# Algebraic Multigrid for Edge Elements *

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**Abstract**

This paper presents an algebraic multigrid method for the efficient solution of the linear system arising from a finite element discretization of variational problems in $H_0(\mathrm{rot}, \Omega)$. The finite element spaces are generated by Nédélec-elements (Whitney-1-forms or further referenced to as edge elements).

A coarsening technique is presented in order to construct a suitable coarse spaces and according grid transfer operators. The prolongation operator is designed such that coarse grid kernel functions of the rot-operator are mapped to fine grid kernel functions. Furthermore, coarse grid rot-free functions are discrete gradients.

The smoothers by Hiptmair [11] and Arnold/Falk/Winther [1] can be directly used in the algebraic framework.

Numerical studies are presented for 3D problems to show the efficiency of the proposed technique.

**Keywords** Maxwell's equations, finite element method, edge elements, algebraic multigrid

## 1 Introduction

This paper is concerned with the solution of the linear equation

$$K_h^e u_h = f_h \,, \tag{1}$$

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where $K_h^e \in \mathbb{R}^{N_h^e \times N_h^e}$ is a symmetric positive definite (SPD) sparse system matrix, $u_h \in \mathbb{R}^{N_h^e}$ the solution vector, and $f_h \in \mathbb{R}^{N_h^e}$ the load vector. Additionally $K_h^e$ results from the finite element (FE) discretization of the bilinear form

$$\int_\Omega \nu \ \mathrm{rot}\,\vec{A} \cdot \mathrm{rot}\,\vec{v}\,dx + \int_\Omega \sigma\,\vec{A} \cdot \vec{v}\,dx \quad \forall\,\vec{v} \in H_0(\mathrm{rot},\Omega)\,, \tag{2}$$

which is discretized with $H_0(\mathrm{rot},\Omega)$-conforming finite elements introduced by Nédélec in [17], i.e., edge elements (the index "e" indicates the discretization by edge elements). Consequently $f_h$ stems from the linear form

$$\int_\Omega \vec{J} \cdot \vec{v}\,dx \quad \forall\,\vec{v} \in H_0(\mathrm{rot},\Omega)\,,$$

where $\vec{J}$ is a given function. The computational domain $\Omega \subset \mathbb{R}^3$ is assumed to be bounded with a Lipschitz boundary $\partial\Omega$. Furthermore, $\nu$ and $\sigma$ are piecewise constant, strictly positive functions. The number $N_h^e$ of unknowns in (1) behaves asymptotically as $N_h^e = O(h^{-d})$ with the mesh size parameter $h$, and thus the linear system is usually very large.

If $\sigma \equiv 0$ then (2) represents a static, magnetic field problem arising from the Maxwell's equations (see [12]). Therein $\vec{A}$ is related to a magnetic vector potential, that means $\mathrm{rot}\,\vec{A}$ represents the magnetic flux. In this case a gauge condition has to be assigned to $\vec{A}$, i.e., $\mathrm{div}\,\vec{A} = 0$ and the right hand side is assumed to be divergence free, i.e., $\mathrm{div}\,\vec{J} = 0$. This is necessary to ensure uniqueness of the solution. In this paper we assume $\sigma > 0$ in order to circumvent the gauge condition. Let us additionally mention that the convergence rate of the proposed method do not depend on a lower bound of $\sigma$.

Solving (1) by means of multilevel methods the kernel of the rot-operator, have to be taken into account carefully. A geometric multilevel method was set up by R. Hiptmair in [11] the first time. An other approach was discussed in [1]. For applications on the geometric multigrid technique in the function space $H(\mathrm{rot},\Omega)$ we refer to [11, 3, 1, 16, 20].

An algebraic multigrid approach for the solution of (1) requires in addition to the available components of the geometric multigrid also a proper coarsening strategy. In spite of the fact that the FE-matrix $K_h^e$ is SPD, the classical approaches of [5, 6, 7, 8, 15, 18, 19, 22] and variants of it fail for the problem at hand. All these methods are designed for SPD problems which either stems from an FE-discretization for $H^1$-elliptic problems or needs beside the SPD property special characteristics of the system matrix (e.g. M-matrix property). A first AMG approach to solve (1) can be found in [2]. The key idea of Beck was to split the $H(\mathrm{rot},\Omega)$ function into a $(H^1)^3$ function and a gradient function, and apply classical AMG for all components. This differs from our approach, since we apply the coarsening directly for the one space $H(\mathrm{rot},\Omega)$.

