MacMahon's Partition Analysis V: Bijections, Recursions, and Magic Squares

George E. Andrews[1]*, Peter Paule[2], Axel Riese[3]**, and Volker Strehl[4]***

[1] Department of Mathematics
The Pennsylvania State University
University Park, PA 16802, USA
andrews@math.psu.edu

[2] Research Institute for Symbolic Computation
Johannes Kepler University Linz
A–4040 Linz, Austria
Peter.Paule@risc.uni-linz.ac.at

[3] Research Institute for Symbolic Computation
Johannes Kepler University Linz
A–4040 Linz, Austria
Axel.Riese@risc.uni-linz.ac.at

[4] Computer Science Institute-Informatik 8
Friedrich-Alexander-Universität Erlangen-Nürnberg
D–91058 Erlangen, Germany
strehl@informatik.uni-erlangen.de

Dedicated to Professor A. Kerber at the occasion of his 60th birthday

Abstract. A significant portion of MacMahon's famous book "Combinatory Analysis" is devoted to the development of "Partition Analysis" as a computational method for solving problems in connection with linear homogeneous diophantine inequalities and equations, respectively. Nevertheless, MacMahon's ideas have not received due attention with the exception of work by Richard Stanley. A long range object of a series of articles is to change this situation by demonstrating the power of MacMahon's method in current combinatorial and partition-theoretic research. The renaissance of MacMahon's technique partly is due to the fact that it is ideally suited for being supplemented by modern computer algebra methods. In this paper we illustrate the use of Partition Analysis and of the corresponding package `Omega` by focusing on three different aspects of combinatorial work: the construction of bijections (for the Refined Lecture Hall Partition Theorem), exploitation of recursive patterns (for Cayley compositions), and finding nonnegative integer solutions of linear systems of diophantine equations (for magic squares of size 3).

* Partially supported by National Science Foundation Grant DMS-9870060.
** Supported by SFB-grant F1305 of the Austrian FWF.
*** Supported by a visiting researcher grant of the J. Kepler University Linz.

1 Introduction

The initial motive for the revival of MacMahon's Partition Analysis was the beautiful refinement of a classic result due to Euler – the number of partitions of N into distinct parts equals the number of partitions of N into odd parts [1, p. 5] – that was discovered by M. Bousquet-Mélou and K. Eriksson [7] only recently:

Theorem 1 ("Lecture Hall Partition Theorem"). *The number of partitions of N of the form $N = b_1 + b_2 + \cdots + b_n$ wherein*

$$\frac{b_n}{n} \geq \frac{b_{n-1}}{n-1} \geq \cdots \geq \frac{b_1}{1} \geq 0$$

equals the number of partitions of N into odd parts each $\leq 2n-1$.

Note that in lecture hall partitions some parts can be 0. For example, if $N = 13$ and $n = 3$ we have the 10 lecture hall partitions $0+0+13$, $0+1+12$, $0+2+11$, $0+3+10$, $1+2+10$, $0+4+9$, $1+3+9$, $0+5+8$, $1+4+8$, and $2+4+7$. On the other hand there are also 10 partitions of $N = 13$ into parts from $\{1,3,5\}$, namely $1^0 3^1 5^2$, $1^3 3^0 5^2$, $1^2 3^2 5^1$, $1^5 3^1 5^1$, $1^8 3^0 5^1$, $1^1 3^4 5^0$, $1^4 3^3 5^0$, $1^7 3^2 5^0$, $1^{10} 3^1 5^0$, and $1^{13} 3^0 5^0$.

The same authors also derived a further refinement of Euler's classic result; see the "Refined Lecture Hall Partition Theorem" (Theorem 17) below. Section 2 will be devoted to the construction of a bijective proof of it.

In [7] Bousquet-Mélou and Eriksson gave two different proofs of this theorem, one using Bott's formula for the affine Coxeter group $\tilde{C}_n$, and one of bijective-combinatorial nature. In [3] the first named author presented a proof following an entirely different approach. Namely, an approach which is based on the observation that MacMahon's Partition Analysis, surveyed in [12, Vol. 2, Sect. VIII, pp. 91–170], is perfectly tailored for theorems of this kind. In order to illustrate this point, recall the definition of MacMahon's Omega operator $\Omega_{\geqq}$.

Definition 2. The operator $\Omega_{\geqq}$ is given by

$$\underset{\geqq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r} \ ,$$

where the domain of the $A_{s_1,\ldots,s_r}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to annuli of the form $1-\epsilon < |\lambda_i| < 1+\epsilon$.

Remark 3. It is convenient to treat everything involved analytically rather than formally because the method relies on unique Laurent series representations of a variety of rational functions. For a more detailed discussion of this aspect, the interested reader is referred to [4].

Let us now consider the instance $n = 3$ of Theorem 1. Obviously, the coefficient of q^N in

$$\frac{1}{(1-q)(1-q^3)(1-q^5)} \tag{1}$$

equals the number of partitions of N into odd parts each ≤ 5. On the other hand, the coefficient of q^N in

$$\underset{\geqq}{\Omega} \sum_{b_1,b_2,b_3 \geq 0} \lambda_1^{2b_3-3b_2} \lambda_2^{b_2-2b_1} q^{b_1+b_2+b_3} \tag{2}$$

gives exactly the number of the desired lecture hall partitions for n being fixed to 3. Note that due to the Omega operator $\Omega_{\geqq}$ only those partitions $b_1 + b_2 + b_3 = N$ are counted for which $2b_3 - 3b_2 \geq 0$ and $b_2 - 2b_1 \geq 0$. By geometric series expansion the triple sum can be brought into product form, which means that expression (2) can be rewritten as

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda_1^2 q)\big(1-\frac{\lambda_2 q}{\lambda_1^3}\big)\big(1-\frac{q}{\lambda_2^2}\big)} \quad . \tag{3}$$

where the factors in the denominator correspond to b_3, b_2, b_1 in this order.

Therefore all what remains for proving the Lecture Hall Partition Theorem for $n = 3$ is to show equality of the generating function expressions (1) and (3). For doing so, MacMahon introduced a catalogue of rules that describe the elimination of the λ-parameters involved. As an example, we state one of these rules in form of a lemma.

Lemma 4. *For any integer $s \geq 0$,*

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda x)\big(1-\frac{y}{\lambda^s}\big)} = \frac{1}{(1-x)(1-x^s y)} \quad .$$

Proof. By geometric series expansion the left hand side equals

$$\underset{\geqq}{\Omega} \sum_{i,j\geq 0} \lambda^{i-sj} x^i y^j = \underset{\geqq}{\Omega} \sum_{j,k\geq 0} \lambda^k x^{sj+k} y^j \quad ,$$

where the summation parameter i has been replaced by $sj + k$. But now $\Omega_{\geqq}$ sets λ to 1, which completes the proof. □

With this lemma in hand, the proof of "(1) = (3)" reduces to successive elimination of the $\Omega_{\geqq}$-parameters λ_1 and λ_2.

Proof (of the Lecture Hall Partition Theorem for $j = 3$). Split (3) additively into two parts by applying partial fraction decomposition $1/((1-\lambda_1 z)(1+\lambda_1 z)) = 1/(2(1-\lambda_1 z)) + 1/(2(1+\lambda_1 z))$ to the term $1/(1-\lambda_1^2 q)$. Then by using Lemma 4 eliminate from both summands the $\Omega_{\geqq}$-parameter λ_1. For the last step one observes that Lemma 4 can be applied again in order to eliminate λ_2; this way one arrives at (1). □

Already this particular example suggests that MacMahon's approach is an ideal candidate for being supplemented by modern computer algebra methods. But rather than implementing tables of rules – as, for example, the list of twelve fundamental evaluations given by MacMahon [12, Vol. II, pp. 102–103] – in [4, Theorem 2] we explain how this can be achieved "in one stroke" by a fairly general setting based on the "fundamental recurrence" for the $\Omega_{\geqq}$ operator. Using the procedures from the Mathematica package `Omega`, the problem of showing "(1) = (3)" is solved as follows:

We put the file `Omega.m` (together with the file `Readme.txt`) in the same directory in which we run our Mathematica session. After invoking Mathematica we load the package:

```
In[1]:= <<Omega.m
        Axel Riese's Omega implementation version 1.3 loaded
```

Now the proof of Theorem 1 for $n = 3$ can simply be done as follows. First we input the expression the $\Omega_{\geqq}$ operator acts on; see (3). Then we call the `OR`-procedure to eliminate the λ-variables $\lambda 1$ and $\lambda 2$:

```
In[2]:= f = 1 / ((1-λ1^2 q)(1-λ2 q/λ1^3)(1-q/λ2^2))
```

Out[2]=

$$\frac{1}{\left(1-\frac{\lambda 2\,q}{\lambda 1^3}\right)\left(1-\lambda 1^2\,q\right)\left(1-\frac{q}{\lambda 2^2}\right)}$$

```
In[3]:= OR[f, λ1]
```

Out[3]=

$$\frac{1+\lambda 2\,q^3}{(1-q)\left(1-\frac{q}{\lambda 2^2}\right)\left(1-\lambda 2^2\,q^5\right)}$$

```
In[4]:= OR[%, λ2]
```

Out[4]=

$$\frac{1}{(1-q)\left(1-q^3\right)\left(1-q^5\right)}$$

This proves the equality in question.

More information (theoretical background, usage, etc.) about the `Omega` package can be found in [4]. In this paper we focus on concrete applications concerning the construction of bijections, exploitation of recursive patterns, and finding nonnegative integer solutions of linear systems of diophantine equations.

In Sect. 2 we illustrate how the `Omega` package and, more generally, Partition Analysis can be used for the construction of a bijective proof, Proposition 14 and Theorem 19, for the Refined Lecture Hall Partition Theorem (Theorem 17), a refinement of Theorem 1 above. To this end we need another essential ingredient, namely an involution, defined in Sect. 2.5, that is equivalent to an involution discovered by Bousquet-Mélou and Eriksson in [8, Prop. 3.4].

In Sect. 3 we deal with a classical type of compositions that have been introduced and studied by A. Cayley [9]. Here the application of the `Omega` package leads to the discovery of a recursive pattern that enables to find a surprising generating function representation, Theorem 29, which encodes the solution to Cayley's problem.

Finally, in Sect. 4 we briefly describe how MacMahon's technique works for finding nonnegative integer solutions of systems of linear homogeneous diophantine equations. Instead of following MacMahon's table look-up approach, we again achieve elimination "in one stroke" by deriving a suitable analogue (Theorem 33) of the "fundamental recurrence" developed in [4, Theorem 2] for diophantine inequalities. We illustrate our Mathematica implementation by revisiting a section of MacMahon's book [12, Vol. 2, Sect. 407, p. 161].

2 A Lecture Hall Bijection

Following [3], for $n \geq 2$ let us define

$$f_n(y_1, \ldots, y_n) = \sum_{\frac{b_n}{n} \geq \frac{b_{n-1}}{n-1} \geq \cdots \geq \frac{b_1}{1} \geq 0} y_1^{b_1} y_2^{b_2} \cdots y_n^{b_n} \ . \tag{4}$$

We note that in [3] the notation $F_{n-1,0}(y_n, y_{n-1}, \ldots, y_1)$ is used instead of $f_n(y_1, \ldots, y_n)$.

The generating function version of Theorem 1 then is

$$f_n(q, \ldots, q) = \prod_{k=1}^{n} \frac{1}{1 - q^{2k-1}} \tag{5}$$

which was proved in [3] by using Partition Analysis. In order to give a flavor of the mechanism of MacMahon's method, we briefly sketch how this has been achieved.

By introducing parameters λ_1 up to λ_{n-1} where the jth lecture hall condition $(j-1)\, b_j \geq j\, b_{j-1}$ is represented by a factor $\lambda_{n-j+1}^{(j-1)\, b_j - j\, b_{j-1}}$, $(2 \leq j \leq n)$, the generating function expression (4), is encoded as an $\Omega_{\geqq}$-expression:

$$\begin{aligned} f_n(y_1, \ldots, y_n) = \underset{\geqq}{\Omega} & \frac{1}{(1 - \lambda_1^{n-1} y_n)\big(1 - \frac{\lambda_2^{n-2}}{\lambda_1^{n}}\, y_{n-1}\big)} \\ & \cdot \frac{1}{\big(1 - \frac{\lambda_3^{n-3}}{\lambda_2^{n-1}}\, y_{n-2}\big) \cdots \big(1 - \frac{\lambda_{n-1}}{\lambda_{n-2}^{3}}\, y_2\big)\big(1 - \frac{y_1}{\lambda_{n-1}^{2}}\big)} \ . \end{aligned}$$

For example, similar to the computation in the introduction, for $n = 2$ one has

$$
\begin{aligned}
f_2(y_1, y_2) &= \sum_{\frac{b_2}{2} \geq \frac{b_1}{1} \geq 0} y_1^{b_1} y_2^{b_2} \\
&= \underset{\geqq}{\Omega} \sum_{b_1, b_2 \geq 0} \lambda_1^{b_2 - 2b_1} y_1^{b_1} y_2^{b_2} \\
&= \underset{\geqq}{\Omega} \frac{1}{(1 - \lambda_1 y_2)\left(1 - \frac{y_1}{\lambda_1^2}\right)} \\
&= \frac{1}{(1 - y_1 y_2^2)(1 - y_2)} \quad .
\end{aligned} \tag{6}
$$

Since only one λ-variable is involved, elimination in the last line is elementary and can be done, for instance, by applying Lemma 4 with $s = 2$. This computation can be found also in MacMahon's book [12].

Setting y_1 and y_2 to q results in the desired product form $1/((1-q)(1-q^3))$, which proves instance $n = 2$ of (5).

