A Computer Algebra solution to a planar newtonian 4-body problem with unequal masses

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Abstract

We apply Computer Algebra methods and techniques, to the solution of a mathematical problem in Celestial Mechanics.

1 Introduction

The N-body problem is the most widely studied problem of Celestial Mechanics. Newton's equations of motion form a system of differential equations which describes the motion of N point masses moving under the influence of their mutual gravitational attraction. This system is in general not integrable, but there is an important special class of solutions that can be computed analytically. These solutions are called central configurations. The enumeration of all such solutions for $N > 4$, is already too difficult for the theoretical methods and computational tools available today.

The complete classification of central configurations of the newtonian planar 4-body problem with equal masses has been done by Albouy in [Alb95] and [Alb96]. These configurations are of three types : the square, the equilateral triangle with its barycenter and the pseudo-centered isosceles triangle. Moreover there is only one configuration with a unique symmetry axis.

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A very general and well-known conjecture in Celestial Mechanics has been established for the newtonian N-body problem with arbitrary masses. The conjecture asserts that for any fixed choice of the masses the number of central configurations is finite. Under the assumption of having a finite number of central configurations, upper bounds for this number are known, but computational difficulties prevent us from being able to explicitly enumerate central configurations. Strong supporting evidence toward the truth of the conjecture, is also provided from the fact that for all choices of the masses except possibly a set of measure zero, the number of central configurations is finite.

A new formulation of the equations of central configurations in terms of polynomial equations, has been established by Albouy-Chenciner in [AC98], using a construction in Linear Algebra. This formulation allows us to apply modern Computer Algebra techniques and methods for polynomial system solving, in the search for central configurations. Some interesting results exploiting this idea have been obtained in [KL98], [Kot98] and [FK99]. For a survey of the basic new results in this direction, see [Kot00].

In this paper, we address a first special case of the conjecture: a newtonian planar 4-body problem with one Albouy-type symmetry. This problem has been suggested to the first author by Albouy.

The Albouy-Chenciner formulation of the problem is too complex to be approached by general solution methods such as Gröbner basis [Buc65,Buc85], or cylindrical decomposition [Col75,CH91].

Fortunately, we could eliminate a lot of variables by hand, (but of course with the help of a Computer Algebra System, namely Maple, see [CGG+91], that was used interactively). Some of them can be eliminated by exploiting the symmetry. A whole group of variables appears only quadratically, in the last column of a matrix whose determinant is set to zero. By careful row operations, we achieve that the elements in the last column have a common linear factor which can be divided out. This leads to an equivalent system of algebraic equations that are linear in the respective group of variables. Now, these variables can be eliminated by standard techniques of linear algebra. This leads to a system with three variables and one parameter. The proof of the conjecture is reduced to showing that the system has only finitely many solutions for each positive value of the parameter.

There are algorithms for computing Gröbner bases in the presence of parameters [Wei92,Mon99]. However, in our case we succeeded to show finiteness using just ordinary Gröbner bases computations and some simple algebraic theorems, which are related to the shape of a Gröbner basis of a zero-dimensional ideal (see [BW93]).

The success of the theory of Gröbner bases in this case suggests that it can be used to obtain further results in this direction.

2 The problem

In this section we describe the general setting of the newtonian planar 4-body problem, and we define the concept of central configurations. Then we mention briefly Albouy's symmetry theorem for the planar newtonian 4-body problem with equal masses and the Finiteness of the number of central configurations. Finally we describe the problem that we consider in this paper.

2.1 Generalities

Consider 4 particles of masses $m_i, i = 1, \ldots, 4$ moving in the plane under their mutual gravitational attraction. Choose a point as the origin and denote by $r_i, i = 1, \ldots, 4$ the position vector of the i-th particle. Define the newtonian potential energy function as :

$$U = \sum_{1 \leq i < j \leq 4} \frac{m_i m_j}{|| \, r_i - r_j \, ||}$$

where $|| \cdot ||$ is the usual euclidean norm.