The challenge for the construction of an AMG method is to cope with the kernel of the rot-operator. Therfore we propose the following technique:

1. Construct a "node to edge" map in order to know which edges belong to a node.

2. Fix the coarse grid unknowns by a coarsening technique.

3. Define a prolongation operator which maps the kernel of the coarse space into the kernel of the fine space.

4. Calculate the coarse grid matrix by Galerkin's method.

5. Take an appropriate smoother for the regarded problem class.

A pivotal point is the construction of the "node to edge" map, to be able to define the coarse edges and to construct the prolongation and the smoother for $K_h^e$. A possibility to construct a "node to edge" map is to define an auxiliary matrix $K_h^n \in \mathbb{R}^{N_h^n \times N_h^n}$ which stems from the bilinear form

$$\int_\Omega \nu \operatorname{grad} u \cdot \operatorname{grad} v \, dx + \int_\Omega \sigma u \cdot v \, dx \,, \tag{3}$$

using linear nodal FE-functions. In addition, the matrix $K_h^n$ is calculated on the same computational domain $\Omega$ (i.e., on the same FE-mesh, where only a simplicial one is used in this paper) and parameters $\nu, \sigma$ defined in (2). The index "n" indicates a nodal FE-discretization. For description we always use a two grid method and therefore the indices $h$ and $H$ are related to the fine and coarse grid quantities, respectively. Each off-diagonal entry of $K_h^n$ is related to an edge in the FE-mesh. Consequently, a "node to edge" map is given in a natural way. In order to get a matrix hierarchy for $K_h^e$ we perform a setup (i.e., coarse grid selection, construction of a prolongation operator and a coarse grid operator) for the auxiliary matrix $K_h^n$. The resulting coarse grid matrix $K_H^n$ gives rise to a "node to edge" map on the coarse level. These coarse edges are degrees of freedom on the coarse grid for $K_H^e$. Consequently, an appropriate prolongation operator for the edge FE-space and a suitable smoothing iteration can be defined with the benefit of the "node to edge" map. The coarse grid matrix $K_H^e$ is computed by Galerkin's method. By recursion, the multigrid method is defined as usual.

The paper is organized as follows: In Sec. 2 a brief overview on Maxwell's equation and its discretization by edge elements is given. In Sec. 3 the AMG method is motivated for SPD matrices which stem from nodal FE-functions. After that an AMG method for (1) is presented, i.e., the coarsening process, the definition of the prolongation operator and the smoother are designed in a pure algebraic way. Numerical studies are presented in Sec. 4 which show the efficiency of the proposed technique. Finally, further remarks are given and conclusions are drawn.

## 2 Problem Formulation

Let us consider the partial differential equation

$$\operatorname{rot} \nu \operatorname{rot} \vec{A} + \sigma \vec{A} = \vec{J} \qquad \text{in } \Omega_i \subset \mathbb{R}^3 \tag{4}$$

with boundary conditions

$$\vec{A} \times \vec{n} = 0 \qquad \text{on } \Gamma_B$$

and appropriate interface condition on $\Gamma_I$. In (4) $\Omega = \Omega_1 \cup \Omega_2$ is the bounded computational domain with sufficient smooth boundary $\partial\Omega = \Gamma_B$ and interface $\Gamma_I$ (see Fig. 1). The piecewise constant functions $\nu$ and $\sigma$ are assumed to be strictly positive. $\vec{n}$ is dedicated to the unit outward vector and $\vec{J}$ is related to an appropriate given right hand side. As we mentioned it very briefly in the

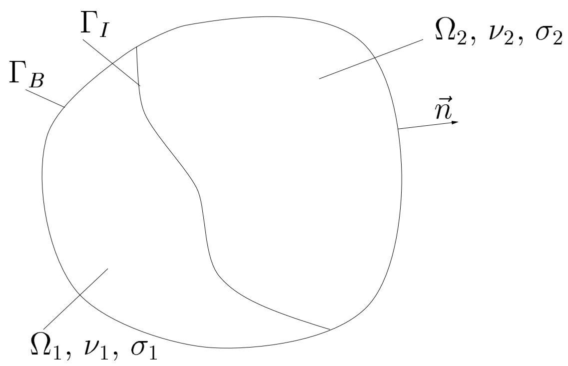


Figure 1: Principle structure of the considered problem class.

introduction this setting is closely related to 3D static magnetic field problems arising from the Maxwell's equations. This can be seen by introducing a magnetic vector potential for the magnetic flux $\vec{B}$, i.e.,

$$\vec{B} = \operatorname{rot} \vec{A}\,.$$

If the high frequency displacement current and all time derivatives are neglected and furthermore $\sigma \equiv 0$ then we end up with the magnetostatic equation. By interpreting $\nu$ as the reluctivity and $\vec{J}$ as the current density the physical relation is given. For further discussion we always have this physical interpretation in mind but set $\sigma > 0$ to a small value compared to $\nu$. This regularization guarantees a unique solution, and does not much affect the quantity of interest, the magnetic flux $\vec{B}$.