The general situation turns out to be considerably more involved; nevertheless it can be handled by Partition Analysis as follows. The λ-elimination rules that are used for carrying out induction with respect to n, reveal that f_n must be of the form

$$
f_n(y_1, \ldots, y_n) = \frac{P_n(y_1, \ldots, y_n)}{(1 - y_1 y_2^2 \cdots y_n^n)(1 - y_2^2 \cdots y_n^n) \cdots (1 - y_{n-1}^{n-1} y_n^n)(1 - y_n)} \quad , \tag{7}
$$

where $P_n(y_1, \ldots, y_n)$ is a polynomial in $y_1, \ldots, y_n$ with integer coefficients, which is defined by a recursive pattern. This recursive description then is used to prove that for the specialization $y_i = q$ the following factorization property [3, Lemma 1] holds:

$$
p_n(q) := P_n(q, q, \ldots, q) = \prod_{k=2}^{n} \frac{1 - q^{kn - \binom{k}{2}}}{1 - q^{2k-1}} \quad (n \geq 2) \quad . \tag{8}
$$

Substituting this expression into the corresponding specialization $y_i = q$ of (7) gives the Lecture Hall Partition Theorem in the form of (5).

We want to remark that from representation (8) the polynomial structure of $p_n(q)$ is not entirely obvious. However, it is fully revealed by the following elementary bijection.

Definition 5. For fixed positive $n \in \mathbb{N}$ we define the following permutation of $\{1, \ldots, n\}$:

$$
\sigma_n : j \mapsto \begin{cases} n - 2j + 1, & \text{if } 1 \leq 2j \leq n, \\ 2j - n, & \text{if } n < 2j \leq 2n \quad . \end{cases}
$$

Example 6. In order to illustrate the structure of the permutation σ_n we give explicit presentations for the first few n:

$$\sigma_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1\,2 \\ 1\,2 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix},$$
$$\sigma_4 = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,1\,2\,4 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 4\,2\,1\,3\,5 \end{pmatrix} \quad . \qquad \square$$

We will need the following property of σ_n.

Lemma 7. *For $1 \le j \le n$ we have*

$$j \cdot (2n - 2j + 1) = \binom{n+1}{2} - \binom{\sigma_n(j)}{2} \quad .$$

Proof. By elementary computation,

$$\begin{aligned} j \cdot (2n - 2j + 1) &= \begin{cases} \frac{(n-(n-2j+1)+1)(n+(n-2j+1))}{2}, & \text{if } 1 \le 2j \le n, \\ \frac{(n+(2j-n))(n-(2j-n)+1)}{2}, & \text{if } n < 2j \le 2n \end{cases} \\ &= \begin{cases} \binom{n+1}{2} - \binom{n-2j+1}{2}, & \text{if } 1 \le 2j \le n, \\ \binom{n+1}{2} - \binom{2j-n}{2}, & \text{if } n < 2j \le 2n \end{cases} \\ &= \binom{n+1}{2} - \binom{\sigma_n(j)}{2} \quad . \qquad \square \end{aligned}$$

Now it is easy to see that $p_n(q)$ is indeed a polynomial. Namely, since $kn - \binom{k}{2} = \binom{n+1}{2} - \binom{n+1-k}{2}$, by Lemma 7 we obtain

$$\begin{aligned} \left\{ kn - \binom{k}{2} : 1 \le k \le n \right\} &= \left\{ \binom{n+1}{2} - \binom{\sigma_n(k)}{2} : 1 \le k \le n \right\} \\ &= \{ (n-k+1)(2k-1) : 1 \le k \le n \} \quad . \end{aligned}$$

In other words, we have

$$\begin{aligned} p_n(q) &= \frac{1-q}{1-q^n} \cdot \prod_{k=1}^{n} \frac{1 - q^{kn - \binom{k}{2}}}{1 - q^{2k-1}} = \frac{1-q}{1-q^n} \cdot \prod_{k=1}^{n} \frac{1 - q^{(n-k+1)(2k-1)}}{1 - q^{2k-1}} \\ &= \prod_{k=2}^{n} \frac{1 - q^{(n-k+1)(2k-1)}}{1 - q^{2k-1}} \quad ; \end{aligned}$$

in particular, this gives

$$p_2(q) = 1, \quad p_3(q) = 1 + q^3, \quad p_4(q) = (1 + q^5)(1 + q^3 + q^6)$$

and in general for $n \ge 3$,

$$\begin{aligned} p_n(q) = {} & (1 + q^{2n-3}) \\ & \cdot (1 + q^{2n-5} + q^{2(2n-5)}) \cdots (1 + q^3 + q^{2\cdot 3} + \cdots + q^{(n-2)\cdot 3}) \quad . \end{aligned}$$

After having recalled the Partition Analysis approach to the Lecture Hall Partition Theorem (Theorem 1) we devote the remaining part of this section to the question how Partition Analysis can help in the task of constructing a bijective proof for the Refined Lecture Hall Partition Theorem (Theorem 17) which is a substantially stronger result.

However, equipped with the `Omega` package, in Sect. 2.1 we will first consider lecture hall bijections for special cases. This study then leads to an algebraic representation of lecture hall partitions (Proposition 14) that will be proved in Sect. 2.2. In Sect. 2.3 this parametrization is used for a further refinement of the problem (Theorem 19) which extends to the case of the Refined Lecture Hall Partition Theorem (Theorem 17). The task of finding a bijective proof of this theorem then is reduced to the task of finding bijections σ_n (derived from Definition 5) and τ_n. The first task is solved in Sect. 2.4 where we use a crucial permutation that has been suggested by the use of the `Omega` package. The second task is solved in Sect. 2.5 by using a fundamental involution that was also discovered by Bousquet-Mélou and Eriksson, but who had used it in a different direction. Finally, combining the mappings σ_n and τ_n results in the desired refined lecture hall bijection.

2.1 Lecture Hall Bijections for Special Cases

Before we construct a general lecture hall bijection for Theorem 1 and its refined version, Theorem 17 below, we examine what Partition Analysis can do for us in various special cases.

The key to making constructive use of Partition Analysis is based on the fact that it delivers a parametrized representation of all nonnegative solutions of a given system of linear homogeneous diophantine inequalities.

Let us consider the case $n = 2$. Starting from (6),

$$f_2(y_1, y_2) = \sum_{\frac{b_2}{2} \geq \frac{b_1}{1} \geq 0} y_1^{b_1} y_2^{b_2} = \frac{1}{(1 - y_1 y_2^2)(1 - y_2)} \quad ,$$

geometric series expansion,

$$\frac{1}{(1 - y_1 y_2^2)(1 - y_2)} = \sum_{\alpha_1, \alpha_2 \geq 0} y_1^{\alpha_1} y_2^{2\alpha_1 + \alpha_2} \quad ,$$

reveals that the diophantine solution set

$$\mathcal{L}_2 := \left\{ \langle b_1, b_2 \rangle \in \mathbb{N}^2 : \tfrac{b_2}{2} \geq \tfrac{b_1}{1} \geq 0 \right\}$$

is identical with the set

$$\mathcal{L}_2^* := \{ \langle \alpha_1, 2\alpha_1 + \alpha_2 \rangle : \langle \alpha_1, \alpha_2 \rangle \in \mathbb{N}^2 \} \quad ; \tag{9}$$

i.e.,

$$\mathcal{L}_2 = \mathcal{L}_2^* \quad . \tag{10}$$

In other words, given a listing algorithm for $\mathbb{N}^2$, $\mathcal{L}_2$ can be constructed via the bijection

$$\omega_2 : \mathbb{N}^2 \to \mathcal{L}_2 \ : \ \langle \alpha_1, \alpha_2 \rangle \mapsto \langle \alpha_1, 2\alpha_1 + \alpha_2 \rangle \ ;$$

i.e.,

$$\mathcal{L}_2 = \omega_2(\mathbb{N}^2) = \mathcal{L}_2^* \ .$$

From now on the following mappings on partitions will play a significant rôle; it will be convenient to consider them as linear functionals on the vector space $\mathbb{Q}^n$.

Definition 8. We define the following three linear functionals on $\mathbb{Q}^n$:

$$\begin{aligned} |\langle x_1, x_2, \ldots, x_n \rangle| &:= x_1 + x_2 + \cdots + x_n \ , \\ \|\langle x_1, x_2, \ldots, x_n \rangle\| &:= (2n-1) \cdot x_1 + (2n-3) \cdot x_2 + \cdots + 1 \cdot x_n \ , \end{aligned}$$

and

$$|||\langle x_1, x_2, \ldots, x_n \rangle||| := x_n - x_{n-1} + x_{n-2} - x_{n-3} + \cdots \pm x_1 \ .$$

We need also a couple of definitions for partitions with odd parts.

Definition 9 (Parametrizing partitions with odd parts). Let $\mathcal{O}_n$ denote the set of partitions with odd parts, all parts $\leq 2n-1$. For each $a = \langle a_1, a_2, \ldots, a_n \rangle \in \mathbb{N}^n$ let $\Psi_n(a)$ denote the partition where parts with size $2j-1$ occur with multiplicity a_{n-j+1}, i.e.,

$$\Psi_n(a) = 1^{a_n} 3^{a_{n-1}} \ldots (2j-1)^{a_{n-j+1}} \ldots (2n-1)^{a_1} \ .$$

If $\mathcal{O}_n(N)$ denotes the partitions from $\mathcal{O}_n$ with sum of parts equal to N, then

$$\mathcal{O}_n(N) = \Psi_n\{a \in \mathbb{N}^n : \|a\| = N\} \ .$$

If $\mathcal{O}_n(N,K)$ denotes the partitions from $\mathcal{O}_n$ with K parts and with sum of parts equal to N, then

$$\mathcal{O}_n(N,K) = \Psi_n\{a \in \mathbb{N}^n : \|a\| = N, |a| = K\} \ .$$

Considering the case $n = 2$ first, we denote for fixed N the corresponding set of lecture hall partitions by

$$\mathcal{L}_2(N) := \{\langle b_1, b_2 \rangle \in \mathcal{L}_2 : |\langle b_1, b_2 \rangle| = N\} \ . \tag{11}$$

But from (10) we know that

$$\mathcal{L}_2(N) = \mathcal{L}_2^*(N)$$

where

$$\mathcal{L}_2^*(N) := \{\langle \alpha_1, 2\alpha_1 + \alpha_2 \rangle \in \mathcal{L}_2^* : |\langle \alpha_1, 2\alpha_1 + \alpha_2 \rangle| = N\} \ .$$

Now, since we need to map any lecture hall partition onto a partition into odd parts, we only need to rewrite N, the sum of the b_i, as a linear combination of the corresponding parameters α_i; namely as

$$N = b_1 + b_2 = \alpha_1 + (2\alpha_1 + \alpha_2) = 3\alpha_1 + \alpha_2 \ ,$$

and we are immediately led to the bijection

$$\gamma_2 : \ \mathcal{L}_2^*(N) \to \mathcal{O}_2(N) \ : \ \langle \alpha_1, 2\alpha_1 + \alpha_2 \rangle \mapsto 1^{\alpha_2} 3^{\alpha_1} \ .$$

Before following the same approach for the higher cases, we introduce notation for the underlying lecture hall partition sets.

Definition 10. For integers $N, K \geq 0$ and $n \geq 1$, let

$$\begin{aligned} \mathcal{L}_n &:= \left\{ \langle b_1, b_2, \ldots, b_n \rangle \in \mathbb{N}^n : \tfrac{b_n}{n} \geq \tfrac{b_{n-1}}{n-1} \geq \cdots \geq \tfrac{b_1}{1} \geq 0 \right\} \ , \\ \mathcal{L}_n(N) &:= \{ b \in \mathcal{L}_n : |b| = N \} \ , \end{aligned}$$

and

$$\mathcal{L}_n(N, K) := \{ b \in \mathcal{L}_n : |b| = N, \ |||b||| = K \} \ .$$

We remark that we will need $\mathcal{O}_n(N, K)$ and $\mathcal{L}_n(N, K)$ for the Refined Lecture Hall Partition Theorem in Sect. 2.3.

For the case $n = 3$ we start out again by computing the required parameter representation with help of the `Omega` package:

$$\begin{aligned} f_3(y_1, y_2, y_3) &= \underset{\geqq}{\Omega} \frac{1}{(1 - \lambda_1^2 y_3)\left(1 - \frac{\lambda_2}{\lambda_1^3} y_2\right)\left(1 - \frac{y_1}{\lambda_2^2}\right)} \\ &= \frac{P_3(y_1, y_2, y_3)}{(1 - y_1 y_2^2 y_3^3)(1 - y_2^2 y_3^3)(1 - y_3)} \ , \end{aligned} \tag{12}$$

with

$$P_3(y_1, y_2, y_3) = 1 + y_2 y_3^2 \ . \tag{13}$$

Analogously to above, from (12) and (13) we can read off the desired parameter representation; namely

$$\mathcal{L}_3 = \mathcal{L}_3^* \ , \tag{14}$$

where

$$\begin{aligned} \mathcal{L}_3^* := \{ & \langle \alpha_1, 2\alpha_1 + 2\alpha_2 + r_2, 3\alpha_1 + 3\alpha_2 + \alpha_3 + 2r_2 \rangle : \\ & \langle \alpha_1, \alpha_2, \alpha_3 \rangle \in \mathbb{N}^3 \text{ and } r_2 \in \{0, 1\} \} \ . \end{aligned} \tag{15}$$

In other words, given a listing algorithm for $\mathbb{N}^3 \times \{0, 1\}$, $\mathcal{L}_3$ can be constructed via the bijection

$$\begin{aligned} \omega_3 : \ & \mathbb{N}^3 \times \{0, 1\} \to \mathcal{L}_3 \\ & \langle \alpha_1, \alpha_2, \alpha_3, r_2 \rangle \mapsto \langle \alpha_1, 2\alpha_1 + 2\alpha_2 + r_2, 3\alpha_1 + 3\alpha_2 + \alpha_3 + 2r_2 \rangle \ ; \end{aligned}$$

i.e.,

$$\mathcal{L}_3 = \omega_3(\mathbb{N}^3 \times \{0,1\}) = \mathcal{L}_3^* \ .$$

Therefore from (14) we learn that

$$\mathcal{L}_3(N) = \mathcal{L}_3^*(N)$$

where

$$\begin{aligned}\mathcal{L}_3^*(N) := \{\langle \alpha_1, 2\alpha_1 + 2\alpha_2 + r_2, 3\alpha_1 + 3\alpha_2 + \alpha_3 + 2r_2\rangle \in \mathcal{L}_3^* :\\ \alpha_1 + (2\alpha_1 + 2\alpha_2 + r_2) + (3\alpha_1 + 3\alpha_2 + \alpha_3 + 2r_2) = N\} \ .\end{aligned}$$

