FEM–BEM COUPLING FOR 3D EXTERIOR MAGNETIC FIELD PROBLEMS

M. KUHN
Johannes Kepler University Linz,
SFB "Numerical and Symbolic Scientific Computing",
A-4040 Linz, Austria
E-mail: kuhn@sfb013.uni-linz.ac.at

O.STEINBACH
University of Stuttgart, Mathematisches Institut A,
Pfaffenwaldring 57,
D-70569 Stuttgart, Germany
E-mail: Olaf.Steinbach@mathematik.uni-stuttgart.de

We propose a concept for the coupling of Finite Element Methods (FEM) based on a vector potential and Boundary Element Methods (BEM) based on a scalar potential for the solution of stationary 3D magnetic field problems. Hereby, the symmetric Galerkin BEM relying on Calderon's equations is used. Numerical results are presented.

1 Introduction

Finite Element Methods (FEM) and Boundary Element Methods (BEM) are the most popular discretization techniques for the numerical solution of partial differential equations. Each method has its own advantages. So, nonlinearities can be modelled more easily by FEM whereas exterior problems fit nicely into the framework of BEM since only the boundary has to be discretized. Coupled FEM–BEM discretizations based on non-overlapping Domain Decomposition (DD) methods have been applied successfully for the solution of 2D magnetic field problems [4,5]. Here, we are going to consider 3D problems which are more difficult since the equations cannot be reduced to scalar unknowns as in the 2D case. Instead, we derive a formulation which is based on vector–valued FEM (vector potential) and scalar BEM (scalar potential). For simplicity we consider stationary problems. Then, the current density J, which is divergence–free, i.e.,

$$\mathrm{div}(J) = 0, \tag{1}$$

determines the magnetic field strength H and the magnetic induction B by

$$\mathrm{curl}(H) = J, \quad H = \nu B, \quad \mathrm{div}(B) = 0, \tag{2}$$

where μ is the permeability and $\nu := 1/\mu$. We allow the coefficient ν to be piecewise constant with $\nu \geq \nu_{\min} > 0$. If permanent magnets are present, J has to be replaced by $J + \mathrm{curl}(H_0)$, where H_0 is the initial (permanent) magnetic field intensity. We mention that the normal component of the B-field and the tangential component of the H-field are continuous across the interfaces.

In Sec. 2, we state the formal exterior magnetic field problem and derive the coupled variational formulation. Then, in Sec. 3, we discuss the discretization and propose an iterative solver. Numerical results are given in Sec. 4.

2 An FE–BE Formulation for Unbounded Domains

We consider the unbounded domain $\Omega = \mathbf{R}^3$ which is decomposed into an interior domain Ω_- and the exterior domain Ω_+ such that $\overline{\Omega} = \overline{\Omega}_- \cup \overline{\Omega}_+$. We assume that both, Ω_- and Ω_+ are simply connected, non-overlapping, i.e., $\Omega_- \cap \Omega_+ = \emptyset$ and the coupling boundary $\Gamma_+ := \partial\Omega_- = \overline{\Omega}_- \cap \overline{\Omega}_+$ is a closed surface which is Lipschitz continuous. The domain Ω_+ together with the coupling boundary Γ_+ is defined such that $J = 0$ in Ω_+ and $\nu(x) = \nu_+ > 0$ in Ω_+. Moreover, we assume $\nu = \nu_- > 0$ in Ω_-. Hence we have $\mathrm{div}(B) = 0$ in Ω and $\mathrm{curl}(H) = 0$ in Ω_+. Consequently we can choose the following representation

$$B = curl(u) \text{ in } \Omega_- \quad \text{and} \quad H = \nabla\Phi \text{ in } \Omega_+. \tag{3}$$

Since the vector potential u is not unique we introduce the Coulomb gauging $\mathrm{div}(u) = 0$. From the interface conditions on Γ_+

$$B_- \cdot n_- = B_+ \cdot n_- \quad \text{and} \quad H_- \times n_- = H_+ \times n_-,$$

where n_- is the outward normal with respect to Ω_-, we obtain the formal exterior magnetic field problem: Find (u, Φ) that satisfy formally

