An Infinite Family of Engel Expansions of Rogers-Ramanujan Type

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Dedicated to the memory of John Knopfmacher 1937-1999,
the inventor of Engel Expansions for q-series

Abstract

The Extended Engel Expansion is an algorithm that leads to unique series expansions of q-series. Various examples related to classical partition theorems, including the Rogers-Ramanujan identities, have been given recently. The object of this paper is to show that the new and elegant Rogers-Ramanujan generalization found by Garrett, Ismail, and Stanton also fits into this framework. This not only reveals the existence of an infinite, parameterized family of extended Engel expansions, but also provides an alternative proof of the Garrett, Ismail, and Stanton result. A finite version of it, which finds an elementary proof, is derived as a by-product of the Engel approach.

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1 Introduction

Let $\mathbb{C}((q))$ denote the field of formal Laurent series over the complex numbers. If

$$A = \sum_{n=\nu}^{\infty} c_n q^n \in \mathbb{C}((q)),$$

we call $\nu = \nu(A)$ the *order* of A and define the *norm* of A to be

$$\|A\| = 2^{-\nu(A)}.$$

Note that this norm induces the standard notion of convergence for sequences, infinite series and products of formal Laurent series. In addition, we define the *integral part* $[A] \in \mathbb{C}[q^{-1}]$ of A by

$$[A] = \sum_{\nu \le n \le 0} c_n q^n.$$

As described by Perron [9, sect. 34], Engel originally defined a series expansion for real numbers. In [7], this concept was extended to formal Laurent series in the following way:

Definition 1 ("Engel sequence"). Given $A \in \mathbb{C}((q))$. Set $A_0 = A$ and $a_0 = [A]$ and recursively define for $n \ge 0$:

$$A_{n+1} = a_n A_n - 1$$

where

$$a_n = \left[\frac{1}{A_n}\right] \quad (n \ge 1).$$

We call $(a_n)_{n\ge 0}$ the *Engel sequence associated to* A.

In [7], the following two theorems are proved.

Theorem 1 ("Extended Engel Expansion (EEE)"). *(i) Given $A \in \mathbb{C}((q))$ with associated Engel sequence $(a_n)_{n\ge 0}$. Then*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 \dots a_n} \tag{1}$$

holds in $\mathbb{C}((q))$ where the series converges with respect to the above norm. This expansion is finite if and only if $A \in \mathbb{C}(q)$.
(ii) For $n \ge 0$:

$$\nu(a_n) \le -n \quad \text{and} \quad \nu(a_{n+1}) \le \nu(a_n) - 1. \tag{2}$$

The extended Engel expansion turns out to be unique in the following sense:

Theorem 2 (Uniqueness of EEE). *Given $A \in \mathbb{C}((q))$. Let $(a_n)_{n\ge 0}$ be a sequence of Laurent polynomials from $\mathbb{C}[q^{-1}]$ with $a_0 = [A]$. If $(a_n)_{n\ge 0}$ satisfies (1) and (2) then it is the Engel sequence associated to A.*

In [3] and [4] various classical q-series identities are shown to be examples of extended Engel expansions. For instance, in [3] one finds a detailed proof that the celebrated Rogers-Ramanujan identities [2] fit exactly into this pattern. In the present article we show that they form the basis of an *infinite* collection of extended Engel expansions.

Only recently, T. Garrett, M. Ismail, and D. Stanton have found a new and elegant generalization [5, (3.5)]. It involves an extra parameter — the parameter m in identity (8) from the next section — which ranges over the nonnegative integers. In particular, the choices $m = 0$ and $m = 1$ result in the classic Rogers-Ramanujan case.

It is the object of this paper to show that this infinite family of Rogers-Ramanujan type identities fits into the EEE pattern. In other words, in the next section we prove that for arbitrary choice of the parameter m, the resulting identity is an extended Engel expansion. For doing so, we need a few facts from q-series.