This time we can rewrite the b_i-sum N as follows,

$$\begin{aligned}N &= b_1 + b_2 + b_3\\ &= \alpha_1 + (2\alpha_1 + 2\alpha_2 + r_2) + (3\alpha_1 + 3\alpha_2 + \alpha_3 + 2r_2)\\ &= \alpha_3 + 3(2\alpha_1 + r_2) + 5\alpha_2 \ ,\end{aligned}$$

which leads us directly to the bijection

$$\begin{aligned}\gamma_3 :\ &\mathcal{L}_3^*(N) \to \mathcal{O}_3(N)\\ &\langle \alpha_1, 2\alpha_1 + 2\alpha_2 + r_2, 3\alpha_1 + 3\alpha_2 + \alpha_3 + 2r_2\rangle \mapsto 1^{\alpha_3} 3^{2\alpha_1 + r_2} 5^{\alpha_2} \ .\end{aligned} \tag{16}$$

For the case $n = 4$ things work analogously. For instance, one computes with the `Omega` package

$$f_4(y_1, y_2, y_3, y_4) = \frac{P_4(y_1, y_2, y_3, y_4)}{(1 - y_1 y_2^2 y_3^3 y_4^4)(1 - y_2^2 y_3^3 y_4^4)(1 - y_3^3 y_4^4)(1 - y_4)}$$

with

$$P_4(y_1, y_2, y_3, y_4) = 1 + y_3 y_4^2 + y_3^2 y_4^3 + y_2 y_3^2 y_4^3 + y_2 y_3^3 y_4^4 + y_2 y_3^4 y_4^6 \ .$$

However proceeding in this manner, the computation of the parametrized representation $\mathcal{L}_n^*$ of $\mathcal{L}_n$ that is needed for the construction of a lecture hall bijection, i.e., a bijection between $\mathcal{L}_n^*$ and $\mathcal{O}_n$, gets more and more involved. Nevertheless, guided by Partition Analysis this task can be accomplished by looking at the structure from an algebraic perspective as shown in the next section.

2.2 The Algebraic Structure of Lecture Hall Partitions

In view of the rational function representation (7) the parametrized representation $\mathcal{L}_n^*$ of $\mathcal{L}_n$ for arbitrary n has to combine two different aspects: the contribution emerging from the nice pattern of the denominator product, and the contribution of the numerator P_n. The latter one is much more involved since the polynomials P_n are growing rapidly with respect to the number of monomials involved; see Proposition 14 together with (17). This growth is also made explicit by the sufficiently complicated recursive scheme spelled out in [3]; however, we don't make use of this scheme in the algebraic approach explained below.

Before stating the main result, Proposition 14, of this section, it is necessary to introduce a couple of definitions.

Definition 11. For $n \in \mathbb{N}, n \geq 1$ we define sets of integer vectors

$$\mathcal{I}_n := \{\langle r_1, r_2, \ldots, r_n\rangle : 0 \leq r_j < j \ (1 \leq j \leq n)\} \ ,$$

and

$$\mathcal{I}_n^0 := \{r \in \mathcal{I}_n : r_n = 0\} \ .$$

The letter $\mathcal{I}$ is a mnemonic for *inversion vectors* as used in the combinatorial study of permutations, although we will not use this significance.

Definition 12. For $n \in \mathbb{N}, n \geq 1$ we define the n linearly independent vectors

$$\delta^j := \langle \underbrace{0, \ldots, 0}_{j-1}, j, \underbrace{0, \ldots, 0}_{n-j}\rangle \ (1 \leq j < n) \quad \text{and} \quad \delta^n := \langle \underbrace{0, \ldots, 0}_{n-1}, 1\rangle \ ,$$

and

$$\mathcal{D}_n^0 := \sum_{1 \leq j \leq n} \mathbb{N}\, \delta^j$$

which is the free $\mathbb{N}$-semimodule generated by the δ^j.

By the division property of the integers, each vector $a = \langle a_1, a_2, \ldots, a_n\rangle \in \mathbb{N}^n$ has a unique presentation

$$\begin{aligned} a &= \langle \alpha_1 \cdot 1 + r_1, \alpha_2 \cdot 2 + r_2, \ldots, \alpha_{n-1} \cdot (n-1) + r_{n-1}, \alpha_n \cdot 1 + r_n\rangle \\ &= \langle r_1, \ldots, r_n\rangle + \sum_{1 \leq j \leq n} \alpha_j\, \delta^j \end{aligned}$$

where $\langle r_1, \ldots, r_n\rangle \in \mathcal{I}_n^0$ and $\langle \alpha_1, \ldots, \alpha_n\rangle \in \mathbb{N}^n$, namely $\alpha_j = \mathtt{quot}(a_j, j)$ and $r_j = \mathtt{rem}(a_j, j)$ for $1 \leq j < n$, and $r_n = 0, \alpha_n = a_n$. In short,

$$\mathbb{N}^n = \mathcal{I}_n^0 \oplus \mathcal{D}_n^0 \ .$$

A parametrized representation of $\mathcal{L}_n$, the set of lecture hall partitions with n parts, is provided by the bijective mapping

$$\omega \ : \ \mathbb{N}^n \to \mathcal{L}_n \ : \ a = \langle a_1, \ldots, a_n\rangle \mapsto \langle b_1, \ldots, b_n\rangle$$

where

$$b_1 := a_1, \quad b_j := a_j + \left\lceil \frac{j \cdot b_{j-1}}{j-1} \right\rceil \ (1 < j \leq n) \ .$$

This parametrization is straight-forward from the inequality constraints between the b_i. For reasons of legibility we omit indexing ω by n since its definition is essentially independent of n.

Note that ω is not linear in general, but certain linearity properties of ω can be exhibited by choosing a second set of basis vectors. Besides the linearly independent vectors δ^j $(1 \leq j \leq n)$ introduced above we define another set of independent vectors $\epsilon^1, \epsilon^2, \ldots, \epsilon^n \in \mathbb{N}^n$ by applying ω to the δ^j.

Definition 13. For $n \in \mathbb{N}, n \geq 1$ we define the n linearly independent vectors

$$\epsilon^j := \omega(\delta^j) = \langle \underbrace{0, \ldots, 0}_{j-1}, j, j+1, \ldots, n \rangle \ (1 \leq j < n) \quad \text{and} \quad e^n := \omega(\delta^n) = \delta^n \ ,$$

and

$$\mathcal{E}_n^0 := \sum_{1 \leq j \leq n} \mathbb{N} \, \epsilon^j$$

which is the free $\mathbb{N}$-semimodule generated by the ϵ^j.

It is easily checked that for any $a \in \mathbb{N}^n$ and $1 \leq j \leq n$

$$\omega(a + \delta^j) = \omega(a) + \epsilon^j$$

and hence in general

$$\omega(a + \textstyle\sum_j \alpha_j \, \delta^j) = \omega(a) + \textstyle\sum_j \alpha_j \, \epsilon^j \ .$$

In particular we have that ω is an isomorphism of semimodules

$$\omega \ : \ \mathcal{D}_n^0 \to \mathcal{E}_n^0 \ .$$

We summarize in form of a proposition.

Proposition 14. *For $n \in \mathbb{N}, n \geq 1$ we have the semilinear presentation*

$$\mathbb{N}^n = \mathcal{I}_n^0 \oplus \mathcal{D}_n^0 \ .$$

Moreover, if we put

$$\mathcal{R}_n^0 := \omega(\mathcal{I}_n^0)$$

we arrive at a semilinear presentation of $\mathcal{L}_n$, namely

$$\mathcal{L}_n = \omega(\mathbb{N}^n) = \omega(\mathcal{I}_n^0) \oplus \omega(\mathcal{D}_n^0) = \mathcal{R}_n^0 \oplus \mathcal{E}_n^0$$

where ω is an isomorphism between the semimodules $\mathcal{D}_n^0$ and $\mathcal{E}_n^0$.

Example 15. Let $n = 2$ and $a = \langle a_1, a_2 \rangle \in \mathbb{N}^2$. The $\mathcal{I}_2^0 \oplus \mathcal{D}_2^0$-decomposition in this case is trivial, namely $a = r + \alpha \in \mathcal{I}_2^0 \oplus \mathcal{D}_2^0$ with

$$r = \langle 0, 0 \rangle \quad \text{and} \quad \alpha = \langle \alpha_1, \alpha_2 \rangle = \alpha_1 \delta^1 + \alpha_2 \delta^2 \quad (= a) \ .$$

Then

$$\begin{aligned} \omega(a) &= \omega(r) + \omega(\alpha) = \langle 0, 0 \rangle + (\alpha_1 \epsilon^1 + \alpha_2 \epsilon^2) = \alpha_1 \langle 1, 2 \rangle + \alpha_2 \langle 0, 1 \rangle \\ &= \langle \alpha_1, 2\alpha_1 + \alpha_2 \rangle \ , \end{aligned}$$

in accordance with the definition (9) of $\mathcal{L}_2^*$. □

Example 16. Let $n = 3$ and $a = \langle a_1, a_2, a_3 \rangle \in \mathbb{N}^3$. By Euclidean division one computes

$$a = \langle \alpha_1 \cdot 1 + 0, \alpha_2 \cdot 2 + r_2, \alpha_3 \cdot 1 + 0 \rangle \quad \text{where } r_2 \in \{0, 1\} \ .$$

This means, the $\mathcal{I}_3^0 \oplus \mathcal{D}_3^0$-decomposition $a = r + \alpha \in \mathcal{I}_3^0 \oplus \mathcal{D}_3^0$ comes with

$$r = \langle 0, r_2, 0 \rangle \ \text{ and } \ \alpha = \langle \alpha_1, 2\alpha_2, \alpha_3 \rangle = \alpha_1 \delta^1 + \alpha_2 \delta^2 + \alpha_3 \delta^3 \ .$$

We obtain,

$$\begin{aligned} \omega(a) &= \omega(r) + \omega(\alpha) = \langle 0, r_2, 2r_2 \rangle + (\alpha_1 \epsilon^1 + \alpha_2 \epsilon^2 + \alpha_3 \epsilon^3) \\ &= \langle 0, r_2, 2r_2 \rangle + \alpha_1 \langle 1, 2, 3 \rangle + \alpha_2 \langle 0, 2, 3 \rangle + \alpha_3 \langle 0, 0, 1 \rangle \\ &= \langle \alpha_1, 2\alpha_1 + 2\alpha_2 + r_2, 3\alpha_1 + 3\alpha_2 + \alpha_3 + 2r_2 \rangle \ , \end{aligned}$$

in accordance with the definition (15) of $\mathcal{L}_3^*$. □

We conclude this section by the remark that the algebraic decomposition $\mathcal{L}_n = \mathcal{R}_n^0 \oplus \mathcal{E}_n^0$ is linked to the generating function presentation (7) in an obvious way. Namely, $\mathcal{R}_n^0$ corresponds to the numerator polynomial P_n, whereas $\mathcal{E}_n^0$ reflects the structure of the denominator product. More precisely, using the convention $y^a = y_1^{a_1} y_2^{a_2} \ldots y_n^{a_n}$ whenever $a = \langle a_1, \ldots, a_n \rangle \in \mathbb{N}^n$ we have

$$P_n(y_1, \ldots, y_n) = \sum_{R \in \mathcal{R}_n^0} y^R \tag{17}$$

and

$$(1 - y_1 y_2^2 \cdots y_n^n)(1 - y_2^2 y_3^3 \cdots y_n^n) \cdots (1 - y_{n-1}^{n-1} y_n^n)(1 - y_n) = \prod_{j=1}^{n} \left(1 - y^{\epsilon^j}\right) \ .$$

2.3 The Refined Lecture Hall Partition Theorem

In the special cases $n = 2$ and $n = 3$ the mappings γ_2 and γ_3 have an important additional property. To illustrate this, let us fix two positive integers N and K.

Take $1^{\alpha_2}3^{\alpha_1} \in \mathcal{O}_2(N,K)$, i.e., $\|\langle\alpha_1,\alpha_2\rangle\| = N$ and $|\langle\alpha_1,\alpha_2\rangle| = K$. Then for $\langle b_1,b_2\rangle := \gamma_2^{-1}(1^{\alpha_2}3^{\alpha_1}) \in \mathcal{L}_2(N)$ we observe also that

$$|||\langle b_1,b_2\rangle||| = |||\langle\alpha_1, 2\alpha_1+\alpha_2\rangle||| = |\langle\alpha_1,\alpha_2\rangle| = K \ .$$

This means, γ_2 is not only a bijection between $\mathcal{L}_2(N)$ and $\mathcal{O}_2(N)$ but also between $\mathcal{L}_2(N,K)$ and $\mathcal{O}_2(N,K)$.

Similarly, let $1^{\alpha_3}3^{2\alpha_1+r_2}5^{\alpha_2} \in \mathcal{O}_3(N,K)$ with $r_2 \in \{0,1\}$, i.e.,

$$\|\langle\alpha_2, 2\alpha_1+r_2, \alpha_3\rangle\| = N \quad \text{and} \quad |\langle\alpha_2, 2\alpha_1+r_2,\alpha_3\rangle| = K \ .$$

Then for $\langle b_1,b_2,b_3\rangle := \gamma_3^{-1}(1^{\alpha_3}3^{2\alpha_1+r_2}5^{\alpha_2}) \in \mathcal{L}_3(N)$ we observe also that

$$\begin{aligned} |||\langle b_1,b_2,b_3\rangle||| &= |||\langle\alpha_1, 2\alpha_1+2\alpha_2+r_2, 3\alpha_1+3\alpha_2+\alpha_3+2r_2\rangle||| \\ &= |\langle\alpha_2, 2\alpha_1+r_2,\alpha_3\rangle| = K. \end{aligned}$$

This means, γ_3 is not only a bijection between $\mathcal{L}_3(N)$ and $\mathcal{O}_3(N)$ but also between $\mathcal{L}_3(N,K)$ and $\mathcal{O}_3(N,K)$.

