ABOUT THE NON-MINIMALITY OF THE OUTPUTS OF ZEILBERGER'S ALGORITHM

FRÉDÉRIC CHYZAK

March 8, 2000

ABSTRACT. We report on case studies where Zeilberger's fast algorithm and the other holonomy-based algorithms known so far for definite hypergeometric summation fail to find the minimal annihilating recurrence satisfied by the sum. To explain the phenomenon we propose a new elimination paradigm, together with a promising heuristic method which we hope to turn into an algorithm in the future. The approach applies to ∂-finite functions as well and extends to the related algorithms.

AN ANSATZ THAT ZEILBERGER MISSED

Zeilberger's fast definite hypergeometric summation algorithm [13] inputs a holonomic hypergeometric sequence $h_{n,k}$ and outputs a linear recurrence with respect to n for the sum

$$H_n = \sum_{k\in\mathbb{Z}} h_{n,k}.$$

More specifically the recurrence obtained is represented by a linear recurrence operator with coefficients that are rational functions in n. Based on the holonomic nature of the input, the algorithm searches for an annihilating operator N of the special form

$$N = P + (S_k - 1)B, \tag{1}$$

where P is a non-zero k-free operator of the Ore algebra $\mathbb{C}(n)[S_n; S_n, 0]$ of linear recurrence operators in the shift S_n with rational function coefficients in n, and B is a rational function of $\mathbb{C}(n,k)$. Upon application of N to the summand $h_{n,k}$ and summation over k, the method of creative telescoping [15] proves (in the case of natural boundaries) that the sum H_n is annihilated by the operator P; indeed, the part in $S_k - 1$ yields a telescoping sum which turns out to be zero. Chyzak has extended the algorithm to the case of a ∂-finite summand [3] by replacing the rational function B with a general operator of $\mathbb{C}(n,k)[S_n; S_n, 0][S_k; S_k, 0]$.

In cases where Zeilberger's or Chyzak's algorithm applies, the operator P is not garanteed to be of the smallest possible degree for an annihilating operator of H. For instance, Paule and Schorn gave a parametrized family of example where Zeilberger's algorithm and provably no other holonomy-based algorithm that searches for an operator of the form (1) can compute the minimal annihilating operator [10]. The order gap can furthermore be made as large as wished. This fact has a geometrical interpretation: the operator P in (1) is taken from

$$(\operatorname{Ann} h + (S_k - 1)\mathbb{C}(n,k)[S_n; S_n, 0][S_k; S_k, 0]) \cap \mathbb{C}(n)[S_n; S_n, 0],$$

where $\operatorname{Ann} h$ is the left ideal in $\mathbb{C}(n,k)[S_n; S_n, 0][S_k; S_k, 0]$ of operators annihilating h. This intersection does not contain any operator of the minimal order in the case of the example by Paule and Schorn.

Based on the observation of the case studies in Part I, we propose to replace the special form (1) for N by the new special form

$$N = P + (S_k - 1)B + Z, \tag{2}$$

where P is a k-free operator from $\mathbb{C}(n)[S_n; S_n, 0]$ and B and Z are rational functions from $\mathbb{C}(n,k)$ with the essential constraint that

$$\sum_{k \in \mathbb{Z}} Z(n,k) f_{n,k} = 0. \tag{3}$$

In the case of a ∂-finite summand, Z is replaced with a general operator of $\mathbb{C}(n,k)[S_n; S_n, 0][S_k; S_k, 0]$. Our observation is that non-minimality occurs in the cases of Part I because there exists a non-zero Z modulo $S_k - 1$ on the left. Such an operator Z is neither accomodated by Zeilberger's algorithm [13] nor by Chyzak's algorithm [3]. As proved in Section 1, the order gap between the minimal annihilating operator for the sum and the output of Zeilberger's algorithm is exactly the order of the output of Zeilberger's algorithm applied to sum Zf for suitable Z.

After Section 1, which provides with an interpretation of the order gap in the form of a lemma, Part I reports on a few case studies which shed light on the non-minimality behaviour of Zeilberger's algorithm; in Part II two heuristics to obtain triples (P, B, Z) satisfying (2) are then detailed.

Acknowledgment. The question of non-minimality of the output of Zeilberger's algorithm was raised already a few years ago by Peter Paule and Volker Strehl. Peter Paule also pointed to the examples I develop in Part I. Bruno Salvy has helped by stimulating discussions.

This work was mainly done as I was visiting RISC-Linz and was supported by the SFB grant F1305 of the Austrian Science Foundation (FWF).

1. Interpretation of the Order Gap

Lemma 1. *Given a non-zero holonomic ∂-finite summand $f_{n,k}$ for ∂-finiteness with respect to the operator algebra $\mathbb{C}(n,k)[S_n; S_n, 0][S_k; S_k, 0]$, let*

$$P + (S_k - 1)B + Z \quad \text{and} \quad P_0 + (S_k - 1)B_0$$

be two annihilating operators of f where P and P_0 are non-zero elements of $\mathbb{C}(n)[S_n; S_n, 0]$, B, B_0, and Z are elements of $\mathbb{C}(n,k)[S_n; S_n, 0][S_k; S_k, 0]$ with the constraint that the sums over k of

$$P \cdot f, \quad P_0 \cdot f, \quad (S_k - 1)B \cdot f, \quad (S_k - 1)B_0 \cdot f, \quad \text{and} \quad Z \cdot f$$

are well-defined, and that additionally P is of minimal order amongst the possible P in the tuples (P, B, Z), that P_0 of minimal order amongst the possible P_0 in the tuples (P_0, B_0), and that

$$\sum_{k\in\mathbb{Z}} (Z \cdot f)(n,k) = 0.$$

Consider then an annihilating operator

$$P_1 + (S_k - 1)B_1$$

of $Z \cdot f$ where P_1 is a non-zero element of $\mathbb{C}(n)[S_n; S_n, 0]$, B_1 is an element of $\mathbb{C}(n,k)[S_n; S_n, 0][S_k; S_k, 0]$ with the constraint that the sums over k of

$$P_1 Z \cdot f \quad \text{and} \quad (S_k - 1)B_1 Z \cdot f$$

are well-defined, and that additionally P_1 is of minimal order amongst the possible P_1 in the tuples (P_1, B_1).

Then if f and $Z \cdot f$ generate the same $\mathbb{C}(n,k)[S_n; S_n, 0][S_k; S_k, 0]$-module, the order gap between P_0 and P is exactly the order of P_1.

Before proving this lemma, let us remark that (P_0, B_0) and (P_1, B_1) are the outputs of the holonomy-based Zeilberger-Chyzak-like algorithms applied to f and $Z \cdot f$, respectively. Furthermore, the last hypothesis

$$\mathbb{C}(n,k)[S_n; S_n, 0][S_k; S_k, 0] \cdot f = \mathbb{C}(n,k)[S_n; S_n, 0][S_k; S_k, 0] Z \cdot f$$

is equivalent to saying that Z is invertible modulo the annihilating operators of f. This always holds when f is non-zero hypergeometric.

