The Parametrization Problem for Algebraic Surfaces
A Survey

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Abstract

The parameterization problem asks for a parameterization of an implicitly given surface, in terms of rational functions in two variables. In this survey paper, we give an overview on the existing results, with a strong emphasize on algorithmic questions.

Introduction

The parameterization problem is the following. Given is the implicit representation of a surface S, in terms of an equation in three variables over a field k of characteristic zero. Wanted is a parametrization of S, in terms of two rational functions of two independant parameters s, t. Such a parametrization may not exist; so, a first step in an algorithmic solution is to decide the existence. In the affirmative case, we say that S is *unirational*. A parametrization which is birational (as a rational map from the st-plane to S) is called *proper*. A surface is called *rational* iff it has a proper parametrization.

A solution to the problem has applications in CAD/CAM, where there is a need to represent curves and surfaces via such parametrizations, at least piecewise (see [21, 45]). It is obvious that parametric varieties are easier to visualize than implicitly given varieties.

The parametrization problem for curves has been solved in [66, 1, 54, 46, 62, 35, 34, 55, 63]; actually, we will need the curve parametrization algorithm as a subalgorithm in surface parametrization.

For complex surfaces, it is well-known that rationality and unirationality are equivalent, and Castelnuovo's criterion $p_a = P_2 = 0$ (see [13, 67, 56]) is necessary and sufficient. Since the numbers p_a and P_2 can be computed (see section 5), we can decide (uni)rationality over the complex.

In the general case (characteristic zero), Castelnuovo's criterion is still necessary. It is interesting to ask how much the surface S can be birationally simplified, provided that Castelnuovo's criterion is fulfilled. This question is answered by the theorem of Enriques and Manin [19, 36]. A surface with $p_a = P_2 = 0$ has a pencil of rational curves or is birationally equivalent to a del Pezzo surface.

Computationally, a pencil of rational curves is already a big step towards the final result, because it gives one parameter t in terms of the coordinate functions x, y, z. It remains to find a parametrization of the rational curves $t =$ constant, in terms of the second parameter s. This leads to the problem of parametrizing a curve over the function field $k(t)$. It is well-known (see [55]) that this problem can be reduced to the problem of parametrizing a conic over $k(t)$, or a surface with an equation of the type $A(t)x^2 + B(t)y^2 + C(t) = 0$, where $A, B, C \in k[t]$ (such a surface is also called a *tubular*.)

The class of tubular surfaces is also interesting because many surfaces arising in CAD/CAM and other geometric applications, such as conical surfaces [16, 17], canal surfaces [44, 43], or generalized offset surfaces of curves (see [3]) can be birationally reduced to it. Its investigation is closely related to the theory of quadratic forms over function fields. The main problem is to find a cross section to the pencil, i.e. a rational curve on S with parametrization $(X(t), Y(t), t)$, where $X, Y \in k(t)$.

In the complex case, there exists always a cross section. The existence proof was given by [42]. This result was later generalized in [60, 61, 31] and is known as the theorem of Tsen-Lang. An efficient algorithm for computing a cross section can be found in in [49].

In the real case, there are tubular surface without cross section. A necessary and sufficient condition for the existence was given by [15]. In the affirmative case, we have algorithms to compute cross sections [43, 48]. The algorithm given in [25] works for arbitrary fields K, provided that we can factor univariate polynomials and decide the existence of nontrivial solutions of ternary quadratic forms over K. We give a new improved algorithm and a new algorithmic criterion for existence (theorem 3.6).

The non-existence of a cross section does not imply the non-existence of a parametrization. For instance, in the real case we know that all tubular surfaces are unirational (see [26, 48]). In the real case, we can also decide rationality and compute proper parametrizations (see [53]).

The second case in the reduction of Enriques and Manin is the case of del Pezzo surfaces. In the complex case, del Pezzo surfaces have several pencils of conics, and can therefore be reduced to the tubular case above in several ways. Over the fields $\mathbb{Q}$ and $\mathbb{R}$, the parametrization problem for del Pezzo surfaces is still open. See [59] and [38, 14] for an overview on the existing results in the real case and in the rational case.

In the case of positive characteristic, Castelnuovo's criterion is equivalent to rationality over the algebraic closure, but it is not necessary for unirationality. A counterexample has been given in [58]. The class of Zariski surfaces (see [7]) contains many counterexamples. If Castelnuovo's criterion is fulfilled, then there is no parametrization which is separable as a rational map.

We turn to the question: how can we effectively compute the reduction asserted by the theorem of Enriques and Manin? The answer is: by the method of adjoints. A polynomial is an adjoint if it vanishes with a certain prescribed order at the singularities of the surface. The vector spaces of all adjoints of some particular degree define the rational maps which achieve the Enriques/Manin reduction.

