = V_{m-1} \quad \text{and} \quad \tilde{V}_m \oplus \tilde{W}_m = \tilde{V}_{m-1}$$

we can write the projections of ϕ_{m-1}^{edge} and $\tilde{\phi}_{m-1}^{\text{edge}}$ onto W_m and $\tilde{W}_m$ as

$$\psi_m^{\text{half}} := \mathbb{P}_{W_m}\,\phi_{m-1}^{\text{edge}} = \phi_{m-1}^{\text{edge}} - \mathbb{P}_{V_m}\,\phi_{m-1}^{\text{edge}}\,,$$
$$\tilde{\psi}_m^{\text{half}} := \mathbb{P}_{\tilde{W}_m}\,\tilde{\phi}_{m-1}^{\text{edge}} = \tilde{\phi}_{m-1}^{\text{edge}} - \mathbb{P}_{\tilde{V}_m}\,\tilde{\phi}_{m-1}^{\text{edge}}\,,$$

and consequently

$$\psi_m^{\text{half}} = G^{\text{half}}\,\phi_{m-1}^{\text{edge}} \quad \text{and} \quad \tilde{\psi}_m^{\text{half}} = \tilde{G}^{\text{half}}\,\tilde{\phi}_{m-1}^{\text{edge}}\,,$$

where

$$G^{\text{half}} = I - {\tilde{H}^{\text{edge}}}{}^t\,H^{\text{edge}} \quad \text{and} \quad \tilde{G}^{\text{half}} = I - {H^{\text{edge}}}{}^t\,\tilde{H}^{\text{edge}}\,.$$

In order to biorthogonalize the families of functions ψ_m^{half} and $\tilde{\psi}_m^{\text{half}}$ we set

$$\psi_m^{\text{edge}} := B\psi_m^{\text{half}} \quad \text{and} \quad \tilde{\psi}_m^{\text{edge}} := \tilde{B}\tilde{\psi}_m^{\text{half}}\,, \tag{49}$$

where the matrices B and $\tilde{B}$ satisfy

$$\left(B^{-1}\right)\left(\tilde{B}^{-1}\right)^t = \Lambda_w\,,$$

where

$$\Lambda_w := \psi_m^{\text{half}}\,{\tilde{\psi}_m^{\text{half}}}{}^t = G^{\text{half}}\,{\tilde{G}^{\text{half}}}{}^t.$$

The construction presented above reveals that there is more freedom in generating biorthogonal wavelets on $\mathbb{R}^+$ than for the the construction of orthogonal wavelets. The choice of the matrices T and $\tilde{T}$ determines the properties of the multiresolution analyses. As in the orthogonal case, any T and $\tilde{T}$, compatible with H and $\tilde{H}$ in the sense of (48) can be used to construct biorthogonal wavelets on the interval. The choices of the biorthogonalizations (46) and (49) affect the scaling functions and wavelets, but not the multiresolution and wavelet spaces. Dahmen et al. [9] suggested transformations T and $\tilde{T}$ for the construction of biorthogonal wavelet bases with certain polynomial exactness.

4.4 (Bi-)orthogonal wavelets with staggered support

The matrix analytical point of view of constructing wavelets on the half line clearly indicates how to impose additional properties on the wavelets and scaling functions. In the construction above we have not paid any attention to preserve staggered support of the scaling functions and wavelets. In the following we show how to construct (bi-)orthogonal wavelets and scaling functions with staggered support. To our knowledge biorthogonal wavelets on the interval with staggered support have not been considered in the literature so far.

The following lemma guarantees existence of a basis ϕ_m^{stag} of V_m^{half} with staggered support.

Lemma 3. *Let $\boldsymbol{T} \in \mathbb{R}^{K\times(2N-1)}$, $K \leq 2N-1$ and let $\boldsymbol{T}_1$ be the $K \times K$ submatrix consisting of the last K columns of $\boldsymbol{T}$. If $\boldsymbol{T}_1$ is invertible, then there exists a invertible matrix $\boldsymbol{S} \in \mathbb{R}^{K\times K}$ such that $\boldsymbol{ST}$ is of lower triangular form.*