Proof. Since by construction both P and P_0 cancel $\sum_{k\in\mathbb{Z}} f$ and P is of minimal order, there exists Q in $\mathbb{C}(n)[S_n; S_n, 0]$ such that $P_0 = QP$. Then, we obtain another annihilator of f,

$$QZ + (S_k - 1)(QB - B_0),$$

and $\sum_{k\in\mathbb{Z}} QZ \cdot f = Q \cdot \sum_{k\in\mathbb{Z}} Z \cdot f = 0$. By the inversibility of Z, there exists H in $\mathbb{C}(n,k)[S_n; S_n, 0][S_k; S_k, 0]$ such that

$$(QB - B_0) \cdot f = HZ \cdot f.$$

Now, the operator $Q + (S_k - 1)H$ cancels $Z \cdot f$, so that by the minimality of P_1, there exists Q_1 in $\mathbb{C}(n,k)[S_n; S_n, 0][S_k; S_k, 0]$ such that $Q = Q_1 P_1$. We contend that Q_1 is invertible, i.e., is a non-zero rational function from $\mathbb{C}(n,k)$, so that the order gap between P_0 and P, which is also the degree of Q, is exactly the order of P_1, as was to be proved.

Indeed, it follows from the definitions of P and P_1 that

$$P_1 P + (S_k - 1)(P_1 B - B_1 Z)$$

annihilates f. Then, by the minimality of P_0, there exists some Q_2 to be found in $\mathbb{C}(n)[S_n; S_n, 0]$ such that $P_1 P = Q_2 P_0$. Since $P_0 = QP = Q_1 P_1 P$, this entails that $P_1 P = Q_2 Q_1 P_1 P$. We thus have the relation $1 = Q_2 Q_1$, whence that Q_1 is invertible. ∎

The following lemma shows that any multiplier Z that appears in (3), and takes the role of L_0 below, has to be found in an annihilator N of the form (2). It is a new form for the "Fundamental Lemma of Holonomic Summation and Integration," already given in the hypergeometric case in [11, 12], and whose essence is an elimination lemma in the Weyl algebra [14, Lemma 4.1]. The kind of a result which is still missing is that minimal order operator P that cancels a sum gives rise to a relation of the form (2) for a suitable Z.

Lemma 2. *Let $f_{n,k}$ be a holonomic ∂-finite summand for ∂-finiteness with respect to the operator algebra $\mathbb{C}(n,k)[S_n;S_n,0][S_k;S_k,0]$ and L_0 be any element of $\mathbb{C}[n,k][S_n;S_n,0][S_k;S_k,0]$. Then there exists a non-zero operator L from $\mathbb{C}[n][S_n;S_n,0][S_k;S_k,0]$, free of k and such that $L+L_0$ annihilates f.*

Note that the result is obvious if L_0 is non-zero and already free of k (just take $L=-L_0$), and that any L in question can uniquely be rewritten in the form $P+(S_k-1)B$, for a P which does not involve S_k.

Proof. We proceed to prove that the map

$$\phi : \mathbb{C}[n][S_n;S_n,0][S_k;S_k,0] \mapsto \mathbb{C}[n,k][S_n;S_n,0][S_k;S_k,0]\cdot f$$

defined by $\phi(L)=(L+L_0)\cdot f$ is not injective, whence that there exists an L that satisfies the claim. To this end, introduce the total degree filtration $(\mathcal{F}_p)_{p\geq 0}$ of the algebra $\mathbb{C}[n,k][S_n;S_n,0][S_k;S_k,0]$. In other words, $\mathcal{F}_p$ is the vector space of all operators of total degree in (n,k,S_n,S_k) less than or equal to p. For each $p\geq 0$, this induces a restriction map

$$\phi_p : \mathbb{C}[n][S_n;S_n,0][S_k;S_k,0]\cap\mathcal{F}_p \mapsto \mathcal{F}_p\cdot f.$$

Now, $\mathbb{C}[n][S_n;S_n,0][S_k;S_k,0]\cap\mathcal{F}_p$ has dimension $O(p^3)$ as a $\mathbb{C}$-vector space, whereas $\mathcal{F}_p\cdot f$ has dimension $O(p^2)$ by the holonomy of f. So for large enough p, the map ϕ_p is not injective, and neither is ϕ. ☐

Part I. Case Studies Showing the Non-Minimality Behaviour

The programs that have been used for these case studies are the Paule-Schorn implementation *zb* of Zeilberger's algorithm [10], the Paule-Riese implementation *qZeilb* of the q-analogue algorithm [9], and our package *Mgfun* for other interactive manipulations. The packages can be obtained: at the URL `http://www.risc.uni-linz.ac.at/software/` for *zb* and *qZeilb*; at the URL `http://algo.inria.fr/chyzak/mgfun.html` for *Mgfun*.

2. A Non-Minimality Example by Paule and Schorn

2.1. Classical Approach. The identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{tk}{n}=(-t)^n \tag{4}$$

for integers $t \geq 2$ and $n \geq 0$ is immediate from the summation theorem [6]

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{i\geq 0}c_i\binom{k}{i} = (-1)^n c_n,$$

after noting that

$$\binom{tk}{n} = t^n\binom{k}{n} + \text{ lower order terms.}$$

Surprizingly enough, the first order operator $S_n + t$ is not obtained by Zeilberger's fast algorithm for $t \geq 3$ [10]. This corresponds to the fact that the result of the application of $S_n + t$ to the summand is not Gosper summable for those values of t.

Throughout the remainder of this section, let

$$f_{n,k;t} = (-1)^k\binom{n}{k}\binom{tk}{n} \quad \text{and} \quad F_{n;t} = \sum_{k=0}^{n} f_{n,k;t}.$$

2.2. **A Weird Contiguity Relation.** Before treating identity (4) from the point of view of our new paradigm, we proceed to rederive it by proving a mysterious contiguity relation that could well be related to a hidden symmetry.

Distinguishing the k that appear in each of the binomials of $f_{n,k;t}$, we consider $(-1)^{k'}\binom{n}{k'}\binom{tk}{n}$. Setting $k' = k - p$ leads to the study of

$$f_{n,k;t,p} = (-1)^{k+p}\binom{n}{k-p}\binom{tk}{n}.$$

We proceed to prove that the bivariate sum

$$F_{n;3,p} = \sum_{k=0}^{n} f_{n,k;3,p} \tag{5}$$

equals $(-3)^n$ independently from the integer $p \in \mathbb{Z}$. This extends the original identity, which corresponds to the case $p = 0$.

