# MacMahon's Partition Analysis III: The Omega Package

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**Abstract**

In his famous book "Combinatory Analysis" MacMahon introduced Partition Analysis ("Omega Calculus") as a computational method for solving problems in connection with linear homogeneous diophantine inequalities and equations, respectively. The objective of this paper is to show that partition analysis is ideally suited for being supplemented by modern computer algebra methods. We developed the computer algebra package `Omega` which implements various aspects of MacMahon's ideas. In addition to an introduction to basic facts of "Omega Calculus", we present a variety of applications that illustrate the usage of the package.

## 1 Introduction

Recall the beautiful refinement of Euler's classic result [1, p. 5] that was discovered by M. Bousquet-Mélou and K. Eriksson [5] only recently:

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**Theorem 1 ("Lecture Hall Partition Theorem").** *The number of partitions of $n$ of the form $n = b_j + b_{j-1} + \cdots + b_1$ wherein*

$$\frac{b_j}{j} \geq \frac{b_{j-1}}{j-1} \geq \cdots \geq b_1 \geq 0$$

*and*

$$b_j - b_{j-1} + \cdots + (-1)^{j-1} b_1 = m$$

*equals the number of partitions of $n$ into exactly $m$ odd parts each $\leq 2j-1$.*

In [5] Bousquet-Mélou and Eriksson gave two different proofs of this theorem, one using Bott's formula for the affine Coxeter group $\tilde{C}_n$, and one of bijective-combinatorial nature. In [2] the first named author presented a proof following an entirely different approach. Namely, an approach which is based on the observation that MacMahon's Partition Analysis, surveyed in [7, Vol. II, Sec. VIII, pp. 91–170], is perfectly tailored for theorems of this kind. In order to illustrate this point, recall the definition of MacMahon's Omega operator $\Omega_{\geqq}$.

**Definition 1.** The operator $\Omega_{\geqq}$ is given by

$$\underset{\geqq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r},$$

where the domain of the $A_{s_1,\ldots,s_r}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the $\lambda_i$ are restricted to annuli of the form $1-\epsilon < |\lambda_i| < 1+\epsilon$.

It is convenient to treat everything involved analytically rather than formally because the method relies on unique Laurent series representations of a variety of rational functions. For instance, if we were to proceed formally, then

$$\underset{\geqq}{\Omega} \sum_{n=0}^{\infty} q^n \lambda^n = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$$

while

$$-\underset{\geqq}{\Omega} \sum_{n=1}^{\infty} q^{-n} \lambda^{-n} = 0.$$

But if we allow formal application of the geometric series, then both initial expressions are

$$\frac{1}{1-\lambda q}.$$

More precisely, instead of fixing suitable annuli for convergence, we then would be forced to fix suitable algebraic domains — which, in addition, may vary from problem to problem. For instance, we then have to decide whether the rational function $1/(1-\lambda q)$ is interpreted as a formal power series from the ring $\mathbb{C}(q)[[\lambda]]$ or from $\mathbb{C}(q)[[\lambda^{-1}]]$.

We can ensure that Definition 1 is mathematically well-posed provided we require that functions to which we apply $\Omega_{\geqq}$ have no singularities in the $\lambda_i$ within the annuli $1-\epsilon < |\lambda_i| < 1+\epsilon$. While this suggests major problems in the abstract, it provides no difficulties whatsoever in practice because the only arguments of $\Omega_{\geqq}$ that ever arise are all of the form

$$\frac{P(x_1,\ldots,x_n;\lambda_1,\ldots,\lambda_r)}{\prod_{i=1}^{n}(1-x_i\,\lambda_1^{v_1(i)}\cdots\lambda_r^{v_r(i)})} \tag{1}$$

where $P$ is a Laurent polynomial in the $n+r$ variables and the $v_h(i)$ are integers not necessarily positive. As long as the $x_i$ are restricted to a small neighborhood of 0, say $|x_i| < \delta$ where $\delta$ is chosen sufficiently small that

$$|\delta\,(1 \pm \epsilon)^{v_1(i)+\cdots+v_r(i)}| < 1,$$

$1 \le i \le n$, then we are guaranteed that we have avoided any singularities inside the annuli that provide the domain for the $\lambda_i$.

Let us now consider the instance $j = 3$ of Theorem 1. Obviously, the coefficient of $x^m q^n$ in

$$\frac{1}{(1-qx)(1-q^3x)(1-q^5x)} \tag{2}$$

equals the number of partitions of $n$ into exactly $m$ odd parts each $\le 5$. On the other hand, the coefficient of $x^m q^n$ in

$$\underset{\geqq}{\Omega} \sum_{b_1,b_2,b_3 \ge 0} \lambda_1^{2b_3-3b_2} \lambda_2^{b_2-2b_1} x^{b_3-b_2+b_1} q^{b_1+b_2+b_3} \tag{3}$$

gives exactly the number of the desired lecture hall partitions for $j$ being fixed to 3. Note that due to the Omega operator $\Omega_{\geqq}$ only those partitions $b_1 + b_2 + b_3 = n$ are counted for which $2b_3 - 3b_2 \ge 0$ and $b_2 - 2b_1 \ge 0$. By geometric series expansion the triple sum can be brought into product form, which means that expression (3) can be rewritten as

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda_1^2 qx)(1-\frac{\lambda_2 q}{\lambda_1^3 x})(1-\frac{qx}{\lambda_2^2})}. \tag{4}$$

Therefore all what remains for proving the Lecture Hall Partition Theorem for $j = 3$ is to show equality of the generating function expressions (2) and (4). For doing so we need the following lemma.

**Lemma 1.** *For any integer $s \ge 0$,*

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda x)(1-\frac{y}{\lambda^s})} = \frac{1}{(1-x)(1-x^s y)}.$$

*Proof.* By geometric series expansion the left hand side equals

$$\underset{\geqq}{\Omega} \sum_{i,j \ge 0} \lambda^{i-sj} x^i y^j = \underset{\geqq}{\Omega} \sum_{j,k \ge 0} \lambda^k x^{sj+k} y^j,$$

where the summation parameter $i$ has been replaced by $i = sj + k$. But now $\Omega_{\geqq}$ sets $\lambda$ to 1 which completes the proof. $\square$

With this lemma in hand, the proof of "(2) = (4)" reduces to successive elimination of the $\Omega_{\geqq}$-parameters $\lambda_1$ and $\lambda_2$.

*Proof of the Lecture Hall Partition Theorem for $j = 3$.* Split (4) additively into two parts by applying partial fraction decomposition $1/((1-\lambda_1 z)(1+\lambda_1 z)) = 1/(2(1-\lambda_1 z)) + 1/(2(1+\lambda_1 z))$ to the term $1/(1-\lambda_1^2 qx)$. Then by using Lemma 1 eliminate from both summands the $\Omega_{\geqq}$-parameter $\lambda_1$. For the last step one observes that Lemma 1 can be applied again in order to eliminate $\lambda_2$; this way one arrives at (2). $\square$

Already this particular example suggests that MacMahon's approach is an ideal candidate for being supplemented by modern computer algebra methods. In Section 2 we explain how this can be achieved in a fairly general setting based on the "fundamental recurrence". In Sections 3.1 and 3.2 a description of the corresponding `Omega` package is given. This package is a collection of procedures that implement the method in the computer algebra system Mathematica. In addition to the introductory examples of how to use the package, in the remaining sections the reader finds further applications that illustrate the powerful combination of MacMahon's classical Partition Analysis with modern computer algebra tools.