- the partial differential equations

$$\mathrm{curl}(\nu_- \, \mathrm{curl}(u)) = J \quad \text{in } \Omega_- \tag{4}$$

$$\mathrm{div}(\frac{1}{\nu_+}\nabla\Phi) = 0 \quad \text{in } \Omega_+, \tag{5}$$

- the gauging condition

$$\mathrm{div}(u) = 0 \quad \text{in } \Omega_-, \tag{6}$$

- the interface conditions

$$(\mathrm{curl}(u) \cdot n_-) = \frac{1}{\nu_+}(\nabla\Phi \cdot n_-) \quad \text{on } \Gamma_+, \tag{7}$$

$$\nu_- \,(\mathrm{curl}(u) \times n_-) = (\nabla\Phi \times n_-) \quad \text{on } \Gamma_+. \tag{8}$$

- the radiation conditions

$$|\Phi(x)| = O(1/|x|) \quad \text{for } |x| \to \infty. \tag{9}$$

Remark 2.1 *We note that (3) implies* $\mathrm{div}(B) = 0$ *in* Ω_- *and* $\mathrm{curl}(H) = 0$ *in* Ω_+*, whereas* $\mathrm{curl}(H) = f$ *in* Ω_- *follows from (4) and* $\mathrm{div}(B) = 0$ *in* Ω_+ *follows from (5).*

The canonical weak formulation of equation (4) is: Find $u \in H(\mathrm{curl}, \Omega_-)$ such that

$$\int_{\Omega_-} \nu\, \mathrm{curl}(u) \cdot \mathrm{curl}(v)\, dx = \int_{\Omega_-} J \cdot v\, dx \quad \forall\, v \in H(\mathrm{curl}, \Omega_-), \tag{10}$$

where $H(\mathrm{curl}, \Omega_-) := \{v \in L^2(\Omega_-)^3 |\ \mathrm{curl}(v) \in L^2(\Omega_-)^3\}$. Treating the weak formulation of the Coulomb gauging (6) as an equality constraint, we arrive at the following mixed weak formulation of (4) and (6): Find $(u,p) \in \mathbf{X} \times \mathbf{M} := \mathbf{H}(\mathrm{curl}, \boldsymbol{\Omega}_-) \times \mathbf{H}^1_\star(\boldsymbol{\Omega}_-)$ such that

$$a(u,v) + b(v,p) = \langle f, v\rangle \quad \forall\, v \in H\,(\mathrm{curl}, \Omega_-), \tag{11}$$

$$b(u,q) = 0 \qquad \forall\, q \in H^1_\star(\Omega_-), \tag{12}$$

where $a(u,v) := \int_{\Omega_-} \nu\, \mathrm{curl}(u) \cdot \mathrm{curl}(v)\, dx$, $b(v,p) := \int_{\Omega_-} v \cdot \nabla p\, dx$, $\langle f, v\rangle := \int_{\Omega_-} J \cdot v\, dx$, and $H^1_\star(\Omega_-) := \{v \in L^2(\Omega_-) |\ \nabla v \in L^2(\Omega_-)^3,\ \langle v, 1\rangle = 0\}$. Choosing $v = \nabla p \in H\,(\mathrm{curl}, \Omega_-)$ in (11), we immediately observe from (1) that $p = 0$. Hence we can add an arbitrary symmetric and positive definite bilinear form $c(\cdot,\cdot) : H^1_\star(\Omega_-) \times H^1_\star(\Omega_-) \to \mathbf{R}^1$ to the second equation of (11) - (12), and arrive at the equivalent mixed variational formulation: Find $(u,p) \in \mathbf{X} \times \mathbf{M}$ such that

$$a(u,v) + b(v,p) = \langle f, v\rangle \quad \forall\, v \in H\,(\mathrm{curl}, \Omega_-), \tag{13}$$