A natural choice for the weak formulation is the function space $V = H_0(\mathrm{rot}, \Omega)$ which is defined by

$$H_0(\mathrm{rot}, \Omega) = \{\vec{v} \in (L^2(\Omega))^3 \mid \operatorname{rot} \vec{v} \in (L^2(\Omega))^3 \text{ and } \vec{v} \times \vec{n} = 0\}\,.$$

Therewith a weak formulation of (4) reads as: Find $\vec{A} \in V$ such that

$$\int_\Omega \nu \cdot \operatorname{rot} \vec{A} \cdot \operatorname{rot} \vec{v}\, dx + \int_\Omega \sigma \cdot \vec{A} \cdot \vec{v}\, dx = \int_\Omega \vec{J} \cdot \vec{v}\, dx \qquad \forall\, \vec{v} \in V\,. \tag{5}$$

It is well known that the kernel of the rot-operator

$$V_0 = \{\vec{v} \in V \mid \operatorname{rot} \vec{v} = 0\}$$

are described by gradient fields in the case of simple connected domains $\Omega$. For multiple connected regions $\Omega$ this is true up to a finite dimensional space. By defining $Q = H_0^1(\Omega)$, the gradient fields are exactly the rot-free functions, i.e.,

$$V_0 = \operatorname{grad} Q.$$

During the last years edge elements became very popular for FE-discretization of $V$ (see [12, 17]). Besides the fact that there exists other possibilities for an FE-discretization (see [4, 21]) we are concerned with edge elements in this paper. Therfore we assume a spatial discretization of $\Omega$ by a simplicial mesh which fulfills the shape regularity in the sense of [9]. The index set of finite elements is denoted by $\tau_h$. The FE-space constructed by the edge elements on $\tau_h$ is abbreviated by $\mathbf{V_h}$ and $\mathbf{V_h} \subset V$ holds. Consequently, a conforming FE-space is constructed for $Q$ by nodal piecewise linear finite elements, which is denoted by $\mathbf{Q_h}$. We will use the Galerkin isomorphism

$$G^e : V_h \to \mathbf{V_h}$$

and

$$G^n : Q_h \to \mathbf{Q_h}$$

with $V_h = \mathbb{R}^{N_h^e}$ and $Q_h = \mathbb{R}^{N_h^n}$. Thus the discrete kernel of the rot-operator is defined by

$$V_{h,0} = \{v_h \in V_h \mid \operatorname{rot} G^e v_h = 0\} \tag{6}$$

and the discrete gradient operator $\operatorname{grad}_h : Q_h \to V_{h,0}$ is given for $q_h \in Q_h$ by

$$\operatorname{grad}_h q_h = (G^e)^{-1} \operatorname{grad} G^n q_h\,. \tag{7}$$

The edge elements have the property that the tangential component is continuous while they let the normal component free to jump, i.e., an edge element discretization is $H_0(\mathrm{rot}, \Omega)$-conform. This is important for (5) in the case of nonconvex domains $\Omega$ or if the coefficient function $\nu$ has a jump to get a "good approximation" of the continuous solution. Further applications in nonlinear or time dependent problems are out of the scope of this paper, and we refer to the extensive literature, see [3, 16, 20].

# 3 Construction of an AMG Method

In this section the ingredients of an AMG method are recalled and especially an approach for edge elements is proposed. Thus we are concerned with the pure algebraic construction of a multilevel hierarchy of coarse matrices for $K_h^e$. Therefore it is assumed that $K_h^e \in \mathbb{R}^{N_h^e \times N_h^e}$ and $K_h^n \in \mathbb{R}^{N_h^n \times N_h^n}$ arises from the same FE-mesh of tetrahedra in $\mathbb{R}^3$. The bilinear form belonging to $K_h^e$ and $K_h^n$ are given in (2) and (3), respectively. The number of edges and nodes of the underlying mesh is given by $N_h^e$ and $N_h^n$, respectively. First, we briefly describe an AMG method for $K_h^n$. Afterwards, we are able to construct a matrix hierarchy for $K_h^e$ with the help of the auxiliary matrix hierarchy of $K_h^n$.

## 3.1 A General Approach to the AMG Method

The tools for an AMG method can be presented rather general in the case of SPD matrices arising from an FE-discretization. The most important points are:

1. Define a coarse grid selection process (see, e.g. [5, 15, 19, 22]).
2. Construct a prolongation operator and use its transposed as restriction operator (see, e.g. [5, 15, 19, 22]).
3. Use Galerkin's method to construct a coarse grid matrix.
4. Take an appropriate smoother for the underlying problem class, i.e., Gauß-Seidel for scalar problems, block Gauß-Seidel for systems of equations.