In Section 3.3 we present an automatic solution to the problem of counting triangles with sides of integer size. Section 3.4 briefly discusses some aspects of using the `Omega` package in the context of systems of linear homogeneous diophantine equations. In Section 4 one finds further `Omega` applications of less elementary nature. Section 4.1 deals with MacMahon's problem of "solid partitions on a cube" which he was able to solve only after the introduction of 18 (!) case distinctions with hairy computations. With the `Omega` package in hand the same problem now finds a straightforward *automatic* solution. Section 4.2 deals with a problem originally raised by Hermite, but whose solution is immediate with the `Omega` package. Based on heuristics extracted from `Omega` computations, we embed Hermite's problem in a general setting which then is treated in a purely combinatorial manner. Finally, in Section 4.3 we introduce "$k$-gon partitions" which provide a general framework for the triangle problem from Section 3.3. In connection with various generating functions computed with the `Omega` package, we conclude with an open problem.

## 2 The Fundamental Recurrence

We have examined MacMahon's use of Partition Analysis with the object of providing a proper algorithmic setting for the problems that he considers. To this end we begin by listing his catalog [7, Vol. II, pp. 102–103] of fundamental evaluations of the Omega operator. Note that the elimination rule described by Lemma 1 is one of the entries in his list.

$$\underset{\geq}{\Omega}\frac{1}{(1-\lambda x)(1-\frac{y}{\lambda})}=\frac{1}{(1-x)(1-xy)},$$

$$\underset{\geq}{\Omega}\frac{1}{(1-\lambda x)(1-\frac{y}{\lambda})(1-\frac{z}{\lambda})}=\frac{1}{(1-x)(1-xy)(1-xz)},$$

$$\underset{\geq}{\Omega}\frac{1}{(1-\lambda x)(1-\lambda y)(1-\frac{z}{\lambda})}=\frac{1-xyz}{(1-x)(1-y)(1-xz)(1-yz)},$$

$$\underset{\geq}{\Omega}\frac{1}{(1-\lambda x)(1-\frac{y}{\lambda^2})}=\frac{1}{(1-x)(1-x^2y)},$$

$$\underset{\geq}{\Omega}\frac{1}{(1-\lambda^2 x)(1-\frac{y}{\lambda})}=\frac{1+xy}{(1-x)(1-xy^2)},$$

$$\underset{\geq}{\Omega}\frac{1}{(1-\lambda x)(1-\lambda y)(1-\frac{z}{\lambda^2})}=\frac{1+xyz-x^2yz-xy^2z}{(1-x)(1-y)(1-x^2z)(1-y^2z)},$$

$$\underset{\geq}{\Omega}\frac{1}{(1-\lambda x)(1-\frac{y}{\lambda^s})}=\frac{1}{(1-x)(1-x^sy)},$$

$$\underset{\geq}{\Omega}\frac{1}{(1-\lambda^s x)(1-\frac{y}{\lambda})}=\frac{1+xy\,\frac{1-y^{s-1}}{1-y}}{(1-x)(1-xy^s)},$$

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda^2 x)(1-\frac{y}{\lambda})(1-\frac{z}{\lambda})} = \frac{1+xy+xz+xyz}{(1-x)(1-xy^2)(1-xz^2)},$$
$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda^2 x)(1-\lambda y)(1-\frac{z}{\lambda})} = \frac{1+xz-xyz-xyz^2}{(1-x)(1-y)(1-yz)(1-xz^2)},$$
$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda x)(1-\lambda y)(1-\lambda z)(1-\frac{w}{\lambda})} = \frac{1-xyw-xzw-yzw+xyzw+xyzw^2}{(1-x)(1-y)(1-z)(1-xw)(1-yw)(1-zw)},$$
$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda x)(1-\lambda y)(1-\frac{z}{\lambda})(1-\frac{w}{\lambda})} = \frac{1-xyz-xyw-xyzw+xy^2zw+x^2yzw}{(1-x)(1-y)(1-xz)(1-xw)(1-yz)(1-yw)}.$$

These twelve formulas and the problems solved by MacMahon using these formulas make clear that we need a general algorithm for the evaluation of

$$\underset{\geqq}{\Omega} \frac{\lambda^a}{(1-x_1\lambda)(1-x_2\lambda)\cdots(1-x_n\lambda)(1-\frac{y_1}{\lambda})(1-\frac{y_2}{\lambda})\cdots(1-\frac{y_m}{\lambda})} \tag{5}$$

(where $m$ and $n$ are nonnegative integers and $a$ is any integer).

At first glance, this appears to be inadequate to cover MacMahon's twelve identities. Some of his identities have expressions of the form $(1-X\lambda^r)$ or $(1-Y\lambda^{-s})$ in the denominator. However these expressions are easily rendered as products that fit our example because

$$(1-X\lambda^r) = \prod_{j=0}^{r-1}(1-\rho^j X^{1/r}\lambda), \tag{6}$$
$$(1-Y\lambda^{-s}) = \prod_{j=0}^{s-1}(1-\sigma^j Y^{1/s}\lambda^{-1}),$$

where $\rho = e^{2\pi i/r}$ and $\sigma = e^{2\pi i/s}$.

Also one can envisage in the numerator of (5) a general Laurent polynomial in $\lambda$. However the linearity of $\Omega_{\geqq}$ shows that again our expression covers these cases as well.

Before we state our result, we must recall the homogeneous symmetric functions, denoted by $h_j(x_1, x_2, \ldots, x_n)$, which are given by

$$\sum_{j=0}^{\infty} h_j(x_1, x_2, \ldots, x_n)\, t^j = \frac{1}{(1-tx_1)(1-tx_2)\cdots(1-tx_n)}.$$

Finally we should dispose of the degenerate cases when either $n$ or $m$ is 0. In these cases, the effect of $\Omega_{\geqq}$ is immediate by inspection:

**Lemma 2.** *For any integer $a$,*

$$\underset{\geqq}{\Omega} \frac{\lambda^a}{(1-x_1\lambda)(1-x_2\lambda)\cdots(1-x_n\lambda)} = \underset{\geqq}{\Omega} \sum_{j=0}^{\infty} h_j(x_1, \ldots, x_n)\, \lambda^{a+j}$$
$$= \begin{cases} \dfrac{1}{(1-x_1)(1-x_2)\cdots(1-x_n)}, & \text{if } a \geq 0, \\ \dfrac{1}{(1-x_1)(1-x_2)\cdots(1-x_n)} - \displaystyle\sum_{j=0}^{-a-1} h_j(x_1, \ldots, x_n), & \text{if } a < 0. \end{cases}$$

Similarly, we have:

**Lemma 3.** *For any integer $a$,*

$$\underset{\geqq}{\Omega} \frac{\lambda^a}{\left(1-\frac{y_1}{\lambda}\right)\left(1-\frac{y_2}{\lambda}\right)\cdots\left(1-\frac{y_m}{\lambda}\right)} = \underset{\geqq}{\Omega} \sum_{j=0}^{\infty} h_j(y_1,\ldots,y_m)\,\lambda^{a-j}$$
$$= \begin{cases} 0, & \text{if } a<0, \\ \sum_{j=0}^{a} h_j(y_1,\ldots,y_m), & \text{if } a\geq 0. \end{cases}$$

The main recurrence for the $\Omega_{\geqq}$-calculus reads as follows.