The equations of motion (in the newtonian formulation) are :

$$m_i \, \ddot{r}_i \;=\; \frac{\partial U}{\partial r_i}, \quad i = 1, \ldots, 4. \tag{1}$$

Note that on the left-hand side we take the derivative of the position vector w.r.t. time, that is the acceleration of the i-th particle, and that the right-hand side represents the total force exercised on the i-th particle by the other three.

The center of mass of the system is defined as :

$$r_c = \frac{m_1 r_1 + m_2 r_2 + m_3 r_3 + m_4 r_4}{m_1 + m_2 + m_3 + m_4} \; .$$

The symmetry of the function U implies that $\ddot{r_c} = 0$, which means that r_c is a linear function of time. This is the principle of the conservation of linear momentum, which says that the center of mass moves in a straight line with uniform velocity. This allows us to shift the center of mass at the origin, without loss of generality and thus it remains to determine the motion relative to the center of mass. Using this and other similar considerations, like the conservation of energy and angular momentum, the order of the system is considerably reduced, but it is still quite difficult to study it.

2.2 Central Configurations

It is well-known that the equations of the 4-body problem are in general not integrable. There exists a special important class of solutions which can be computed analytically. From the point of view of dynamical systems these correspond to the singularities of the dynamical system in question. We give now the mathematical definition of central configurations.

Let the center of mass of the system be at the origin of the coordinate axes. This means that we have $r_c = 0$. In this case, we have the following definition:
Definition 2.1. *A configuration of the n particles is called a central configuration when the acceleration vector of each particle is a common scalar multiple of his position vector. In mathematical terms this means that there exists a scalar λ such that we have $\ddot{r}_i = \lambda r_i$ for $i = 1, \ldots, 4$.*

Central configurations are very rare solutions of the 4-body problem, but their study is important mainly because of the following two reasons :

(1) Central configurations can be used for analyzing actual configurations of celestial bodies in the Universe.
(2) Central configurations correspond to solutions which are the only possible solutions that can be formed asymptotically by the bodies in the neighbourhood of a multiple collision (see [Che96])

Central configurations have been an object of intensive study for more than two centuries. For an extensive (but not exhaustive) catalog of bibliographical references that can serve as a starting point to the vast literature on the subject, see [Alb97] and [Kot98].

2.3 The equations of central configurations

In this paragraph we will be interested in the central configurations of the planar 4-body problem. We will give a characterization of the central configurations by a system of algebraic equations, proposed by O. Dziobek (see [Dzi00]). The innovative idea of Dziobek was to use the mutual distances between the bodies as the coordinates to write the equations of motion. The resulting algebraic system is tractable by symbolic computation techniques after some preprocessing using Linear Algebra.

Conserving the notations of the previous paragraph we let $r_{ij} = \| r_i - r_j \|$ for $1 \leq i < j \leq 4$ and $s_{ij} = r_{ij}^2$. So s_{ij} is the square of the euclidean distance of the ith particle from the jth particle. There are ten such mutual distances. Finally, let $\Delta_1,\ \Delta_2,\ \Delta_3,\ \Delta_4$ be the oriented areas of the four triangles formed by the points $(2,3,4),\ (4,3,1),\ (1,2,4),\ (3,2,1)$ respectively. They verify the relation:

$$\sum_{i=1}^{4} \Delta_i = 0. \tag{2}$$

Central configurations are then characterized, up to multiplication of the Δ_i with a common nonzero scalar, by the following two groups of equations:

$$\boxed{\begin{array}{ll} (A) & \displaystyle\sum_{j\neq i}\Delta_j s_{ij} = X, \qquad \text{for some } X \text{ independent of } i \\ (B) & \text{there are two real numbers } x \text{ and } y \text{ such that} \\ & \qquad\qquad s_{ij}^{-3/2} = x + y\,\frac{\Delta_i\Delta_j}{m_i m_j} \end{array}}$$

The group (A) equations are due to A. Albouy (see [Alb95]). The group (B) equations for the central configurations of the planar 4-body problem with arbitrary masses appears in [Dzi00]. The general principles developed in [AC98] for the N-body problem in general, establish both groups of equations using a construction in Linear Algebra.