The matrix $K_h^n$ stems from an FE-discretization with nodal linear FE-functions and therfore we can interpret the $i^{th}$ matrix row of $K_h^n$ as follows: The diagonal entry $k_{ii}^n$ is related to the grid point $i$ and an entry $k_{ij}^n$ is related to an edge $(i,j)$ (see Fig. 2). In the case of a scalar problem we are able to identify "grid point" with "unknown". The set of grid points is denoted by $\omega_s^n = \{1, 2, \ldots, N_s^n\}$, with cardinality $\mathrm{card}(\omega_s^n) = N_s^n$ on level $s$ for the matrix $K_s^n \in \mathbb{R}^{N_s^n \times N_s^n}$. Below the neighborhood of a node $i \in \omega_s^n$, the set of edges on level $s$, and the set of edges belonging to a node $i \in \omega_s^n$ are given by

$$\begin{aligned} \omega_s^e &= \{(i,j) \,|\, (K_s^n)_{ij} \neq 0, i \neq j\}, \\ N^i &= \{j \,|\, (K_s^n)_{ij} \neq 0, i \neq j\}, \\ T^i &= \{(i,j) \,|\, j \in N^i\}, \end{aligned}$$

respectively. In addition the graph of a matrix $K_h^n$ is defined as a 2-tuple of nodes and edges, i.e.,

$$\mathrm{graph}(K_h^n) = (\omega_h^n, \bigcup_{i \in \omega_h^n} T^i)\,. \tag{8}$$

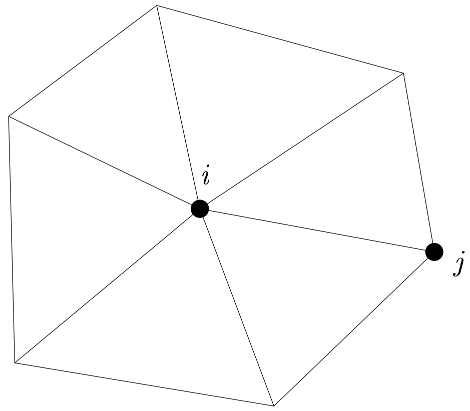


Figure 2: Clipping of an FE-grid.

Motivated from an FE-grid, we see that a "grid" or "set of unknowns" can be split up into two disjoint subsets, i.e.,

$$\omega_h^n = C \cup F\,, \qquad C \cap F = \emptyset$$

with $C$ and $F$ denotes the coarse and fine nodes, respectively. The coarse grid is defined by identifying each coarse grid node $j \in C$ with an index $k \in \omega_{s+1}^n$. This is express by the index map ind(.) as

$$\omega_{s+1}^n = \mathrm{ind}(C)\,.$$

The local fine and coarse grid sets around a node $i \in \omega_s^n$ are defined by

$$\begin{aligned} F^i &= F \cap N^i \\ C^i &= C \cap N^i\,. \end{aligned}$$

In terms of two consecutive levels $s$ and $s+1$ it is assumed that each coarse level variable $u_i^{s+1}$, $i \in \omega_{s+1}^n$, is used to directly correct a fine grid variable $u_k^s$, $k \in F^{j(i)}$ and $j(i) \in \omega_s^n$. The "setup process" (for an algorithm see Alg. 1) constructs a hierarchy of matrices with suitable (i.e., problem dependent) prolongation operators and possible preparatory work for the smoothing operator (e.g. block smoother). Therefore a multilevel cycle can be setup in the usual way (see Alg. 2).

So far, the basic tools for AMG are presented. Next, the AMG method for edge elements is discussed in more detail. First, it should be noted, that an application of the classical AMG approaches (which are designed for nodal basis functions) fails, in spite of the fact that the system matrix $K_h^e$ is SPD. Thus the construction of a matrix hierarchy with suitable prolongation operators and a smoothing iteration is required.

**Algorithm 1** Setup process for AMG: **Setup($K_l^n$,l)**

// COARSEGRID is defined before this function call
**if** card($\omega_l^n$) $> COARSEGRID$ **then**
Split $\omega_l^n$ into disjoint sets $C$ and $F$
Set $\omega_{l+1}^n = \mathrm{ind}(C)$
Define the interpolation operator $P_l^n$, $R_l^n = (P_l^n)^T$
$K_{l+1}^n = R_l^n K_l^n P_l^n$
**Setup($K_{l+1}$,l+1)**
**else**
Perform a factorization of $K_l^n$
COARSELEVEL = l
**end if**

**Algorithm 2** V($\nu_F$,$\nu_B$)-cycle: **MG($u_l$, $f_l$,l)**

**if** $l = COARSELEVEL$ **then**
$u_l = (K_l^n)^{-1} f_l$ with a direct solver
**else**
Smooth $\nu_F$ times on $K_l^n u_l = f_l$
Calculate the defect $d_l = f_l - K_l^n u_l$
Restrict the defect to the next coarser level $l+1$: $d_{l+1} = R_l^n d_l$
Set $u_{l+1} \equiv 0$
Apply **MG($u_{l+1}$,$d_{l+1}$,l+1)**
Prolongate the correction $s_l = P_l^n u_{l+1}$
Update the solution $u_l = u_l + s_l$
Smooth $\nu_B$ times on $K_l^n u_l = f_l$
**end if**

## 3.2 The Coarsening Process

In order to perform a coarsening for the edge element matrix $K_h^e$ and therewith the construction of a prolongation operator and a smoother we use an auxiliary matrix arising from an FE-discretization (linear, nodal FE-functions) of the weak formulation (3) which results in an SPD matrix $K_h^n \in \mathbb{R}^{N_h^n \times N_h^n}$. For this type of matrices the setup process can be done in the usual way (see [5, 15, 19, 22]).