The observation concerning this extra property with respect to the parameter K has already been made by Bousquet-Mélou and Eriksson. More precisely, they refined their theorem accordingly as follows [7].

Theorem 17 ("Refined Lecture Hall Partition Theorem"). *The number of partitions of N of the form $|\langle b_1,\ldots,b_n\rangle| = N$ wherein*

$$\frac{b_n}{n} \geq \frac{b_{n-1}}{n-1} \geq \cdots \geq \frac{b_1}{1} \geq 0$$

and

$$|||\langle b_1,\ldots,b_n\rangle||| = K \tag{18}$$

equals the number of partitions of N into exactly K odd parts each $\leq 2n-1$. In short,

$$\#\mathcal{L}_n(N,K) = \#\mathcal{O}_n(N,K) \ .$$

Example 18. From the partition sets listed after Theorem 1 we see, for instance, that

$$\mathcal{L}(13,5) = \{\langle 2,4,7\rangle, \langle 1,4,8\rangle, \langle 0,4,9\rangle\}$$

and

$$\mathcal{O}(13,5) = \{1^1 3^4 5^0, 1^2 3^2 5^1, 1^3 3^0 5^2\} \ . \qquad \square$$

Our goal for the rest of the paper is to construct a lecture hall bijection Λ_n that takes also condition (18) into account. More precisely, for arbitrary nonnegative integers N and K we will construct a map

$$\Lambda_n \ : \ \mathcal{O}_n(N,K) \to \mathcal{L}_n(N,K)$$

that is a bijection.

We already know that both $\mathcal{O}_n$ and $\mathcal{L}_n$ are parametrized by $\mathbb{N}^n$. Hence, given

$$1^{a_n} 3^{a_{n-1}} \ldots (2n-1)^{a_1} \in \mathcal{O}_n(N, K) \ ,$$

an obvious first step is to apply Ψ_n^{-1}; i.e.,

$$a := \langle a_1, a_2, \ldots, a_n \rangle = \Psi_n^{-1}(1^{a_n} 3^{a_{n-1}} \ldots (2n-1)^{a_1}) \in \mathbb{N}^n \ . \tag{19}$$

Next, by Euclidean division on the parts, we decompose a according to $\mathbb{N}^n = \mathcal{I}_n^0 \oplus \mathcal{D}_n^0$; see the first statement of Proposition 14. In other words, we compute r and α such that

$$a = r + \alpha \tag{20}$$

where

$$r = \langle r_1, \ldots, r_n \rangle \in \mathcal{I}_n^0$$

and

$$\alpha = \langle \alpha_1, 2\alpha_2, \ldots, (n-1)\,\alpha_{n-1}, \alpha_n \rangle = \sum_{j=1}^{n} \alpha_j \delta^j \in \mathcal{D}_n^0 \ .$$

Finally, in view of $\mathcal{L}_n = \omega(\mathbb{N}^n) = \mathcal{R}_n^0 \oplus \mathcal{E}_n^0$, the second part of Proposition 14, we need bijections

$$\tau_n \ : \ \mathcal{I}_n^0 \to \mathcal{R}_n^0 \quad \text{and} \quad \sigma_n \ : \ \mathcal{D}_n^0 \to \mathcal{E}_n^0$$

that respect the various functional relations with respect to N and K. More precisely, for any $r \in \mathcal{I}_n^0$ we must have

$$|r| = |||\tau_n(r)||| \quad \text{and} \quad \|r\| = |\tau_n(r)| \ , \tag{21}$$

and for any $\alpha \in \mathcal{D}_n^0$ we need

$$|\alpha| = |||\sigma_n(\alpha)||| \quad \text{and} \quad \|\alpha\| = |\sigma_n(\alpha)| \ . \tag{22}$$

Because then we would achieve our goal as follows:

For a as in (19), i.e., $\|a\| = N$ and $|a| = K$, and such that $a = r + \alpha$ as in (20) we have,

$$\tau_n(r) + \sigma_n(\alpha) \in \mathcal{R}_n^0 \oplus \mathcal{E}_n^0 = \mathcal{L}_n$$

with

$$|\tau_n(r) + \sigma_n(\alpha)| = |\tau_n(r)| + |\sigma_n(\alpha)| = \|r\| + \|\alpha\| = \|r + \alpha\| = \|a\| = N$$

and

$$|||\tau_n(r) + \sigma_n(\alpha)||| = |||\tau_n(r)||| + |||\sigma_n(\alpha)||| = |r| + |\alpha| = |r + \alpha| = |a| = K \ .$$

We summarize these considerations in form of a theorem.

Theorem 19. *Let $n \in \mathbb{N}$, $n \geq 1$. Suppose we have bijections*

$$\tau_n \ : \ \mathcal{I}_n^0 \to \mathcal{R}_n^0 \quad \text{and} \quad \sigma_n \ : \ \mathcal{D}_n^0 \to \mathcal{E}_n^0$$

satisfying (21) *and* (22). *Let Γ_n be the bijection defined as*

$$\Gamma_n \ : \ \mathbb{N}^n = \mathcal{I}_n^0 \oplus \mathcal{D}_n^0 \to \mathcal{L}_n = \mathcal{R}_n^0 \oplus \mathcal{E}_n^0 \ : \ r + \alpha \mapsto \tau_n(r) + \sigma_n(\alpha) \ .$$

Then the map

$$\Lambda_n := \Gamma_n \circ \Psi_n^{-1} \ : \ \mathcal{O}_n(N,K) \to \mathcal{L}_n(N,K) \tag{23}$$

is a bijection.

2.4 The Bijection σ_n

The construction of the map σ_n is suggested by Partition Analysis. More precisely, we generalize the pattern that emerges from the special cases $n = 2$ and $n = 3$ as follows.

Definition 20. For $n \in \mathbb{N}$, $n \geq 1$, we define the linear transformation

$$\sigma_n \ : \ \mathbb{Q}^n \to \mathbb{Q}^n$$

on the basis vectors by setting

$$\sigma_n(\delta^j) := \epsilon^{\sigma_n(j)} \quad (1 \leq j \leq n) \ ,$$

where the σ_n on the right hand side is the permutation from Definition 5.

Since the meaning will be always clear from the context, for the linear transformation we use the same symbol σ_n as for the corresponding permutation.

Lemma 21. *The linear transformation σ_n provides a semimodule isomorphism between $\mathcal{D}_n^0$ and $\mathcal{E}_n^0$, i.e.,*

$$\sigma_n \ : \ \mathcal{D}_n^0 \to \mathcal{E}_n^0 \ ,$$

which satisfies the conditions (22).

Proof. The first part of the statement is obvious. Concerning property (22) we have,

$$\|\delta^j\| = j \cdot (2n - 2j + 1) = \binom{n+1}{2} - \binom{\sigma_n(j)}{2} = |\epsilon^{\sigma_n(j)}| \quad (1 \leq j \leq n) \ ,$$

$$|\delta^j| = \left.\begin{cases} j, & \text{if } 1 \leq j < n, \\ 1, & \text{if } j = n \end{cases}\right\} = |||\epsilon^{\sigma_n(j)}||| \quad (1 \leq j \leq n)$$

as a consequence of Lemma 7. By linearity it follows that for any $m = \sum_{j=1}^n m_j \, \delta^j \in \mathcal{D}_n^0$,

$$|\sigma_n(m)| = \|m\| \quad \text{and} \quad |||\sigma_n(m)||| = |m| \ . \qquad \square$$

We conclude this section by re-examining the special cases $n = 2$ and $n = 3$.

Example 22. For $n = 2$ the maps τ_2 and σ_2 are nothing but the corresponding ω-mappings. (Note that $\mathcal{I}_2^0 = \{\langle 0,0\rangle\}$ and $\sigma_2 = \left(\begin{smallmatrix}1&2\\1&2\end{smallmatrix}\right)$, the trivial permutation of $\{1,2\}$.) $\square$

Example 23. In the case $n = 3$ the map τ_3 can again be chosen as the corresponding ω-mapping. But now the isomorphism σ_3 is non-trivial since $\sigma_3 = \left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right)$ as a permutation of $\{1,2,3\}$. Suppose we are given

$$a = \langle a_1, a_2, a_3\rangle = \Psi_3^{-1}(1^{a_3}3^{a_2}5^{a_1}) \in \mathbb{N}^3$$

with $\mathcal{I}_3^0 \oplus \mathcal{D}_3^0$- decomposition

$$a = r + \alpha = \langle 0, r_2, 0\rangle + \langle \alpha_2, 2\alpha_1, \alpha_3\rangle = \langle 0, r_2, 0\rangle + \alpha_2\delta^1 + \alpha_1\delta^2 + \alpha_3\delta^3 \quad .$$

Then

$$\begin{aligned}\Gamma_3(a) &= \tau_3(r) + \sigma_3(\alpha) = \omega(r) + \alpha_2\epsilon^2 + \alpha_1\epsilon^1 + \alpha_3\epsilon^3\\ &= \langle 0, r_2, 2r_2\rangle + \alpha_2\langle 0,2,3\rangle + \alpha_1\langle 1,2,3\rangle + \alpha_3\langle 0,0,1\rangle\\ &= \langle \alpha_1, 2\alpha_1 + 2\alpha_2 + r_2, 3\alpha_1 + 3\alpha_2 + \alpha_3 + 2r_2\rangle \quad ,\end{aligned}$$

which corresponds to the mapping γ_3; see (16). $\square$

Finally, let us consider an example for $n = 4$. Now $\sigma_4 = \left(\begin{smallmatrix}1&2&3&4\\2&1&3&4\end{smallmatrix}\right)$ as a permutation of $\{1,2,3,4\}$. Take, for instance, $1^1 3^0 5^1 7^1 \in \mathcal{O}_4(13,3)$ and

$$a = \langle 1,1,0,1\rangle = \Psi_4^{-1}(1^1 3^0 5^1 7^1) \in \mathbb{N}^4 .$$

As in the case $n = 3$ let us choose again $\tau_4 = \omega$. Consider the $\mathcal{I}_4^0 \oplus \mathcal{D}_4^0$-decomposition

$$a = r + \alpha = \langle 0,1,0,0\rangle + 1\cdot\delta^1 + 0\cdot\delta^2 + 0\cdot\delta^3 + 1\cdot\delta^4 \quad .$$

Then we obtain

$$\begin{aligned}\Gamma_4(a) &= \tau_4(r) + \sigma_4(\alpha) = \omega(r) + 1\cdot\epsilon^3 + 0\cdot\epsilon^1 + 0\cdot\epsilon^2 + 1\cdot\epsilon^4\\ &= \langle 0,1,2,3\rangle + 1\cdot\langle 0,0,3,4\rangle + 1\cdot\langle 0,0,0,1\rangle\\ &= \langle 0,1,2,3\rangle + \langle 0,0,3,5\rangle = \langle 0,1,5,8\rangle \quad ;\end{aligned}$$

but $\langle 0,1,5,8\rangle \in \mathcal{L}_4(14,4)$! In other words, if we choose $\tau_4 = \omega$ then at least one of the functional properties we need with respect to N and K is violated. In this particular example we have

$$|r| = 1,\ |||\omega(r)||| = 2 \quad \text{and} \quad \|r\| = 5,\ |\omega(r)| = 6 \quad .$$

Hence in general we need some extra effort to construct the bijection τ_n in a suitable manner. This is accomplished by an involutive approach described in the next section.

2.5 The Bijection τ_n

A Local Involution. We consider again the ω-mapping form Sect. 2.2 which affords a parametrization of lecture hall partitions:

$$\omega \ : \ \mathbb{N}^n \to \mathbb{N}^n \ : \ r = \langle r_1, r_2, \ldots, r_n \rangle \mapsto \langle R_1, R_2, \ldots, R_n \rangle = \omega(r) =: R$$

where

$$R_1 = r_1 , \ R_k = r_k + \left\lceil \frac{k}{k-1} R_{k-1} \right\rceil \ (1 < k \leq n) \ .$$

Now the equation relating R_k and R_{k-1} can be written as

$$(R_k - r_k)(k-1) = kR_{k-1} + \varepsilon_{k-1}$$

where $0 \leq \varepsilon_{k-1} < k-1$. Yet another way of writing this equation is

$$\Delta_k = r_k(k-1) + \varepsilon_{k-1}$$

where we put[1]

$$\Delta_k := (k-1)R_k - kR_{k-1} \ .$$

Division by $k-1$ allows to recover (r_k, ε_{k-1}) from Δ_k:

$$r_k = \mathtt{quot}(\Delta_k, k-1), \ \varepsilon_{k-1} = \mathtt{rem}(\Delta_k, k-1) = \mathtt{rem}(-R_{k-1}, k-1) \ .$$

Note that ε_{k-1} depends only on R_{k-1}, not on R_k.