The exponent $-3/2$ in the group (B) equations reflects the fact that we are working with the newtonian potential. In the rest of the paper and in order to keep up with standard notation, we will use the following notation:

$$a = s_{12}, \quad b = s_{13}, \quad c = s_{14}, \quad d = s_{23}, \quad e = s_{24}, \quad f = s_{34} \tag{3}$$

and

$$A = s_{12}^{-3/2},\ B = s_{13}^{-3/2},\ C = s_{14}^{-3/2},\ D = s_{23}^{-3/2},\ E = s_{24}^{-3/2},\ F = s_{34}^{-3/2}. \tag{4}$$

It follows that we have the following relations between the six capital and the six lower-case letters:

$$A^2a^3 = 1,\ B^2b^3 = 1,\ C^2c^3 = 1,\ D^2d^3 = 1,\ E^2e^3 = 1,\ F^2f^3 = 1. \tag{5}$$

These equations of degree 5 will not appear in the final system, because we will use a suitable parameterization to eliminate them.

2.4 The matrix formulation

A new formulation for the group (B) equations has been introduced in [KL98] and [Kot98]. This formulation allows us to perform the elimination of variables systematically and to clearly identify the symmetries.

Definition 2.2. *Let M_6 denote the transpose of the following 3×6 matrix :*

$$\begin{bmatrix} A & B & C & D & E & F \\ \\ \dfrac{1}{\Delta_1\ \Delta_2} & \dfrac{1}{\Delta_1\ \Delta_3} & \dfrac{1}{\Delta_1\ \Delta_4} & \dfrac{1}{\Delta_2\ \Delta_3} & \dfrac{1}{\Delta_2\ \Delta_4} & \dfrac{1}{\Delta_3\ \Delta_4} \\ m_1\ m_2 & m_1\ m_3 & m_1\ m_4 & m_2\ m_3 & m_2\ m_4 & m_3\ m_4 \end{bmatrix}. \tag{6}$$

The two propositions that follow, relate the solvability of the group (B) equations to a condition on the rank of the matrix M_6.

Proposition 2.1. *If the group (B) equations have a solution in the unknowns x, y, then*

$$rank\ M_6 \leq 2.$$

The proof of the above proposition is clear from Linear Algebra, if we look at the group (B) equations as a system of linear equations in the unknowns x, y.

Proposition 2.2. *If $rank\ M_6 \leq 2$ and relation (2) holds and the Δ_i are not all zero, then the group (B) equations have a solution in x, y.*

The proof of the above proposition can be found in [KL98] or [Kot98].

2.5 Albouy's Symmetry Theorem and the Finiteness conjecture

In this section we will state Albouy's Symmetry Theorem and the Finiteness conjecture. We intentionally avoid the rigorous statements of both results and prefer to give two references which contain detailed accounts and related bibliography.

For the original proof of Albouy's Symmetry Theorem see [Alb95]. For a recent proof using Gröbner Bases, see [FK99]. The Finiteness conjecture is mentioned in the classical Celestial Mechanics book [Win41]. A recent reference to the Finiteness conjecture together with a survey of modern results are contained in [Sma98]. The Finiteness conjecture is the sixth of a list of eighteen problems in that paper.

Theorem 2.1 (Albouy). *Every central configuration of the newtonian planar 4-body problem with equal masses has at least one symmetry.*

Algebraically this theorem asserts that in every real (and physically meaningful) solution of the preceding system of equations there will be at least two equal Δ_i. This fact will be used in the resolution of the system. In particular we will suppose that

$$\Delta_3 = \Delta_4 \tag{7}$$

which implies that equation (2) becomes :

$$\Delta_1 + \Delta_2 + 2\ \Delta_3 = 0. \tag{8}$$

There is a very general and well-known conjecture concerning the number of central configurations in the case of n arbitrary masses. In the case of the newtonian planar 4-body problem, the conjecture may be formulated as follows:

Conjecture 2.1. *In the newtonian planar 4-body problem, for any arbitrarily fixed choice of the four masses, the number of central configurations is finite.*

The remarkable results established by Albouy in [Alb95] and [Alb96], show that the conjecture is true in the case of the planar newtonian 4-body problem with equal masses.