**Theorem 2 ("Fundamental Recurrence").** *For $n$ and $m$ positive integers and $a$ any integer,*

$$\underset{\geqq}{\Omega} \frac{\lambda^a}{(1-x_1\lambda)(1-x_2\lambda)\cdots(1-x_n\lambda)\left(1-\frac{y_1}{\lambda}\right)\left(1-\frac{y_2}{\lambda}\right)\cdots\left(1-\frac{y_m}{\lambda}\right)}$$
$$= \frac{P_{n,m,a}(x_1,\ldots,x_n;y_1,\ldots,y_m)}{\prod_{i=1}^{n}(1-x_i)\prod_{i=1}^{n}\prod_{j=1}^{m}(1-x_iy_j)},$$

*where for $n>1$,*

$$P_{n,m,a}(x_1,\ldots,x_n;y_1,\ldots,y_m)$$
$$= \frac{1}{x_n-x_{n-1}}\Big\{x_n(1-x_{n-1})\prod_{j=1}^{m}(1-x_{n-1}y_j)\,P_{n-1,m,a}(x_1,\ldots,x_{n-2},x_n;y_1,\ldots,y_m)$$
$$-x_{n-1}(1-x_n)\prod_{j=1}^{m}(1-x_ny_j)\,P_{n-1,m,a}(x_1,\ldots,x_{n-2},x_{n-1};y_1,\ldots,y_m)\Big\}$$

*and for $n=1$,*

$$P_{1,m,a}(x_1;y_1,\ldots,y_m)$$
$$= \begin{cases} x_1^{-a}, & \text{if } a\leq 0, \\ x_1^{-a}+\prod_{j=1}^{m}(1-x_1y_j)\sum_{j=0}^{a}h_j(y_1,\ldots,y_m)(1-x_1^{j-a}), & \text{if } a>0. \end{cases}$$

*Proof.* The proof of the main recurrence follows simply from the fact that

$$\frac{1}{(1-x_n\lambda)(1-x_{n-1}\lambda)} = \frac{1}{x_n-x_{n-1}}\left(\frac{x_n}{1-x_n\lambda}-\frac{x_{n-1}}{1-x_{n-1}\lambda}\right).$$

It is sufficient to carry out the proof under the assumption $x_i\neq x_j$ $(i\neq j)$, because the general case is an immediate consequence of the following elementary fact: if $T(x_1,\ldots,x_n;\lambda_1,\ldots,\lambda_r)$ is a term of the form as in (1) then

$$\underset{\geqq}{\Omega}\lim_{x_i\to x_j} T(x_1,\ldots,x_n;\lambda_1,\ldots,\lambda_r) = \lim_{x_i\to x_j}\underset{\geqq}{\Omega}\, T(x_1,\ldots,x_n;\lambda_1,\ldots,\lambda_r).$$

(The limit is understood to be taken within the corresponding domain of convergence.)

Hence for $n > 1$

$$
\begin{aligned}
&\Omega_{\geqq} \frac{\lambda^a}{(1-x_1\lambda)(1-x_2\lambda)\cdots(1-x_n\lambda)\left(1-\frac{y_1}{\lambda}\right)\left(1-\frac{y_2}{\lambda}\right)\cdots\left(1-\frac{y_m}{\lambda}\right)}\\
&\quad = \Omega_{\geqq} \frac{x_n\lambda^a}{(x_n-x_{n-1})(1-x_1\lambda)\cdots(1-x_{n-2}\lambda)(1-x_n\lambda)\left(1-\frac{y_1}{\lambda}\right)\left(1-\frac{y_2}{\lambda}\right)\cdots\left(1-\frac{y_m}{\lambda}\right)}\\
&\quad - \Omega_{\geqq} \frac{x_{n-1}\lambda^a}{(x_n-x_{n-1})(1-x_1\lambda)\cdots(1-x_{n-2}\lambda)(1-x_{n-1}\lambda)\left(1-\frac{y_1}{\lambda}\right)\left(1-\frac{y_2}{\lambda}\right)\cdots\left(1-\frac{y_m}{\lambda}\right)}.
\end{aligned}
$$

This is exactly the main recurrence once the expressions involving $P_{n,m,a}$ have been substituted and the left denominator cleared.

For the $n = 1$ case, we see that

$$
\begin{aligned}
\frac{P_{1,m,a}(x_1;y_1,\ldots,y_m)}{(1-x_1)(1-x_1y_1)\cdots(1-x_1y_m)} &= \Omega_{\geqq} \frac{\lambda^a}{(1-x_1\lambda)\left(1-\frac{y_1}{\lambda}\right)\left(1-\frac{y_2}{\lambda}\right)\cdots\left(1-\frac{y_m}{\lambda}\right)}\\
&= \Omega_{\geqq} \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} h_j(y_1,\ldots,y_m)\, x_1^n \lambda^{n+a-j}.
\end{aligned}
$$

Now if $a \le 0$, then this expression is

$$
\begin{aligned}
\sum_{j=0}^{\infty}\sum_{n=j-a}^{\infty} h_j(y_1,\ldots,y_m)\, x_1^n &= \frac{\sum_{j=0}^{\infty} h_j(y_1,\ldots,y_m)\, x_1^{j-a}}{1-x_1}\\
&= \frac{x_1^{-a}}{(1-x_1)(1-x_1y_1)\cdots(1-x_1y_m)},
\end{aligned}
$$

and so for $a \le 0$

$$
P_{1,m,a}(x_1;y_1,\ldots,y_m) = x_1^{-a}.
$$

If $a > 0$, then we have

$$
\begin{aligned}
&\Omega_{\geqq} \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} h_j(y_1,\ldots,y_m)\, x_1^n \lambda^{n+a-j}\\
&= \Omega_{\geqq} \sum_{n=0}^{\infty}\Big(\sum_{j=0}^{a} + \sum_{j=a+1}^{\infty}\Big) h_j(y_1,\ldots,y_m)\, x_1^n \lambda^{n+a-j}\\
&= \frac{\sum_{j=0}^{a} h_j(y_1,\ldots,y_m)}{1-x_1} + \sum_{j=a+1}^{\infty}\sum_{n=j-a}^{\infty} h_j(y_1,\ldots,y_m)\, x_1^n\\
&= \frac{\sum_{j=0}^{a} h_j(y_1,\ldots,y_m)}{1-x_1} + \frac{\sum_{j=a+1}^{\infty} h_j(y_1,\ldots,y_m)\, x_1^{j-a}}{1-x_1}\\
&= \frac{\sum_{j=0}^{a} h_j(y_1,\ldots,y_m)}{1-x_1} + \frac{x_1^{-a}}{(1-x_1)(1-x_1y_1)\cdots(1-x_1y_m)} - \frac{\sum_{j=0}^{a} h_j(y_1,\ldots,y_m)\, x_1^{j-a}}{1-x_1}\\
&= \frac{1}{1-x_1}\sum_{j=0}^{a} h_j(y_1,\ldots,y_m)(1-x_1^{j-a}) + \frac{x_1^{-a}}{(1-x_1)(1-x_1y_1)\cdots(1-x_1y_m)}\\
&= \frac{x_1^{-a} + \prod_{j=1}^{m}(1-x_1y_j)\sum_{j=0}^{a} h_j(y_1,\ldots,y_m)(1-x_1^{j-a})}{(1-x_1)(1-x_1y_1)\cdots(1-x_1y_m)},
\end{aligned}
$$

which gives the desired formula for $P_{1,m,a}$. □

# 3 The Mathematica Implementation

The object of this section is to describe the usage of the `Omega` package which has been implemented by the third author. In order to illustrate how the package is used in practice, a few tutorial examples are given. In Section 4 the reader finds further applications of less elementary nature.