$$b(u,q) - c(p,q) = 0 \qquad \forall\, q \in H^1_\star(\Omega_-). \tag{14}$$

Let us define the operator $T_n : H(\mathrm{curl}, \Omega_-) \to H^{-1/2}(\Gamma_+)$ by $T_n\, u := \mathrm{curl}(u) \cdot n_-$, Then we conclude from the equations (5), (7), (9) that the pair $(\Phi,\, T_n\, u)$ has to satisfy Calderon's equations (see [4] and references therein)

$$\begin{pmatrix} \Phi \\ T_n\, u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\mathcal{I} + \mathcal{K} & -\nu_+\, \mathcal{V} \\ -\frac{1}{\nu_+}\, \mathcal{D} & \frac{1}{2}\mathcal{I} - \mathcal{K}' \end{pmatrix} \begin{pmatrix} \Phi \\ T_n\, u \end{pmatrix}. \tag{15}$$

Here, the integral operators are defined with respect to n_- by

$$\begin{aligned}
\mathcal{V}\lambda(x) &:= \int_{\Gamma_+} \mathcal{E}(x,y)\lambda(y)\, ds_y, \\
\mathcal{K}u(x) &:= \int_{\Gamma_+} \partial_{n_y}\mathcal{E}(x,y)u(y)\, ds_y, \\
\mathcal{D}u(x) &:= -\partial_{n_x} \int_{\Gamma_+} \partial_{n_y}\mathcal{E}(x,y)u(y)\, ds_y.
\end{aligned} \tag{16}$$

$\mathcal{E}(x,y) := \frac{1}{4\pi\,|x-y|}$ is the fundamental solution of the Laplacian in $\mathbf{R^3}$, ∂_{n_x} denotes the (weak) normal derivative with respect to the variable x.

Combining (13) - (14) and (15) together with the interface conditions (8) gives the following coupled variational formulation: Find $(u,p,\Phi) \in \mathbf{X}\times\mathbf{M}\times\mathbf{W}$ such that

$$a(u,v) + a_B(u,v,\Phi,\Psi) + b(v,p) = \langle f, v\rangle \quad \forall (v,\Psi) \in \mathbf{X}\times \mathbf{H}^{1/2}_\star(\boldsymbol{\Gamma}_+) \tag{17}$$

$$b(u,q) - c(p,q) = 0 \qquad \forall q \in \mathbf{M} \tag{18}$$

where

$$\begin{aligned}
a_B(u,v,\Phi,\Psi) := {}& \nu_+ \langle \mathcal{V}\,T_n\,u,\, T_n\,v\rangle_{\Gamma_+} - \frac{1}{2}\langle \Phi,\, T_n\,v\rangle_{\Gamma_+} - \langle \mathcal{K}\Phi,\, T_n\,v\rangle_{\Gamma_+} \\
& + \frac{1}{2}\langle \Psi,\, T_n\,u\rangle_{\Gamma_+} + \langle \Psi, \mathcal{K}'\,T_n\,u\rangle_{\Gamma_+} + \frac{1}{\nu_+}\langle \Psi, \mathcal{D}\Phi\rangle_{\Gamma_+},
\end{aligned}$$

and $a(.,.)$, $b(.,.)$, $c(.,.)$, $\langle f, .\rangle$ as before, and $\mathbf{W} := H^{1/2}_\star(\Gamma_+) := \{\Phi \in H^{1/2}(\Omega) : \langle \Phi, 1\rangle = 0\}$. The variational formulation (17)–(18) has a unique solution. Moreover, this solution satisfies the corresponding interface conditions. The latter we conclude from integration by parts together with the vector identity $\int_{\Gamma_+}(\nabla\,\Phi \times n_-)\,v\,ds = \int_{\Gamma_+} \Phi\,T_n\,v\;ds$ which holds for sufficiently smooth functions Φ and v and a closed surface Γ_+.

3 Discretization and Iterative Solver

Let now be $\mathbf{X_h} \subset \mathbf{X}$ and $\mathbf{M_h} \subset \mathbf{M}$ the lowest order edge element space (see [7]) and the space of piecewise linear nodal elements on a shape-regular tetrahedral triangularization with mesh width h of Ω_-, respectively. The space $\mathbf{W_h} \subset \mathbf{W}$ is defined by the traces on Γ_+ of piecewise linear nodal elements.