2.6 Precise statement of the problem

Albouy's symmetry theorem does not hold in the case of unequal masses, but it provides a good starting point to tackle special cases of the Finiteness conjecture. Thus it is reasonable to impose the hypothesis of a symmetry axis as well as some restrictions on the masses and try to prove the conjecture.

The following special case has been suggested to the first author by Albouy in 1998.

Consider a newtonian planar 4-body problem, in the sense of the previous paragraphs, with a symmetry axis given by $\Delta_3 = \Delta_4$. Suppose that the two masses on either side of the axis are equal. This means that $m_3 = m_4$. Without loss of generality we can assume that $m_3 = m_4 = 1$. Therefore we are left with two parameters m_1, m_2. In this paper we consider the case $m_1 = m_2$ and we denote this common value by m.

We want to prove that for every (strictly positive) value of m the number of central configurations, subject to the above requirements, is finite. Using algebraic vocabulary, this statement means that the corresponding system of polynomial equations should be of dimension zero for every (strictly positive) value of the parameter m.

We will prove that this result is indeed true, using the theory of Gröbner Bases.

3 Simplification

In this section we study the special form taken by the general equations of central configurations, in the case depicted in section 2.6. Then we apply the matrix formalism of paragraph 2.4 in order to obtain the corresponding simplified polynomial system. The final system will consist of three equations in three unknowns and the parameter m.

To clarify the meaning of the group (A) and group (B) equations we will now write these equations in the case of the planar newtonian 4-body problem with arbitrary masses, using the standard notation introduced at the end of section 2.3.

The group (A) gives the equations:

$$\begin{cases} \Delta_2\ a + \Delta_3\ b + \Delta_4\ c - X = 0 \\ \Delta_1\ a + \Delta_3\ d + \Delta_4\ e - X = 0 \\ \Delta_1\ b + \Delta_2\ d + \Delta_4\ f - X = 0 \\ \Delta_1\ c + \Delta_2\ e + \Delta_3\ f - X = 0 \end{cases} \tag{9}$$

The equations (9) are linear in $X, \Delta_1, \Delta_2, \Delta_3, \Delta_4$. The group (B) gives the equations :

$$\begin{cases} A = x + y\ \dfrac{\Delta_1\ \Delta_2}{m^2}, \quad B = x + y\ \dfrac{\Delta_1\ \Delta_3}{m}, \quad C = x + y\ \dfrac{\Delta_1\ \Delta_4}{m}, \\ \\ D = x + y\ \dfrac{\Delta_2\ \Delta_3}{m}, \quad E = x + y\ \dfrac{\Delta_2\ \Delta_4}{m}, \quad F = x + y\ \Delta_3\ \Delta_4. \end{cases} \tag{10}$$

Since our symmetry hypothesis is that $\Delta_3 = \Delta_4$, it is clear that we have $B = C$ and $D = E$. By the equations (5), we also get $b = c$ and $d = e$. This eliminates the variables C, E, c, e.

By propositions (2.1) and (2.2), the solvabilty of equations (10) is equivalent to the statement that the matrix $M_4(m)$ below, has rank ≤ 2.

$$M_4(m) := \begin{bmatrix} A & 1 & \dfrac{\Delta_1\,\Delta_2}{m^2} \\ B & 1 & \dfrac{\Delta_1\,\Delta_3}{m} \\ D & 1 & \dfrac{\Delta_2\,\Delta_3}{m} \\ F & 1 & {\Delta_3}^2 \end{bmatrix} \tag{11}$$

(the matrix $M_4(m)$ is obtained by the matrix M_6 after deletion of two obviously equal rows).

If $B = D$, then it follows from equations (5) that $b = d$, and from equations (10) that $\Delta_1 = \Delta_2 = 0$. Now equation (8) shows that we also have $\Delta_3 = 0$. Since the areas of the triangles formed by the four points are zero, the four points are collinear, and this is one of the well-known solutions.

From now on we assume $B \neq D$, and consequently $\Delta_1 \neq \Delta_2$, by equations (10).