By careful investigation of the resulting parametrization algorithm (see [49]), the author could give polynomial bounds for the size of the parametrization in [50].

Computing the adjoints is the most expensive part of the parametrization algorithm, because it requires a *desingularization*, i.e. a birational proper map $f:\tilde{S}\rightarrow S$ such that $\tilde{S}$ is nonsingular. It suffices to know $\tilde{S}$ by local patches, but already the local problem is complicated enough.

We do have algorithms that computes a desingularization by blowing up [64, 5, 65, 6, 18, 10]. Unfortunately, these algorithms are far too expensive to be useful for the computation of adjoints. A more efficient way is to resolve the discriminant by blowing up, thereby reducing to the quasi-ordinary case. In the quasi-ordinary case, the singularities are resolved using two-dimensional Puiseux expansions [2, 40, 22]. This approach is presently taken in a joint work with W. Stöcher (see [52]).

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1 The Problem

Let S be a surface given by an equation $F(x,y,z)=0$, where k is a field, and $F\in k[x,y,z]$ is irreducible. Algebraically, a parametrization is a triple $(X,Y,Z)\in k(s,t)$, such that $F(X,Y,Z)=0$, and the Jacobian matrix

$$\begin{pmatrix} X & Y & Z \\ \partial_s X & \partial_s Y & \partial_s Z \\ \partial_t X & \partial_t Y & \partial_t Z \end{pmatrix}$$

has maximal rank 2 (both conditions are supposed to hold in the field $k(s,t)$). A parametrization is called proper iff there exist rational functions $S,T\in k(x,y,z)$, such that the composites $S(X,Y,Z)$ and $T(X,Y,Z)$ are defined in $k(s,t)$, and are equal to s and t. (The existence of such rational functions imply the Jacobian condition.)

Example 1.1. An easy example is the parametrization of the unit sphere $x^2+y^2+z^2-1=0$ by

$$(X,Y,Z)=(\frac{2s}{s^2+t^2+1},\frac{2t}{s^2+t^2+1},\frac{s^2+t^2-1}{s^2+t^2+1}).$$

This is indeed a proper parametrization, its inverse is the stereographic projection:

$$(S,T) = (\frac{x}{1-z}, \frac{y}{1-z})$$

The *parametrization problem* is the following:

Given: F, the implicit equation of a surface S;

Find: A parametrization (X, Y, Z) of S, if exists;
`NotExist` otherwise.

The *proper parametrization problem* is the following:

Given: F, the implicit equation of a surface S;

Find: If exists, a parametrization (X, Y, Z) of S, and its inverse (S, T);
`NotExist` otherwise.

The common terminology is to use *unirational* as synonym for properly parametrizable and *rational* as synonym for properly parametrizable.

2 Some theoretical results

For the definition of the arithmetic genus p_a and the plurigeni P_n of an algebraic variety, we refer to [56]. Here, we will only need the facts that these numbers are birational invariants, and that their definition does not depend on the choice of the field.

In the real case, the number h_0 of connected components is also a birational invariant (see [9]). Note that the components have to be counted in the projective setting after desingularization. For instance, a two-sheeted hyperbola has $h_0 = 1$.

The following theorems are very helpful for deciding existence.

Theorem 2.1 (Castelnuovo). *Let S be a surface. Over the complex numbers, the following are equivalent.*
a) S is rational.
b) S is unirational.
c) $p_a(S) = P_2(S) = 0$.

Theorem 2.2 (Comesatti). *Let S be a real surface. Over the real numbers, rationality is equivalent to $p_a(S) = P_2(S) = h_0(S) - 1 = 0$.*

For the computation of p_a and P_2, we refer to section 5. For its computation of h_0, we refer to [11].

The next theorem is even more relevant, because it gives a strategy for computing a parametrization. It holds for arbitrary fields.

Theorem 2.3 (Enriques,Manin). *A surface with $p_a = P_2 = 0$ has a pencil of rational curves or is birationally equivalent to a del Pezzo surface.*

In the following sections, we discuss these two cases separately.

Over algebraically closed fields of positive characteristic, Castelnuovo's criterion is equivalent to rationality, but unirationality is a strictly weaker property. For example, the "Zariski surface" with equation $z^p - F(x,y) = 0$, where $F \in (\mathbb{Z}/p)[x,y]$, has the parametrization

$$(X,Y,Z) = (s^p, t^p, F(s,t)),$$

but it has $P_2 > 0$ for generic F of degree greater than $2p$.