The hypergeometric term $f_{n,k;3,p}$ being holonomic in (n, k, p), it is possible to simulaneously eliminate the indeterminates k and S_k from a description of its annihilating ideal in the algebra

$$\mathbb{C}[n, p, k][S_n; S_n, 0][S_p; S_p, 0][S_k; S_k, 0];$$

this can be performed by a Gröbner basis calculation and it turns out that $f_{n,k;3,p}$ is annihilated by the k-free operator

$$C = (n + 1)S_n + (2n + 3p + 3)S_p + n - 3p.$$

At this point note that by setting S_p to 1, this operator C becomes

$$(n + 1)(S_n + 3),$$

a multiple of the annihilating operator of $F_{n;3} = F_{n;3,0}$ which we are hunting for. Thus, if we knew the independence of $F_{n;3,p}$ from p, we would have obtained the minimal annihilating operator of $F_{n;3}$. Applying C to $f_{n,k;3,p}$ and summing over k yields the contiguity relation

$$(n+1)F_{n+1;3,p} + (2n+3p+3)F_{n;3,p+1} + (n-3p)F_{n;3,p} = 0. \tag{6}$$

For any integer $p \in \mathbb{Z}$, the equality of (5) to $(-3)^n$ is easily verified in the case $n = 0$; it is then obtained by induction on $n \in \mathbb{N}$ from the previous contiguity relation.

Of course, the same k-free operator annihilates

$$(-1)^k \binom{n}{k} \binom{3(k+p)}{n}.$$

On this form, the identity can also be obtained directly as an application of the classical method of the previous section. Also note that the contiguity relation (6) can easily be obtained by the variant of Takayama's algorithm described in [4].

2.3. **The New Paradigm.** In the case $t = 3$ it was noted that Zeilberger's algorithm fails to obtain the first order operator $S_n + 3$ for the sum (5). However, it is recovered once one notices

$$\sum_{k=0}^{n} \tilde{f}_{n,k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{3k}{n} (3k - n(n+2)) = 0, \tag{7}$$

for the modified summand

$$\tilde{f}_{n,k} = (3k - n(n+2)) f_{n,k} = (-1)^k \binom{n}{k} \binom{3k}{n} (3k - n(n+2)),$$

and non-negative integers n, and that the operator N of the form (2) obtained when

$$P = 2(n+1)(2n+1)(S_n+3), \tag{8}$$

$$B = -\frac{27k^3 - 27(n+1)k^2 + 3(3n^2+6n+2)k - n(n+1)(n+2)}{k-n-1}, \tag{9}$$

$$Z = 9(3k - n(n+2)), \tag{10}$$

annihilates the original summand f. On the other hand, note that the sum in (7) is non-terminating and divergent for non-integer n.

We obtained the operator Z above from the knowledge of the operator P and the heuristic described in Section 6. Conversely, the knowledge of Z and the nullity of the modified sum

$$\tilde{F}_n = \sum_{k=0}^{n} \tilde{f}_{n,k}$$

suffices to obtain the annihilating operator N, including the operator P of minimal order; more specifically, modifying Zeilberger's algorithm by replacing the ansatz

$$\sum_{i=0}^{L} \eta_i(n) S_n^i$$

with the new form

$$9(3k - n(n+2)) + \sum_{i=0}^{L} \eta_i(n) S_n^i$$

returns the values for the $\eta_i(n)$ when $0 \le i \le 1 = L$ and for the rational function B. Applying to $\tilde{f}_{n,k}$ and summing over k yields

$$\begin{aligned} 0 &= \sum_{k=0}^{n} \left(N \cdot \tilde{f}\right)(n,k) \\ &= \left(2(n+1)(2n+1)(S_n+3) \cdot \tilde{F}\right)(n) + 0 - \left[B(n,k)\tilde{f}_{n,k}\right]_{k=0}^{k=n}, \end{aligned}$$

where the last term actually is zero. For this summation to be over natural boundaries, a crucial point is to argue that the denominator $k-n-1$ of B actually introduces no singularity; this is readily proven in view of the identity

$$B\tilde{f} = -(3k-n-1)(3k-n-2)S_n \cdot f.$$

For this matter, any linear combination of f and $\tilde{f}$ with coefficients that are any difference operators with polynomial coefficients would do for the right-hand side.

Again, the modified summand $\tilde{f}$ is not Gosper summable, so that Zeilberger's algorithm does not find the expected trivial recurrence operator of order 0 satisfied by the zero sum $\tilde{F}$, namely the identity 1. Indeed, it returns the operator

$$N' = 2(2n+1)S_n + 3(n+2)$$

of order 1, with "parasitic" solution

$$(-3)^n (n+1) \binom{2n}{n}^{-1}.$$

Up to a polynomial left factor from $\mathbb{C}[n]$, the product $N'P$ is the operator

$$2(2n+3)S_n^2 + 3(5n+7)S_n + 9(n+1) = (2(2n+3)S_n + 3(n+1))(S_n+3)$$

of order two which is obtained by Zeilberger's algorithm. This could in fact be predicted:

$$N'N = N'P + (S_k - 1)N'B + N'Z = N'P + (S_k-1)(N'B + CZ), \tag{11}$$

where C is the rational function delivered by Zeilberger's algorithm together with N', and precisely such that

$$N'Z \cdot f = N' \cdot \tilde{f} = (S_k - 1)C \cdot \tilde{f} = (S_k - 1)CZ \cdot f.$$

Since $N'N$ annihilates h and $N'P \neq 0$, the right-hand side in (11) is the operator obtained by Zeilberger's algorithm.

Remark: the equality of the modified sum to zero rewrites

$$ {}_7F_3\left(\left.\begin{matrix} -n, -n(n+2)/3, 1/3, 2/3, 1, 1, 1 \\ 1/3 - n/3, 2/3 - n/3, 1 - n/3 \end{matrix}\right| 1\right) = 0 $$

for $n \in \mathbb{N}$. Again, it is divergent for non-integer n.

2.4. **The Case $t = 4$.** Similarly when $t = 4$, Zeilberger's algorithm applied to $f_{n,k;4}$ returns the operator

$$ \begin{aligned} (12)\quad & 3(n+3)(3n+4)(3n+7)(3n+8)S_n^3 \\ & \quad + 4(n+3)(3n+4)(37n^2+180n+218)S_n^2 \\ & \quad + 16(n+2)(n+3)(33n^2+125n+107)S_n \\ & \qquad\qquad + 64(n+1)(n+2)(n+3)(3n+7) \end{aligned} $$

of order three, instead of the order one operator $P = S_n + 4$.