Then the mixed finite element – boundary element approximation to the regularized coupled mixed variational problem (17) - (18) reads as follows: Find $(u_h, p_h, \Phi_h) \in \mathbf{X_h}\times\mathbf{M_h}\times\mathbf{W_h}$ such that

$$a(u_h,v_h) + a_B(u_h,v_h,\Phi_h,\Psi_h) + b(v_h,p_h) = \langle f, v_h\rangle \quad \forall\,(v_h,\Psi_h), \tag{19}$$

$$b(u_h,q_h) - c(p_h,q_h) = 0 \qquad \forall\, q_h. \tag{20}$$

Again, using the Brezzi-Babuška theory, we conclude that the mixed finite element scheme (19) - (20) has a unique solution. Similarly to the continuous case, we observe that $p_h = 0$ since $q_h \in \mathbf{M_h}$ implies $\nabla q_h \in \mathbf{X_h}$.
Let us choose the standard finite element basis in the spaces $\mathbf{X_h}$, $\mathbf{M_h}$ and $\mathbf{W_h}$. Then the coupled mixed finite–boundary element scheme (19) - (20) is equivalent to the system

$$\begin{pmatrix} D & -\widetilde{K}^T & 0 \\ \widetilde{K} & A_F + A_B & B^T \\ 0 & B & -C \end{pmatrix} \begin{pmatrix} \underline{\Phi}_h \\ \underline{u}_h \\ \underline{p}_h \end{pmatrix} = \begin{pmatrix} 0 \\ \underline{f}_h \\ 0 \end{pmatrix} \tag{21}$$

of linear finite element equations, where the matrices are defined by

$$\begin{aligned} (D\underline{\Phi}_h, \underline{\Psi}_h) &= \frac{1}{\nu_+}\langle \mathcal{D}\Phi_h, \Psi_h\rangle_{\Gamma_+} \quad \Phi_h, \Psi_h \in \mathbf{W_h} \\ (\widetilde{K}\underline{\Phi}_h, T_n\underline{v}_h) &= \langle(-\frac{1}{2}\mathcal{I} - \mathcal{K})\Phi_h, T_n v_h\rangle_{\Gamma_+} \quad \Phi_h \in \mathbf{W_h}, \mathbf{v_h} \in \mathbf{X_h}, \\ ((A_F + A_B)\underline{u}_h, \underline{v}_h) &:= a(u_h, v_h) + \nu_+\langle \mathcal{V}T_n u_h, T_n v_h\rangle_{\Gamma_+} \quad u_h, v_h \in \mathbf{X_h} \end{aligned}$$

The matrices B and C and the component $\underline{f}_h$ of the right-hand side are derived from the bilinear forms $b(\cdot,\cdot)$, $c(\cdot,\cdot)$, and the linear form $\langle f, \cdot\rangle$, respectively.

Let Λ_h be the space of piecewise constants defined on the surface mesh of Γ_+. Then it is easy to see that $T_n : \mathbf{X_h} \to \mathbf{\Lambda_h}$, i.e., T_n can be realized exactly. Furthermore the integral operators in (15) have to be computed for the discrete product space $\mathbf{W_h} \times \mathbf{\Lambda_h}$.

Since $p_h = 0$, we can replace C by any $C_C > 0$. In particular we choose C_C such that $\underline{p}_h = -C_C^{-1}B\underline{u}_h$ can be eliminated directly. We obtain the Schur complement system

$$\begin{pmatrix} D & -\widetilde{K}^T \\ \widetilde{K} & \widetilde{A}_F + A_B \end{pmatrix} \begin{pmatrix} \underline{\Phi}_h \\ \underline{u}_h \end{pmatrix} = \begin{pmatrix} 0 \\ \underline{f}_h \end{pmatrix} \tag{22}$$

where $\widetilde{A}_F := A_F + B^T C_C^{-1} B$.