**Remark 3.1.**

1. *The coarse grid selection can be done by several different coarsening strategies. On the one hand side a pure matrix graph based method can be used, or on the other hand side a coarsening method depending on the matrix entries. Only the latter case has chances to detect parameter jumps and anisotropies.*

2. *We can think of other auxiliary matrices which defines an appropriate "node to edge" map, but this is not the content of this paper.*

In order to get a "useful" set of coarse grid edges $\omega_s^e$ we invest in a special prolongation operator $P^n$ for the auxiliary matrix $K_h^n$. The prolongation operator $P^n$ is constructed such that each fine grid variable prolongates exactly from one coarse grid variable. We extend the index map $\mathrm{ind} : C \to \omega_H^n$ defined above onto the whole fine space $\omega_h^n$ by assigning the coarse grid index of the representant of the cluster

$$\mathrm{ind} : \omega_h^n \to \omega_H^n .$$

A consequence is that

$$\mathrm{ind}(i) = \mathrm{ind}(j) \quad \text{iff} \quad i, j \in \omega_h^n \text{ prolongate from the same coarse grid variable.}$$

We denote the set of an agglomerate around a grid point $i$ by

$$I_i = \{ j \in \omega_s^n \mid \mathrm{ind}(j) = \mathrm{ind}(i) \}$$

and

$$\omega_{s+1}^n = \{ \mathrm{ind}(i) \mid i \in \omega_s^n \} .$$

Therefore the prolongation operator $P^n$ has only 0 and 1 entries by construction, i.e.,

$$(P^n)_{ij} = p_{ij}^n = \begin{cases} 1 & i \in \omega_h^n,\ j = \mathrm{ind}(i) \\ 0 & \text{otherwise} . \end{cases} \tag{9}$$

The coarse grid matrix $K_H^n$ calculated by Galerkin's method, i.e., $K_H^n = (P^n)^T K_h^n P^n$, which is equivalent to the formulae

$$(K_H^n)_{\tilde{k}\tilde{l}} = \sum_{i \in I_k} \sum_{j \in I_l} p_{i\tilde{k}}^n \cdot (K_h^n)_{ij} \cdot p_{i\tilde{l}}^n \tag{10}$$

with $\tilde{k} = \mathrm{ind}(k)$, $\tilde{l} = \mathrm{ind}(l)$, and $k, l \in \omega_{s+1}^n$. $K_H^n$ has useful properties, because of the prolongation operator defined in (9). This is the content of the next lemma.

**Lemma 3.2.** *Let $k, l \in C$, $k \neq l$ and $\tilde{k} = \mathrm{ind}(k) \in \omega_H^n$, $\tilde{l} = \mathrm{ind}(l) \in \omega_H^n$. Further let $K_h^n$ be given from (3) and $P^n$ be defined by (9). $K_H^n = (P^n)^T K_h^n P^n$.
If for all $i \in I_k$ and for all $j \in I_l$*

$$(K_h^n)_{ij} = 0$$

*then*

$$(K_H^n)_{\tilde{k}\tilde{l}} = 0 .$$

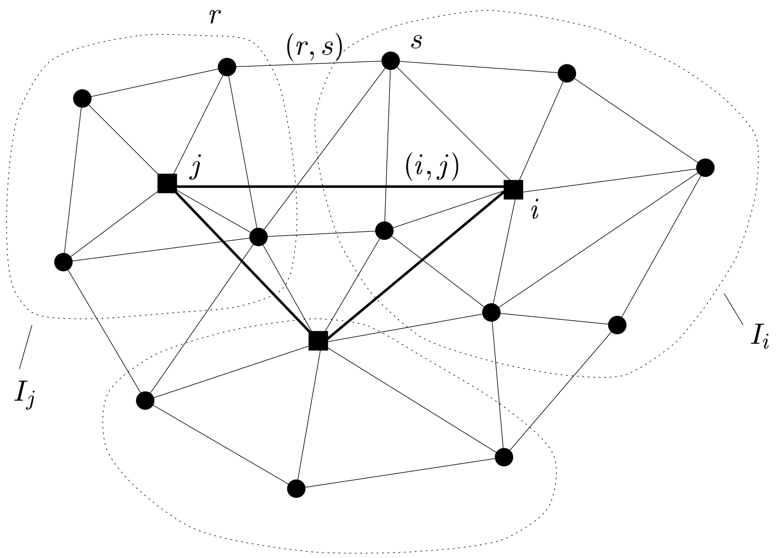


Figure 3: FE-grid and agglomeration.