We will now say that the sequence R is *reduced at position k*, or *k-reduced*, if $0 \leq r_k < k$. Obviously we have the equivalence

$$0 \leq r_k < k \ \Leftrightarrow \ \Delta_k < k(k-1) \ \Leftrightarrow \ r_k = \mathtt{rem}(R_k + \varepsilon_{k-1}, k) \ . \tag{24}$$

Now note that the relation between R_{k-1} and R_k can also be seen "from the right", i.e., one can write

$$R_{k-1} = \left\lfloor \frac{k-1}{k} R_k \right\rfloor - s_{k-1}$$

where $s_{k-1} \geq 0$. This can be rewritten as

$$kR_{k-1} + ks_{k-1} = (k-1)R_k - \delta_k$$

where $0 \leq \delta_k < k$, or as

$$\Delta_k = s_{k-1}k + \delta_k \ .$$

It follows that the k-reducibility condition (24) transforms into another equivalent statement:

$$0 \leq s_{k-1} < k-1 \ .$$

[1] Note: the Lecture Hall Condition is equivalent to saying "$\Delta_k \geq 0$" for $1 \leq k \leq n$. The condition for $k = 1$ is void, of course.

Again, the pair (s_{k-1}, δ_k) can be recovered from Δ_k:

$$s_{k-1} = \texttt{quot}(\Delta_k, k),\ \delta_k = \texttt{rem}(\Delta_k, k) = \texttt{rem}(-R_k, k) \ .$$

This shows that δ_k only depends on R_k, not on R_{k-1}.

Now assume that the sequence R is both k-reduced and $(k+1)$-reduced; we will say k^+*-reduced* for short. This means that we have equations

$$\begin{aligned} \Delta_k &= r_k(k-1) + \varepsilon_{k-1}, \qquad &&\text{where } 0 \le r_k < k \\ \Delta_{k+1} &= s_k(k+1) + \delta_{k+1}, \qquad &&\text{where } 0 \le s_k < k \ . \end{aligned}$$

This suggests the following: since the conditions put on r_k and s_k are precisely the same, and since ε_{k-1} depends only on R_{k-1} and δ_{k+1} depends only on R_{k+1}, we may simply exchange the rôles of r_k and s_k – thus producing a new k^+-reduced sequence R' which differs from R only in position k, namely, we define

$$R'_k := R_k - r_k + s_k, \quad R' = \langle R_1, \ldots, R_{k-1}, R'_k, R_{k+1}, \ldots, R_n \rangle$$

and we have

$$\begin{aligned} R_k &= r_k + \left\lceil \frac{k}{k-1}\, R_{k-1} \right\rceil = \left\lfloor \frac{k}{k+1}\, R_{k+1} \right\rfloor - s_k \ , \\ R'_k &= s_k + \left\lceil \frac{k}{k-1}\, R_{k-1} \right\rceil = \left\lfloor \frac{k}{k+1}\, R_{k+1} \right\rfloor - r_k \end{aligned}$$

and an immediate consequence of this is

$$R_k + R'_k = \left\lceil \frac{k}{k-1}\, R_{k-1} \right\rceil + \left\lfloor \frac{k}{k+1}\, R_{k+1} \right\rfloor \ . \tag{25}$$

It is clear from the exchange argument that this mapping $\phi_k : R \mapsto R'$ is an involution on the set of k^+-reduced sequences for any fixed $k < n$.

Global Involution for Reduced Sequences. A sequence $R = \langle R_1, R_2, \ldots, R_n \rangle$ is *reduced* if it is k-reduced for all $k \le n$. Note that this is equivalent to saying that $R \in \mathcal{R}_n$ where

$$\mathcal{R}_n := \omega(\mathcal{I}_n) \ .$$

For reduced sequences the involutive procedure $\phi_k : R \mapsto R'$ at position k, as described in the previous section, can be simultaneously executed[2] for all positions $k = n - 2j - 1$, $(0 \le j < \lfloor n/2 \rfloor)$. Each of the ϕ_k keeps all the R_{n-2i}, $(0 \le i < \lfloor n/2 \rfloor)$ fixed, so that these actions commute with each other. We denote by $R \mapsto \Phi_n(R)$ this simultaneous involution on $\mathcal{R}_n$.

[2] In order to deal with the case $k = 1$ consistently one has to put formally $R_0 = 0$.

From the fact that the action of Φ_n keeps all the R_{n-2i} $(0 \le i < \lfloor n/2 \rfloor)$ fixed it follows that

$$|R| + |||R||| = |\Phi_n(R)| + |||\Phi_n(R)|||$$

or equivalently

$$|||R - \Phi_n(R)||| = |\Phi_n(R) - R| \ . \tag{26}$$

The main property of Φ_n w.r.t. reduced sequences is

$$|||R + \Phi_n(R)||| = \left\lceil \frac{n+1}{n} R_n \right\rceil \ . \tag{27}$$

The *proof* uses the result (25) from the previous section and a simple rearrangement of terms:

$$\begin{aligned}
&|||R + \Phi_n(R)||| \\
&\quad = 2\sum_{k\ge 0} R_{n-2k} - \sum_{k\ge 0} \left(R_{n-2k-1} + R'_{n-2k-1}\right) \\
&\quad = 2\sum_{k\ge 0} R_{n-2k} - \sum_{k\ge 0} \left(\left\lceil \frac{n-2k-1}{n-2k-2} R_{n-2k-2} \right\rceil + \left\lfloor \frac{n-2k-1}{n-2k} R_{n-2k} \right\rfloor \right) \\
&\quad = 2\sum_{k\ge 0} R_{n-2k} - \sum_{k> 0} \left(\left\lceil \frac{n-2k+1}{n-2k} R_{n-2k} \right\rceil + \left\lfloor \frac{n-2k-1}{n-2k} R_{n-2k} \right\rfloor \right) \\
&\qquad - \left\lfloor \frac{n-1}{n} R_n \right\rfloor \\
&\quad = 2\sum_{k\ge 0} R_{n-2k} - \sum_{k> 0} 2R_{n-2k} - \left\lfloor \frac{n-1}{n} R_n \right\rfloor \\
&\quad = 2R_n - \left\lfloor \frac{n-1}{n} R_n \right\rfloor = \left\lceil \frac{n+1}{n} R_n \right\rceil \ .
\end{aligned}$$

The Extension Step. Let $r \in \mathcal{I}_n$, $r' = \langle r, r_{n+1} \rangle \in \mathcal{I}_{n+1}$ and let $S = \langle S_1, \ldots, S_n \rangle \in \mathcal{R}_n$ such that

$$|r| = |||S||| \quad \text{and} \quad \|r\| = |S| \ .$$

Then for $S' := \langle \Phi_n(S), r_{n+1} + \lceil \frac{n+1}{n} S_n \rceil \rangle \in \mathcal{R}_{n+1}$:

$$|r'| = |||S'||| \quad \text{and} \quad \|r'\| = |S'| \ .$$

The sequence S' belongs indeed to $\mathcal{R}_{n+1}$: from $S \in \mathcal{R}_n$ we have $\Phi_n(S) \in \mathcal{R}_n$ because Φ_n is an involution on $\mathcal{R}_n$. Reducibility at position $n+1$ follows from the fact that $\Phi_n(S)$ has the same last element as S, namely S_n. By the same argument: if $r' \in \mathcal{I}^0_{n+1}$, i.e., if $r_{n+1} = 0$, then $S' \in \mathcal{R}^0_{n+1}$.

The *proof* of the asserted properties of S' is by simple verification, using the properties (26) and (27) mentioned in the previous section.

$$\begin{aligned}|||S'||| &= r_{n+1} + \left\lceil \frac{n+1}{n}\, S_n \right\rceil - |||\Phi_n(S)||| \\ &= r_{n+1} + \left\lceil \frac{n+1}{n}\, S_n \right\rceil + |||S||| - \left\lceil \frac{n+1}{n}\, S_n \right\rceil \\ &= r_{n+1} + |r| = |r'| \ ,\end{aligned}$$

$$\begin{aligned}|S'| &= |\Phi_n(S)| + r_{n+1} + \left\lceil \frac{n+1}{n}\, S_n \right\rceil \\ &= |S| + |||S - \Phi_n(S)||| + r_{n+1} + |||S + \Phi_n(S)||| \\ &= |S| + 2|||S||| + r_{n+1} \\ &= \|r\| + 2|r| + r_{n+1} = \|r'\| \ .\end{aligned}$$

We will use the extension step in the following obvious way.

Iterative Construction of τ_n.

Suppose that $\tau_n : \mathcal{I}_n \to \mathcal{R}_n$ is a bijection that satisfies

$$|r| = |||\tau_n(r)||| \quad \text{and} \quad \|r\| = |\tau_n(r)|$$

then, writing $S = \tau_n(r)$,

$$\tau_{n+1} \ : \ \langle r, r_{n+1}\rangle \mapsto \left\langle \Phi_n(S), r_{n+1} + \left\lceil \tfrac{n+1}{n}\, S_n \right\rceil \right\rangle$$

is a bijection $\tau_{n+1} : \mathcal{I}_{n+1} \to \mathcal{R}_{n+1}$ that satisfies

$$|r'| = |||\tau_{n+1}(r')||| \quad \text{and} \quad \|r'\| = |\tau_{n+1}(r')| \ .$$

Since the existence of such a bijection can easily be checked for small values of n, it follows that such a bijection $\tau_n : \mathcal{I}_n \to \mathcal{R}_n$ and $\tau_n : \mathcal{I}_n^0 \to \mathcal{R}_n^0$ exists for all $n \in \mathbb{N}$.

In particular, we can construct a specific sequence τ_n by starting with $\tau_2 = \omega$ and $\tau_3 = \omega$.

Example 24. Choosing $\tau_3 = \omega$ one can easily check that the iterative construction then gives

$$\begin{aligned}\tau_4 \ : \ &\mathcal{I}_4 \to \mathcal{R}_4 \\ &\langle r_1, r_2, r_3, r_4\rangle \mapsto \\ &\quad \left\langle r_1, 2r_1 + \left\lfloor \tfrac{2r_2+r_3}{3} \right\rfloor, 2r_2 + r_3, 2r_2 + r_3 + r_4 + \left\lceil \tfrac{2r_2+r_3}{3} \right\rceil \right\rangle \ . \qquad \square\end{aligned}$$

Example 25. Consider

$$\mathcal{O}_4(13,3) = \{1^1 3^0 5^1 7^1, 1^0 3^2 5^0, 1^0 3^1 5^2 7^0\}$$

and

$$\mathcal{L}_4(13,3) = \{\langle 0,0,5,8\rangle, \langle 0,1,5,7\rangle, \langle 1,2,4,6\rangle\} \ .$$

The corresponding bijective map Λ_4 between these two sets details as follows:

$$\Psi_4^{-1}(1^1 3^0 5^1 7^1) = \langle 1,1,0,1\rangle = r + \alpha \in \mathcal{I}_4^0 \oplus \mathcal{D}_4^0$$

where

$$r = \langle 0,1,0,0\rangle \quad \text{and} \quad \alpha = 1\cdot\delta^1 + 0\cdot\delta^2 + 0\cdot\delta^3 + 1\cdot\delta^4 \ ;$$

this gives

$$\Gamma_4(\langle 1,1,0,1\rangle) = \tau_4(r) + \sigma_4(\alpha) = \langle 0,0,2,3\rangle + \langle 0,0,3,5\rangle = \langle 0,0,5,8\rangle \ .$$

The second entry:

$$\Psi_4^{-1}(1^0 3^2 5^0 7^1) = \langle 1,0,2,0\rangle = r + \alpha \in \mathcal{I}_4^0 \oplus \mathcal{D}_4^0$$

where

$$r = \langle 0,0,2,0\rangle \quad \text{and} \quad \alpha = 1\cdot\delta^1 + 0\cdot\delta^2 + 0\cdot\delta^3 + 1\cdot\delta^4 \ ;$$

this gives

$$\Gamma_4(\langle 1,0,2,0\rangle) = \tau_4(r) + \sigma_4(\alpha) = \langle 0,1,2,3\rangle + \langle 0,0,3,4\rangle = \langle 0,1,5,7\rangle \ .$$

The third entry:

$$\Psi_4^{-1}(1^0 3^1 5^2 7^0) = \langle 0,2,1,0\rangle = r + \alpha \in \mathcal{I}_4^0 \oplus \mathcal{D}_4^0$$

where

$$r = \langle 0,0,1,0\rangle \quad \text{and} \quad \alpha = 0\cdot\delta^1 + 1\cdot\delta^2 + 0\cdot\delta^3 + 0\cdot\delta^4 \ ;$$

this gives

$$\Gamma_4(\langle 0,2,1,0\rangle) = \tau_4(r) + \sigma_4(\alpha) = \langle 0,0,1,2\rangle + \langle 1,2,3,4\rangle = \langle 1,2,4,6\rangle \ . \qquad \square$$

We conclude this section by the remark that all our bijections can be reverted easily.

2.6 Concluding Remarks

With the Benefit of Hindsight. If we take variables $y_1, y_2, \ldots, y_n$ and define as in Sect. 2.2 for $r = \langle r_1, r_2, \ldots, r_n\rangle \in \mathbb{N}^n$ the monomial

$$y^r := y_1^{r_1} y_2^{r_2} \cdots y_n^{r_n}$$

then we have from the semilinear representation of lecture hall partitions

$$f_n(y_1, y_2, \ldots, y_n) = \sum\{y^b : b \in \mathcal{L}_n = \mathcal{R}_n^0 \oplus \mathcal{E}_n^0\}$$
$$= \frac{\sum\{y^R : R \in \mathcal{R}_n^0\}}{\prod_{1 \le j \le n}(1 - y^{\epsilon^j})} \ .$$

Multiplying the numerator and the denominator of this last fraction by $1-y_n = 1 - y^{\omega(n\,\delta^n)}$ leads (thanks to a finite geometric series) to

$$f_n(y_1, y_2, \ldots, y_n) = \frac{\sum\{y^R : R \in \mathcal{R}_n\}}{\prod_{1 \le j < n}(1 - y^{\epsilon^j}) \cdot (1 - y^{\omega(n\,\delta^n)})}$$
$$= \sum\{y^b : b \in \mathcal{L}_n = \mathcal{R}_n \oplus \mathcal{E}_n\}$$

where $\mathcal{E}_n = \omega(\mathcal{D}_n)$, and where $\mathcal{D}_n$ is the free semimodule generated by $\delta^1, \ldots, \delta^{n-1}$ and $n\,\delta^n = \langle 0, \ldots, 0, n\rangle (= \omega(n\,\delta^n))$. Obviously: $\mathbb{N}^n = \mathcal{I}_n \oplus \mathcal{D}_n$. This avoids the special treatment of the last basis vector.