Now, the system (22) can be transformed into a symmetric positive definite system using Bramble/Pasciak's [3] transformation. It requires a preconditioner for the D–block which satisfies

$$\underline{\gamma}_D C_D \le D \le \overline{\gamma}_D C_D \tag{23}$$

with $\underline{\gamma}_D > 1$. Then, the system

$$\begin{pmatrix} C_D^{-1}D & -C_D^{-1}\widetilde{K}^T \\ -\widetilde{K}C_D^{-1}(D - C_D) & \widetilde{A}_{FB} + \widetilde{K}C_D^{-1}\widetilde{K}^\top \end{pmatrix} \begin{pmatrix} \underline{\Phi}_h \\ \underline{u}_h \end{pmatrix} = \begin{pmatrix} 0 \\ \underline{f}_h \end{pmatrix} \tag{24}$$

is symmetric and positive definite with respect to the scalar product

$$\left[\left(\begin{array}{c}\underline{\Phi}_h\\ \underline{u}_h\end{array}\right),\left(\begin{array}{c}\underline{\Psi}_h\\ \underline{v}_h\end{array}\right)\right] := ((D - C_D)\underline{\Phi}_h, \underline{\Psi}_h) + (\underline{u}_h, \underline{v}_h).$$

We can apply the CG algorithm for solving (24). For this purpose we require a preconditioner $C_{\widetilde{A}_F + A_B}$ which satisfies

$$\underline{\gamma}_{\widetilde{A}_F+A_B} C_{\widetilde{A}_F+A_B} \leq \widetilde{A}_F + A_B \leq \overline{\gamma}_{\widetilde{A}_F+A_B} C_{\widetilde{A}_F+A_B}$$

with constants $\underline{\gamma}_{\widetilde{A}_F+A_B}$, $\overline{\gamma}_{\widetilde{A}_F+A_B}$ independent of the discretization parameter h (meshwidth).

4 Preconditioners and a Numerical Example

In order to obtain (22) we have to specify C_C. We choose $c(p,q) := 1/\nu_- \int_{\Omega_-} \nabla p \nabla q \, dx$ and $C_C := C(I - M_C)^{-1}$, where C is the matrix corresponding to $c(.,.)$ and M_C is the multigrid iteration operator w.r.t. C. In particular we apply a V–cycle with 2 pre– and 2 postsmoothing Gauss–Seidel steps. Moreover, we define M by $(M\underline{u}_h, \underline{v}_h) := \nu_- \int_{\Omega_-} u_h \, v_h \, dx$. Defining further $H := A_F + M$ and $C_H := H(I - M_H)^{-1}$, we conclude that $\widetilde{A}_F$, A_F, H and C_H are pairwise spectrally equivalent, see [5]. Again, M_H is the multigrid iteration operator w.r.t. H. Here we apply a V–cycle with 1 pre– and 1 postsmoothing step. We use the block smoothers introduced in [2] yielding robust convergence in h.

Furthermore, we conclude $A_F \leq A_F + A_B \leq c\, A_F$ from trace theorems in $H(\mathrm{curl})$ (see [1]) and mapping properties of $\mathcal{V}$. Hence, C_H is a preconditioner for $\widetilde{A}_F + A_B$ with constants independent of h.

It is well known that $D \cdot \underline{1} = 0$. In order to define C_D we introduce the projection $P_D : \mathbf{R}^{\mathbf{dim}(\mathbf{W_h})} \rightarrow \mathbf{R}^{\mathbf{dim}(\mathbf{W_h})}$ where $P_D \underline{\Phi}_h := \Psi_h$ with $\Psi_{h,i} := \Phi_{h,i} - \sigma$ and $\sigma := 1/dim(\mathbf{W}_h) \cdot \sum_{i=1}^{dim(\mathbf{W}_h)} \Phi_{h,i}$. Let M_D be the multigrid iteration operator w.r.t. D and $\widetilde{C}_D := D(I - M_D)^{-1}$. Then we choose $C_D^{-1} := \delta^{-1} P \widetilde{C}_D^{-1} P$ which satisfies (23) with constants independent of h (see [4] and references therein). The scaling δ is chosen such that $\underline{\gamma}_D > 1$, in practice $\delta := 0.8$. M_D corresponds to a V–cycle with 1 pre– and 1 postsmoothing Jacobi step.