First recall the standard definition of the q-shifted factorials:

$$(a;q)_k = \begin{cases} (1-a)(1-aq)\cdots(1-aq^{k-1}), & \text{if } k > 0, \\ 1, & \text{if } k = 0, \\ 1/((1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{k})), & \text{if } k < 0, \end{cases}$$

and

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The Gaussian polynomials are defined as usual as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

It is well-known that I. Schur [10] independently rediscovered the Rogers-Ramanujan identities for which he gave two different proofs. In one of these proofs certain polynomial families $(d_m)_{m\ge 0}$ and $(e_m)_{m\ge 0}$ play a fundamental role. Schur [10, (29.)] gave a presentation in terms of Gaussian polynomials as follows: For $m \ge 1$,

$$d_m = \sum_k (-1)^k q^{k(5k-3)/2} \begin{bmatrix} m-1 \\ \lfloor \frac{m+1-5k}{2} \rfloor \end{bmatrix} \tag{3}$$

and

$$e_m = \sum_k (-1)^k q^{k(5k+1)/2} \begin{bmatrix} m-1 \\ \lfloor \frac{m-1-5k}{2} \rfloor \end{bmatrix}, \tag{4}$$

where the initial values are defined as $d_0 = 1$ and $e_0 = 0$.

In particular, Schur [10, (20.)] made use of the following fundamental properties: *Both polynomial sequences (d_m) and (e_m) satisfy the recurrence*

$$c_{m+2} = c_{m+1} + q^m c_m \quad (m \ge 0); \tag{5}$$

in addition, in the limit $m \to \infty$ one has

$$d_\infty = (q;q)_\infty^{-1} \sum_k (-1)^k q^{k(5k-3)/2} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \tag{6}$$

and

$$e_\infty = (q;q)_\infty^{-1} \sum_k (-1)^k q^{k(5k+1)/2} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}. \tag{7}$$

We remark that the proof of the crucial property (5) nowadays can be done automatically with the computer [8]. Also, note that the convertion of the series into the product representation is by Jacobi's triple product identity. (See, e.g., [2, p. 21, eq. (2.2.10)].)

We conclude this section by listing the first sequence entries explicitly. Namely,

$$d_0 = 1, d_1 = 0, d_2 = 1, d_3 = 1, d_4 = 1+q^2, d_5 = 1+q^2+q^3, d_6 = 1+q^2+q^3+q^4+q^6,$$

and

$$e_0 = 0, e_1 = 1, e_2 = 1, e_3 = 1+q, e_4 = 1+q+q^2, e_5 = 1+q+q^2+q^3+q^4,$$
$$e_6 = 1+q+q^2+q^3+2q^4+q^5+q^6.$$

2 Engel Expansion

First we recall the elegant Rogers-Ramanujan generalization found by T. Garrett, M. Ismail, and D. Stanton [5, (3.5)]: For $m \geq 0$,

$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n} = \frac{(-1)^m q^{-\binom{m}{2}} d_m}{(q;q^5)_\infty (q^4;q^5)_\infty} - \frac{(-1)^m q^{-\binom{m}{2}} e_m}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{8}$$

The authors of [5] proved this identity by evaluating an integral involving q-Hermite polynomials in two different ways and equating the results. In this section we will show that given the right hand side of (8), the left hand side of (8) is nothing but the corresponding extended Engel expansion. Our proof will only use elementary properties of the Schur polynomials d_m and e_m, and thus provides an alternative verification of (8).

More precisely, we will prove the following statement:

Theorem 3. *Let m be a nonnegative integer and A be the right hand side of* (8). *Then the sequence $(a_n)_{n\geq 0}$ defined as*

$$a_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{-(2n+m-1)} - q^{-(n+m-1)}, & \text{if } n \geq 1, \end{cases}$$

is the Engel sequence associated to A.

This proves identity (8) since (1) then implies

$$A = a_0 + \sum_{n\geq 1} \frac{1}{a_1 \dots a_n} = 1 + \sum_{n\geq 1} \frac{q^{(m+1)+(m+2)+\cdots+(m+2n-1)}}{(q;q)_n}$$
$$= 1 + \sum_{n\geq 1} \frac{q^{n^2+mn}}{(q;q)_n}.$$

The proof of Theorem 3 for the cases $m = 0$ and $m = 1$ can be found in [3]. Also for $m \geq 2$ we essentially follow the same steps; however for doing so, we need information about the expressions $d_m e_{m+k} - d_{m+k} e_m$ for $k \geq 1$.

2.1 A finite version of the MacMahon-Schur identity

The extra ingredients we need for the proof of Theorem 3 are listed below. For instance, the asymptotic estimate (11) will be used in order to extract the integral part.