3 Surfaces with a Pencil of Rational Curves

A pencil is a family of curves arising as the fibers of a rational map $f : S \to \mathbf{P}^1$. Except in a finite number of base points which are common to all, the curves in a pencil are pairwise disjoint, and their union is equal to S. In this section, we assume that S has such a pencil, and the generic curve of the pencil is an irreducible rational curve.

Let $k(t)$ be a simple transcendental extension of k. Then $(t:1)$ is a generic point of $\mathbf{P}^1_k$. The generic fiber $f^{-1}((t:1))$ is a curve defined over the field $k(t)$. We know that this curve is irreducible and rational over the algebraic closure $\overline{k(t)}$.

Typical examples are surfaces with an equation $A(t)x^2 + B(t)y^2 + C(t) = 0$, where $A, B, C \in k[t]$. Such surfaces are called *tubular surfaces*. The pencil is formed by the horizontal sections $t =$ constant. The generic horizontal section is an irreducible conic. For special values of t, the conic may split into two lines.

From a birational point of view, any surface with a pencil of rational curves may be reduced to a tubular surface. To obtain this result, we recall a well-known theorem from the theory of algebraic curves.

Theorem 3.1 (Noether). *Let K be an arbitrary field. Let C be a curve which is irreducible and rational over the algebraic closure of K. Then C is rational over K, or there is a birational equivalence, defined over K, transforming C to a conic.*

For a modern treatment and an algorithmic version of this theorem, see [55].

Applying theorem 3.1 to the field $K = k(t)$, we get the desired result.

Example 3.2. Let S be the surface with equation $x^4 + z^4 + (xy + z^2)^3 = 0$. The sections of the planes through the triple line $x = z = 0$ cut out cubic plane curves. The structure map is

$$f : S \to \mathbf{P}^1, (x,y,z) \mapsto (x:z).$$

The generic fiber has equations

$$x - tz = (t^4 + 1)z + (ty + z)^3 = 0.$$

This curve has a double point at infinity. By the genus formula (see [66]) it is rational. A parametrization is

$$(X, Y, Z) = (\frac{-s^3 t}{t^4+1}, \frac{s^3 + st^4 + s}{t^5 + t}, \frac{-s^3}{t^4+1}).$$

The same functions yield a parametrization of the surface S.

Example 3.3. Let S be the surface with equation

$$x^2 + 2xz^2 + 2xyz + 2y^3 + 2z^3 y + x^4 + y^4 + z^4 = 0.$$

A rational pencil is given by the structure map

$$f : S \to \mathbf{P}^1, (x, y, z) \mapsto (x : y).$$

The generic fiber, with equations

$$x - ty = x^2 + 2xz^2 + 2xyz + 2y^3 + 2z^3 y + x^4 + y^4 + z^4 = 0$$

is a quadric plane curve with an double point at the origin and two infinitely near double points. It is therefore a rational curve, but it is not easy to construct a parametrization.

We can, however, compute a birational map to the conic over $k(t)$ with equation

$$-2u^2 + tv^2 + 2tuv + 2v + (t^5 + t) = 0$$

(which is also the equation of a tubular surface T). It is given by

$$(x, y, z) \mapsto (U, V) = (\frac{z}{y}, \frac{ty + z^2}{y^2}),$$

$$(u, v) \mapsto (X, Y, Z) = (\frac{t^2}{v - u^2}, \frac{t}{v - u^2}, \frac{tu}{v - u^2}).$$

The same functions induce a birational map between S and T, keeping t fixed.

3.1 Cross Sections

In order to parametrize a given tubular surface $S : A(t)x^2 + B(t)y^2 + C(t) = 0$, the main problem is to find a *cross section*, i.e. a curve on S with a parametrization $(X(t), Y(t), t)$. (A cross section intersects any horizontal section in precisely one point.)

Regarding t as a constant, a cross section is nothing but a point on the conic $A(t)x^2 + B(t)y^2 + C(t) = 0$, defined over $K(t)$. The conic can then be parametrized by secants through this point with varying slope s (see [66]). Interpreting t as a variable again, we have found a parametrization of S by the parameters s, t.

The existence of cross sections depends heavily on the choice of the ground field k.

Theorem 3.4 (Noether). *In the complex case, any tubular surface has a cross section.*

A proof can be found in [42] or [31]. Nöther's proof is algorithmic, but the proposed algorithm is practically useless. A better algorithm was given in [49]. For a good performance, one has to compute with floating point numbers, because symbolic factorization into linear factors is too expensive if the degree is larger than 4.

Theorem 3.5. *In the real case, a tubular surface has a cross section if and only if every horizontal cross section is non-empty.*

This theorem is also due to Comesatti. A proof can be found in [26]. Algorithms for computing cross sections can be found in [43, 49].