Our goal is to eliminate the variables $X, \Delta_1, \Delta_2, \Delta_3$. This seems to be cumbersome because $\Delta_1, \Delta_2, \Delta_3$ occur quadratically in the third column of the matrix $M_4(m)$. However, by performing elementary row operations on the matrix $M_4(m)$, we obtain submatrices where a linear factor can be divided out in the third column. Thus we obtain subdeterminants in which the Δ_i appear only linearly.

Equation (2) can be solved immediately. We do this by expressing Δ_1 and Δ_2 in terms of Δ_3 and a new variable $\Delta_M = \Delta_1 - \Delta_2$:

$$\Delta_1 = \frac{\Delta_M - 2\Delta_3}{2}, \text{ and } \Delta_2 = \frac{-\Delta_M - 2\Delta_3}{2}. \tag{12}$$

Substituting the values of Δ_1, Δ_2, from (12) into the matrix (11) we have:

$$M_4(m) = \begin{bmatrix} A & 1 & \frac{{\Delta_3}^2}{m^2} - \frac{1}{4}\frac{{\varDelta_M}^2}{m^2} \\ B & 1 & -\frac{{\Delta_3}^2}{m} + \frac{1}{2}\frac{\Delta_3\,\varDelta_M}{m} \\ D & 1 & -\frac{{\Delta_3}^2}{m} - \frac{1}{2}\frac{\Delta_3\,\varDelta_M}{m} \\ F & 1 & {\Delta_3}^2 \end{bmatrix}$$

Now, we add suitable multiples of the last row to the other rows in order to eliminate the summand Δ_3^2: We now introduce a notation that will be used throughout this paper to perform elementary row operations with rows of matrices.

(1) The notation **add i j c** will mean multiply the i-th line of the matrix by the coefficient c and add the result to the j-th line.
(2) The notation **mul i c** will mean multiply the i-th line of the matrix by the coefficient c.

We make the convention that when more than one of add or mul operations appear together, the order of application is from top to bottom.

Here are the row operations:

$$M_4(m) \overset{\begin{pmatrix} \texttt{add}\ 4\ 1\ -1/m^2 \\ \texttt{add}\ 4\ 2\ 1/m \\ \texttt{add}\ 4\ 3\ 1/m \end{pmatrix}}{\sim} = \begin{bmatrix} -\frac{F}{m^2} + A & -\frac{1}{m^2} + 1 & -\frac{1}{4m^2}{\varDelta_M}^2 \\ \frac{F}{m} + B & \frac{1}{m} + 1 & \frac{1}{2m}\Delta_3\,\varDelta_M \\ \frac{F}{m} + D & \frac{1}{m} + 1 & -\frac{1}{2m}\Delta_3\,\varDelta_M \\ F & 1 & {\Delta_3}^2 \end{bmatrix} \tag{13}$$

The resulting matrix is denoted by $M_4'(m)$.

As row operations preserve the rank, the equations (10) are equivalent to the statement that $M_4'(m)$ has rank less than or equal to 2.

We claim that this statement is equivalent to the vanishing of the determinants of the 3×3 submatrices $M_4'(m)_{123}$ and $M_4'(m)_{234}$ consisting of the rows 1,2,3 and 2,3,4, respectively.

One implication is immediate: if the rank is less than or equal to 2, then the two determinants must vanish. Vice versa, assume that both determinants vanish. Since the rows 2 and 3 are linearly independent, the first row is a linear combination of rows 2 and 3 (by the vanishing of the first determinant), and the second row likewise (by the vanishing of the second determinant). Thus, the vector space of rows is spanned by two rows, hence the rank is 2.