## 3.1 The `Omega` package

The package consists of the Mathematica file `Omega.m` and the small help file `Readme.txt`; both can be downloaded from the `Omega` homepage at

`http://www.risc.uni-linz.ac.at/research/combinat/risc/software/Omega/`

After loading the package with `<<Omega.m`, the function `OR` (for `O`mega `R`ule) is provided. According to the fundamental recurrence given in Theorem 2 it applies the Omega operator with respect to a certain variable, say $\lambda$, to an expression. The calling syntax is

$$\texttt{OR[}expr\texttt{, }\lambda\texttt{]}$$

where $\lambda$ is a variable (i.e., a Mathematica symbol) and *expr* is a rational function of the form

$$\frac{L(\lambda)}{(1 \pm x_1\lambda^{c_1})\cdots(1 \pm x_n\lambda^{c_n})(1 \pm y_1/\lambda^{d_1})\cdots(1 \pm y_m/\lambda^{d_m})}$$

with

| | |
|---|---|
| $L(\lambda)$ | a Laurent polynomial of $\lambda$ over $\mathbb{Q}(\kappa_1, \ldots, \kappa_l)$, where the $\kappa_i$ are indeterminates different from $\lambda$, |
| $x_i, y_j$ | power products (with integral exponents) in $\kappa_1, \ldots, \kappa_l$, |
| $c_i, d_j$ | positive integers. |

For involved applications it turns out that the numerator sometimes gets so complicated that it cannot be factored by Mathematica in reasonable time. In this case, calling

$$\texttt{OR[}expr\texttt{, }\lambda\texttt{, FactorProc->FactorSquareFree]}$$

performs only square free factorization. If this still does not solve the problem, factorization can be reduced once more by calling

$$\texttt{OR[}expr\texttt{, }\lambda\texttt{, FactorProc->None].}$$

However note that in both cases the numerator and denominator of the result might contain common factors.

From the programmer's point of view it is worth remarking that the main difficulties concerning the implementation were caused by decomposition (6), since Mathematica is not able to handle roots of unity efficiently. For instance, consider the factorization of $1 - x\lambda^5$ into $\prod_{i=0}^{4}(1 - x^{(i)}\lambda)$, where the $x^{(i)}$ denote the fifth roots of $x$. Then the only way to reconstruct the original term $1 - x\lambda^5$ from this decomposition is to apply the Mathematica functions `Expand` and `FullSimplify`, which results in an incredibly bad runtime behavior even for simple applications. We could finally overcome this problem by observing that after evaluating such polynomials numerically, the imaginary parts of the coefficients vanish immediately and the remaining real (integer!) parts can be reestablished easily. We have been studying this problem carefully and are hopeful that recent algebraic discoveries will mitigate our problems with roots of unity.

### 3.2 How to use the Omega package in practice

We put the file **Omega.m** (together with the file **Readme.txt**) in the same directory in which we run our Mathematica session. After invoking Mathematica we load the package,

```
In[1]:= <<Omega.m
Axel Riese's Omega implementation version 1.3 loaded
```

Now the proof of Theorem 1 for $j = 3$ can simply be done as follows. First we input the expression the $\Omega_{\geqq}$ operator acts on; see (4). Then we call the OR-procedure to eliminate the $\lambda$-variables `l1` and `l2`:

```
In[2]:= f = 1 / ((1-l1^2 q x)(1-l2 q/(l1^3 x))(1-q x/l2^2))

                              1
Out[2]= -----------------------------------
              l2 q          2           q x
        (1 - -----) (1 - l1  q x) (1 - ---)
                3                        2
              l1  x                    l2

In[3]:= OR[f, l1]

                                 3
                     1 + l2 q  x
Out[3]= ----------------------------------
                          q x         2  5
        (1 - q x) (1 - ---) (1 - l2  q  x)
                          2
                        l2

In[4]:= OR[%, l2]

                          1
Out[4]= -------------------------------
                          3           5
        (1 - q x) (1 - q  x) (1 - q  x)
```

This proves the equality in question.

Alternatively, we can prove "(2) = (4)" by reversing the order of elimination:

```
In[5]:= OR[f, l2]

                               1
Out[5]= -------------------------------------
                               3
                q           q              2
        (1 - -----) (1 - -----) (1 - l1  q x)
                3           6
              l1  x       l1  x
```

```
In[6]:= OR[%, l1]

                             1
Out[6]= ------------------------------
                              3          5
        (1 - q x) (1 - q  x) (1 - q  x)
```

This means, if we reverse the order in which the $\lambda$-variables are eliminated, we obtain a different intermediate result. This fact is of particular importance in more involved situations. In other words, in applications where we need to eliminate several variables a certain order of elimination might turn out to be optimal with respect to running time. See, for instance, the generating function for solid partitions on a cube in Section 4.1.

## 3.3 Triangles with sides of integer size

As another illustration of the use of the `Omega` package let us consider an exercise from Stanley's book [9, Ch. 4, Ex. 16]; see also [3]:

**Problem 1.** *Let $t(n)$ be the number of non-congruent triangles whose sides have integer length and whose perimeter is $n$. For instance, $t(9) = 3$, corresponding to $3+3+3$, $2+3+4$, $1+4+4$. Find $\sum_{n\geq 3} t(n)x^n$.*

In other words, we have to count all positive integer triples $(a, b, c)$ satisfying $a \leq b \leq c$, $a + b > c$, and $a + b + c = n$. This number is the coefficient of $x^n$ in

$$\underset{\geqq}{\Omega} \sum_{a,b,c\geq 1} \lambda_1^{b-a}\lambda_2^{c-b}\lambda_3^{a+b-c-1}x^{a+b+c} = \underset{\geqq}{\Omega} \frac{x^3}{\left(1 - x\frac{\lambda_3}{\lambda_1}\right)\left(1 - x\frac{\lambda_1\lambda_3}{\lambda_2}\right)\left(1 - x\frac{\lambda_2}{\lambda_3}\right)},$$

and the corresponding elimination of the $\lambda_i$'s is performed by the `Omega` package as follows:

```
In[2]:= t = x^3 / ((1-x l3/l1)(1-x l1 l3/l2)(1-x l2/l3))

                              3
                             x
Out[2]= ----------------------------------
             l2 x        l3 x       l1 l3 x
        (1 - ----) (1 - ----) (1 - -------)
             l3          l1          l2

In[3]:= OR[t, l1]

                              3
                             x
Out[3]= --------------------------------
                                     2  2
             l2 x        l3 x       l3  x
        (1 - ----) (1 - ----) (1 - ------)
             l3          l2          l2

In[4]:= OR[%, l2]
```

```
                  3
                 x
Out[4]= -----------------------------
             x          2           3
        (1 - --) (1 - x ) (1 - l3 x )
             l3
```

```
In[5]:= OR[%, l3]
```

```
                 3
                x
Out[5]= --------------------------
              2         3         4
        (1 - x ) (1 - x ) (1 - x )
```

From this a closed form representation of $t(n)$ can be extracted easily.