For arbitrary fields, we can reduce to the case where A, B, C are squarefree and pairwise relatively prime (it is easy to compute such a reduction, see [49]). Here is a necessary and sufficient criterion for the existence of a cross section.

Theorem 3.6. *Let A, B, C squarefree and relatively prime. Then the tubular surface $A(t)x^2+B(t)y^2+C(t) = 0$ has a cross section if and only if the following conditions are fulfilled.*

a) $-AB$ is a square in the ring $k[t]/\langle C\rangle$.

b) $-BC$ is a square in the ring $k[t]/\langle A\rangle$.

c) $-AB$ is a square in the ring $k[t]/\langle B\rangle$.

d) If the parities of $\deg(A), \deg(B), \deg C$ are not all equal, then the quadratic form $\mathrm{lc}(A)x^2 + \mathrm{lc}(B)y^2 + \mathrm{lc}(C)z^2$ has a nontrivial solution.

Proof. It is obvious that the existence of a cross section is equivalent to the existence of polynomials $X, Y, Z \in k[t]$, not all equal to zero, satisfying $AX^2 + BY^2 + CZ^2 = 0$.

Necessity of (a): we may assume that X, Y, Z do not have a common factor. Let U be a common factor of X and C. Since U does not divide B, it must divide Y. Hence U^2 divides CZ^2. Since C is squarefree, U divides Z. This is a contradiction, hence X and C are relatively prime and X is a unit in $k[t]/\langle C\rangle$.

Modulo C, we get $-AB = (BYX^{-1})^2$.

Necessity of (b,c): analogously.

Necessity of (d): Let $a := \deg(A)$, $b := \deg(B)$, $c := \deg(C)$. We may assume that a, b, c have equal parity.

Let $N := \max(2\deg(X) + a, 2\deg(Y) + b, 2\deg(Z) + c)$. Let

$$l := \frac{N-a}{2}, m := \frac{N-b}{2}, n := \frac{N-c}{2}.$$

Let x be the coefficient of t^l in X. Let y be the coefficient of t^m in Y. Let z be the coefficient of t^n in Z. Then $lc(a)x^2 + lc(b)y^2 + lc(c)z^2$ is the coefficient of t^N in the polynomial $AX^2 + BY^2 + CZ^2$, hence it is zero. Moreover, at least one of x, y, z is different from 0, hence (d).

To show sufficiency, we generalize the algorithm from [48]. We say that we are in the "even case" if the parities of a, b, c are all equal, and in the "uneven case" otherwise. Let

$$l := \frac{b+c}{2}, m := \frac{a+c}{2}, n := \frac{a+b}{2}$$

in the even case, and

$$l := \lfloor \frac{b+c-1}{2} \rfloor, m := \lfloor \frac{a+c-1}{2} \rfloor, n := \lfloor \frac{a+b-1}{2} \rfloor,$$

in the uneven case. Assume that l, m, n are non-negative.

Let X, Y, Z be indeterminate polynomials of degree l, m, n, respectively, with coefficients $x_l, \ldots, x_0$, $y_m, \ldots, y_0$, $z_n, \ldots, z_0$. Let $U \in k[t]$ such that $U^2 = -AB$ modulo C. The congruence $AX = UY$ can be satisfied by imposing c linear conditions on the coefficients $x_l, \ldots, y_0$, corresponding to the c coefficients of the remainder of $AX - UY$ modulo C. This congruence implies

$$0 = (AX - UY)(AX + UY) = A^2X^2 + ABY^2 = A(AX^2 + BY^2 + CZ^2)$$

modulo C. Since A is not a zero divisor, it follows that $AX^2 + BY^2 + CZ^2$ is divisible by C.

Similarily, one can enforce that $AX^2 + BY^2 + CZ^2$ be divisible by A and B by imposing $a + b$ additional conditions on $x_m, \ldots, z_0$. The total number of linear equations is $a+b+c$, and the number of unknowns is $l+m+n+3$, which is equal to $a+b+c$ in the even case and $a+b+c-2$ in the uneven case.

In the even case, let (u, v, w) be a nontrivial zero of the quadratic form $lc(A)x^2 + lc(B)y^2 + lc(C)z^2$. We impose three additional linear equations

$$uy_m - vx_l = uz_n - wx_l = vz_n - wy_m = 0,$$

which are linearly dependant, i.e. they reduce the rank of the solution space by 2. Then by counting the number of equations and the number of unknowns, we see that there is a nontrivial solution in both cases.

Now, we substitute this solution into X, Y, Z. Then $P := AX^2 + BY^2 + CZ^2$ is divisible by ABC. In the uneven case, the formal degree of P is $a+b+c-1$. In the even case, the formal degree is $a+b+c$, but the formal leading coefficient is zero. Thus, we have $\deg(P) \leq a+b+c-1$ in both cases, hence $AX^2 + BY^2 + CZ^2 = 0$.