Yet, to recover this first order operator, it suffices to prove that

$$ \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{4k}{n}(64k^2 + 4(4-5n-3n^2)k - n(15+23n+10n^2)) = 0, $$

and to consider the annihilator N of $f = (f_{n,k;4})$ given in the form (2) by

$$ \begin{aligned} P &= S_n + 4, \\ B &= \frac{(4k-n)(4k-n-1)(4k-n-2)(4k-n-3)}{3(n+1)(3n+1)(3n+2)(n+1-k)}, \\ Z &= \frac{8(64k^2 + 4(4-5n-3n^2)k - n(15+23n+10n^2))}{3(n+1)(3n+1)(3n+2)}. \end{aligned} $$

As in the case $t = 3$, we obtained this operator Z from the operator P by using the heuristic described in Section 6. So far, we do not understand whether the congruence modulo 3 in the above denominators has any significance in relation to $t = 4$. To see that the singularity induced by B is no problem, note that

$$ Bf = \frac{(4k-n-1)(4k-n-2)(4k-n-3)}{3(n+1)(3n+1)(3n+2)} S_n \cdot f. $$

The same singularity can be made obvious for P on the corresponding form:

$$ P = \frac{3n+4}{n+1-k} \mod \operatorname{Ann} f. $$

Zeilberger's algorithm applied to Zf now returns the operator

$$ \begin{aligned} 3(3n+4)(3n+7)(3n+8)S_n^2 + 20(n+2)(2n+5)(3n+4)S_n& \\ +16(n+1)(n+2)(3n+7)& \end{aligned} $$

of order two, instead of the trivial identity operator (or order zero).

Remark: another candidate for Z is

$$Z = 8k - n(3n+5)$$

and was obtained by the heuristic described in Section 7; Zeilberger's algorithm applied to Zf now returns the operator

$$\begin{aligned}3n(3n+4)(3n+5)S_n^2 + 4(n+3)(10n^2+15n+2)S_n \\ +16(n+1)(n+2)(n+3)\end{aligned}$$

of order two, instead of the trivial identity operator (or order zero). The corresponding B is

$$B = -3\frac{n(n+1)(3n+4)(4k-n)(4k-n-1)(4k-n-2)(4k-n-3)}{(k-n-1)(k-n-2)(8k-n(3n+5))}.$$

2.5. **Hidden Symetry?** For $t=3$, let us apply the usual heuristic of creative symmetrizing to try and reduce the order of the ouputs of Zeilberger's algorithm. To this end, introduce

$$\bar{f}_{n,k} = \frac{1}{2}(f_{n,k;3} + f_{n,n-k;3})$$

whose sum equals that of the original $f_{n,k;3}$. Applying Chyzak's ∂-finite extension of Gosper's algorithm shows that $(S_n+3)\cdot\bar{f}$ has no indefinite sum in the module (2-dimensional vector space) generated by $\bar{f}$, a failure of the creative symmetrizing heuristic. However, this raises the question of another hidden symmetry: it could still be possible that a symmetry be hidden in the form of an expression

$$\bar{f}_{n,k} = P(n, S_k)\cdot f_{n,k;3} + B(n, S_k)\cdot f_{n,n-k;3}$$

for which $(S_n+3)\cdot\bar{f}$ would be indefinitely summable by the ∂-finite extension of Gosper's algorithm (in the module generated by this $\bar{f}$).

2.6. **Obvious But Useless Other Candicates For Z.** Observing the hypergeometric nature of f and starting from the identity

$$q(n,k)f_{n,k+1;t} = p(n,k)f_{n,k;t}$$

for any two suitable polynomials p and q satisfying

$$\frac{f_{n,k+1;t}}{f_{n,k;t}} = \frac{p(n,k)}{q(n,k)},$$

one is tempted to introduce a generic (hypergeometric) term g_k and the multiplicative twister

$$Z_{n,k} = p(n,k)g_{k+1} - q(n,k-1)g_k,$$

for then the sum of the hypergeometric summand Zf is zero:

$$\sum_{k=0}^{n} Z_{n,k} f_{n,k} = \sum_{k=0}^{n} (p(n,k)g_{k+1} - q(n,k-1)g_k) f_{n,k;t}$$
$$= \sum_{k=-1}^{n} (p(n,k)f_{n,k;t} - q(n,k)f_{n,k+1;t}) g_{k+1} = 0,$$

after shifting the term in q and using the recurrence equation satisfied by f. In particular for $g_k = g_{k+1} = 1$, the twister

$$Z_{n,k} = p(n,k) - q(n,k-1)$$

first appears to be a good candidate to explain non-minimality, but is immediately rejected in view of the fact that Zf is then Gosper summable:

$$Z_{n,k} f_{n,k} = (p(n,k) - q(n,k-1)) f_{n,k} = q(n,k) f_{n,k+1} - q(n,k-1) f_{n,k}.$$

2.7. **Gould-Type Inverse Relations.** The Gould class of inverse relations [7, Sec. 1.3] relates the definition

$$f_n = \sum_{k\in\mathbb{Z}} (-1)^k \binom{n}{k} \binom{a+bk}{n} g_k \tag{13}$$

of a sequence f in terms of another sequence g and its inverse relation

$$g_n = \sum_{k\in\mathbb{Z}} (-1)^k \binom{a+bn}{n}^{-1} \frac{a+bk-k}{a+bn-k} \binom{a+bn-k}{n-k} f_k, \tag{14}$$

which gives g in terms of f. For pairs (a,b) such that the denominator $a+bn-k$ could vanish, we use the variant relation

$$g_n = \sum_{k\in\mathbb{Z}} (-1)^k \binom{a+bn}{n}^{-1} \frac{a+bk-k}{a+bn-n} \binom{a-1+bn-k}{n-k} f_k.$$

Composing both inverse relations in different orders yields two associated orthogonality relations,

$$\sum_{k\in\mathbb{Z}} (-1)^{m+k} \binom{a+(b-1)k}{n-k} \frac{a+bm-m}{a+bk-k} \binom{a-1+bk-m}{k-m} = \delta_{n,m} \tag{15}$$

and

$$\sum_{k\in\mathbb{Z}} (-1)^{m+k} \binom{k}{m} \binom{a+bm}{k} \binom{a+bn}{n}^{-1} \times \frac{a+bk-k}{a+bn-n} \binom{a-1+bn-k}{n-k} = \delta_{n,m}. \tag{16}$$

Our focus is on the case $(a,b) = (0,3)$, and on the pair of sequences defined by

$$f_n = (-3)^n \quad \text{and} \quad g_n = 1.$$

We observed that the Zeilberger's algorithm does not return the minimal (order one) recurrence for the direct relation (13); on the other hand, studying the sum

$$\sum_{k\in\mathbb{Z}} t_k \quad \text{for} \quad t_k = 3^k \frac{2k}{3n-k}\binom{3n}{n}^{-1}\binom{3n-k}{n-k}$$

involved in the reverse relation (14), we obtain that the summand t_k is Gosper summable:

$$t_k = (S_k - 1)\cdot\left(-\frac{3n-k}{2k}t_k\right).$$

The equality to 1 follows upon summation over k, but the interesting point is that the annihilating operator obtained in this way *is that of minimal order*, namely $S_n - 1$. Turning to the orthogonality identities, we first note that the summand s_k in the first form (15) is Gosper summable, even for generic a and b:

$$s_k = (S_k - 1)\cdot -\frac{(k-m)(a+bn-k)}{(a+bk-k)(n-m)}s_k;$$

on the other hand, the summand in the second form (16) is not Gosper summable.