Lemma 1. *For integers $m \geq 0$ and $k \geq 1$:*

$$d_m e_{m+1} - d_{m+1} e_m = (-1)^m q^{\binom{m}{2}}, \tag{9}$$

$$d_m e_{m+2} - d_{m+2} e_m = (-1)^m q^{\binom{m}{2}}, \tag{10}$$

and

$$d_m e_{m+k} - d_{m+k} e_m = (-1)^m q^{\binom{m}{2}} \big(1 + O(q^{m+1})\big). \tag{11}$$

The statements of Lemma 1 are easy induction exercises using (5). However, it is more illuminating to prove Lemma 1 as a corollary of the following proposition which is a q-analog of an identity for Fibonacci numbers

$$F_{m-1} F_{m+k} - F_{m+k-1} F_m = (-1)^m F_k. \tag{12}$$

In fact, (12) is a specialization of Euler's generalization of Cassini's identity.(E.g., eq. (6.134) in [6] with $k \to m-2$, $m \to k$, $n \to m-1$, and all $x_i \to 1$.)

Proposition 1. *For integers $m \geq 0$ and $k \geq 1$:*

$$d_m e_{m+k} - d_{m+k} e_m = (-1)^m q^{\binom{m}{2}} \sum_{j \geq 0} \begin{bmatrix} k-1-j \\ j \end{bmatrix} q^{j^2+mj} \tag{13}$$

Proof. Let us denote the left side of (13) by $f_{m,k}$ and the right side of (13) by $g_{m,k}$. Clearly,

$$g_{m,1} = g_{m,2} = (-1)^m q^{\binom{m}{2}}. \tag{14}$$

Furthermore for $k > 2$,

$$\begin{aligned} g_{m,k} - g_{m,k-1} &= (-1)^m q^{\binom{m}{2}} \sum_{j \geq 0} \left(\begin{bmatrix} k-1-j \\ j \end{bmatrix} - \begin{bmatrix} k-2-j \\ j \end{bmatrix} \right) q^{j^2+mj} \\ &= (-1)^m q^{\binom{m}{2}} \sum_{j \geq 0} q^{k-1-2j} \begin{bmatrix} k-2-j \\ j-1 \end{bmatrix} q^{j^2+mj} \\ &= (-1)^m q^{\binom{m}{2}+m+k-2} \sum_{j \geq 0} \begin{bmatrix} k-3-j \\ j \end{bmatrix} q^{j^2+mj} \\ &= q^{m+k-2} g_{m,k-2}. \end{aligned} \tag{15}$$

To complete the proof we need only show that $f_{m,k}$ satisfies the initial conditions in (14) and the recurrence in (15).

We note

$$f_{m,k} = \begin{vmatrix} d_m & d_{m+k} \\ e_m & e_{m+k} \end{vmatrix} = \begin{vmatrix} d_m & d_{m+k-1} + q^{m+k-2} d_{m+k-2} \\ e_m & e_{m+k-1} + q^{m+k-2} e_{m+k-2} \end{vmatrix}$$
$$= f_{m,k-1} + q^{m+k-2} f_{m,k-2}, \tag{16}$$

which replicates the recurrence in (15).

Furthermore it is obvious that $f_{m,0} = 0$, and

$$f_{m,1} = \begin{vmatrix} d_m & d_{m+1} \\ e_m & e_{m+1} \end{vmatrix} = \begin{vmatrix} d_m & d_m + q^{m-1} d_{m-1} \\ e_m & e_m + q^{m-1} e_{m-1} \end{vmatrix} = -q^{m-1} \begin{vmatrix} d_{m-1} & d_m \\ e_{m-1} & e_m \end{vmatrix}$$
$$= -q^{m-1} f_{m-1,1}. \tag{17}$$

From (17) we may deduce by induction on m that

$$f_{m,1} = (-1)^m q^{\binom{m}{2}}, \tag{18}$$

and by (16) with $k = 2$, we see that

$$f_{m,2} = (-1)^m q^{\binom{m}{2}} \tag{19}$$

as well.

Consequently $f_{m,k}$ and $g_{m,k}$ fulfill the same two initial values and second order recurrence. So $f_{m,k} = g_{m,k}$ as desired. □

For the sake of completeness we state:

Proof of Lemma 1. Equations (9) and (10) are simply the cases $k = 1$ and $k = 2$ of Proposition 1; also (11) is a direct consequence of the sum representation in (13). □

Remark. (i) One of us had wondered for decades whether there were q-analogs of effectively bi-linear expressions like (12). It was only when we were forced to estimate the left side of (13) that this q-analog emerged. Note that while the right side of (13) is, indeed, a q-analog of $(-1)^m F_k$, it mixes the variables m and k in such a way that $(-1)^m$ is the only trace of m left when $q = 1$.