It remains to discuss the case when one of l, m, n is negative[1]. Suppose $l < 0$. Then we must be in the uneven case, and $b = c = 0$ (i.e. B, C are constants), hence a is odd. Let α be a root of C in the algebraic closure of k. By (b), $-BC$ is a square modulo A, hence there is a polynomial $Q \in k[t]$ such that $Q(\alpha)^2 = -BC$. The algebraic number $\beta := Q(\alpha)$ is in $k(\alpha)$, hence it has a minimal polynomial of odd degree. On the other hand, it satisfies the quadratic polynomial $\beta^2 + BC = 0$. Thus, $\beta \in k$. Then $(X, Y, Z) = (0, \beta, B)$ is a nontrivial solution. $\square$

[1] The proof for this case is due to W. Stöcher.

The proof of theorem 3.6 is actually an algorithm that reduces the problem of finding a cross section (and, of course, deciding existence) to two easier subproblem: finding solutions of ternary forms over k and finding a square root modulo a squarefree polynomial. The algorithm by Hillgarter in [25] does precisely the same reductions, but the number of required instances of these subproblems is much larger there: this number grows proportional to the degree in [25], but it is constant in our algorithm.

If k is a number field, then both subproblems can be solved efficiently. For the first problem, we refer to [12]. For the second problem, we reduce to the case that the squarefree polynomial is irreducible, by the Chinese remainder theorem. For irreducible polynomials, the problem is equivalent to factoring a quadratic polynomial over a finite extension of k: U is a squareroot of $-AB$ in the field $k[t]/\langle C\rangle$ if and only if $x-U$ is a factor of x^2+AB. For the problem of factoring over number fields, we refer to [32, 30].

It should be pointed out that the non-existence of a cross section does not imply the non-existence of a parametrization! Counterexamples can be found in the next subsection.

3.2 The Real Case

If k is a real closed field, then we have Comesatti's criterion 2.2 for rationality: a surface is rational if and only if $p_a = P_2 = 0$ and $h_0 = 1$, where the number h_0 of connected components has to be computed in the projective setting after a desingularization. For tubular surfaces, we always have $p_a = P_2 = 0$ because there is a complex cross section yielding a proper parametrization (see theorem 3.4). In order to compute h_0, we introduce the concept of the *spine*, which is the image of the projection to the t-axis after disregarding isolated points. If the spine is unbounded, we add the point ∞ of the real projective line.

Obviously, the spine is a finite union of intervals bounded by real zeroes of the polynomials A, B, C. It is easy to see that the number of components of the spine is equal to h_0. We can distinguish three cases.

The spine is full (i.e. the whole projective line). In this case, we already know that we can find a cross section by theorem 3.5 (or the algorithm in the proof of theorem 3.6). The cross section allows to compute a proper parametrization easily, as above.

The spine is connected, but not full. By theorem 3.5, we do not have a cross section. Nevertheless, we can construct a proper parametrization by the algorithm [53]. The main idea is to construct a tubular surface S' with the same spine and with an equation of the type $x^2+y^2+F(t)=0$, where F is a polynomial of degree 1 or 2, and to use the following theorem (also due to Comesatti).

Theorem 3.7. *If two tubular surfaces have the same spine, then they are birationally equivalent over the reals.*

The surface S' is a quadric surface, which can be properly parametrized by the well-known method of stereographic projection. By composing with the birational equivalence above, we get the desired parametrization.

The spine is disconnected. Since $h_0 > 1$, we know that S is not rational. However, we can construct an improper parametrization by the following trick (see also [48]).

Let $[a, b]$ be an interval which is contained in the spine. By substituting $t = \frac{as^2+b}{s^2+1}$ in the equation of S and clearing denominators, we get a tubular surface S' and a rational map

$$f : S' \to S, (x, y, s) \mapsto (x, y, \frac{as^2 + b}{s^2 + 1}).$$

The tubular surface S' has full spine, therefore it has a cross section by theorem 3.5. Thus, S' has a parametrization $p : \mathbb{R}^2 \to S'$. Then the composition $f \circ p : \mathbb{R}^2 \to S$ is a parametrization of S.

We have just shown the following theorem.

Theorem 3.8. *In the real case, any tubular surface is unirational.*

3.3 Arbitrary Fields

In a series of papers [26, 27, 28] (see also [29, 14]), Iskovskih gives a necessary criterion for rationality and shows that any surface fulfilling this criterion can be birationally reduced to a del Pezzo surface (see the next section). These results are theoretical, there are no algorithms available.