But more can be done by taking the following refinement of the generating function:

$$\underset{\geqq}{\Omega} \sum_{a,b,c\geq 1} \lambda_1^{b-a}\lambda_2^{c-b}\lambda_3^{a+b-c-1} x_1^a x_2^b x_3^c = \underset{\geqq}{\Omega} \frac{x_1 x_2 x_3}{\left(1 - x_1 \frac{\lambda_3}{\lambda_1}\right)\left(1 - x_2 \frac{\lambda_1\lambda_3}{\lambda_2}\right)\left(1 - x_3 \frac{\lambda_2}{\lambda_3}\right)}. \tag{7}$$

```
In[6]:= T = x1 x2 x3 / ((1-x1 l3/l1)(1-x2 l1 l3/l2)(1-x3 l2/l3))
```

```
                    x1 x2 x3
Out[6]= --------------------------------------
             l3 x1       l1 l3 x2       l2 x3
        (1 - -----) (1 - --------) (1 - -----)
              l1            l2            l3
```

```
In[7]:= OR[T, l1]
```

```
                    x1 x2 x3
Out[7]= ---------------------------------------
                            2
             l3 x2       l3  x1 x2       l2 x3
        (1 - -----) (1 - ---------) (1 - -----)
              l2            l2             l3
```

```
In[8]:= OR[%, l2]
```

```
                    x1 x2 x3
Out[8]= --------------------------------------
             x3
        (1 - --) (1 - x2 x3) (1 - l3 x1 x2 x3)
             l3
```

```
In[9]:= OR[%, l3]
```

```
                      x1 x2 x3
Out[9]= ------------------------------------------
```

```
                                              2
        (1 - x2 x3) (1 - x1 x2 x3) (1 - x1 x2 x3 )
```

This way we computed a parameterized representation of all positive integer triples $(a, b, c)$ satisfying the diophantine inequalities $a \leq b \leq c$ and $a + b > c$. Namely, by geometric series expansion in summation parameters $\alpha$, $\beta$ and $\gamma$, we can directly extract from the generating function that

$$a = \beta + \gamma + 1, \quad b = \alpha + \beta + \gamma + 1, \quad c = \alpha + \beta + 2\gamma + 1,$$

where $\alpha$, $\beta$ and $\gamma$ run through all nonnegative integers. Then the triples counted by $t(n)$ are those with $2\alpha + 3\beta + 4\gamma + 3 = n$. (See also [9, Ch. 4, Solution to Ex. 16] and [3].)

## 3.4 Linear homogeneous diophantine equations

For a final introductory application we want to have a look at linear homogeneous diophantine *equations*. Also this problem area has been studied by MacMahon extensively. To this end MacMahon defined the following variant of the Omega operator.

**Definition 2.** The operator $\Omega_=$ is given by

$$\underset{=}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := A_{0,\ldots,0}.$$

This means, all nontrivial power-products in the $\lambda$'s are killed by the $\Omega_=$-operator.

As already pointed out by MacMahon [7, Vol. II, Sec. VIII, p. 104], this operator is related to $\Omega_{\geqq}$, for instance, as follows:

$$\underset{=}{\Omega} F(\lambda) = \underset{\geqq}{\Omega} F(\lambda) + \underset{\geqq}{\Omega} F(1/\lambda) - F(1). \tag{8}$$

We use exactly this relation in order to find a parameterized representation of all tuples $(a_1, a_2, a_3, a_4)$ of nonnegative integers satisfying $a_1 + a_2 - a_3 - a_4 = 0$. (See Stanley [9, Ch. 4, Example 4.6.15].) Equivalent to this is the computation of the corresponding generating function, i.e.,

$$\underset{=}{\Omega} \sum_{a_1,a_2,a_3,a_4 \geq 0} \lambda^{a_1+a_2-a_3-a_4} x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} = \underset{=}{\Omega} \frac{1}{(1-x_1\lambda)(1-x_2\lambda)\left(1-\frac{x_3}{\lambda}\right)\left(1-\frac{x_4}{\lambda}\right)}.$$

Using (8) the elimination is done as follows:

```
In[2]:= f = 1 / ((1-x1 l)(1-x2 l)(1-x3/l)(1-x4/l))

                               1
Out[2]= ----------------------------------------
                                   x3        x4
        (1 - l x1) (1 - l x2) (1 - --) (1 - --)
                                   l         l

In[3]:= Factor[ OR[f, l] + OR[f /. l->1/l, l] - (f /. l->1) ]

                           -1 + x1 x2 x3 x4
Out[3]= -(---------------------------------------------------)
          (-1 + x1 x3) (-1 + x2 x3) (-1 + x1 x4) (-1 + x2 x4)
```

As another example from Stanley's book [9, Ch. 4, Prop. 4.6.21], let us count the number $S_3(r)$ of $3 \times 3$ *symmetric* matrices with nonnegative integer entries such that every row (and column) sum equals $r$. If $(a_1, a_2, a_3)$ stands for the first, $(a_2, a_4, a_5)$ for the second, and $(a_3, a_5, a_6)$ for the third row, then the corresponding linear system of homogeneous equations is

$$a_1 + a_2 + a_3 - r = 0,\ a_2 + a_4 + a_5 - r = 0,\ \text{and } a_3 + a_5 + a_6 - r = 0.$$

The solution in generating function form can be written down immediately by means of the $\Omega_=$-operator,

$$\begin{aligned}&\underset{=}{\Omega} \sum_{a_1,a_2,a_3,a_4,a_5,a_6,r \geq 0} \lambda_1^{a_1+a_2+a_3-r} \lambda_2^{a_2+a_4+a_5-r} \lambda_3^{a_3+a_5+a_6-r} x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} x_5^{a_5} x_6^{a_6} y^r \\ &\quad = \underset{=}{\Omega} \frac{1}{(1 - x_1\lambda_1)(1 - x_2\lambda_1\lambda_2)(1 - x_3\lambda_1\lambda_3)(1 - x_4\lambda_2)(1 - x_5\lambda_2\lambda_3)(1 - x_6\lambda_3)\left(1 - \frac{y}{\lambda_1\lambda_2\lambda_3}\right)}.\end{aligned}$$

For the elimination we use the procedure "`OEqR`" that encodes relation (8):

```
In[4]:= SymMagicSquares = 1 / ((1-l1 x1)(1-l1 l2 x2) *
        (1-l1 l3 x3)(1-l2 x4)(1-l2 l3 x5)(1-l3 x6)(1-y/(l1 l2 l3)))

Out[4]= 1 / ((1 - l1 x1) (1 - l1 l2 x2) (1 - l1 l3 x3) (1 - l2 x4)

                                                y
>        (1 - l2 l3 x5) (1 - l3 x6) (1 - --------))
                                             l1 l2 l3

In[5]:= OEqR[SymMagicSquares, l1, {x1,x2,x3,x4,x5,x6,y}]

(* The third argument in the OEqR-expression consists of a list
   of all those variables that are treated as constants in the
   course of the elimination. *)

Out[5]=

                                         1
> ---------------------------------------------------------------------
                                                x1 y        x2 y       x3 y
  (1 - l2 x4) (1 - l2 l3 x5) (1 - l3 x6) (1 - -----) (1 - ----) (1 - ----)
                                                l2 l3        l3         l2

In[6]:= OEqR[%, l2, {x1,x2,x3,x4,x5,x6,y}]

                                2
Out[6]= -((-1 + x1 x3 x4 x5 y ) /

                         x2 y       x1 x4 y
>       ((1 - l3 x6) (1 - ----) (1 - -------) (1 - x3 x4 y) (1 - x1 x5 y)
                          l3          l3

>         (1 - l3 x3 x5 y)))
```

```
In[7]:= OEqR[%, l3, {x1,x2,x3,x4,x5,x6,y}]

                                            3
Out[7]= -((-1 + x1 x2 x3 x4 x5 x6 y ) /

>       ((1 - x3 x4 y) (1 - x1 x5 y) (1 - x2 x6 y) (1 - x1 x4 x6 y)

                              2
>         (1 - x2 x3 x5 y )))

In[8]:= % /. {x1->1,x2->1,x3->1,x4->1,x5->1,x6->1}

                        3
                  -1 + y
Out[8]= -(-----------------)
                   4       2
           (1 - y)  (1 - y )
```