We run Chyzak's multivariate extension of Zeilberger's algorithm to perform the first summation summation (15) over k in the case $(a,b) = (0,3)$. We obtain the Gröbner basis

$$\begin{aligned} P_1 &= 2(m+1)(m-n)(2m-2n-1) + 3m(m-n+1)(m-n)S_m \\ &\qquad +2(n+1)(2n+1)(m+1)S_n, \\ P_2 &= 2(2m+1)(m+2)(m+1) + (m+2)(7m-n+5)(m-n+1)S_m \\ &\qquad +3(m+1)(m+2-n)(m-n+1)S_m^2, \end{aligned}$$

which annihilates the sum. More specifically, the rational multipliers B_i such that for each i, $P_i + (S_k - 1)B_i$ annihilates the summand are:

$$B_1 = \frac{(m+1)(3n+2-m)}{n+1-m}B_0 \quad \text{and} \quad B_2 = \frac{(m+1)(m+2)(2m+1)}{m(n-m)}B_0$$

for

$$B_0 = \frac{(k-m)(3k-n)(3k-n-1)(3k-n-2)}{(3k-m-1)(3k-m-2)}.$$

This B_0 introduces no singularity, owing to the relation

$$B_0(n,m,k)t_k = (k-n-1)(3k-m-3)S_k^{-1}\cdot t_k,$$

and therefore each $(S_k-1)\cdot B_i(n,m,k)t_k$ can be summed over natural boundaries. However, the multipliers of B_0 in B_1 and B_2 contain denominators that seemingly cannot be removed by combining shifts. It entails that the system $\{P_1, P_2\}$ is only valid when $m(n-m)(n+1-m) \neq 0$. As a consequence, the sum can be proved to be zero for $n > m$, but the zone $n < m$

seemingly cannot be related to the zone $n \geq m$. A natural problem still to be understood is to understand if this relates to non-minimality.

2.8. **q-Analogue.** The identity (4) of this section

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{tk}{n} = (-t)^n,$$

for positive integers $t \geq 2$ and $n \geq 0$, has the following q-analogue, that was recently invented by Abramov, Paule, and Petkovšek [1]: for positive integers t and n, we have

$$\sum_{k=0}^{n} (-1)^k q^{t\binom{n-k}{2}} \binom{n}{k}_{q^t} \binom{tk}{n}_q = (-1)^n q^{(t-1)\binom{n}{2}} \frac{(q^t; q^t)_n}{(q;q)_n}$$

where

$$(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{and} \quad \binom{n}{m}_q = \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}.$$

The same non-minimality behaviour was observed in [1] for this q-case when $t = 3$. It should be possible to study this q-identity the way we do for the Rogers-Ramanujan identity in Section 4 of the present paper.

3. Hagen's Identity

3.1. **Origin of the Problem.** The identity

$$\sum_{k=0}^{n} \frac{1}{tk+1} \binom{tk+1}{k} \binom{t(n-k)}{n-k} = \frac{tn+1}{(t-1)n+1} \binom{tn}{n} = \binom{tn+1}{n}$$

for integers $t \geq 0$ and $n \geq 0$ is known as Hagen's identity [5, p. 80]; it is also mentioned in [11, p. 117]. When $t = 3$, Zeilberger's algorithm applied to the summand

$$g_{n,k;t} = \frac{1}{tk+1} \binom{tk+1}{k} \binom{t(n-k)}{n-k} = \frac{1}{(t-1)k+1} \binom{tk}{k} \binom{t(n-k)}{n-k}$$

returns the recurrence operator of order 2

$$\begin{aligned} 4(n+2)(2n+3)(2n+5)S_n^2 - 12(2n+3)(9n^2+27n+22)S_n \\ +81(n+1)(3n+2)(3n+4) \\ = (2(n+3)S_n - 27(n+1))(2(n+1)(2n+3)S_n - 3(3n+2)(3n+4)), \end{aligned} \tag{17}$$

whereas a recurrence of order 1 only is expected.

Of course, the summand $g'_{n,k;t} = (g_{n,k;t} + g_{n,n-k;t})/2$, which by construction is symmetric about $k = n/2$, shares the same sum as the original $g_{n,k;t}$.

Petkovšek additionally noticed that g' is again hypergeometric:

$$\begin{aligned} g'_{n,k;t} &= \frac{g_{n,k;t} + g_{n,n-k;t}}{2} \\ &= \frac{(t-1)n+2}{2} \frac{1}{(t-1)(n-k)+1} \binom{t(n-k)}{n-k} \frac{1}{(t-1)k+1} \binom{tk}{k}, \end{aligned}$$

and that applying Zeilberger's algorithm to the symmetrized version $g'_{n,n;3}$ now returns the right-hand factor of order 1 in the factorization (17) above.

3.2. **New Ansatz.** For $t = 3$, we derived the annihilating operator $N = P + (S_k - 1)B + Z$ of the summand

$$g_{n,k} = \frac{1}{3k+1} \binom{3k+1}{k} \binom{3(n-k)}{n-k}$$

for

$$\begin{aligned} P &= S_n - \frac{3(3n+2)(3n+4)}{2(n+1)(2n+3)}, \\ B &= \frac{3k(18k^3 - 9(4n+1)k^2 + (18n^2-5)k + (3n+1)(3n+2))}{2(n+1)(2n+3)(k-n-1)}, \\ Z &= -\frac{3(2k-n)(18k^2 - 18nk - (8n+3))}{2(n+1)(2n+3)(2k-2n-1)}. \end{aligned}$$

These operators were obtained by a derivation following the heuristic of Section 6. This time, Z is not a polynomial in k, but a rational function. However, its only pole introduces no singularity at integer points $(n, k) \in \mathbb{Z}^2$. One verifies that the sum of Zg is zero, or in other words,

$$\sum_{k=0}^{n} -\frac{3(2k-n)(18k^2 - 18nk - (8n+3))}{2(n+1)(2n+3)(2k-2n-1)(3k+1)} \binom{3k+1}{k} \binom{3(n-k)}{n-k} = 0.$$

This is obtained for instance by using Zeilberger's algorithm, which returns a recurrence of order one only, and by checking a single initial conditions.