4 Del Pezzo Surfaces

A surface is called del Pezzo surfaces iff its generic plane section is elliptic and it satisfies the condition $p_a = P_2 = 0$. Typical examples are the nonsingular cubic and the nonsingular intersections of two quadrics in $\mathbf{P}^4$. The degree of a del Pezzo surfaces is quite small (less than or equal to 9).

The complex case Over algebraically closed fields, a del Pezzo surface has several pencils of conics (see [51]). For example, a nonsingular cubic has 27 lines; for each line L, the sections with planes through L cut out a pencil of conics. Another example is the torus, which has 4 pencils of conics: one formed by the rotating circle, one formed by the orbits of the rotation, and two cut out by tangent planes through the center (each such tangent plane intersects the torus in a union of two circles).

To construct a proper parametrization, one can just use one of those pencils and proceed as in the previous section. For a detailed computational treatment of the cubic surface, we refer to [4].

The real case Here is Comesatti's classification of real del Pezzo surfaces.

Theorem 4.1. *A del Pezzo surface has at most five components.*

Any connected del Pezzo surface is rational.

A del Pezzo surface with two or three components is not rational, but it has a pencil of conics.

A del Pezzo surface with four components does not have a pencil of rational curves (and is therefore not rational), but it is unirational.

A del Pezzo surface with five components does not have a pencil of rational curves.

A readable proof is contained in [59]. Logically, theorem 2.2 is a consequence of theorem 3.7 and the above classification.

The question of unirationality is still open (the problem consists of the del Pezzo surfaces with five components).

Arbitrary Fields In [36], Manin gave a birational invariant, based on Galois cohomology, which could in principle be computed by a constant number of field operations and factorizations of polynomials of constant degree. Manin and Iskovskih computed this invariant for various subclasses, but an algorithm has not yet been worked out in detail.

Using Galois cohomology, one can also sometimes give lower bounds for the degree of unirationality, and sometimes find two points on S which can not be both in the image of a parametrization (R-equivalence and Brauer equivalence). An introduction to this approach can be found in [37].

An obvious necessary condition for (uni)rationality is the existence of k-rational points. For number fields, this is a complicated problem because diophantine equations are undecidable in general by [39]. For subclasses of del Pezzo surfaces, decision can be achieved by the Hasse principle: a solution exists if and only if there exists a solution in the completion of k with respect to any possible valuation. The survey papers [38, 14] give an overview on results of this kind.

5 Adjoints

The construction of a pencil of rational curves, the construction of a birational reduction to a del Pezzo surface, and the computation of the numbers p_a and P_2 can be accomplished with one single tool: adjoints. Some references for the theory of adjoints are [20, 68, 8, 41, 49]; however, I believe that the full power of this tool has not been exploited yet.

Here is an algorithmic definition.

Definition 5.1. A polynomial is an m-adjoint iff it vanishes with order at least $m(r-1)$ at each r-fold singular curve, and it vanishes with order at least $m(r-2)$ at each r-fold singular point.

Infinitely near singularities have to be taken into account. We consider only polynomials that are reduced with respect to the equation of S, i.e. we compute modulo the ideal of the surface.

For any two natural numbers n, m, we define the vector space $V_{n,m}$ as the space of all m-adjoints of degree at most $n + m(d - 4)$, where d is the degree of the given surface S.

For any m, the dimension of the vector-space $V_{0,m}$ is a birational invariant, the m-th *plurigenus* P_m. Another birational invariant, the arithmetic genus p_a, can be computed by the formula

$$p_a = d + 2\dim(V_{1,1}) - \dim(V_{2,1}) - 1,$$

where d is the degree of the surface S (see [49]).

For each n, m, such that $\dim(V_{n,m}) > 0$, we define a rational map

$$f_{n,m} : S \to \mathbf{P}^{\dim V_{n,m} - 1},$$

$$(x, y, z) \mapsto (p_1(x, y, z) : \ldots : p_{\dim V_{n,m}}(x, y, z)),$$

where $p_1, \ldots, p_{\dim V_{n,m}}$ is a basis for $V_{n,m}$. The choice of the basis is not important.

Theorem 5.2 (Schicho). *If $p_a = P_2 = 0$, then one can find n, m such that one of the following holds.*

1. *$f_{n,m}$ is a map to a rational curve with rational curves as fibers.*
2. *$f_{n,m}$ is a birational map to a del Pezzo surface.*
3. *$f_{n,m}$ is a birational map to a projective plane.*
4. *$f_{n,m}$ is a birational map to a quadric surface in $\mathbf{P}^3$.*

More precisely, let μ be the smallest integer such that $V_{1,\mu+1}$ is the zero space. (One can show that such a μ always exists.) Then (n, m) is one of $(1, \mu)$, $(1, \mu - 1)$, or $(2, 2\mu + 1)$.