Obviously, $S_3(r)$ is the coefficient of $y^r$ after setting all the $x_i$'s to 1 in the computed $\lambda$-free generating function expression. This specialization, `Out[8]`, can be found in [9, Ch. 4, after Prop. 4.6.21].

Finally, we remark that there are more efficient ways than using (8) to implement the $\Omega_=$-operator. For more details we refer the interested reader to the forthcoming article [4].

# 4 Further Applications

In this section we present further `Omega` applications that are of less elementary nature.

## 4.1 Solid partitions on a cube

MacMahon devoted *Art. 98* of [6, Sec. 7] to the consideration of the simplest "lattice in solido"; namely, the lattice "in which the points are the summits of a cube and the branches the edges of the cube." In other words, following MacMahon let us put nonnegative integer weights $a_i$ $(1 \leq i \leq 8)$ on the vertices of a cube as described by Figure 1 below.

The vertices of the cube in Figure 1 are connected by *directed* edges that are interpreted as inequalities. For instance, the directed edge from the vertex with weight $a_1$ to the vertex with weight $a_2$ corresponds to the inequality relation $a_1 \geq a_2$, and so on. This way we have introduced 12 diophantine inequalities to which we relate the generating function

$$\sum q^{a_1+\cdots+a_8},$$

where the sum runs over all nonnegative integer tuples $(a_1, \ldots, a_8)$ with entries satisfying the 12 inequalities induced by the cube.

It is easy to check that this generating function is nothing but the $\Omega_{\geqq}$-operator applied (with respect to the `l`-variables) to the expression

```
In[2]:= f = 1 / ((1-q l1 l2 l3)(1-q l4 l5/l1)(1-q l6 l7/l2) *
            (1-q l8/(l4 l6))(1-q l9 l10/l3)(1-q l11/(l5 l9)) *
            (1-q l12/(l7 l10))(1-q/(l8 l11 l12)));
```

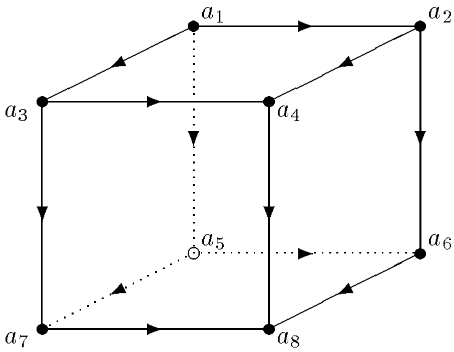


Figure 1: MacMahon's solid cube

from which we first eliminate `l12`:

```
In[3]:= OR[f, l12]
                                    l4 l5 q        q          l6 l7 q
Out[3]= 1 / ((1 - l1 l2 l3 q) (1 - -------) (1 - ------) (1 - -------)
                                      l1         l10 l7         l2

                                                              2
            l8 q        l11 q       l10 l9 q                 q
>      (1 - -----) (1 - -----) (1 - --------) (1 - -------------))
            l4 l6       l5 l9          l3           l10 l11 l7 l8
```

Now one proceeds by eliminating the remaining `l`-variables in some convenient order, for instance, in the order `(l11,l10,l1,l8,l7,l5,l9,l6,l3)`.

This way one keeps the numerator equal to 1 until `OR[%, l9]`. For readers who want to use the `Omega` package in practice we point to the important fact that other orders of elimination might result in (much) more involved numerator polynomials that might slow down the computation tremendously. A good heuristic principle for finding a "good" order is to apply `OR` with respect to a $\lambda$-variable that occurs less frequently than others.

After eliminating `l3` the last two steps are:

```
In[13]:= OR[%, l4]

                      2    3    4          7          8
Out[13]= ((1 - q + q  - q  + q ) (-l2 + q ) (-l2 + q )

             2        2       4       4     5     7       7       9       10
>      (1 - q  + l2 q  + 2 q  - l2 q  + q  - q  - l2 q  - l2 q  - l2 q   -

            12      2  12      2  14       15       2  15       17
>       l2 q   - l2  q   + l2  q   - l2 q   + 2 l2  q   + l2 q   -

          2  17      2  19
>       l2  q   + l2  q   )) /
```

```
        2              q                           2           3         4
>    (12  (1 - q) (1 - --) (1 - 12 q) (1 - 12 q ) (1 - 12 q ) (1 - q )
                       12

                                                              7         8
          4            4          6          7          q         q          10
>      (1 + q ) (1 - 12 q ) (1 - q ) (1 - q ) (1 - --) (1 - --) (1 - q   ))
                                                             12        12

In[14]:= Factor[ OR[%, 12] ]

                     2       3       4       5       6       7       8       9
Out[14]= (1 + 2 q  + 2 q  + 3 q  + 3 q  + 5 q  + 4 q  + 8 q  + 4 q  +

            10       11       12       13       14     16
>      5 q   + 3 q   + 3 q   + 2 q   + 2 q   + q  ) /

                8         4        2 2           2            2 2       4
>    ((-1 + q)  (1 + q)  (1 + q )  (1 - q + q ) (1 + q + q )  (1 + q )

                    2    3    4             2    3    4    5    6
>      (1 + q + q  + q  + q ) (1 + q + q  + q  + q  + q  + q ))
```

Finally it can be easily verified that the denominator expression is equal to

$$(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)(1-q^7)(1-q^8).$$

This way we computed the generating function $\sum q^{a_1+\cdots+a_8}$ in purely routine manner — and in few seconds of computation time.

At the times of MacMahon the situation was quite a different one. It is instructive to compare how he managed to solve the $\Omega_{\geqq}$-elimination problem. Namely, MacMahon divided the calculation into eighteen (!) parts (according to all possible inequality relations between weights corresponding to non-adjacent vertices [6, Art. 98]) whose sum then gives the desired generating function. We conclude this section with the footnote added by MacMahon after the description of his computation, "Mr. A.B. KEMPE, Treas. R.S., has verified this conclusion by a different and most ingenious method of summation, which also readily yields the result for any desired restriction on the part-magnitude."

## 4.2 A problem of Hermite

In their famous book [8, Ex. 31], Pólya and Szegö posed the following problem:

**Problem 2.** *For an integer $n$ greater than 2, let $h(n)$ be the number of positive integer triples $(a,b,c)$ such that $a+b+c=n$ and $a \le b+c$, $b \le a+c$, and $c \le a+b$. Show that $h(n)=(n+8)(n-2)/8$, if $n$ is even, and $h(n)=(n^2-1)/8$, if $n$ is odd.*

Pólya and Szegö remark that the problem originally was posed by Ch. Hermite in 1868 and solved by V. Schlegel in 1869. (In [8, Solution to Ex. 31] the exact references can be found.)