Note that the summand Zg satisfies the antisymmetry relation:

$$(Zg)(n, n-k) = -(Zg)(n, k).$$

In this case, going from the non-minimal recurrence of order one to the minimal recurrence of order zero is thus a degenerate instance of creative symmetrizing.

As a last remark, also note that operating the heuristic of Section 6 in the localized algebra $\mathcal{L}$ described there is essential to get the rational function Z above with numerator of degree 2: working in the usual algebra $\mathbb{C}(n)[k]$ of polynomials in k yields a different candidate Z with numerator of higher degree.

3.3. **Another Candidate.** Again for $t = 3$, another candidate for Z is

$$Z = (2k + 1)(2k - n)$$

and was obtained by the heuristic described in Section 7 with the ansatz of a polynomial in n and k of total degree 2; Zeilberger's algorithm applied to Zg now returns the operator

$$2(2n + 1)S_n - 27(n + 1)$$

of order one, instead of the trivial identity operator (or order zero).

Additionally, Z is a good candidate, and we found the triple solution:

$$\begin{aligned} P &= 2(n + 1)(2n + 3)S_n - 3(3n + 2)(3n + 4), \\ B &= -\frac{3k(2k + 1)(3k - 1 - 3n)(3k - 2 - 3n)(4k - 5 - 6n)}{(2n + 1)(k - 1 - n)(2k - 1 - 2n)}, \\ Z &= \frac{27(2k + 1)(2k - n)}{2n + 1}, \end{aligned}$$

with the property that Bf is summable over natural boundaries, owing to

$$Bf = -\frac{1}{9}(3k - 1)(3k - 2)(4k - 5 - 6n)S_k \cdot f.$$

4. A Finite Version of Rogers-Ramanujan Identity

$$\sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k}(q; q)_{n+k}} = \sum_{k=0}^{n} \frac{q^{k^2}}{(q; q)_k(q; q)_{n-k}}$$

for an integer $n \in \mathbb{N}$.

4.1. **Creative Symmetrizing.** The q-version of Zeilberger's algorithm applied to the sum

$$\sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k}(q; q)_{n+k}}$$

(where $(x; q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$) returns a recurrence operator of order 5, although the sum also satisfies an operator of order 2 only. Paule's creative symmetrizing makes it possible to obtain this operator of order 2 by noting the antisymmetry of the summand when k goes to $-k$, and therefore that the initial sum equals the other sum

$$\sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k}(q; q)_{n+k}} \frac{1 + q^k}{2}.$$

Now, the q-version of Zeilberger's algorithm applied to this last sum returns the operator

$$\left(1 - q^{n+2}\right) S_n^2 - \left(1 + q - q^{n+2} + q^{2n+3}\right) S_n + q,$$

which is of the desired order [8].

A consequence of the equality between the sum and its symmetrized version is the nullity

$$\sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}(q^k-1)=0.$$

This suggests q^k-1 as a candidate for Z in the new ansatz.

4.2. **New Ansatz.** With a variant heuristic not described in Part II but better suited to the q-case, we got the following values

$$\begin{aligned}
P &= \left(1-q^{n+2}\right)S_n^2-\left(1+q-q^{n+2}+q^{2n+3}\right)S_n+q,\\
B &= \frac{q^{2n-2k+4}(1-q^{n+k+2})B'}{(1+q^{n+2})(1-q^{2n+3})(1+q^{2n+1})}S_n^2 \qquad \operatorname{mod}\ \operatorname{Ann} h,\\
Z &= \frac{q^{n+3}Z_1'}{(1+q^{n+2})(1-q^{2n+3})(1+q^{2n+1})}S_n\\
&\quad+\frac{q^{n+3}Z_0'}{(1+q^{n+2})(1-q^{2n+3})(1+q^{2n+1})} \qquad \operatorname{mod}\ \operatorname{Ann} h,
\end{aligned}$$

where h is the original summand $h_{n,k}=(-1)^k q^{(5k^2-k)/2}(q;q)_{n-k}^{-1}(q;q)_{n+k}^{-1}$ and

$$\begin{aligned}
B' &= q^{5n+6}-q^{4n+k+5}+q^{3n+5}-q^{4n+k+4}+q^{3n+4}+q^{3n+2k+3}\\
&\quad+q^{3n+k+3}-q^{2n+k+3}+q^{3n+k+2}-q^{2n+k+2}-q^{2n+2}-q^{2n+1}\\
&\quad-q^{2n+2k+1}+q^{n+k+1}+q^{n+k}-1,\\
Z_0' &= -1+q^n-q^{2n+3k+3}+q^{n+3k+1}+q^{n+1}-q^{2n+1}+q^{3n+2}-q^{2n+2},\\
Z_1' &= 1+q^{4n+3}-2q^{3n+2}+2q^{2n+2}+q^{2n+1}+q^{5n+5}-q^{5n+4}\\
&\quad+q^{3n+4}-q^{3n+3}-2q^{n+1}+q^{3n+3k+2}-q^{4n+3k+4}-q^n+q^{2n+k+2}\\
&\quad+q^{4n+k+3}-q^{5n+k+5}-q^{3n+k+4}+q^{2n+2k+2}+q^{5n+2k+5}-q^{3n+k+3}\\
&\quad+q^{2n+k+1}-q^{3n+2k+2}-q^{n+2k+1}.
\end{aligned}$$

(Here to avoid length, we do not explicitly show the rational functions B and Z, which are uniquely determined by the above relations.) It is obvious on this presentation of P, B, and Z that no cancellation of any denominator arises of any value of n and k, so that the summation over all k is possible. Beside this, one proves

$$\sum_{k=-n}^{n} Z\frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}=0.$$

Using Riese's implementation of the q-variant of Zeilberger's fast algorithm, we obtained a recurrence of order 3 and no recurrence of order 2, *as was hoped*.

So far, we have been unable to relate the multiplier Z in this zero sum with the "predicted" multiplier q^k-1; nor have we found any symmetry of the summand Zg which would explain the nullity of the sum.

5. Andrews' Identity (Pfaff's Method)

The identity

$$ {}_5F_4\left(\begin{matrix}-2n-1, 2x+2n+2, 2x-z+1/2, 2x+n+1, z+n+1\\ x+1/2, x+1, 2z+2n+2, 4x-2z+1\end{matrix}\middle|1\right)=0 $$

for an integer $n \geq 0$ is proved by contiguity methods in [2, Eq. (1.6)] (with $(i, x/2)$ replaced with (n, x)). There it is used as a key step to derive a new evaluation of the Mills-Robbins-Rumsey determinant by Pfaff's method.