Clearly, theorem 5.2, together with the algorithms discussed in previous sections and with some auxiliary algorithms such as the inversion of birational maps (see [57, 47]) provides a parametrization algorithm for surfaces over algebraically closed fields. The implementation of Heiss [24] provides all subalgorithms that are needed in the four cases, except for the computation of the adjoint spaces $V_{n,m}$.

By careful analysis of the algorithm, one can derive bounds for the size of the output parametrization (see section 7).

The full proof of theorem 5.2 is too long to be reproduced here, so we refer to [49]. Instead, we give the "secret definition" of adjoints, which gives an idea why the birational behavior of adjoints is so good.

Definition 5.3. A rational differential form is said to be *of the first kind* iff its integral over any compact set is finite.

Differential forms of the first kind are regular at all regular points, but they may have poles of restricted order at the singularities.

Example 5.4. Let S be the quadric cone $xy - z^2 = 0$. Then

$$\frac{dx \wedge dy}{z} = x^{-1/2} y^{-1/2} dx \wedge dy$$

is a differential form of the first kind. Since

$$\frac{dx \wedge dy}{z} = \frac{2dx \wedge dz}{x} = \frac{2dz \wedge dy}{y},$$

we see that the singularity has an isolated pole at the vertex $(x, y, z) = (0, 0, 0)$.

Theorem 5.5. *Let $F(x, y, z)$ be the equation of the surface S. Then the polynomial $G(x, y, z)$ is a 1-adjoint iff*

$$\frac{G dx \wedge dy}{\partial F / \partial z}$$

is a differential form of the first kind.

In the above example, we verified that $\frac{dx \wedge dy}{\partial F / \partial z}$ is a differential form of the first kind. Because a polynomial multiple of a differential form of the first kind is again of the first kind, we see that any polynomial is a 1-adjoint of the quadric cone. This agrees with the definition, because the only singularity of the quadric cone is a double point, which yields no restriction in the condition of adjunction.

Differential forms of the first kind have very good birational behavior.

Theorem 5.6. *Any proper birational map induces an isomorphism of the vector-spaces of rational differential forms of the first kind.*

Proof. Let $f : S_1 \to S_2$ be a proper birational map. The pullback f^* of rational differential forms is an isomorphism. Thus, it suffices to show that ω is of the first kind if and only if $f^*(\omega)$ is of the first kind.

If $f^*(\omega)$ is of the first kind, then

$$\int_K \omega = \int_{f^{-1}(K)} f^*(\omega)$$

is finite for all compact K (note that $f^{-1}(K)$ is compact by properness). Vice versa, any compact set $K \subset S_1$ is contained in the compact set $f^{-1}(f(K))$, hence $\int_K f^*(\omega)$ is dominated by

$$\int_{f^{-1}(f(K))} f^*(\omega) = \int_{f(K)} \omega$$

. □

The properness assumption is automatically fulfilled for projective varieties. This is a partial explanation for the nice global birational properties of adjoints.

6 Desingularization

The computation of adjoints is the most expensive part of the parametrization algorithm. The problem is that one has to resolve the singularities, in order to detect the infinitely near singularities.

Here is a recursive algorithm that computes adjoints.

Input: A surface F in 3-space, given implicitly.
A basis for a vectorspace V of polynomials.
An integer $m > 0$.
Output: A basis for the subvectorspace of m-adjoints.

```
Algorithm AdjointSubspace(F, V, m):
  if F is smooth then
    return V;
  else
    choose a singularity S;
    r := the multiplicity of F at S;
    if S is a curve then
      p := m(r − 1);
    else (S is a point)
      p := m(r − 2);
    V1 := the subvectorspace of the elements that vanish with order ≥p at S;
    (F̃, e) := (BlowUp(F, S), equation of the exceptional divisor);
    f : V1→{forms on F̃} := division by e^p ∘ pullback;
    V2 := AdjointSubspace(F̃, f(V1), m);
    return f^{-1}(V2);
```

The problem of choosing a singularity to be blown up in such a way that one eventually achieves a nonsingular variety is called the problem of *resolution of singularities*. It is in itself a famous problem.

The existence was proven by Walker and Zariski in the 30's. Hironaka gave an existence proof for arbitrary dimension (characteristic zero). Abhyankar proved existence for surfaces in positive characteristic (here the case of arbitrary dimension is still open). For an introduction into this problem, see [33]. For an overview on the difficulties in positive characteristic, consult [23].