(ii) In fact, Proposition 1 is a natural generalization of the polynomial versions of the Rogers-Ramanujan identities given in [1]. (A computer proof can be found in [8, Theorem 2].) Proposition 1 itself is a polynomial version of the Garrett, Ismail and Stanton result; namely, sending k to infinity in (13) immediately results in (8).

Now we are ready for the proof of Theorem 3.

2.2 Proof of Theorem 3

Proof of Theorem 3. Define A to be the right hand side of (8) and set $A_0 = A$. For $n \geq 1$ set

$$A_n = (-1)^m q^{-\binom{m}{2}-(m-1)(n-1)} \sum_{j=m+1}^{\infty} q^{jn} (d_m e_j - d_j e_m). \tag{20}$$

Given $(a_n)_{n\geq 0}$ as in the statement of Theorem 3, the proof according to Definition 1 and Theorem 1 splits into two parts: (i) verifying the relation

$$a_n A_n = 1 + A_{n+1} \quad (n \geq 0) \tag{21}$$

and (ii) showing that

$$a_0 = [A] \quad \text{and} \quad a_n = \left[\frac{1}{A_n}\right] \quad (n \geq 1). \tag{22}$$

Part (i): First we treat the case $n = 0$. Applying (5) we observe that for any integer N $(\geq m+1)$,

$$\begin{aligned}
d_m \sum_{j=m+1}^{N} q^j e_j - e_m \sum_{j=m+1}^{N} q^j d_j &= d_m \sum_{j=m+1}^{N} (e_{j+2} - e_{j+1}) - e_m \sum_{j=m+1}^{N} (d_{j+2} - d_{j+1}) \\
&= d_m (e_{N+2} - e_{m+2}) - e_m (d_{N+2} - d_{m+2}) = d_m e_{N+2} - d_{N+2} e_m - (d_m e_{m+2} - d_{m+2} e_m) \\
&= d_m e_{N+2} - d_{N+2} e_m - (-1)^m q^{\binom{m}{2}},
\end{aligned}$$

where the last line is by (10). In the limit $N \to \infty$, and after multiplication by $(-1)^m q^{-\binom{m}{2}}$, this turns into

$$\begin{aligned}
A_1 &= (-1)^m q^{-\binom{m}{2}} \left(d_m \sum_{j=m+1}^{\infty} q^j e_j - e_m \sum_{j=m+1}^{\infty} q^j d_j \right) \\
&= (-1)^m q^{-\binom{m}{2}} d_m e_\infty - (-1)^m q^{-\binom{m}{2}} e_m d_\infty - 1 \\
&= \frac{(-1)^m q^{-\binom{m}{2}} d_m}{(q;q^5)_\infty (q^4;q^5)_\infty} - \frac{(-1)^m q^{-\binom{m}{2}} e_m}{(q^2;q^5)_\infty (q^3;q^5)_\infty} - 1 = a_0 A_0 - 1,
\end{aligned}$$

since $A_0 = A$ and $a_0 = 1$.

For $n \geq 1$ we compute

$$\begin{aligned}
a_n A_n &= \left(q^{-(2n+m-1)} - q^{-(n+m-1)} \right) A_n \\
&= (-1)^m q^{-\binom{m}{2}-(m+1)n} \sum_{j \geq m+1} q^{jn} (d_m e_j - d_j e_m) - \\
&\qquad (-1)^m q^{-\binom{m}{2}-mn} \sum_{j \geq m+1} q^{jn} (d_m e_j - d_j e_m) \\
&= (-1)^m q^{-\binom{m}{2}-mn} \sum_{j \geq m} q^{jn} (d_m e_{j+1} - d_{j+1} e_m) - \\
&\qquad (-1)^m q^{-\binom{m}{2}-mn} \sum_{j \geq m+1} q^{jn} (d_m e_j - d_j e_m) \\
&= (-1)^m q^{-\binom{m}{2}} (d_m e_{m+1} - d_{m+1} e_m) + \\
&\qquad (-1)^m q^{-\binom{m}{2}-mn} \sum_{j \geq m+1} q^{jn} \left(d_m (e_{j+1} - e_j) - (d_{j+1} - d_j) e_m \right) \\
&= 1 + (-1)^m q^{-\binom{m}{2}-mn} \sum_{j \geq m+1} q^{jn+j-1} (d_m e_{j-1} - d_{j-1} e_m) \\
&\quad \text{(by (9) and (5), respectively)} \\
&= 1 + (-1)^m q^{-\binom{m}{2}-(m-1)n} \sum_{j \geq m} q^{j(n+1)} (d_m e_j - d_j e_m)
\end{aligned}$$