For the first case, $n = 3$, it is obvious that $(1,1,1)$ is the only solution, thus $h(3) = 1$. If $n = 4$ we have three solutions, namely $(1,1,2)$, $(1,2,1)$, and $(2,1,1)$; this means, $h(4) = 3$. The general case can be settled *automatically* with the **Omega** package as follows.

First we encode the situation in form of a generating function the $\Omega_{\geqq}$ operator acts on,

$$\sum_{n=3}^{\infty} h(n)\, q^n = \underset{\geqq}{\Omega} \sum_{a,b,c \geq 1} \lambda_1^{b+c-a} \lambda_2^{a+c-b} \lambda_3^{a+b-c} q^{a+b+c} = \underset{\geqq}{\Omega} \frac{\lambda_1 \lambda_2 \lambda_3\, q^3}{\left(1 - \frac{\lambda_2 \lambda_3 q}{\lambda_1}\right)\left(1 - \frac{\lambda_1 \lambda_3 q}{\lambda_2}\right)\left(1 - \frac{\lambda_1 \lambda_2 q}{\lambda_3}\right)}.$$

Then we successively eliminate the $\lambda$-variables:

```
In[2]:= hfu = l1 l2 l3 q^3 / ((1-l2 l3 q/l1)(1-l1 l3 q/l2)(1-l1 l2 q/l3))

                                  3
                      l1 l2 l3 q
Out[2]= -----------------------------------------
              l1 l2 q        l1 l3 q        l2 l3 q
        (1 - -------) (1 - -------) (1 - -------)
                l3             l2             l1

In[3]:= OR[hfu, l1]

                    3                    2  2      2  2
          l2 l3 q  (-1 - l2 l3 q + l2   q  + l3   q )
Out[3]= -(------------------------------------------------)
            l2 q        l3 q          2  2          2  2
          (1 - ----) (1 - ----) (1 - l2   q ) (1 - l3   q )
             l3          l2

In[4]:= OR[%, l2]

             3                2
        l3 q  (1 + 2 l3 q - q )
Out[4]= ------------------------------
          q          2          2  2
        (1 - --) (1 - q ) (1 - l3   q )
             l3

In[5]:= OR[%, l3]

          3               2
         q  (-1 - 2 q + 2 q )
Out[5]= -(---------------------)
                         2 2
          (1 - q) (1 - q  )
```

Finally, from the following partial fraction representation of `Out[5]` the explicit formulae for $h(n)$ can be read off almost directly,

$$\sum_{n \geq 3} h(n)\, q^n = \frac{q^3(1 + 2q - 2q^2)}{(1-q)(1-q^2)^2} = \frac{q^3}{(1-q^2)^3} + \frac{q^4}{(1-q^2)^3} + \frac{2q^4}{(1-q^2)^2}.$$

For the rest of this section we consider a natural generalization of Hermite's problem. For instance, as the next case we consider positive integer tuples $(a_1, a_2, a_3, a_4)$ such that $a_1 + a_2 + a_3 + a_4 = n$ where each $a_i$ is less or equal the sum of the others; or equivalently, where we have $a_i \le n - a_i$ for all $i \in \{1, 2, 3, 4\}$.

**Definition 3.** As the set of "compositions of $n$ into $k$ positive parts" we define

$$C_k(n) := \{(a_1, \ldots, a_k) \in \mathbb{Z}^k \mid a_i \ge 1 \text{ for all } i, \text{ and } a_1 + \cdots + a_k = n\}.$$

As the set of "$k$-gon compositions of $n$ into positive parts" we define

$$H_k(n) := \{(a_1, \ldots, a_k) \in C_k(n) \mid a_i \le n - a_i \text{ for all } i\}.$$

Note that $h(n) = |H_3(n)|$. With the **Omega** package in hand it is as easy as in the case $k = 3$ to compute the generating functions of $|H_k(n)|$ for the next higher values. For instance, for $k = 4$ and $k = 5$ one obtains

$$\sum_{n \ge 4} |H_4(n)|\, q^n = \frac{q^4}{(1-q)^4} - 4\frac{q^7}{(1-q)^4(1+q)^3}$$

and

$$\sum_{n \ge 5} |H_5(n)|\, q^n = \frac{q^5}{(1-q)^5} - 5\frac{q^9}{(1-q)^5(1+q)^4}.$$

We leave the verification as an **Omega** exercise.

If one brings the generating function for $|H_3(n)|$ into the analogous form, i.e.,

$$\sum_{n \ge 3} |H_3(n)|\, q^n = \sum_{n \ge 3} h(n)\, q^n = \frac{q^3}{(1-q)^3} - 3\frac{q^5}{(1-q)^3(1+q)^2},$$

the underlying pattern already becomes obvious. We state the resulting conjecture as a theorem, because — once discovered — its proof causes no further difficulty.

**Theorem 3.** *Let $k$ be an integer greater or equal to 3, then*

$$\sum_{n \ge k} |H_k(n)|\, q^n = \frac{q^k}{(1-q)^k} - k\frac{q^{2k-1}}{(1-q)^k(1+q)^{k-1}}.$$

We prove the statement combinatorially; to this end we need a bit of preparatory work.

The first quotient is nothing but the generating function for compositions, i.e.,

$$\sum_{n \ge k} |C_k(n)|\, q^n = \left(\frac{q}{1-q}\right)^k. \tag{9}$$

For the combinatorial interpretation of the second quotient we introduce a suitable composition set.

**Definition 4.** For integers $n \ge 3$ and $k \ge 2$ the set of compositions of $n$ into $k$ positive parts where at most one part is odd can be described by the set

$$O_k(n) := \{(e_1, \ldots, e_{k-1}, e_k) \in \mathbb{Z}^k \mid e_i \ge 1 \text{ for all } i, \text{ and } 2e_1 + \cdots + 2e_{k-1} + e_k = n\}.$$

Its generating function is very close to the second quotient.

**Lemma 4.** *For integers $k \geq 2$ and $n \geq 3$,*

$$\sum_{n \geq 2k-1} |O_k(n)|\, q^n = \frac{q^{2k-1}}{(1-q)^k (1+q)^{k-1}}.$$

*Proof.* The proof is obvious from

$$\sum_{n \geq 2k-1} |O_k(n)|\, q^n = \left(\frac{q^2}{1-q^2}\right)^{k-1} \cdot \frac{q}{1-q}. \qquad \square$$

For the proof of Theorem 3 it is also convenient to introduce the following lemma.

**Lemma 5.** *Let $(a_1, \ldots, a_k) \in C_k(n)$. If $a_i \geq n - a_i$ for some $i \in \{1, \ldots, k\}$ then $a_j < n - a_j$ for all $j \in \{1, \ldots, k\} \setminus \{i\}$.*

*Proof.* It suffices to prove the lemma for the case $i = k$. Suppose $a_j \geq n/2$ for some $j \in \{1, \ldots, k-1\}$. Then, since all parts are positive, we have $n = (a_1 + \cdots + a_{k-1}) + a_k > n/2 + n/2 = n$, a contradiction. $\square$

After this preparatory work we are ready for the desired proof.