Now we will present results for some model problem using the code FEPP [6]. The model problem represents some permanent magnet ($0.2 \times 0.2 \times 0.4$) with $H_0 = (0, 0, 1)^\top$ situated at the center of a cube of edge-length 2. The coarsest mesh ($\ell = 1$) consists of 144 tetrahedra, 92 surface elements and 48 nodes at the coupling boundary Γ_+. Finer meshes are obtained by adaptive refinement using the Zienkiewicz-Zhu error estimator with respect to the finite element fluxes in Ω_- (see, e.g., [8]). Fig. 1 on the left shows the boundary elements of the finest tetrahedral mesh corresponding to 6 refinement steps ($\ell = 7$) with 175529 tetrahedra, 1350 surface elements and 677 nodes on Γ_+. The coefficients are given by $\nu_- = \nu_+ = 1$. We obtain the magnetic induction in the middle plane as shown in Fig. 1 on the right.

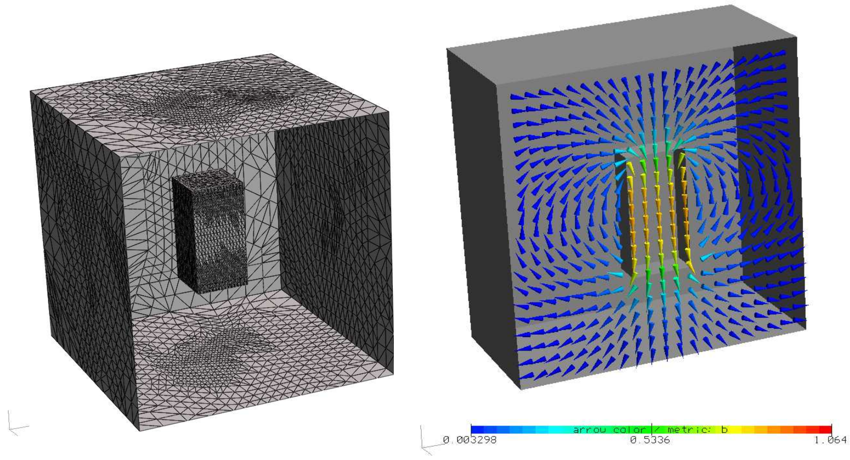

Figure 1. Geometry, final mesh (left), resulting magnetic flux density B (right).

Table 1 shows the corresponding results on an SGI Octane R10000 250 MHz. The Galerkin BEM matrices have been computed semi-analytically. We observe that the iteration numbers given for an relative accuracy of $\varepsilon = 10^{-4}$ are independent of h. Hence our iterative solver is of optimal complexity.

The coupling method can be applied to more general situations within the framework of non-overlapping Domain Decomposition methods. Then, BEM can be used for interior domains as well, see [5] for details. Moreover, DD allows naturally the parallelization of the method similar to the 2D case [4].

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Table 1. Dimension of the discrete spaces, number of iterations (It.) ($\varepsilon = 10^{-4}$), CPU time in seconds for FEM/BEM matrix generation and solver (Sol.). Overall time for each level (Σ) including mesh refinement.

ℓ	$dim(\mathbf{W_h})$	$dim(\Lambda_h)$	$dim(\mathbf{X_h})$	It.	FEM	BEM	Sol.	Σ
1	48	92	246	7	0.1	1.7	0.1	1.8
2	70	136	667	8	0.2	2.9	0.2	3.4
3	129	254	2538	9	0.7	7.0	0.9	9.5
4	159	314	8452	10	2.6	9.6	3.8	19
5	257	510	27416	10	9.4	19.9	14.5	55
6	409	814	79547	11	31	42	54	160
7	677	1350	218508	12	81	101	191	486

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