$$= 1 + A_{n+1}.$$

Part (ii): Again we treat the case $n = 0$ first. Applying (11) we obtain

$$\begin{aligned}
[A] &= \left[(-1)^m q^{-\binom{m}{2}} d_m e_\infty - (-1)^m q^{-\binom{m}{2}} e_m d_\infty\right] \\
&= \left[(-1)^m q^{-\binom{m}{2}} \left((-1)^m q^{\binom{m}{2}} (1 + O(q^{m+1}))\right)\right] = 1 = a_0.
\end{aligned}$$

In order to prove the case $n \geq 1$, i.e., before extracting the integral part of A_n, we first derive a suitable asymptotic representation of A_n. To this end we introduce a positive integer parameter α as follows:

$$\begin{aligned}
A_n &= (-1)^m q^{-\binom{m}{2}-(m-1)(n-1)} \sum_{j \geq m+1} q^{jn} \left(d_m e_j - d_j e_m\right) \\
&= (-1)^m q^{-\binom{m}{2}-(m-1)(n-1)} \sum_{k \geq 1} q^{(m+k)n} \left(d_m e_{m+k} - d_{m+k} e_m\right) \\
&= (-1)^m q^{-\binom{m}{2}-(m-1)(n-1)} \\
&\quad \Big(q^{(m+1)n}(d_m e_{m+1} - d_{m+1} e_m) + q^{(m+2)n}(d_m e_{m+2} - d_{m+2} e_m) + \\
&\quad \sum_{k=3}^{\alpha} q^{(m+k)n}(d_m e_{m+k} - d_{m+k} e_m) + (-1)^m q^{\binom{m}{2}} O(q^{(m+\alpha+1)n})\Big) \\
&\quad \text{(by (11))} \\
&= q^{-(m-1)(n-1)} \Bigg(q^{(m+1)n} + q^{(m+2)n} + \\
&\quad \sum_{k=3}^{\alpha} \left(q^{(m+k)n} \left(1 + O(q^{m+1})\right)\right) + O(q^{(m+\alpha+1)n}) \Bigg) \\
&\quad \text{(by (9), (10) and (11), respectively)} \\
&= q^{2n+m-1} \left(\sum_{k=0}^{\alpha-1} q^{kn} + \sum_{k=2}^{\alpha-1} O(q^{kn+m+1}) + O(q^{\alpha n}) \right) \\
&= q^{2n+m-1} \left(\frac{1 - q^{\alpha n}}{1 - q^n} + O(q^{2n+m+1}) + O(q^{\alpha n}) \right) \\
&= q^{2n+m-1} \frac{1 + O(q^{2n+m+1}) + O(q^{\alpha n})}{1 - q^n}.
\end{aligned}$$

Note that this estimate for A_n indeed holds for any positive integer α and $m \geq 0$.

Consequently,

$$\begin{aligned}
\left[\frac{1}{A_n}\right] &= \left[\frac{1 - q^n}{q^{2n+m-1}} \left(1 + O(q^{2n+m+1}) + O(q^{\alpha n})\right)\right] \\
&= \left[\frac{1}{q^{2n+m-1}} \left(1 - q^n + O(q^{2n+m+1}) + O(q^{\alpha n})\right)\right] \\
&= \left[q^{-2n-m+1} - q^{-n-m+1} + O(q^2) + O(q^{(\alpha-2)n-m+1})\right]
\end{aligned}$$

$$= q^{-2n-m+1} - q^{-n-m+1} \quad \text{if} \quad n \geq \frac{m-1}{\alpha-2}.$$

Therefore, if we choose, for instance,

$$\alpha = m+1, \text{ if } m \geq 2, \text{ and } \alpha = 3, \text{ if } m = 0 \text{ or } m = 1,$$

we have proven that

$$\left[\frac{1}{A_n}\right] = a_n \quad (n \geq 1).$$

This completes the proof of Theorem 3. □

We conclude by recalling the classic Rogers-Ramanujan identities, namely the instances $m = 0$ and $m = 1$ of (8). As already mentioned, in [3] these identities are shown to be extended Engel expansions. If one specializes m to 0 or to 1 in the derivation given above, one essentially recovers the proofs presented there.

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