Proof. Since $\boldsymbol{T}_1$ is invertible, $\left(\boldsymbol{T}_1^{-1}\right)^t$ exists and can be decomposed by a LU-factorization into

$$P\left(\boldsymbol{T}_1^{-1}\right)^t = LU\,,$$

where L and U are lower and upper triangular matrices, respectively, and P is a permutation matrix. Thus

$$L^t P^t \boldsymbol{T}_1 = \left(U^{-1}\right)^t$$

is a lower triangular matrix (note that U^t is lower triangular and thus also $\left(U^{-1}\right)^t$ and L^tP^t is invertible. Let $\boldsymbol{T} = (\boldsymbol{T}_0, \boldsymbol{T}_1)$, then as a consequence

$$L^tP^t\boldsymbol{T} = L^tP^t(\boldsymbol{T}_0, \boldsymbol{T}_1) = \left(L^tP^t\boldsymbol{T}_0, (U^{-1})^t\right)$$

is lower triangular. Thus the assertion is proved with $\boldsymbol{S} := L^tP^t$. $\square$

T in (34) is of the form

$$T = \begin{pmatrix} \boldsymbol{T} & 0 \\ 0 & I \end{pmatrix}.$$

Let $\boldsymbol{S}$ be defined as in the lemma above, then

$$T^{\text{stag}} := \begin{pmatrix} \boldsymbol{S} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \boldsymbol{T} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \boldsymbol{ST} & 0 \\ 0 & I \end{pmatrix}$$

is of lower triangular form and thus $\phi_m^{\text{stag}} := T^{\text{stag}}\phi_m^{\text{half}}$ has staggered support.

Proposition 4.3 in [8] guarantees the existence of wavelets with staggered support.

The above considerations can be easily carried over to biorthogonal wavelets. In order to get biorthogonal wavelets and scaling functions with staggered support an LU-factorization of (47) has to be performed, since only this factorization guarantees that the staggered support is preserved during the biorthogonalization procedure.

4.5 An Algorithm for the Calculation of the Refinement matrices

For the readers convenience we summarize the computational steps for calculating the refinement matrices H^{edge} and G^{edge} in the orthogonal case. The modifications of this algorithm to calculate the refinement matrices in the biorthogonal case are obvious.

Each step of the proposed algorithm can either be performed numerically or symbolically.

Given the filter sequence h_k of a compactly supported orthonormal wavelet family on $\mathbb{R}$ with $h_k = 0$ if $k \le -N$ or $k \ge N+1$ and the matrix $T \in \mathbb{R}^{K\times(2N-1)}$ ($T = I \in \mathbb{R}^{(2N-1)\times(2N-1)}$ for Meyer's construction and $(T_{k,l}) = \binom{N-l-1}{N-k-1} \in \mathbb{R}^{N\times(2N-1)}$ for the construction of Cohen et al.)

1. Define the filter matrix $H := (H_{k,l}) \in \mathbb{R}^{(2N-1)\times(4N-2)}$, with
$$H_{k,l} = h_{l-2k}/\sqrt{2}$$
for $-N+1 \le k \le N-1$ and $-N+1 \le l \le 3N-2$.
2. Solve
$$\Lambda = H\Lambda^{\mathrm{ext}}H^t$$
with $\Lambda \in \mathbb{R}^{(2N-1)\times(2N-1)}$ and
$$\Lambda^{\mathrm{ext}} = \begin{pmatrix} \Lambda & 0 \\ 0 & I \end{pmatrix} \in \mathbb{R}^{(4N-2)\times(4N-2)} .$$
3. Compute the matrix of inner products
$$\tilde{\Lambda} = T\,\Lambda T^t \in \mathbb{R}^{K\times K} .$$
4. Compute $A \in \mathbb{R}^{K\times K}$ from the Cholesky decomposition
$$\tilde{\Lambda} = \left(A^{-1}\right)\left(A^{-1}\right)^t .$$
5. The dilation matrix for the edge scaling functions is then given by
$$H^{\mathrm{edge}} = A\,TH\left(T^{\mathrm{ext}}\right)^{\dagger}\left(A^{\mathrm{ext}}\right)^{-1} \in \mathbb{R}^{K\times(K+2N-1)} ,$$
where
$$A^{\mathrm{ext}} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \in \mathbb{R}^{(K+2N-1)\times(K+2N-1)} .$$
and
$$T^{\mathrm{ext}} = \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix} \in \mathbb{R}^{(K+2N-1)\times 4N-2} ,$$
and $\left(T^{\mathrm{ext}}\right)^{\dagger}$ is a right inverse of T^{ext}.