Introduce

$$ u_n = {}_5F_4\left(\begin{matrix}-2n-1, 2x+2n+2, 2x-z+1/2, 2x+n+1, z+n+1\\ x+1/2, x+1, 2z+2n+2, 4x-2z+1\end{matrix}\middle|1\right). $$

Then for $2n+1 \in \mathbb{N}$, the series u_n is terminating, therefore over natural boundaries; it evaluates to 0 for $n \in \mathbb{N}$, and to a non-zero rational function of x and z for $n \in \mathbb{N}+1/2$. Certainly, the recurrence obtained by Zeilberger's algorithm cannot make the distinction between these two cases. It follows that the recurrence has to be of order 1 at least, which is non-minimal for the case $n \in \mathbb{N}$. Indeed, Zeilberger's algorithm fails (in short time) to find any recurrence of order 1 and 2; as to the order 3, it seems to be beyond the computational capability of our computer.

We have not been able so far to decide if this example can be explained by our new paradigm. However, it suggests one reason for the occurence of non-minimality.

Part II. Heuristics

6. Rational Heuristic by a Singularity Analysis

For this heuristic, we assume to be given a hypergeometric term f for which the Zeilberger algorithm fails to obtain a minimal order operator. We also start with a known annihilator $P(n, S_n)$ of the sum $\sum_{k\in\mathbb{Z}} f_{n,k}$ of lower order than what the Zeilberger algorithm gets. The goal is to obtain rational functions B and Z in $\mathbb{C}(n, k)$ such that

$$ (P + (S_k - 1)B + Z) \cdot f = 0 \quad \text{and} \quad \sum_{k\in\mathbb{Z}} Z f_{n,k} = 0. $$

Additionally, we need to check that both Bf and Zf can be summed, i.e., that the rational functions B and Z do not introduce any nasty singularity.

In the remainder of the section, the variable n will only be viewed as a parameter, which will not be shifted, so that we freely drop the dependency in n in all the rational and hypergeometric functions. Since f is hypergeometric, write

$$ S_k \cdot f = \frac{\alpha(k)}{\beta(k)} f \quad \text{and} \quad P \cdot f = \frac{\gamma(k)}{\delta(k)} f $$

for polynomials α, β, γ, and δ such that both rational functions α/β and δ/γ are reduced. In this way we obtain Z as a rational function modulo the annihilator of f. We introduce $\bar{B} = B - \gamma/\delta$, so that (modulo $\operatorname{Ann} f$):

$$\begin{aligned} Z &= -B(k+1)S_k + B(k) - P \\ &= -\frac{\alpha(k)}{\beta(k)}\left(\bar{B}(k+1) + \frac{\gamma(k+1)}{\delta(k+1)}\right) + \bar{B}(k) \\ &= -\frac{\alpha(k)}{\delta(k+1)}\left(\frac{\delta(k+1)\bar{B}(k+1) + \gamma(k+1)}{\beta(k)}\right) + \bar{B}(k). \end{aligned}$$

We view Z as depending on $\bar{B}$.

To avoid nasty singularity, we try to satisfy the constraint that each of the three rational functions

$$\bar{B}, \quad \frac{\alpha(k)}{\delta(k+1)}, \quad \text{and} \quad \frac{\delta(k+1)\bar{B}(k+1) + \gamma(k+1)}{\beta(k)} \tag{18}$$

is a polynomial in k, or at least introduces no pole which would disallow the evaluation of the summand for some values of k and n. To this end, we slightly change the notation to incorporate the "nice" poles into the numerators α and γ and keep the "nasty" poles only in the denominators. Consider the localization $\mathcal{L} = \mathcal{S}^{-1}\mathbb{C}(n)[k]$ where $\mathcal{S}$ is the multiplicatively stable set consisting of the polynomials $s \in \mathbb{C}(n)[k]$ such that for each sufficiently large integer n, the polynomial $s(n,k) \in \mathbb{C}[k]$ has no integer root k. (This set $\mathcal{L}$ contains more general objects than polynomials in $\mathbb{C}(n)[k]$ but less general that rational functions in $\mathbb{C}(n,k)$: while $2n-2k-1$ is in $\mathcal{S}$, $2n-3k$ is not, so that $(2n-2k-1)^{-1}$ is in $\mathcal{L}$, whereas $(2n-3k)^{-1}$ is not.) Now, we assume that α and γ are in $\mathcal{L}$, but that β and δ are in $\mathbb{C}(n)[k] \setminus \mathcal{S}$. We ensure that the entries of (18) are elements of $\mathcal{L}$ by making the two following assumptions, observed to hold in all of the cases studied so far.

Assumption for the rational heuristic:

1. $\delta(k+1) \in \mathbb{C}(n)[k]$ divides $\alpha(k) \in \mathcal{L}$ (as elements of the ring $\mathcal{L}$);
2. the gcd of $\beta(k)$ and $\delta(k+1)$ (as polynomials of $\mathbb{C}(n)[k]$) divides $\gamma(k+1)$ (as elements of the ring $\mathcal{L}$).

The first assumption ensures that the second element of (18) wears no nasty pole. Considering the third element of (18), let us write

$$\lambda(k)\beta(k) = \delta(k+1)\bar{B}(k+1) + \gamma(k+1)$$

where we wish λ to be in $\mathcal{L}$, and use the second assumption to get the Bézout relation:

$$u(k)\beta(k) + v(k)\delta(k+1) = \frac{\gamma(k+1)}{\rho(k)} \in \mathbb{C}(n)[k] \tag{19}$$

for polynomials u and v in $\mathbb{C}(n)[k]$, and ρ in $\mathcal{L}$. Setting $\bar{B}(k+1) = -\rho(k)v(k)$ yields

$$\rho(k)u(k)\beta(k) = \delta(k+1)\bar{B}(k+1) + \gamma(k+1),$$

whence the relation $\lambda = \rho u \in \mathcal{L}$ by identification.

As a conclusion, both assumptions 1 and 2 above combined with the minimal order operator P yield good candidates for Z, in the sense that both summations of Bf and Zf are with no singularity and over natural boundaries. There only remains to decide by inspection whether Zf sums to 0. In a situation where Zeilberger's algorithm fails to return an operator of minimal order and where the preceding heuristic succeeds in returning a non-zero Z for which $\sum_{k\in\mathbb{Z}} Zf = 0$, Zf cannot be viewed as the finite difference of a term $B'f$ for another rational function $B' \in \mathbb{C}(n,k)$: this would contradict the failure of Zeilberger's algorithm. In this case, Lemma 1 in Section 1 implies that Zeilberger's algorithm applied to Zf returns an operator whose order is exactly the gap between the order of P and the order of the operator returned when applied to f.

Special Case. By working out an example, we deal with the special case when the gcd in the second assumption is 1, so that $\rho(k) = \gamma(k+1)$, and when additionally δ, therefore also v, are polynomials of degree 1. Explicit computation are of course easier in this setting.