*Proof.* The proof follows immediately by using (10). □

**Remark 3.3.**

1. *The essence of Lemma 3.2 is that a coarse grid edge is constructed only if there exists at least one fine edge connecting the agglomerates $I_i$ and $I_j$ $(i \neq j)$, i.e.,*

$$\exists r \in I_i, \exists s \in I_j \textit{ such that } (r, s) \in \omega_h^e$$

   *(see Fig. 3).*

2. *The graph of $K_H^n$, i.e.,* $\mathrm{graph}(K_H^n)$*, gives rise to a grid on $H$ with coarse edges $\omega_H^e$.*

3. *The decrease of edges in the coarsening process is in fact given, if the average number of nonzero entries of $K_h^n$ grows not too fast.*

In principle a setup for $K_h^e$ can be performed and the setup process Alg. 1 changes to Alg. 3. Before the prolongation operator for $K_h^e$ has to be defined properly.

## 3.3 The Prolongation Operator

The construction of the prolongation operator is delicate because of the kernel of the rot-operator. The prolongation operator $P^e \in \mathbb{R}^{N_h^e \times N_H^e}$ is defined for

**Algorithm 3** Setup process for AMG(edge elements): **Setup**($K_l^e$,$K_l^n$,**l**)

// COARSEGRID is defined before this function call
**if** card($\omega_l^e$) $> COARSEGRID$ **then**
  Split $\omega_l^n$ into disjoint sets $C$ and $F$
  Set $\omega_{l+1}^n = C$
  Define the interpolation operator $P_l^n$, $R_l^n = (P_l^n)^T$
  Calculate the coarse grid matrix $K_{l+1}^n$ by the Galerkin method
  $K_{l+1}^n = R_l^n K_l^n P_l^n$

  Define the index set for the block-smoother (see Sec. 3.4)
  Define the interpolation operator $P_l^e$, $R_l^e = (P_l^e)^T$ (see Sec. 3.3)
  Calculate the coarse grid matrix $K_{l+1}^e$ by the Galerkin method
  $K_{l+1}^e = R_l^e K_l^e P_l^e$

  **Setup**($K_{l+1}^e$,$K_{l+1}^n$,**l+1**)
**else**
  Perform a factorization of $K_l$
  $COARSELEVEL = l$
**end if**

$i = (i_1, i_2) \in \omega_h^e, j = (j_1, j_2) \in \omega_H^e$ as

$$(P^e)_{ij} = \begin{cases} 1 & \text{if } j = (\text{ind}(i_1), \text{ind}(i_2)), \\ -1 & \text{if } j = (\text{ind}(i_2), \text{ind}(i_1)), \\ 0 & \text{otherwise}. \end{cases} \tag{11}$$

You convince yourself that the constructed prolongation operator $P^e$ has full rank, because every coarse grid edge prolongates at least to one fine grid edge.

Next, we note that the operator $\text{grad}_h : Q_h \to V_h$ defined in (7) has the representation (with $i = (i_1, i_2) \in \omega_h^e$ and $q_h \in Q_h$)

$$(\text{grad}_h\, q_h)_i = q_{h,i_2} - q_{h,i_1}. \tag{12}$$

In analogy, we define $\text{grad}_s : Q_s \to V_s$ on the coarser levels. The crux is that $P^e$ prolongates discrete gradients of the coarse space to discrete gradients of the fine space.

Since we use a Galerkin approach, the coarse grid kernel is a subspace of the fine grid kernel, i.e.,

$$V_{H,0} = \{v_H \in V_H \mid P^e v_H \in V_{h,0}\} \tag{13}$$

with $V_{h,0}$ defined in (6) and $V_H = \mathbb{R}^{N_H^e}$.

**Lemma 3.4.** *For $q_H \in Q_H$ there holds*

$$P^e \operatorname{grad}_H q_H = \operatorname{grad}_h P^n q_H. \tag{14}$$

*This means, there holds the commuting diagram*

$$\begin{array}{ccc} Q_H & \xrightarrow{\operatorname{grad}_H} & V_H \\ \downarrow P^n & & \downarrow P^e \\ Q_h & \xrightarrow{\operatorname{grad}_h} & V_h \end{array}$$

*Proof.* We consider the edge $i = (i_1, i_2) \in \omega_h^e$. We have to distinguish two cases. First, let us assume the edge is inside one agglomerate, i.e., $\operatorname{ind}(i_1) = \operatorname{ind}(i_2)$. Then both sides of (14) vanish. The left hand side vanishes by definition of the prolongation operator $P^e$, the right hand side vanishes since $(P^n q_H)_{i_1} = (P^n q_H)_{i_2}$.