6. Compute

$$C = \left(I - H^{\mathrm{edge}^t} H^{\mathrm{edge}}\right) \in \mathbb{R}^{(K+2N-1)\times(K+2N-1)}\,,$$

and define $G^{\mathrm{half}} \in \mathbb{R}^{N\times(K+2N-1)}$ as

$$G^{\mathrm{half}} = (C_{k,l})_{\substack{0\le k\le N-1\\ N-K\le l\le 2N-2}}$$

7. Compute an upper triangular matrix $U \in \mathbb{R}^{(N-1)\times(N-1)}$ such that UG^{half} is a lower triangular block matrix in the sense that $(UG^{\mathrm{half}})_{k,l} = 0$ for $l > N + 2k$. This can be done using the following algorithm:

   (a) Define the matrix $\tilde{C} \in \mathbb{R}^{N\times N}$ by

   $$\tilde{C}_{k,l} = G^{\mathrm{half}}_{k,N+2l} \qquad 0 \le k, l \le N-1\,.$$

   (b) Compute U by the unpivoted LU-decomposition

   $$\tilde{C}^{-1} = LU\,.$$

8. Compute the matrix of inner products

$$\Lambda_w = UG^{\mathrm{half}}G^{\mathrm{half}^t}U^t \in \mathbb{R}^{N\times N}\,.$$

9. Compute the Cholesky decomposition

$$\Lambda_w = \left(B^{-1}\right)\left(B^{-1}\right)^t\,.$$

10. The filter matrix G^{edge} is then given by

$$G^{\mathrm{edge}} = BUG^{\mathrm{half}}\,.$$

The entries of H^{edge} and G^{edge} for $k \ge N$ are given by $H^{\mathrm{edge}}_{k,l} = h_{l-2k}/\sqrt{2}$ and $G^{\mathrm{edge}}_{k,l} = g_{l-2k}/\sqrt{2}$, respectively.

5 Results

In this section we present some closed form representations of the filter coefficients for the Daubechies wavelets (§5.1) and of the refinement matrices for the construction proposed by Cohen, Daubechies & Vial where $N = 2$ (§5.2).

5.1 Mathematica Program and Output

The following MATHEMATICA program calculates the filter coefficients of the Daubechies wavelets.

```
(* The Mathematica program : *)

Polys1[N_]:={Sum[Q[N,j]/2^j,{j,0, N-1}] - 1/2^(2N-2)};
Polys2[N_]:=Table[Sum[Binomial[4N-i-j-2,2N+2l-i-1]*
                      Q[N,i]*Q[N,j],{i,0,N-1},{j,0,N-1}],
                  {l,1,N-1}];
AllPolys[N_]:=Join[Polys1[N],Polys2[N]];
Eqns[N_]:=Map[#==0&,AllPolys[N]];
Unknowns[N_]:=Table[Q[N,j],{j,0,N-1}];
CFSols[N_]:=Solve[Eqns[N],Unknowns[N]]
GB[N_]:=GroebnerBasis[AllPolys[N],Reverse[Unknowns[N]]];

cc[N_,k_]:= Binomial[2N-1,N-k] *
    Sum[Q[N,j] Binomial[N-k,j]/Binomial[2N-1,j],
{j,0,N-1}]

CoefficientTable[N_,rules_]:=
   Table[h[N,k]->Simplify[ cc[N,k] /.rules],{k,-N+1,N}]

(* The Groebner basis in the case N=3: *)
GB[3]
```

$$\{9 - 96\,Q(3,0) - 1536\,Q(3,0)^2 - 4096\,Q(3,0)^3 + 16384\,Q(3,0)^4,$$
$$21\,Q(3,0) + 32\,Q(3,0)^2 - 128\,Q(3,0)^3 + 3\,Q(3,1),$$
$$-3 - 120\,Q(3,0) - 256\,Q(3,0)^2 + 1024\,Q(3,0)^3 + 12\,Q(3,2)\}$$

```
(* A real solution for N=3: *)
rules = Simplify[CFSols[3][[3]]];
CoefficientTable[3, rules]
```