In other words, we could have used the semilinear presentation $\mathcal{L}_n = \mathcal{R}_n \oplus \mathcal{E}_n$ instead of $\mathcal{L}_n = \mathcal{R}_n^0 \oplus \mathcal{E}_n^0$ – and everything would have gone through equally well. In particular, the essential properties of the mapping λ_n are available for both $\lambda_n : \mathcal{I}_n \to \mathcal{R}_n$, and $\lambda_n : \mathcal{I}_n^0 \to \mathcal{R}_n^0$.

The above use of the presentation $\mathcal{L}_n = \mathcal{R}_n^0 \oplus \mathcal{E}_n^0$ was motivated by what Ω-calculus (or better: its implementation) had suggested. Automatic simplification led to cancelling the factor $1-y_n^n$, which made things less homogeneous.

Other Bijections. The involutory approach we have used for the construction of τ_n is essentially equivalent to an involution discovered by Bousquet-Mélou and Eriksson in [8, Prop. 3.4]. However, they applied this tool in a different direction; in addition, our presentation differs very much from that in [8].

We also want to remark that the limiting case $n \to \infty$ of the Refined Lecture Hall Partition Theorem (Theorem 17) finds a much more direct bijective treatment. In fact, before the finite version in form of Theorem 17 had been discovered, in 1994 C. Bessenrodt [6, Prop. 2.2] described a very elegant bijection between the underlying sets of this limiting case. To this end Bessenrodt uses 2-modular Young diagrams in order to formulate a new version of a classic bijection due to Sylvester. Another variant of Sylvester's bijection was given by D. Kim and A. Yee [11] in 1999; they essentially describe the inverse of Bessenrodt's map. This gives rise to the following problem.

Problem 26. Is there any bijective proof of Theorem 17 that in the limit $n \to \infty$ converges to Bessenrodt's bijection?

Being based on an iterative use of the involution Φ_n, our bijection does not have an infinite version. More generally one can ask the following.

Problem 27. Is there any bijective proof of Theorem 17 without using the involution Φ_n?

It might well be possible that there is a simpler bijection in case the refinement condition is dropped. So we conclude by raising another problem.

Problem 28. Is there any simpler lecture hall bijection for the version of Theorem 1, i.e., in case the refinement condition (18) is dropped?

Note added in proof: A.E. Yee [18] developed a bijective approach which is different to our bijection and which seems to solve Problem 26 and Problem 27.

3 Cayley Compositions

3.1 Introduction

In [9], A. Cayley poses and solves the following problem:

> "It is required to find the number of [compositions] into a given number of parts, such that the first part is unity, and that no part is greater than twice the preceding part.
> Commencing to form the [compositions] in question, these are (read vertically):
>
> $$\begin{array}{c|cc|cccccc|} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\ & & & 1 & 2 & 1 & 2 & 3 & 4 \end{array} \quad \&\text{c}\ldots\text{"}$$

We shall call such compositions, Cayley compositions. Let us define $c_j(n)$ to be the total number of Cayley compositions ending in n and having j parts. Thus from Cayley's enumeration we see that $c_1(1) = 1$, $c_2(1) = c_2(2) = 1$, $c_3(1) = c_3(2) = 2$, $c_3(3) = c_3(4) = 1$.

Clearly the Cayley composition with j parts and largest last part is $1, 2, 4, \ldots, 2^{j-1}$. So if we define a Cayley polynomial as

$$\mathcal{C}_j(q) = \sum_{n \geq 0} c_j(n)\, q^n \quad ,$$

then $\mathcal{C}_j(q)$ has degree 2^{j-1}. Returning to Cayley enumeration, we see that

$$\begin{aligned} \mathcal{C}_1(q) &= q \quad , \\ \mathcal{C}_2(q) &= q + q^2 \quad , \\ \mathcal{C}_3(q) &= 2q + 2q^2 + q^3 + q^4 \quad . \end{aligned}$$

Cayley's Theorem. *The number of [Cayley compositions with j parts] is equal to the number of partitions of $2^{j-1}-1$ into the parts $1, 1', 2, 4, \ldots, 2^{j-2}$. Or again, it is equal to twice the sum of the number of partitions of $0, 1, 2, \ldots, 2^{j-2}-1$ respectively into the parts $1, 1', 2, 4, \ldots, 2^{j-3}$ (where the number of partitions of 0 counts for 1).*

Cayley closes [9] with this example:

"... the partitions of $0, 1, 2, 3$, &c. with the parts $1, 1', 2, \ldots$ are

$$\begin{aligned}
&(\cdot)\ ,\\
&1, 1'\ ,\\
&1+1, 1+1', 1'+1', 2\ ,\\
&1+1+1, 1+1+1', 1+1'+1', 1'+1'+1', 2+1, 2+1'\ ,
\end{aligned}$$

the numbers of which are $1, 2, 4, 6$. Hence, by the first part of the theorem, the number of 3-partitions is 6, and by the second part of the theorem, the number of 4-partitions is

$$2(1+2+4+6) = 26\ ."$$

Cayley's proof of his theorem is quite elegant, efficient and elementary. Our object here is not to improve on Cayley. Rather we wish to show that a direct application of the Partition Analysis paradigm developed by P.A. MacMahon [12] (and subsequently implemented in Mathematica [4]) allows one to obtain easily:

Theorem 29. *For $j \geq 2$:*

$$\begin{aligned}
\mathcal{C}_j(q) = &\sum_{h=1}^{j-2} \frac{b_{j-h-1}\,(-1)^{h-1}\,q^{2^h-1}}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2^{h-1}})}\\
&+ \frac{(-1)^j\,q^{2^{j-1}-1}\,(1-q^{2^{j-1}})}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2^{j-2}})}\ ,
\end{aligned} \tag{28}$$

where b_n is the coefficient of q^{2^n-1} in the power series expansion of

$$\frac{1}{1-q}\prod_{m=0}^{\infty}\frac{1}{1-q^{2^m}}\ . \tag{29}$$

It hardly needs to be pointed out that (28) is a surprising representation of a polynomial. Indeed the right-hand side does not look like a polynomial at all. However when $j=3$, we note $b_1 = 2$ and

$$\frac{2q}{1-q} - \frac{q^3\,(1-q^4)}{(1-q)(1-q^2)} = \frac{2q-q^3-q^5}{1-q} = 2q+2q^2+q^3+q^4 = \mathcal{C}_3(q)\ .$$

From Theorem 29, Cayley's Theorem follows as a natural corollary.

In Sect. 3.2, we shall apply Partition Analysis to Cayley compositions. This will yield Theorem 29 quite directly. The short Sect. 3.3 will derive Cayley's Theorem from Theorem 29. In Sect. 3.4 we briefly describe some generalizations and relations to other work.

3.2 Partition Analysis and Cayley Compositions

The following is the only strictly Partition Analysis identity that is required:

$$\underset{\geqq}{\Omega} \frac{\lambda}{(1-\lambda^2 A)(1-B/\lambda)} = \frac{1+B}{(1-A)(1-AB^2)} \ . \tag{30}$$

While (30) is not in MacMahon's fundamental list [12, p.102], it is easily proved:

$$\begin{aligned}
\underset{\geqq}{\Omega} \frac{\lambda}{(1-\lambda^2 A)(1-B/\lambda)} &= \underset{\geqq}{\Omega} \sum_{r,s\geq 0} A^r B^s \lambda^{2r+1-s} \\
&= \sum_{r\geq 0} \sum_{s=0}^{2r+1} A^r B^s \\
&= \sum_{r\geq 0} \frac{A^r(1-B^{2r+2})}{1-B} \\
&= \frac{1}{(1-A)(1-B)} - \frac{B^2}{(1-AB^2)(1-B)} \\
&= \frac{(1-AB^2)-B^2(1-A)}{(1-A)(1-B)(1-AB^2)} \\
&= \frac{1+B}{(1-A)(1-AB^2)} \ .
\end{aligned}$$

We remark that applying the `Omega` package would give (30) in one stroke.

Let us now consider a $j+1$ variable generating function for Cayley compositions:

$$\begin{aligned}
p_j(x_0,x_1,\ldots,x_j) &:= \sum_{\substack{n1,n2,\ldots,n_j\geq 1 \\ n1\leq 2, n_{i+1}\leq 2n_i}} x_0^1\, x_1^{n_1} \cdots x_j^{n_j} \\
&= \underset{\geqq}{\Omega} \sum_{n1,\ldots,n_j\geq 1} x_0^1\, x_1^{n_1} \cdots x_j^{n_j} \lambda_1^{2-n_1} \lambda_2^{2n_1-n_2} \cdots \lambda_j^{2n_{j-1}-n_j} \\
&= \underset{\geqq}{\Omega} \frac{x_0\, x_1 \cdots x_j\, \lambda_1 \lambda_2 \cdots \lambda_j}{\left(1-\frac{\lambda_2^2 x_1}{\lambda_1}\right)\left(1-\frac{\lambda_3^2 x_2}{\lambda_2}\right)\cdots\left(1-\frac{\lambda_j^2 x_{j-1}}{\lambda_{j-1}}\right)\left(1-\frac{x_j}{\lambda_j}\right)} \\
&= \underset{\geqq}{\Omega} \frac{x_0\, x_1 \cdots x_j\, \lambda_1 \lambda_2 \cdots \lambda_{j-1}}{\left(1-\frac{\lambda_2^2 x_1}{\lambda_1}\right)\cdots\left(1-\frac{\lambda_{j-1}^2 x_{j-2}}{\lambda_{j-2}}\right)} \frac{(1+x_j)}{\left(1-\frac{x_{j-1}}{\lambda_{j-1}}\right)\left(1-\frac{x_{j-1}x_j^2}{\lambda_{j-1}}\right)} \\
&\qquad\qquad \text{(by applying (30) to } \lambda_j)
\end{aligned}$$

$$= \underset{\geqq}{\Omega} \frac{x_0 \cdots x_j \, \lambda_1 \cdots \lambda_{j-1}}{\left(1 - \frac{\lambda_2^2 x_1}{\lambda_1}\right) \cdots \left(1 - \frac{\lambda_{j-1}^2 x_{j-2}}{\lambda_{j-2}}\right)} \frac{(1 + x_j)}{(x_{j-1} - x_{j-1} x_j^2)}$$
$$\cdot \left(\frac{x_{j-1}}{1 - \frac{x_{j-1}}{\lambda_{j-1}}} - \frac{x_{j-1} x_j^2}{1 - \frac{x_{j-1} x_j^2}{\lambda_{j-1}}} \right)$$
$$= \frac{x_j}{1 - x_j} \underset{\geqq}{\Omega} \frac{x_0 \cdots x_{j-1} \lambda_1 \cdots \lambda_{j-1}}{\left(1 - \frac{\lambda_2^2 x_1}{\lambda_1}\right) \cdots \left(1 - \frac{\lambda_{j-1}^2 x_{j-2}}{\lambda_{j-2}}\right)} \left(\frac{1}{1 - \frac{x_{j-1}}{\lambda_{j-1}}} - \frac{x_j^2}{1 - \frac{x_{j-1} x_j^2}{\lambda_{j-1}}} \right)$$
$$= \frac{x_j}{1 - x_j} \left(p_{j-1}(x_0, \ldots, x_{j-1}) - p_{j-1}(x_0, \ldots, x_{j-2}, x_{j-1} x_j^2) \right) \ . \qquad (31)$$

We now note that for $j \geq 2$

$$\mathcal{C}_j(q) = p_{j-1}(1, 1, \ldots, 1, q) \ .$$

So (31) implies the recurrence

$$\mathcal{C}_j(q) = \frac{q}{1 - q} \left(\mathcal{C}_{j-1}(1) - \mathcal{C}_{j-1}(q^2) \right) \ . \qquad (32)$$

It is interesting to note that once the recurrence (32) has been found, here by using Ω-calculus, it can be also proved by straight-forward combinatorial reasoning.