We can divide the determinant of $M_4'(m)_{123}$ by Δ_M and the determinant of $M_4'(m)_{234}$ by Δ_3. Actually, the first determinant contains another factor $\frac{m+1}{4m^3}$, and the second contains another factor $\frac{1}{2m}$. Since these factors do not vanish, we divide them out and get the linear equations

$$\begin{aligned} &2\,m\Delta_3 B + 2\,Dm\Delta_3 - 4\,A\Delta_3 m + 4\,\Delta_3 F - B\Delta_M - 2\,\Delta_3 B + D\Delta_M - 2\,D\Delta_3 = 0 \\ &-2\,\Delta_3 B - 2\,m\Delta_3 B - B\Delta_M + 2\,D\Delta_3 + 2\,Dm\Delta_3 - D\Delta_M + 2\,F\Delta_M = 0 \end{aligned} \tag{14}$$

Now, the variables X, Δ_3, Δ_M occur linearly in the equations (9) (after substitution of (12)) and (14). Thus, their elimination is easy. The existence of a nontrivial solution in (X, Δ_3, Δ_M) is equivalent to the statement that the matrix of the system

$$L := \begin{bmatrix} -1 & -a + 2\,b & -\frac{1}{2}\,a \\ -1 & -a + 2\,d & \frac{1}{2}\,a \\ -1 & -b - d + f & \frac{1}{2}\,(b - d) \\ 0 & 2\,Bm + 2\,Dm - 4\,Am + 4\,F - 2\,B - 2\,D & -B + D \\ 0 & -2\,B - 2\,Bm + 2\,D + 2\,Dm & -B - D + 2\,F \end{bmatrix} \tag{15}$$

has rank less than or equal to 2.

We calculate the rank by elementary row operations. The first pivot element is L_{11}.

$$L \overset{\begin{pmatrix} \texttt{add}\ 1\ 2\ -1 \\ \texttt{add}\ 2\ 1\ 1/2 \\ \texttt{add}\ 1\ 3\ -1 \end{pmatrix}}{\sim} = \begin{bmatrix} -1 & b+d-a & 0 \\ 0 & -2\,b+2\,d & a \\ 0 & -2\,b-2\,d+a+f & \frac{1}{2}\,(b-d) \\ 0 & 2\,Bm+2\,Dm-4\,Am+4\,F-2\,B-2\,D & -B+D \\ 0 & -2\,B-2\,Bm+2\,D+2\,Dm & -B-D+2\,F \end{bmatrix}$$

The resulting matrix is called L'.

We could choose L'_{22} as the next pivot element. However, one obtains simpler equations by choosing L'_{23} instead.

$$L' \overset{\begin{pmatrix} \texttt{add}\ 2\ 3\ \frac{d-b}{2a} \\ \texttt{add}\ 2\ 4\ \frac{B-D}{a} \\ \texttt{add}\ 2\ 5\ \frac{B+D-2F}{a} \end{pmatrix}}{\sim} = \begin{bmatrix} -1 & b+d-a & 0 \\ 0 & -2\,b+2\,d & a \\ 0 & \frac{b^2-2\,bd+d^2-2\,ba-2\,da+a^2+fa}{a} & 0 \\ 0 & 2\,\frac{-bB+dB+bD-dD+aBm+aDm-2\,aAm+2\,aF-aB-aD}{a} & 0 \\ 0 & 2\,\frac{-bB+dB-bD+dD+2\,bF-2\,dF-aB-aBm+aD+aDm}{a} & 0 \end{bmatrix}$$

The resulting matrix is called L''. Here, the first two rows are linearly independent. The remaining rows cannot be obtained by linear combinations of them, except for the trivial linear combination. Thus, the rank assertion is equivalent to the vanishing of these lines, or equivalently

$$L''_{32} = L''_{42} = L''_{52} = 0. \tag{16}$$

By expanding and taking numerators, we get the three equations

$$\begin{cases} b^2-2\,bd+d^2-2\,ba-2\,da+a^2+fa=0 \\ -2\,bB+2\,dB+2\,bD-2\,dD+2\,aBm+2\,aDm-4\,aAm+4\,aF-2\,aB-2\,aD=0 \\ 2\,bB+2\,dB-2\,bD+2\,dD+4\,bF-4\,dF-2\,aB-2\,aBm+2\,aD+2\,aDm=0 \end{cases} \tag{17}$$

We normalize by setting $a = A = 1$, and use the equations (5) in order to parametrize the variables

b, d, f, B, D, F by:

$$b = \beta^2,\ B = \frac{1}{\beta^3},\ d = \delta^2,\ D = \frac{1}{\delta^3},\ f = \phi^2,\ F = \frac{1}{\phi^3}.$$