$$\{h(3,-2) \to \frac{1+\sqrt{10}+\sqrt{5+2\sqrt{10}}}{16}, h(3,-1) \to \frac{5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}}{16},$$
$$h(3,0) \to \frac{5-\sqrt{10}+\sqrt{5+2\sqrt{10}}}{8}, h(3,1) \to \frac{5-\sqrt{10}-\sqrt{5+2\sqrt{10}}}{8},$$
$$h(3,2) \to \frac{5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}}{16}, h(3,3) \to \frac{1+\sqrt{10}-\sqrt{5+2\sqrt{10}}}{16}\}$$

```
(* The Groebner basis in case N=4: *)
GB[4]
```

$$\{625 + 16000\,Q(4,0) - 1433600\,Q(4,0)^2 + 22937600\,Q(4,0)^3 + 220200960\,Q(4,0)^4 -$$
$$4697620480\,Q(4,0)^5 - 60129542144\,Q(4,0)^6 -$$
$$137438953472\,Q(4,0)^7 + 1099511627776\,Q(4,0)^8,$$

$$
\begin{aligned}
&125 + 389200\, Q(4,0) - 1469440\, Q(4,0)^2 - 29245440\, Q(4,0)^3 + 124780544\, Q(4,0)^4 + \\
&\quad 2936012800\, Q(4,0)^5 + 7516192768\, Q(4,0)^6 - \\
&\quad 51539607552\, Q(4,0)^7 + 39200\, Q(4,1), \\
&-1875 - 6661200\, Q(4,0) + 57164800\, Q(4,0)^2 + 775864320\, Q(4,0)^3 - \\
&\quad 9064939520\, Q(4,0)^4 - 136113553408\, Q(4,0)^5 - 323196289024\, Q(4,0)^6 + \\
&\quad 2456721293312\, Q(4,0)^7 + 196000\, Q(4,2), \\
&-11625 + 3553200\, Q(4,0) - 42470400\, Q(4,0)^2 - 483409920\, Q(4,0)^3 + \\
&\quad 7817134080\, Q(4,0)^4 + 106753425408\, Q(4,0)^5 + 248034361344\, Q(4,0)^6 - \\
&\quad 1941325217792\, Q(4,0)^7 + 98000\, Q(4,3)\}
\end{aligned}
$$

```
(* The univariate Groebner basis polynomial in case N=5: *)
GB[5][[1]]
```

$$
\begin{aligned}
&2251875390625 - 131766880000000\ Q(5,0) - 57826836480000000\ Q(5,0)^2 - \\
&\quad 4000515620864000000\ Q(5,0)^3 + 92455465857843200000\ Q(5,0)^4 + \\
&\quad 1513668383462129664000\ Q(5,0)^5 - 32198063323420295168000\ Q(5,0)^6 - \\
&\quad 2600406347161414087475200\ Q(5,0)^7 \\
&\quad +15180699036295329711390720\ Q(5,0)^8 \\
&\quad -24345747195367204805253529600\ Q(5,0)^9 \\
&\quad -28222373258903518083515678720\ Q(5,0)^{10} \\
&\quad +12421567725441961014604993658880\ Q(5,0)^{11} \\
&\quad +71032999454195120173711747973120\ Q(5,0)^{12} \\
&\quad -287756686251808074139751627620352\ Q(5,0)^{13} \\
&\quad -38942226439011207213978722469150720\ Q(5,0)^{14} \\
&\quad -83076749736557242056487941267521536\ Q(5,0)^{15} \\
&\quad +1329227995784915872903807060280344576\ Q(5,0)^{16}
\end{aligned}
$$

```
(* The univariate Groebner basis polynomial in case N=6: *)
Timing[ GB[6][[1]] ]
```

$$
\begin{aligned}
&\{19.98\ \ \text{Second}, \\
&\quad 6158129128018216491432748544\,1 + 8007522828051623729812234297344\ Q(6,0) \\
&\quad -1613909870816957102724822638316748\,8\ Q(6,0)^2 + \cdots + 2^{288} Q(6,0)^{32}\}
\end{aligned}
$$