Combinatorial proof of (32). Comparing the coefficients of q^n on both sides of (32) after shifting $j \to j + 1$, we see that (32) is equivalent to

$$\mathcal{C}_j(1) - \sum_{l=1}^{m-1} c_j(l) = \begin{cases} c_{j+1}(2m), & \text{if } n = 2m, \\ c_{j+1}(2m - 1), & \text{if } n = 2m - 1 \ . \end{cases}$$

First, suppose that $n = 2m$ and let

$$\{\langle 1, n_2, \ldots, n_j, 2m \rangle\}$$

be the set of Cayley compositions with $j+1$ parts ending in $2m$. Its cardinality is $c_{j+1}(2m)$. Each part of a Cayley composition is less or equal twice the preceding part. Hence, if we omit the last entry $2m$ from all these tuples, the resulting set

$$\{\langle 1, n_2, \ldots, n_j \rangle\}$$

is running through all Cayley compositions with j parts ending in elements $n_j \geq m$. The cardinality of this set is exactly $\mathcal{C}_j(1) - \sum_{l=1}^{m-1} c_j(l)$. The case $n = 2m - 1$ is analogous. □

Now let us iterate recurrence (32) which implies something very close to (28) namely

$$
\begin{aligned}
\mathcal{C}_j(q) &= \sum_{h=1}^{j-2} \frac{\mathcal{C}_{j-1-h}(1)\,(-1)^{h-1}\,q^{2^h-1}}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2^{h-1}})} \\
&\qquad + \frac{(-1)^j\,q^{2^{j-1}-1}\big(\mathcal{C}_1(1)-\mathcal{C}_1(q^{2^{j-1}})\big)}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2^{j-2}})} \\
&= \sum_{h=1}^{j-2} \frac{\mathcal{C}_{j-1-h}(1)\,(-1)^{h-1}\,q^{2^h-1}}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2^{h-1}})} \\
&\qquad + \frac{(-1)^j\,q^{2^{j-1}-1}(1-q^{2^{j-1}})}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2^{j-2}})} \ .
\end{aligned}
$$

Consequently

$$
\begin{aligned}
q^{2^{j-1}}\mathcal{C}_j(q^{-1}) &= -\sum_{h=1}^{j-2} \frac{\mathcal{C}_{j-1-h}(1)\,q^{2^{j-1}}}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2^{h-1}})} \\
&\qquad + \frac{1}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2^{j-2}})} \ . \qquad (33)
\end{aligned}
$$

Now we observe the magic of (33). The $\mathcal{C}_j(q)$ have degree 2^{j-1} and the lowest power of q appearing is q^1. Consequently $q^{2^{j-1}}\mathcal{C}_j(q^{-1})$ is a polynomial of degree $2^{j-1}-1$. Now let us examine the right-hand side of (33) as an analytic function of q with $|q| < 1$ (even though we know a priori that it is a polynomial of degree $2^{j-1}-1$). The terms in the sum all have $q^{2^{j-1}}$ as the lowest power of q appearing. They, therefore, contribute nothing to this polynomial; i.e. they must be cancelled out by the tail of the expansion of

$$
\frac{1}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2^{j-2}})} \ .
$$

Hence $q^{2^{j-1}}\mathcal{C}_j(q^{-1})$ is the polynomial made up of the first 2^{j-1} terms of the power series expansion of

$$
\prod_{n=0}^{\infty} \frac{1}{1-q^{2^n}} \ .
$$

Therefore $\mathcal{C}_j(1)$ is the coefficient of $q^{2^{j-1}-1}$ in

$$
\frac{1}{1-q}\prod_{n=0}^{\infty} \frac{1}{1-q^{2^n}} \ ,
$$

and this completes the proof of Theorem 29.

3.3 Cayley's Theorem

The first assertion in Cayley's Theorem is equivalent to the last sentence in Sect. 3.2. The second assertion is equivalent to the statement that the sum of the first 2^{j-1} coefficients in

$$\frac{1}{1-q}\prod_{n=0}^{\infty}\frac{1}{1-q^{2^n}}$$

equals the coefficient of q^{2^j-1} in the same series. To see this we note that if

$$F(q)=\frac{1}{1-q}\prod_{n=0}^{\infty}\frac{1}{1-q^{2^n}}\ ,$$

then

$$F(q)=\frac{1+q}{1-q}\,F(q^2)=(1+2q+2q^2+2q^3+\cdots)\,F(q^2)$$

and comparison of the coefficients of q^{2^j-1} on both sides of this identity is the second assertion in Cayley's Theorem.

3.4 Generalizations and Observations

The entire development so far can be generalized by essentially replacing "2" by "k" throughout where $k>1$ is an integer. In doing so, one replaces (30) by

$$\underset{\geqq}{\Omega}\,\frac{\lambda^{k-1}}{(1-\lambda^k A)(1-B/\lambda)}=\frac{1+B+\cdots+B^{k-1}}{(1-A)(1-AB^k)}\ ,$$

and (32) by

$$\mathcal{C}_j(k;q)=\frac{q}{1-q}\,\left(\mathcal{C}_{j-1}(k;1)-\mathcal{C}_{j-1}(k;q^k)\right)\ . \qquad (34)$$

The rest of Sect. 3.2 can be generalized accordingly. For example, $\mathcal{C}_3(3;q)$ is

$$3q+3q^2+3q^3+2q^4+2q^5+2q^6+q^7+q^8+q^9\ ,$$

and the coefficient of q^{9-1} in

$$\frac{1}{1-q}\prod_{n=0}^{\infty}\frac{1}{1-q^{3^n}}$$
$$=1+2q+3q^2+5q^3+7q^4+9q^5+12q^6+15q^7+18q^8+23q^9+\cdots$$

is $18=\mathcal{C}_3(3;1)$.

It should be emphasized that a combinatorial proof of (32) (or more generally (34)) is quite straight-forward. The point here is that Partition

Analysis reveals these recurrences without any combinatorial reasoning on the part of the investigator.

It should be pointed out that H. Minc [13,14] in his work on groupoids studied enumeration problems that are essentially equivalent to Cayley compositions. In a subsequent paper [2], inspired by [14], it was shown that (in our notation)

$$\sum_{j=0}^{\infty} p_j(q, q, \ldots, q) = \frac{q}{1 + \sum_{j=1}^{\infty} \frac{(-1)^j \, q^{2^{j+1}-j-1}}{(1-q)(1-q^3)(1-q^7)\cdots(1-q^{2^j-1})}} \ .$$

This is the generating function for all Cayley compositions classified according to the number being composed (not largest summand).

4 Linear Homogeneous Diophantine Equations

The fundamental step in our construction of a bijective proof for the Refined Lecture Hall Partition Theorem (Theorem 17) was the computation of a parametrized representation of lecture hall partitions; see Sect. 2.2. This was achieved by extrapolation from the first special cases that have been computed by applying the `Omega` package to $\Omega_{\geqq}$-expressions which encode generating functions whose summation parameters satisfy constraints in form of linear homogeneous diophantine inequalities.

In this concluding section we want to explain briefly that generating functions involving constraints in form of linear homogeneous diophantine *equations* can be handled in an analogous fashion as already observed by MacMahon. This does not come as entire surprise since any equation is equivalent to two inequalities. However, for various reasons, e.g., from efficiency point-of-view, it pays off indeed to have a closer look at this aspect of MacMahon's method.

To this end we follow MacMahon and consider:

Definition 30. The operator $\Omega_{=}$ is given by

$$\underset{=}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := A_{0,\ldots,0} \ .$$

This means, all nontrivial power-products in the λ's are killed by the $\Omega_{=}$ operator which, alternatively, can be viewed as a constant term operator.

As already pointed out by MacMahon [12, Vol. 2, Sect. VIII, p. 104], this operator is related to $\Omega_{\geqq}$, for instance, as follows:

$$\underset{=}{\Omega} F(\lambda) = \underset{\geqq}{\Omega} F(\lambda) + \underset{\geqq}{\Omega} F(1/\lambda) - F(1) \ .$$

In other words, the rules for the $\Omega_{\geqq}$ operator in principle would be sufficient to carry out elimination of λ-variables from $\Omega_{=}$-expressions. However, it turns out that the use of special $\Omega_=$-rules that are tailored in the spirit of Lemma 4 is much more convenient – especially with respect to efficiency of computer algebra implementation. Despite having developed his theory long time before the age of computers, this was exactly the program carried out by MacMahon in his book. There he presents a collection of such rules, for instance

$$\underset{=}{\Omega} \frac{1}{(1-\lambda^2 x)\left(1-\frac{y}{\lambda}\right)\left(1-\frac{z}{\lambda}\right)} = \frac{1+xyz}{(1-xy^2)(1-xz^2)} \ ; \tag{35}$$

see [12, Vol. 2, Sect. 351, p. 105]. The proofs of many of these rules are quite elementary but in case of several λ-variables, elimination can be much more cumbersome.

In Sect. 4.2 we present a very general elimination mechanism, Theorem 33. As an application we will have a look at magic squares of size 3. But before doing so, we discuss two elementary examples.

4.1 Introductory Examples

We had mentioned that Partition Analysis has not received due attention with the exception of work by R. Stanley. For instance, in the pioneering paper [15] containing his proof of the Anand-Dumir-Gupta conjecture, Stanley makes essential use of an $\Omega_=$-method that MacMahon describes as "The Method of Elliott"; see [12, Vol. 2, Sects. 358 and 359]. Stanley's interest in the problem of solving linear homogeneous equations for nonnegative integers is also reflected by his book [16] that contains many further references to this problem area. An additional reference is the chapter on rational generating function in Stanley's textbook [17].

Example 31. We illustrate the use of rule (35) by choosing an example from [16, Ex. 3.5]: find all nonnegative integer solutions $\langle a_1, a_2, a_3\rangle \in \mathbb{N}^3$ of

$$a_1 + a_2 - 2a_3 = 0 \ . \tag{36}$$

First we encode the corresponding generating function as an $\Omega_=$ expression,

$$\begin{aligned}\sum_{a_1+a_2-2a_3=0} x_1^{a_1} x_2^{a_2} x_3^{a_3} &= \underset{=}{\Omega} \sum_{a_1,a_2,a_3\geq 0} \lambda^{a_1+a_2-2a_3} x_1^{a_1} x_2^{a_2} x_3^{a_3} \\ &= \underset{=}{\Omega} \frac{1}{(1-\lambda x_1)(1-\lambda x_2)(1-\frac{x_3}{\lambda^2})} \ .\end{aligned}$$

Now due to

$$\underset{=}{\Omega} F(\lambda) = \underset{=}{\Omega} F(1/\lambda) \ ,$$

rule (35) gives

$$\underset{=}{\Omega} \frac{1}{(1-\lambda x_1)(1-\lambda x_2)(1-\frac{x_3}{\lambda^2})} = \frac{1+x_1x_2x_3}{(1-x_1^2x_3)(1-x_2^2x_3)} \ .$$

By geometric series expansion we obtain the desired parametrized representation of the solution set of (36), namely

$$\langle a_1, a_2, a_3\rangle = \{n_1\langle 2,0,1\rangle + n_2\langle 0,2,1\rangle + r\langle 1,1,1\rangle : \langle n_1, n_2\rangle \in \mathbb{N}^2, r \in \{0,1\}\} .$$

This means,

$$\{\langle 2,0,1\rangle, \langle 0,2,1\rangle, \langle 1,1,1\rangle\}$$

is the set of *fundamental solutions*, whereas

$$\{\langle 2,0,1\rangle, \langle 0,2,1\rangle\}$$

is called the set of *completely fundamental solutions*; note that $2\langle 1,1,1\rangle = \langle 2,0,1\rangle + \langle 0,2,1\rangle$. This terminology, together with corresponding ring and module theoretic considerations, traces back to Hilbert's syzygy theorem [10]; for further information consult, e.g., [15], [16] or [17]. □

In order to treat also such $\Omega_=$-problems in a purely automatic fashion, we developed a procedure that has also been implemented in Mathematica.

Example 32. We illustrate its use by taking another example from [16, Ex. 5.14] (see also [17, Ch. 4, Example 4.6.15]): find all nonnegative integer solutions $\langle a_1, a_2, a_3, a_4\rangle \in \mathbb{N}^4$ of

$$a_1 + a_2 - a_3 - a_4 = 0 \quad . \tag{37}$$

Encoding the corresponding generating function as an $\Omega_=$-expression results in

$$\begin{aligned}\sum_{a_1+a_2-a_3-a_4=0} x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4} &= \underset{=}{\Omega} \sum_{a_1,a_2,a_3,a_4\geq 0} \lambda^{a_1+a_2-a_3-a_4} x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4} \\ &= \underset{=}{\Omega} \frac{1}{(1-\lambda x_1)(1-\lambda x_2)(1-\frac{x_3}{\lambda})(1-\frac{x_4}{\lambda})} \quad .\end{aligned}$$

The λ-elimination rule that is needed for this situation is also to find in MacMahon's book; see [12, Vol. 2, Sect. 351, p. 105]. Nevertheless, this time we will apply our procedure:

```
In[2]:= ?OEqR

        OEqR[expr, z] applies the OmegaEq operator to expr
        eliminating the variable z.

In[3]:= f = 1 / ((1-x1 λ)(1-x2 λ)(1-x3/λ)(1-x4/λ))

Out[3]=
```

$$\frac{1}{(1-\lambda\,\mathrm{x1})\,(1-\lambda\,\mathrm{x2})\,\left(1-\frac{\mathrm{x3}}{\lambda}\right)\,\left(1-\frac{\mathrm{x4}}{\lambda}\right)}$$

```
In[4]:= OEqR[f, λ]
```

```
Out[4]=
```

$$\frac{1 - \mathrm{x1\,x2\,x3\,x4}}{(1 - \mathrm{x1\,x3})\,(1 - \mathrm{x2\,x3})\,(1 - \mathrm{x1\,x4})\,(1 - \mathrm{x2\,x4})}$$

By geometric series expansion we again obtain the desired parametrized representation of the solution set of (37), but this time – due to the *minus sign* in the numerator polynomial – the representation in this form is not of the same type as in the previous example. Namely, despite the fact that the fundamental solutions again are immediate from the factors of the denominator, namely

$$\{\langle 1,0,1,0\rangle, \langle 0,1,1,0\rangle, \langle 1,0,0,1\rangle, \langle 0,1,0,1\rangle\} \ ,$$

the numerator monomial gives rise to the syzygy

$$\langle 1,0,1,0\rangle + \langle 0,1,0,1\rangle = \langle 0,1,1,0\rangle + \langle 1,0,0,1\rangle \ .$$

Instead of writing the syzygy additively, its multiplicative version reads as

$$x_1x_2x_3x_4 = (x_1x_3)(x_2x_4) = (x_2x_3)(x_1x_4) \ . \qquad \square$$

One of MacMahon's interests was to use Partition Analysis for discovering syzygetic relations. Below we will give an example in connection with magic squares.

4.2 The Fundamental Recurrence

For automatic elimination of λ-variables with respect to $\Omega_=$ we use essentially the same method as described in [4]. The only difference consists in the replacement of the "fundamental recurrence" [4, Theorem 2] by the corresponding result for the $\Omega_=$ operator which we formulate as Theorem 33.