Now, we assume that $\operatorname{ind}(i_1) \neq \operatorname{ind}(i_2)$. Thus, there exists a coarse grid edge $j = (j_1, j_2)$ such that either $j_1 = \operatorname{ind}(i_1), j_2 = \operatorname{ind}(i_2)$ or $j_1 = \operatorname{ind}(i_2), j_2 = \operatorname{ind}(i_1)$. In both cases there holds $(\operatorname{grad}_H q_H)_j = m(q_{H,j_1} - q_{H,j_2})$, with $m \in \{-1, +1\}$. The sign in the prolongation compensates, such that $(P^e \operatorname{grad}_H q_H)_i = q_{H,\operatorname{ind}(i_1)} - q_{H,\operatorname{ind}(i_2)}$. Evaluating $(\operatorname{grad}_h P^n q_H)_i$ gives the same result. □

**Lemma 3.5.** *The coarse grid kernel functions are exactly gradient functions, i.e., there holds*

$$V_{H,0} = \operatorname{grad}_H Q_H. \tag{15}$$

*Proof.* First, we show the inclusion $\operatorname{grad}_H Q_H \subset V_{H,0}$. We fix a $q_H \in Q_H$ and define $v_H = \operatorname{grad}_H q_H$. Using Lemma 3.4 we obtain

$$P^e v_H = P^e \operatorname{grad}_H q_H = \operatorname{grad}_h P^n q_H.$$

From $\operatorname{grad}_h Q_h = V_{h,0}$ there follows $P^e v_H \in V_{h,0}$, and from definition (13) of $V_{H,0}$ there follows $v_H \in V_{H,0}$.

Now, we verify $V_{H,0} \subset \operatorname{grad}_H Q_H$. Therefore, we fix a $v_H \in V_{H,0}$. Since the kernels are nested, $v_h := P^e v_H$ is in $V_{h,0}$, and thus there exists a $q_h \in Q_h$ such that

$$v_h = \operatorname{grad}_h q_h.$$

By the definition of the prolongation $P^e$, the values of $v_h$ inside an agglomerate vanish, i.e., $(v_h)_i = 0$ for $i = (i_1, i_2)$ and $\operatorname{ind}(i_1) = \operatorname{ind}(i_2)$. Since $(v_h)_i = q_{h,i_1} - q_{h,i_2}$, the potential $q_h$ is constant inside an agglomerate. Thus there exists a $q_H \in Q_H$ such that $q_h = P^n q_H$. Combining the steps and using Lemma 3.4 we obtain

$$v_h = P^e v_H = \operatorname{grad}_h q_h = \operatorname{grad}_h P^n q_H = P^e \operatorname{grad}_H q_H\,.$$

Since $P^e$ has full rank, we can conclude that $v_H = \operatorname{grad}_H q_H$.

□

## 3.4 The Smoothing Iteration

To complete the ingredients for an AMG method we need an appropriate smoother. We consider two different types of smoothers for $K_h^e$. The first one was suggested from Arnold/Falk/Winther. This is a block Gauß-Seidel where all edges smoothed simultaneously which belong to $T^i$ for all $i \in \omega_h^n$ (see Fig. 4).

Another kind of smoother was suggest by Hiptmair. A mathematical equivalent formulation is outlined in Alg. 4. Therein the vector $g_h^{e,i} \in \mathbb{R}^{N_h^e}$ is defined by

$$g_h^{e,i} = \mathrm{grad}_h\, g_h^{n,i} = \begin{cases} \mathrm{orient}((k,l)) & (k,l) \in T^i \\ 0 \quad \text{otherwise} \end{cases}$$

where the vector $g_h^{n,i}$ is defined zero everywhere, except for the $i^{th}$ entry which is 1.

**Algorithm 4** Smoothing operator: **Smooth(**$K_h^e$,$u_h$,$f_h$**)**

Perform a Gauß-Seidel sweep on $K_h^e$
i.e., GS($K_h^e$, $f_h$, $u_h$)
Update the solution $u_h$
**for all** $i \in \omega_h^n$ **do**
  $u_h = u_h + \frac{(g_h^{e,i})^T(f_h - K_h^e u_h)}{(g_h^{e,i})^T K_h^e g_h^{e,i}} \cdot g_h^{e,i}$
**end for**

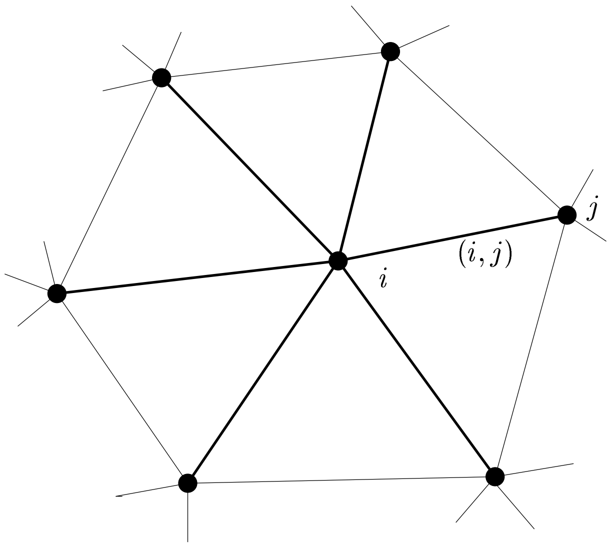


Figure 4: Detail view of graph($K_h^n$)

Both smoothing iterations are performed in the multigrid cycle (Alg. 2) with $\nu_F$ forward sweeps and $\nu_B$ backward sweeps. A detailed discussion of these smoothers is given in [1, 11].