5.2 Refinement matrices for construction proposed by Cohen, Daubechies & Vial

The filter coefficients h_k of the Daubechies wavelets for the case $N = 2$ are given by

$$
h_{-1} = \frac{1+\sqrt{3}}{4}\,, \quad h_0 = \frac{3+\sqrt{3}}{4}\,, \quad h_1 = \frac{3-\sqrt{3}}{4}\,, \quad h_2 = \frac{1-\sqrt{3}}{4}\,.
$$

The entries of the refinement matrices H^{edge} and G^{edge} are the following:

$$
\begin{aligned}
H_{0,0}^{\mathrm{edge}} &= \frac{\sqrt{2}\left(1137-119\sqrt{3}\right)}{2182} \\
H_{0,1}^{\mathrm{edge}} &= \frac{\sqrt{2\left(3969-2184\sqrt{3}\right)\left(242883-140092\sqrt{3}\right)}\left(16589+9619\sqrt{3}\right)}{14283372} \\
H_{0,2}^{\mathrm{edge}} &= -\frac{\sqrt{2\left(3969-2184\sqrt{3}\right)}\left(123+85\sqrt{3}\right)}{13092}
\end{aligned}
$$

$$
\begin{aligned}
H_{1,0}^{\mathrm{edge}} &= \frac{\sqrt{2\left(3969-2184\sqrt{3}\right)\left(242883-140092\sqrt{3}\right)}\left(32238331+17965009\sqrt{3}\right)}{501189240108} \\
H_{1,1}^{\mathrm{edge}} &= \frac{\sqrt{2}\left(999+238\sqrt{3}\right)}{4364} \\
H_{1,2}^{\mathrm{edge}} &= \frac{\sqrt{2\left(242883-140092\sqrt{3}\right)}\left(2963297+1744500\sqrt{3}\right)}{153128396} \\
H_{1,4}^{\mathrm{edge}} &= \frac{\sqrt{3}\sqrt{2\left(242883-140092\sqrt{3}\right)}\left(897+445\sqrt{3}\right)}{280712} \\
H_{1,4}^{\mathrm{edge}} &= -\frac{\sqrt{2\left(242883-140092\sqrt{3}\right)}\left(897+445\sqrt{3}\right)}{280712}
\end{aligned}
$$

$$
\begin{aligned}
G_{0,0}^{\mathrm{edge}} &= \frac{\sqrt{\left(2826138238+1021826769\sqrt{3}\right)\left(3969-2184\sqrt{3}\right)}\left(6136686+2872499\sqrt{3}\right)}{12905163547593} \\
G_{0,1}^{\mathrm{edge}} &= -\frac{\sqrt{\left(2826138238+1021826769\sqrt{3}\right)\left(242883-140092\sqrt{3}\right)}\left(2102042+1389397\sqrt{3}\right)}{8603442365062} \\
G_{0,2}^{\mathrm{edge}} &= \frac{\sqrt{2826138238+1021826769\sqrt{3}}\left(80542-29121\sqrt{3}\right)}{7885831682}
\end{aligned}
$$

$$
\begin{aligned}
G_{1,0}^{\mathrm{edge}} &= \frac{\sqrt{35089\left(80542+29121\sqrt{3}\right)\left(3969-2184\sqrt{3}\right)}\left(605486-295683\sqrt{3}\right)}{8603442365062} \\
G_{1,1}^{\mathrm{edge}} &= \frac{\sqrt{35089\left(80542+29121\sqrt{3}\right)\left(3969-2184\sqrt{3}\right)}\left(1147827+503061\sqrt{3}\right)}{17206884730124} \\
G_{1,2}^{\mathrm{edge}} &= -\frac{\sqrt{35089\left(80542+29121\sqrt{3}\right)}\left(147657-115797\sqrt{3}\right)}{15771663364} \\
G_{1,3}^{\mathrm{edge}} &= -\frac{\sqrt{3}\sqrt{35089\left(80542+29121\sqrt{3}\right)}}{140356} \\
G_{1,4}^{\mathrm{edge}} &= \frac{\sqrt{35089\left(80542+29121\sqrt{3}\right)}}{140356}
\end{aligned}
$$

For $k \geq N$ the entries are given by $H_{k,l}^{\mathrm{edge}} = h_{l-2k}/\sqrt{2}$ and $G_{k,l}^{\mathrm{edge}} = g_{l-2k}/\sqrt{2}$.

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