After canceling denominators and factoring out a linear factor $(\beta - \delta)$ we obtain the following system of equations:

$$\begin{cases} \beta^4 - 2\,\beta^2\delta^2 + \delta^4 - 2\,\beta^2 - 2\,\delta^2 + 1 + \phi^2 = 0 \\ 2\,\delta^5\phi^3 + 2\,\beta^5\phi^3 - 2\,\delta^3\beta^2\phi^3 - 2\,\delta^2\beta^3\phi^3 - 2\,\delta^3\phi^3 - 2\,\beta^3\phi^3 + 4\,\beta^3\delta^3 + 2\,m\delta^3\phi^3 + 2\,m\beta^3\phi^3 - 4\,m\beta^3\delta^3\phi^3 = 0 \\ -2\,\beta^4\phi^3 + 4\,\beta^4\delta^3 - 2\,\delta\,\beta^3\phi^3 + 4\,\delta^4\beta^3 + 2\,\beta^2 m\phi^3 + 2\,\beta^2\phi^3 - 2\,\beta\,\delta^3\phi^3 + 2\,\beta\,\delta\,m\phi^3 + 2\,\beta\,\delta\,\phi^3 - 2\,\delta^4\phi^3 + 2\,\delta^2 m\phi^3 + 2\,\phi^3\delta^2 = 0 \end{cases} \tag{18}$$

This system still turned out to be too complicated for Gröbner basis computation. In order to simplify it further, we apply the fundamental theorem of symmetric functions putting $s = \beta + \delta$ and $p = \beta\delta$. The result is the system:

$$\begin{cases} 1 + \phi^2 + s^4 - 4\,ps^2 - 2\,s^2 + 4\,p = 0 \\ 2\,\phi^3 s^5 + 8\,\phi^3 sp^2 - 10\,\phi^3 s^3 p + 2\,s^3 m\phi^3 + 6\,\phi^3 ps - 2\,s^3\phi^3 - 6\,sm\phi^3 p + 4\,p^3 - 4\,p^3 m\phi^3 = 0 \\ -2\,s^4\phi^3 + 6\,s^2 p\phi^3 + 2\,s^2 m\phi^3 + 2\,\phi^3 s^2 + 4\,p^3 s - 2\,p\phi^3 - 2\,\phi^3 mp = 0 \end{cases} \tag{19}$$

Summing up, we have proved the following fact.
Lemma 3.1. *The following two assertions are equivalent.*

(1) Conjecture 2.1 is true under the additional assumptions $m_3 = m_4$ *and* $\Delta_3 = \Delta_4$.
(2) For each positive value of m*, the system 19 has only a finite number of solutions.*

The system (19) is suitable for processing with a general-purpose Computer Algebra System such as Maple ([CGG+91]).

4 Solving the Algebraic System

In order to solve the system of equations (19) efficiently, we will use the well-known method of Gröbner bases [Buc65,Buc85] and some well-known algebraic facts, for which we will prove some adapted versions in order to use them in our particular situation.

We refer to [BW93] for the definitions of admissible term orders, leading coefficients, leading power product, block order.
Theorem 4.1. *Let* $g_1, \ldots, g_n \in K[\vec{x}]$ *be polynomials such that the leading power product (with respect to some admissible term order) of* g_i *is equal to* $x_i^{d_i}$*, for some* d_i*. Then the system* $g_1 = \ldots = g_n = 0$ *has only finitely many solutions.*

The proof of theorem (4.1) is well known ([BW93], corollary 6.55 shows that the ideal generated by $g_1, \dots, g_n$ is zero-dimensional).