# 4 Numerical Studies

The proposed AMG technique is used as a preconditioner for the preconditioned conjugate gradient (PCG) method (see [10, 14, 13]). The iteration was stopped if an error reduction in the preconditioner energy norm by a factor of $10^{-6}$ has been achieved. All calculations were done on an SGI Octane, 300 MHz, workstation.

Every row of the time table consists of two sub-rows. The first one is directed to a $V(1,1)$ cycle and the second one to a generalized V-cycle with $2^{s-1}$ smoothing steps on level $s$. The domain $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_2 = (0, 0.5)^3$, $\Omega = (0,1)^3$ and $\Omega_1 = \Omega \setminus \Omega_2$. We assume further homogenous Dirichlet boundary conditions on $\partial\Omega$ and an appropriate interface condition inside the cube. In Tab. 1 the used short cuts are listed.

| | |
|---|---|
| edges | number of edges, i.e., $N_h^e$ |
| setup | CPU time of the setup process, i.e., construction of the matrix hierarchy |
| solver | CPU time for the iterations of the PCG method with preconditioner AMG |
| solution | overall CPU time, i.e., setup and solver |
| iteration | number of iterations in the PCG method |

Table 1: Used short cuts for the numerical studies.

The first example is dedicated to a parameter setting of $\mu = 1$ and $\sigma = 10^{-4}$ in $\Omega$. Results therefore are given in Tab. 2 for the smoothing iteration of Arnold/Falk/Winther and in Tab. 3 for the smoother of Hiptmair. In both cases a slight increase can be detected in the number of iterations with respect to the number of unknowns. This might be an effect of the designed prolongation operator. As it can be expected the generalized V-cycle performs better for both smoothers compared to the V(1,1)-cycle. Actually there are no big differences between the smoothers and the cycles with respect to CPU time.

The second example was calculated with a parameter jump. Therefore $\mu = 1$ on $\Omega_1$ and $\mu = 10^3$ on $\Omega_2$. In this case we are only considering the smoother of Arnold/Falk/Winther. The results (see Tab. 4) are very similar to the above ones. It seems to show that the AMG method do not depend on parameter jumps.

# 5 Conclusions and Further Remarks

A new algebraic multigrid approach was proposed for the solution of $H(\mathrm{rot}, \Omega)$-conforming FE-discretization, i.e., edge elements. Therefore a coarsening technique based on an auxiliary matrix was introduced in order to setup a proper

| edges | setup (sec) | solver (sec) | solution (sec) | iteration |
|---|---|---|---|---|
| 4184 | 0.24 | 0.23 | 0.47 | 9 |
| | | 0.27 | 0.51 | 9 |
| 31024 | 1.98 | 4.76 | 6.74 | 14 |
| | | 4.56 | 6.54 | 12 |
| 238688 | 16.83 | 58.15 | 74.98 | 20 |
| | | 53.59 | 70.42 | 16 |

Table 2: Example without parameter jump and smoother of Arnold/Falk/Winther.

| edges | setup (sec) | solver (sec) | solution (sec) | iteration |
|---|---|---|---|---|
| 4184 | 0.15 | 0.30 | 0.45 | 12 |
| | | 0.31 | 0.46 | 11 |
| 31024 | 1.32 | 6.29 | 7.61 | 17 |
| | | 6.21 | 7.53 | 15 |
| 238688 | 11.39 | 67.98 | 79.37 | 21 |
| | | 63.93 | 75.32 | 17 |

Table 3: Example without parameter jump and smoother of Hiptmair.

prolongation operator and an appropriate smoother for the system matrix arising from an edge element discretization.

The numerical studies seem to show that the performance of the method is independent of parameter jumps but unfortunately it depends slightly on the mesh size. Nevertheless it performs much better than standard preconditioners (i.e., incomplete Cholesky preconditioner). An improvement of the proposed prolongation operator should be possible and is under current research, in order to get better convergence rates.

| edges | setup (sec) | solver (sec) | solution (sec) | iteration |
|---|---|---|---|---|
| 3588 | 0.20 | 0.18 | 0.38 | 8 |
| | | 0.18 | 0.38 | 8 |
| 26856 | 1.67 | 3.46 | 5.13 | 12 |
| | | 3.40 | 5.07 | 11 |
| 207696 | 14.06 | 41.99 | 56.05 | 17 |
| | | 42.16 | 56.22 | 15 |

Table 4: Example with parameter jump and smoother of Arnold/Falk/Winther.

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