*Proof of Theorem 3.* From Lemma 5 we have for each $(a_1, \ldots, a_k) \in C_k(n) \setminus H_k(n)$ that $a_i > n - a_i$ for some $i \in \{1, \ldots, k\}$ and $a_j < n - a_j$ for all other indices $j \neq i$. This induces the following partition of the set $C_k(n) \setminus H_k(n)$ into $k$ disjoint subsets, namely, $C_k(n) \setminus H_k(n) = \bigcup_{i \in \{1,\ldots,k\}} CH_k^{(i)}(n)$, where $CH_k^{(i)}(n) := \{(a_1, \ldots, a_k) \in C_k(n) \setminus H_k(n) \mid a_i > n - a_i\}$.

All these sets are of the same cardinality, i.e., for $i, j \in \{1, \ldots, k\}$ we have $|CH_k^{(i)}(n)| = |CH_k^{(j)}(n)|$. This is immediate from the fact that for $i < j$ the map

$$(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_k) \mapsto (a_1, \ldots, a_j, \ldots, a_i, \ldots, a_k)$$

is a bijection.

Hence, again by Lemma 5, we get that for all integers $k \geq 2$ and $n \geq 3$,

$$|C_k(n)| - |H_k(n)| = k\, |CH_k^{(k)}(n)|.$$

In view of (9) and of Lemma 4, the proof of Theorem 3 is completed once we are able to show that $|CH_k^{(k)}(n)| = |O_k(n)|$. To this end we define the map

$$\begin{aligned} \phi: \quad & CH_k^{(k)}(n) \to O_k(n), \\ & (a_1, \ldots, a_k) \mapsto (a_1, \ldots, a_{k-1}, 2a_k - n). \end{aligned}$$

The map $\phi$ is well-defined, since $2(a_1 + \cdots + a_{k-1}) + (2a_k - n) = n$ and $2a_k - n > 0$.

Obviously, $\phi$ is an injective map. Finally we show that $\phi$ is also surjective, i.e., given any $(e_1, \ldots, e_{k-1}, e_k) \in O_k(n)$ one can find $(a_1, \ldots, a_k) \in CH_k^{(k)}(n)$ such that $\phi(a_1, \ldots, a_k) = (e_1, \ldots, e_{k-1}, e_k)$. We will verify that the choice

$$a_i := e_i \text{ for } i \in \{1, \ldots, k-1\} \text{ and } a_k := e_1 + \cdots + e_k$$

does the job. This can be seen as follows:

Obviously, $(a_1, \ldots, a_k)$ is a composition of $n$, since all the $a_i$'s are positive and $a_1 + \cdots + a_k = 2(e_1 + \cdots + e_{k-1}) + e_k = n$. Also, $a_k > n - a_k$, since $2a_k = 2(e_1 + \cdots + e_k) = n + e_k > n$. Thus $(a_1, \ldots, a_k) \in CH_k^{(k)}(n)$. Finally, the fact that

$$\phi(a_1, \ldots, a_k) = (a_1, \ldots, a_{k-1}, 2a_k - n) = (e_1, \ldots, e_{k-1}, e_k),$$

completes the proof that $\phi$ is surjective. Therefore $\phi$ is a bijection, and Theorem 3 is proved. $\square$

## 4.3 $k$-gon partitions

One can view Problem 1, i.e., the study of triangles with sides of integer size (Section 3.3) as a *partition* counterpart to Hermite's problem. More precisely, it gives rise to the following definition.

**Definition 5.** As the set of "non-degenerate $k$-gon partitions of $n$ into positive parts" we define

$$T_k(n) := \{(a_1, \ldots, a_k) \in C_k(n) \mid a_1 \leq a_2 \leq \cdots \leq a_k \text{ and } a_k < n - a_k\}.$$

The term "non-degenerate" refers to the restriction to strict inequality, i.e., to $a_k < n - a_k$. (Note that $n - a_k = a_1 + \cdots + a_{k-1}$.) In Section 3.3 we computed that

$$\sum_{n \geq 3} |T_3(n)|\, q^n = \frac{q^3}{(1-q^2)(1-q^3)(1-q^4)}. \tag{10}$$

With the `Omega` package in hand, we are able to compute the next cases in purely mechanical way. For instance,

$$\sum_{n \geq 4} |T_4(n)|\, q^n = \frac{q^4(1+q+q^5)}{(1-q^2)(1-q^3)(1-q^4)(1-q^6)}, \tag{11}$$

$$\sum_{n \geq 5} |T_5(n)|\, q^n = \frac{q^5(1-q^{11})}{(1-q)(1-q^2)(1-q^4)(1-q^5)(1-q^6)(1-q^8)}, \tag{12}$$

and

$$\sum_{n \geq 6} |T_6(n)|\, q^n = \frac{q^6(1-q^4+q^5+q^7-q^8-q^{13})}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^6)(1-q^8)(1-q^{10})}. \tag{13}$$

From these results we can derive a number of consequences. For example,

$$|T_4(2n)| - |T_4(2n-1)| = g(n)$$

where $g(n)$ is the number of partitions of $n$ into 2's and 3's with at least one 3. This implies that

$$g(n) = \begin{cases} \lceil \frac{n-1}{6} \rceil, & \text{if } n \equiv 3 \pmod 6, \\ \lceil \frac{n-1}{6} \rceil - 1, & \text{otherwise.} \end{cases}$$

Also we can show that

$$|T_5(2n)| - |T_5(2n-1)| = h(n)$$

where $h(n)$ is the number of partitions of $n$ into 1's, 2's, 4's and 5's with at least one 5.

Each of these results is easily derived once we observe that if $a_{-1} := 0$ and

$$f(q) = \sum_{n=0}^{\infty} a_n q^n,$$

then

$$\sum_{n=0}^{\infty} (a_{2n} - a_{2n-1})\, q^{2n} = \frac{1}{2}(1-q)f(q) + \frac{1}{2}(1+q)f(-q).$$

It would be interesting to know if there are any simple combinatorial explanations of these observations.

The verification of the generating function representations (11), (12), and (13) is left to the reader as a routine `Omega` exercise. Moreover, it is also easily checked that

$$\sum_{n \geq k} |T_k(n)|\, q^n = \underset{\geqq}{\Omega} \frac{\lambda_k^{-1}\lambda_{k+1}^{-1}}{\left(1 - \frac{\lambda_1 x_1}{\lambda_{k+1}}\right)\left(1 - \frac{\lambda_2 \lambda_{k+1} x_2}{\lambda_1}\right)\left(1 - \frac{\lambda_3 \lambda_{k+1} x_3}{\lambda_2}\right) c \ldots \left(1 - \frac{\lambda_k \lambda_{k+1} x_k}{\lambda_{k-1}}\right)}$$

is the corresponding $\Omega_{\geqq}$-representation for the generating function in full generality. (Cf. also identity (7), where the case $k = 3$ is presented in an equivalent, but slightly different form.)

Despite the fact that the particular instances of $\sum_{n \geq k} |T_k(n)|\, q^n$ can be computed so easily, we were not able to find a common underlying pattern as in the case of $k$-gon compositions. This suggests to conclude by stating this question as an open problem:

**Problem 3.** *In view of the generating function representations* (10), (11), (12), *and* (13)*: Is it possible to find a common pattern for all possible choices of $k$ as in the case of $k$-gon compositions (Theorem* 3*)?*

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