For an integer $t \geq 2$, consider the hypergeometric term

$$f_{n,k} = (-1)^k \binom{n}{k}\binom{tk}{n}.$$

Its sum over k is $(-t)^n$, which is cancelled by $P = S_n + t$. We obtain:

$$\begin{aligned} \alpha &= t(tk+1)\cdots(tk+t-1)(k-n), \qquad & \gamma &= tn-n+t, \\ \beta &= (tk-n+1)\cdots(tk-n+t), & \delta &= n+1-k. \end{aligned}$$

Since $\beta(k)$ and $\delta(k+1) = n-k$ are relatively prime, we set $\rho(k) = \gamma(k+1)$ and look for cofactors u and v of degree 0 and $t-1$, respectively. Thus, u is a constant: $u(k) = u(n)$. Evaluating (19) at $k = n$ yields:

$$u(k) = \frac{1}{\beta(n)}, \quad \text{whence} \quad v(k) = \frac{\beta(k) - \beta(n)}{\beta(n)(k-n)}.$$

Then,

$$\begin{aligned} \bar{B}(k) &= -\gamma(k)v(k-1) = \gamma(k)\frac{\beta(k-1) - \beta(n)}{\beta(n)\delta(k)}, \\ B(k) &= \bar{B}(k) + \frac{\gamma(k)}{\delta(k)} = \frac{\beta(k-1)\gamma(k)}{\beta(n)\delta(k)} = \frac{\beta(k-1)}{\beta(n)}P, \\ Z(k) &= -\frac{\alpha(k)}{\delta(k+1)}\gamma(k+1)u(k) + \bar{B}(k) \\ &= -\frac{\alpha(k)\gamma(k+1)}{\beta(n)\delta(k+1)} + \frac{\beta(k-1)\gamma(k)}{\beta(n)\delta(k)} - \frac{\gamma(k)}{\delta(k)}. \end{aligned}$$

Remark that despite the equality

$$\left(\frac{\alpha(k)\gamma(k+1)}{\delta(k+1)} - \frac{\beta(k-1)\gamma(k)}{\delta(k)}\right) f_k = \frac{\beta(k)\gamma(k+1)}{\delta(k+1)} f_{k+1} - \frac{\beta(k-1)\gamma(k)}{\delta(k)} f_k,$$

the corresponding sum is not telescoping (over natural boundaries) because of the singularity induced by the pole $1/\delta(k)$.

Observation: for small t, the sum over k of $Z(k)f_k$ is zero for all n, i.e.,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{tk}{n}\left(t\frac{tk+1}{tn-n+1}\cdots\frac{tk+t-1}{tn-n+t-1}\right.$$
$$\left.+\frac{tn-n+t}{n+1-k}\left(\frac{tk-n+1-t}{tn-n+1}\cdots\frac{tk-n}{tn-n+t}-1\right)\right)=0.$$

Additionally, the term in the long parenthesis is by construction a polynomial in k, so that the above summation is over natural boundaries; similarly, the last expression given for B, in terms of P, shows that the sum of $B(k)f_k$ is also over natural boundaries.

7. Polynomial Heuristic by Evaluation of Sums

For each example in Part I it has been possible to find an operator Z which is in fact a polynomial in n and k. This suggests the following heuristic to find candidates Z:

1. choose a maximal degree d and introduce Z under undetermined coefficient form:
$$Z(n,k)=\sum_{i+j\le d}c_{i,j}n^ik^j\in\mathbb{C}[n,k];$$
2. choose a maximal order r and evaluate the sums
$$s_n(c)=\sum_{k\in\mathbb{Z}}Z(n,k)h_{n,k}$$
for n between 0 and r;
3. solve the linear system consisting of the equations $s_n(c)=0$ for values of the $c_{i,j}$'s;
4. if solvable, return the corresponding value for $Z(n,k)$.

Note that the system will always be solvable if $2n>(r+1)(r+2)$; also note that in case the system is solvable, the output is generally parametrized by some of the $c_{i,j}$'s. This parametrization has to be kept so as to allow more flexibility in the modified version of Zeilberger's and Chyzak's algorithms described in Part I.

References

[1] Abramov, S. A., Paule, P., and Petkovšek, M. q-Hypergeometric solutions of q-difference equations. *Discrete Math. 180*, 1-3 (1998), 3–22. Proceedings of the 7th Conference on Formal Power Series and Algebraic Combinatorics (Noisy-le-Grand, 1995).

[2] Andrews, G. E. Pfaff's method. I. The Mills-Robbins-Rumsey determinant. *Discrete Math. 193*, 1-3 (1998), 43–60. Selected papers in honor of Adriano Garsia (Taormina, 1994).

[3] Chyzak, F. An extension of Zeilberger's fast algorithm to general holonomic functions. In *Formal Power Series and Algebraic Combinatorics, 9th Conference* (1997), vol. 1, Universität Wien, pp. 172–183. Conference Proceedings.

[4] Chyzak, F., and Salvy, B. Non-commutative elimination in Ore algebras proves multivariate identities. *J. Symbolic Comput. 26*, 2 (1998), 187–227.

[5] Egorychev, G. P. *Integral Representation and the Computation of Combinatorial Sums*, vol. 59 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, Rhode Island, 1984.

[6] Graham, R. L., Knuth, D. E., and Patashnik, O. *Concrete mathematics*, second ed. Addison-Wesley Publishing Company, Reading, MA, 1994. A foundation for computer science.

[7] Greene, D. H., and Knuth, D. E. *Mathematics for the analysis of algorithms.* Birkhäuser Boston, Mass., 1981.

[8] Paule, P. Short and easy computer proofs of the Rogers-Ramanujan identities and of identities of similar type. *Electron. J. Combin. 1* (1994), Research Paper 10, approx. 9 pp. (electronic).

[9] Paule, P., and Riese, A. A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated approach to q-hypergeometric telescoping. In *Special functions, q-series and related topics (Toronto, ON, 1995).* Amer. Math. Soc., Providence, RI, 1997, pp. 179–210.

[10] Paule, P., and Schorn, M. A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities. *J. Symbolic Comput. 20*, 5-6 (1995), 673–698. Symbolic computation in combinatorics Δ_1 (Ithaca, NY, 1993).

[11] Petkovšek, M., Wilf, H. S., and Zeilberger, D. $A = B$. A K Peters Ltd., Wellesley, MA, 1996. With a foreword by Donald E. Knuth, With a separately available computer disk.

[12] Wilf, H. S., and Zeilberger, D. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. *Invent. Math. 108*, 3 (1992), 575–633.

[13] Zeilberger, D. A fast algorithm for proving terminating hypergeometric identities. *Discrete Math. 80*, 2 (1990), 207–211.

[14] Zeilberger, D. A holonomic systems approach to special functions identities. *J. Comput. Appl. Math. 32*, 3 (1990), 321–368.

[15] Zeilberger, D. The method of creative telescoping. *J. Symbolic Comput. 11*, 3 (1991), 195–204.