Corollary 4.1. *Let $f_1, \dots, f_r$ be polynomials in $\vec{x} = (x_1, \dots, x_n)$ over some coefficient domain $R = K[\vec{t}]$, where $\vec{t} = (t_1, \dots, t_m)$ is a set of parameters. Let $g_1, \dots, g_n$ be in the ideal generated by $f_1, \dots, f_r$ over $R[\vec{x}]$, such that the leading power product of g_i is equal to $x_i^{d_i}$, for some d_i. Let $\tau \in K^m$ be a point in parameter space such that the leading coefficients of any g_i does not vanish at τ. Then the system $f_1(\tau) = \ldots = f_r(\tau) = 0$ has only finitely many solutions.*

Proof 4.1. *By theorem4.1, the system $g_1(\tau) = \ldots = g_n(\tau) = 0$ has only finitely many solutions. Since the $g_i(\tau)$'s are linear combinations of the $f_j(\tau)$'s, any solution of the system $f_1(\tau) = \ldots = f_r(\tau) = 0$ is also a solution of the system $g_1(\tau) = \ldots = g_n(\tau) = 0$.*

In order to apply corollary 4.1 to our situation, we need to find polynomials $g_1, \dots, g_n$ in the ideal $\langle f_1, \dots, f_r \rangle$, such that the leading powers have the desired shape. Here is a theorem that tells how to find such polynomials.

Theorem 4.2. *Let $F = \{f_1, \dots, f_r\}$ be a set of polynomials in $K[\vec{t}][\vec{x}]$. Let G be a Gröbner basis of F with respect to a block order $< := (<_{\vec{x}}, <_{\vec{t}})$. If the system $f_1(\tau) = \ldots = f_r(\tau) = 0$ has only finitely many solutions for generic $\tau \in K^m$, then G contains elements $g_1, \dots, g_n$, such that the leading power product of g_i with respect to $<_{\vec{x}}$ is a power of x_i.*

Proof 4.2. *We claim that G is a Gröbner basis of F over the ring $K(\vec{t})[\vec{x}]$. To show this, let g be an S-polynomial of two elements in G computed in this ring. Because S-polynomials can be computed without division, g is in the ideal generated by G. Thus, g can be reduced with respect to $<$ by some element g_0 in G. Because the leading power product with respect to $<_{\vec{x}}$ is just the $\vec{x}$-monomial of the leading power product with respect to $<$, it follows that g can also be reduced with respect to $<_{\vec{x}}$ by g_0. This proves the claim.*

By the assumption on finitely many solutions for generic τ, it follows that the ideal generated by G over $K(\vec{t})[\vec{x}]$ is zero-dimensional. By a well-known theorem (see [BW93], theorem 6.54), it follows that G contains elements $g_1, \dots, g_n$ of the desired shape.

We apply these algebraic facts to the case $\vec{x} = (s, p, \phi)$, $\vec{t} = (m)$, F is the system (19). Let $<$ be the elimination order which orders (s, p, ϕ) in reverse lexicographical order. The reduced Gröbner bases with respect to $<$ has 13 elements. Its leading monomials with respect to the reverse lexicographical order in (s, p, ϕ) are

$$\{s^4, p^3 m\phi^3, \phi^3 ps^2, \phi^7, p^2 m\phi^5, s^2\phi^5, 2\,p^6 ms, 2\,m^2\,(1+m)\,p^6\phi^2, \phi^6 p^2 s, s^3\phi^2 p^4, 32\,p^8 m^2\phi, 4\,\phi\,p^5 s^3, 512\,p^{10} m^2\}$$

The computation was done using Maple on an Intel 686 and took 7.1 seconds.

We see that the power products $s^4, \phi^7, 512m^2p^{10}$ occur (hence there are only finitely many solutions in the generic case). By corollary 4.1, there are only finitely many solutions in the case $m \neq 0$. By lemma 3.1, it follows that the conjecture 2.1 is true under the assumptions of section 2.6.

5 Conclusion

In this paper we apply Computer Algebra methodologies to a mathematical problem arising in the study of central configurations of a newtonian 4-body problem with unequal masses. This problem is a special case of a very general conjecture concerning the finiteness of the number of central configurations. We prove that the conjecture is true in the special case we are considering.

The theory of Gröbner bases emerges as a promising way of dealing with systems of polynomial equations that arise in the study of central configurations in the N-body problem. Indeed Gröbner bases have been the driving force behind some new results in the problem of central configurations. This paper explores still another such fruitful exchange between Computer Algebra and Celestial Mechanics.

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