Before we state our result, we must recall the homogeneous symmetric functions, denoted by $h_j(x_1, x_2, \ldots, x_n)$, which are given by

$$\sum_{j=0}^{\infty} h_j(x_1, x_2, \ldots, x_n)\, t^j = \frac{1}{(1-tx_1)(1-tx_2)\cdots(1-tx_n)} \ .$$

Theorem 33 ("Fundamental Recurrence"). *For n and m positive integers and a any integer,*

$$\underset{=}{\Omega} \frac{\lambda^a}{(1-A_1\lambda)(1-A_2\lambda)\cdots(1-A_n\lambda)\left(1-\frac{B_1}{\lambda}\right)\left(1-\frac{B_2}{\lambda}\right)\cdots\left(1-\frac{B_m}{\lambda}\right)}$$
$$= \frac{P_{n,m,a}(A_1,\ldots,A_n;B_1,\ldots,B_m)}{\prod_{i=1}^{n}\prod_{j=1}^{m}(1-A_iB_j)} \ ,$$

where for $n > 1$,

$$\begin{aligned}
&P_{n,m,a}(A_1,\dots,A_n;B_1,\dots,B_m)\\
&\quad=\frac{1}{A_n-A_{n-1}}\\
&\qquad\cdot\Big\{A_n\prod_{j=1}^{m}(1-A_{n-1}B_j)\,P_{n-1,m,a}(A_1,\dots,A_{n-2},A_n;B_1,\dots,B_m)\\
&\qquad\quad-A_{n-1}\prod_{j=1}^{m}(1-A_nB_j)\,P_{n-1,m,a}(A_1,\dots,A_{n-2},A_{n-1};B_1,\dots,B_m)\Big\}
\end{aligned}$$

and for $n = 1$,

$$\begin{aligned}
&P_{1,m,a}(A_1;B_1,\dots,B_m)\\
&\quad=\begin{cases} A_1^{-a}, & \text{if } a\le 0,\\ A_1^{-a}-\prod_{j=1}^{m}(1-A_1B_j)\sum_{j=0}^{a-1}A_1^{j-a}h_j(B_1,\dots,B_m), & \text{if } a>0\ .\end{cases}
\end{aligned}$$

Proof. The proof of the recurrence is exactly as the proof in the $\Omega_{\geqq}$-case [4, Theorem 2]. We again use the convenient identity

$$\frac{1}{(1-A_n\lambda)(1-A_{n-1}\lambda)}=\frac{1}{A_n-A_{n-1}}\Big(\frac{A_n}{1-A_n\lambda}-\frac{A_{n-1}}{1-A_{n-1}\lambda}\Big)\ ,$$

and the rest follows as before.

The $n = 1$ case splits into two cases as before:
Case $a \le 0$.

$$\begin{aligned}
&\underset{=}{\Omega}\frac{\lambda^a}{(1-A_1\lambda)\big(1-\frac{B_1}{\lambda}\big)\big(1-\frac{B_2}{\lambda}\big)\cdots\big(1-\frac{B_m}{\lambda}\big)}\\
&\quad=\underset{=}{\Omega}\sum_{h=0}^{\infty}\sum_{n_1,\dots,n_m\ge 0}A_1^hB_1^{n_1}\cdots B_m^{n_m}\lambda^{a+h-n_1-\dots-n_m}\\
&\quad=\sum_{n_1,\dots,n_m\ge 0}A_1^{n_1+\dots+n_m-a}B_1^{n_1}\cdots B_m^{n_m}\\
&\quad=\frac{A_1^{-a}}{(1-A_1B_1)(1-A_1B_2)\cdots(1-A_1B_m)}\ ,
\end{aligned}$$

which means that when $a \le 0$

$$P_{1,m,a}(A_1;B_1,\dots,B_m)=A_1^{-a}\ .$$

Case $a > 0$.

$$
\begin{aligned}
&\underset{=}{\Omega} \frac{\lambda^a}{(1-A_1\lambda)\left(1-\frac{B_1}{\lambda}\right)\left(1-\frac{B_2}{\lambda}\right)\cdots\left(1-\frac{B_m}{\lambda}\right)} \\
&\quad = \underset{=}{\Omega} \sum_{h=0}^{\infty} \sum_{n_1,\dots,n_m \geq 0} A_1^h B_1^{n_1} \cdots B_m^{n_m} \lambda^{a+h-n_1-\cdots-n_m} \\
&\quad = \sum_{\substack{n_1,\dots,n_m \geq 0 \\ n_1+\cdots+n_m \geq a}} A_1^{n_1+\cdots+n_m-a} B_1^{n_1} \cdots B_m^{n_m} \\
&\quad = \left(\sum_{n_1,\dots,n_m \geq 0} - \sum_{\substack{n_1,\dots,n_m \geq 0 \\ n_1+\cdots+n_m < a}} \right) A_1^{n_1+\cdots+n_m-a} B_1^{n_1} \cdots B_m^{n_m} \\
&\quad = \frac{A_1^{-a}}{(1-A_1B_1)(1-A_1B_2)\cdots(1-A_1B_m)} - \sum_{j=0}^{a-1} A_1^{j-a} h_j(B_1,\dots,B_m) \ ,
\end{aligned}
$$

so for $a > 0$

$$
P_{1,m,a}(A_1; B_1,\dots,B_m) = A_1^{-a} - \prod_{j=1}^{m}(1-A_1B_j) \sum_{j=0}^{a-1} A_1^{j-a} h_j(B_1,\dots,B_m) \ ,
$$

as desired. □

Example 34. Starting with Sect. 406, in his book [12, Vol. 2, p. 149] MacMahon studied magic squares of size 3. More precisely, given an arbitrary integer a_{10} he is interested to find all nonnegative solutions a_1 up to a_9 of the corresponding eight equations

$$
\begin{aligned}
a_1 + a_2 + a_3 = a_4 + a_5 + a_6 = a_7 + a_8 + a_9 = a_{10} \ , \\
a_1 + a_4 + a_7 = a_2 + a_5 + a_8 = a_3 + a_6 + a_9 = a_{10} \ , \\
a_1 + a_5 + a_9 = a_3 + a_5 + a_7 = a_{10} \ .
\end{aligned}
\tag{38}
$$

MacMahon then states that the corresponding generating function

$$
f = \sum_{a_1,\dots,a_{10}} x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} x_5^{a_5} x_6^{a_6} x_7^{a_7} x_8^{a_8} x_9^{a_9} x_{10}^{a_{10}} \ ,
$$

where the nonnegative integers a_i satisfy the equations in (38), turns into the $\Omega_=$-expression

$$
\begin{aligned}
f = \underset{=}{\Omega} & \frac{1}{(1-\lambda_1\lambda_4\lambda_7x_1)(1-\lambda_1\lambda_5x_2)(1-\lambda_1\lambda_6\lambda_8x_3)(1-\lambda_2\lambda_4x_4)} \\
& \cdot \frac{1}{(1-\lambda_2\lambda_5\lambda_7\lambda_8x_5)(1-\lambda_2\lambda_6x_6)(1-\lambda_3\lambda_4\lambda_8x_7)} \\
& \cdot \frac{1}{(1-\lambda_3\lambda_5x_8)(1-\lambda_3\lambda_6\lambda_7x_9)(1-x_{10}/(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_7\lambda_8))} \ .
\end{aligned}
$$

We eliminate λ_1 up to λ_6 with our package as follows:

```
In[2]:= f = 1/((1-λ1 λ4 λ7 x1) (1-λ1 λ5 x2) (1-λ1 λ6 λ8 x3) *
              (1-λ2 λ4 x4) (1-λ2 λ5 λ7 λ8 x5) (1-λ2 λ6 x6) *
              (1-λ3 λ4 λ8 x7) (1-λ3 λ5 x8) (1-λ3 λ6 λ7 x9) *
              (1-x10/(λ1 λ2 λ3 λ4 λ5 λ6 λ7 λ8)));

In[3]:= g = OEqR[OEqR[OEqR[OEqR[OEqR[f,λ1],λ6],λ5],λ2],λ3]

Out[3]=
```

$$-(-1+\mathrm{x1\,x10^3\,x2\,x3\,x4\,x5\,x6\,x7\,x8\,x9})\ /\ \Big((1-\lambda8^2\,\mathrm{x10\,x3\,x5\,x7})\left(1-\tfrac{\mathrm{x10\,x2\,x6\,x7}}{\lambda 7}\right)\left(1-\tfrac{\mathrm{x10\,x3\,x4\,x8}}{\lambda 7}\right)\left(1-\tfrac{\mathrm{x1\,x10\,x6\,x8}}{\lambda 8}\right)\left(1-\tfrac{\mathrm{x10\,x2\,x4\,x9}}{\lambda 8}\right)(1-\lambda7^2\,\mathrm{x1\,x10\,x5\,x9})\Big)$$

Note that λ_7 and λ_8 correspond to the last two equations in (38). So setting $\lambda_7 = \lambda_8 = 1$ corresponds to dropping the conditions on the diagonals. Furthermore, if we also set $x_i = x$ for $0 \le i \le 9$ and $x_{10} = y$ we obtain

```
In[4]:= g /. {λ7->1,λ8->1,x1->x,x2->x,x3->x,x4->x,x5->x,x6->x,
             x7->x,x8->x,x9->x,x10->y}

Out[4]=
```

$$-\frac{-1+x^9\,y^3}{(1-x^3\,y)^6}$$

which is the result obtained by MacMahon [12, Vol. 2, Sect. 407, p. 161] who concludes his (hand) computation by the remark, "In this the coefficient of $x^{3n}y^n$ represents the number of squares such that in each column and in each row the sum of the numbers is n. It has the value $3\binom{n+3}{4} + \binom{n+2}{2}$." □

4.3 Concluding Remarks

At the end of his Ω-treatment of the syzygetic theory of magic square enumeration MacMahon writes [12, Vol. 2, Sect. 409, p. 164], "There is no theoretical difficulty in dealing with the squares of higher orders, but even in the case $n = 4$ there is practical difficulty in handling the $\Omega_=$ generating function."

With the Mathematica implementation of the Fundamental Recurrence, Theorem 33 above, it was our hope to be able to treat at least the cases $n = 4$ and $n = 5$ in purely automatic fashion. But it turned out that the problems Mathematica had with the rational function arithmetics were too involved. In order to overcome these computational difficulties we decided to take a different approach which is essentially based on partial fraction decomposition and which proceeds iteratively.

We already mentioned that Stanley [15] had used the $\Omega_=$-method in a way that MacMahon describes as "The Method of Elliott"; see [12, Vol. 2, Sects. 358 and 359]. Also this algorithm proceeds iteratively with basic steps being partial fraction decompositions of the type

$$\frac{1}{(1-x\lambda^A)(1-\frac{y}{\lambda^B})} = \frac{1}{1-xy\lambda^{A-B}}\left(\frac{1}{1-x\lambda^A}+\frac{1}{1-\frac{y}{\lambda^B}}-1\right) \ ,$$

where A and B are positive integers.

Our new $\Omega_=$-algorithm [5] is a variation of this Elliott iteration but uses a different partial fraction decomposition for the fundamental steps. Computations show that concerning efficiency this new strategy is far superior to the method of Elliott and also considerably faster than the implementation based on the Fundamental Recurrence. Moreover, we adapted this new method also to the $\Omega_{\geqq}$-situation where we achieved a similar speed-up. Finally we want to mention that using this new approach the computation of the generating function for general magic squares of order 4 has been reduced to a basic problem of computer algebra, namely to the task of adding 254 rational functions, i.e., to simplify them to a single one. So far we were not able to accomplish this in Mathematica.

References

1. G.E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and Its Applications, Vol. 2, G.-C. Rota ed., Addison-Wesley, Reading, 1976. (Reissued: Cambridge University Press, Cambridge, 1985.)
2. G.E. Andrews, *The Rogers-Ramanujan reciprocal and Minc's partition function*, Pacific J. Math. **95** (1981), 251–256.
3. G.E. Andrews, *MacMahon's partition analysis I: The lecture hall partition theorem*, in "Mathematical essays in honor of Gian-Carlo Rota's 65th birthday" (B.E. Sagan et al., eds.), Prog. Math. **161**, Birkhäuser, Boston, 1998, pp. 1–22.
4. G.E. Andrews, P. Paule and A. Riese, *MacMahon's partition analysis III: The* `Omega` *package*, (to appear).
5. G.E. Andrews, P. Paule and A. Riese, *MacMahon's partition analysis VI: The fast algorithm*, (in preparation).
6. C. Bessenrodt, *A bijection for Lebesgue's identity in the spirit of Sylvester*, Discrete Math. **132** (1994), 1–10.
7. M. Bousquet-Mélou and K. Eriksson, *Lecture hall partitions*, Ramanujan J. **1** (1997), 101–111.
8. M. Bousquet-Mélou and K. Eriksson, *Lecture hall partitions II*, Ramanujan J. **1** (1997), 165–187.
9. A. Cayley, *On a problem in the partition of numbers*, Philosophical Mag. **13** (1857), 245–248 (reprinted: The Coll. Math. Papers of A. Cayley, Vol. III, Cambridge University Press, Cambridge, 1890, pp. 247–249).
10. D. Hilbert, *Über die Theorie der algebraischen Formen*, Math. Ann. **36** (1890), 473–534.
11. D. Kim and A.J. Yee, *A note on partitions into distinct parts and odd parts*, Ramanujan J. **3** (1999), 227–231.
12. P.A. MacMahon, *Combinatory Analysis*, 2 vols., Cambridge University Press, Cambridge, 1915–16 (reprinted: Chelsea, New York, 1960).

13. H. Minc, *The free commutative entropic logarithmetic*, Proc. Roy. Soc. Edinburgh **65** (1959), 177–192.
14. H. Minc, *A problem in partitions: Enumeration of elements of a given degree in the free commutative entropic groupoid*, Proc. Edinburgh Math. Soc. **11** (1959), 223–224.
15. R.P. Stanley, *Linear homogeneous diophantine equations and magic labelings of graphs*, Duke Math. J. **40** (1973), 607–632.
16. R.P. Stanley, *Combinatorics and Commutative Algebra*, Birkhäuser, Boston, 1983.
17. R.P. Stanley, *Enumerative Combinatorics - Volume 1*, Wadsworth, Monterey, California, 1986.
18. A.J. Yee, *On Combinatorics of Lecture Hall Partitions*, (to appear).