An algorithm for resolution in arbitrary dimensions (characteristic zero) has been given in [64, 65, 18] and independently in [5, 6]. Bodnar and the author developed the necessary data structures (see [10]) in order to implement Villamayor's algorithm in Maple. The full program is downloadable at the web site `http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup`. Special thanks go to S. Encinas for helping us to understand the structure of Villamayor's algorithm.

The computed desingularization is represented by an atlas of charts. For each chart, we have a finite set of polynomials describing the ideal of a nonsingular variety. For each pair of charts, we have a chart transition map identifying some principal open subsets. This representation is perfectly suitable for adjoints computation.

Unfortunately, a closer look at the performance of the algorithm shows that the algorithm is only of theoretical interest. No bound for the asymptotical complexity is known. Even for small examples, the computing time and the size of the output are enormous.

Our current approach towards the computation of adjoints (joint work with W. Stöcher, see forthcoming paper) proceeds in a different way: first, we resolve the discriminant by blowing ups, thereby reducing to the case where all singularities are quasi-ordinary. Second, we compute the two-dimensional Puiseux expansions [2] at all singularities. Third, we plug each Puiseux expansion into a general element of the input vectorspace. The adjoints are then characterized by the vanishing of certain coefficients in these expansions.

The fact that we compute with infinite series also requires more sophisticated data types. Power series have to be represented as black box procedures, producing the n-th coefficient when fed with the integer n. An operation on power series is then actually a functor which takes an input procedure and produces an output procedure. Lazy evaluation ensures that one never computes more terms in the expansions than those that are really required in the final output.

7 Degree Bounds

We define the degree of a parametrization $(\frac{X_n(s,t)}{D(s,t)}, \frac{Y_n(s,t)}{D(s,t)}, \frac{Z_n(s,t)}{D(s,t)})$ as the maximum of the total degrees of the four polynomials X_n, Y_n, Z_n, D.

In the curve case, it is easy to show that the degree of a proper parametrization is equal to the degree of the equation of the curve, and that the degree of an improper parametrization is larger. In the surface case, the degree of a parametrization can be lower, as shown by the following example.

Example 7.1. The surface of degree n^2 with equation

$$(x^n - y)^n - z = 0$$

has a parametrization of degree n, namely

$$(X, Y, Z) = (t, s + t^n, s^n).$$

The parametrization in this example reaches a lower bound.

Theorem 7.2. *If P is a parametrization of the surface S, then* $\deg P \geq \sqrt{\deg S}$.

Proof. Suppose that P is an $e : 1$-parameterization of F, i.e. a generic point of F is reached by e points in the parameter space. We may assume that X_n, Y_n, Z_n, D do not have a common divisor (otherwise we may cancel by this divisor and obtain a parameterization of strictly smaller degree). Let E_1, E_2 be two generic planes in k^3. Then the line $E_1 \cap E_2$ intersects F in exactly d points.

The preimage of the set of these points is a set Σ of $e{\cdot}\mathrm{deg}F$ points. On the other hand, Σ is the set of common zeroes of the two linear combinations Q_1, Q_2 of X_n, Y_n, Z_n, D corresponding to E_1 and E_2 minus the set of common zeros of X_n, Y_n, Z_n, D. Since X_n, Y_n, Z_n, D do not have a common factor, and Q_1, Q_2 are generic linear combinations, Q_1 and Q_2 do not have a common factor. By Bezout's theorem, the number of common zeroes is less or equal $(\mathrm{deg}P)^2$. Hence $\mathrm{deg}F \le e{\cdot}\mathrm{deg}F = \mathrm{card}(\Sigma) \le (\mathrm{deg}P)^2$. □

On the other hand, there are rational surfaces which do not have a small parametrization.

Example 7.3. The surface of degree $2n-1$ with equation

$$x^{2n-1} - y^n - z^2 = 0$$

has a parametrization of degree n^2, namely

$$(X,Y,Z) = (\frac{1}{s^n+t^2}, \frac{s}{(s^n+t^2)^2}, \frac{t}{(s^n+t^2)^n}).$$

One can show that then $\mathrm{deg}P \ge \frac{n^2}{2}$ for any parametrization. So, the degree of the smallest parametrization grows quadratically.

By an analysis of the algorithm [49], one can obtain the following upper bound. It is valid for the complex case only.

Theorem 7.4. *Let S be a rational surface. Then there exists a parameterization P of S, such that* $\mathrm{deg}P \le 2(\mathrm{deg}S)^4$.

For the proof, we refer to [50].

There are no known series of rational surfaces where the degree of the smallest possible parametrization grows larger than quadratically. Hence, it is quite possible that the upper bound above can be improved a lot.

Another open question is to compute the smallest possible parametrization for a given rational surface. The algorithm [49] is guaranteed to produce a parametrization which is at most twice as big as the smallest possible parametrization (see [50]).

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