$-\Delta = -\operatorname{grad}\operatorname{div} + \operatorname{rot}\operatorname{rot}$ for matrices, with application to the finite element solution of the Stokes problem

Gisbert Stoyan, Budapest

Abstract: We consider a matrix equivalent of the identity given in the title and prove that it gives rise to an orthogonal decomposition of $\mathbb{R}^n$ into three orthogonal subspaces - in analogy to a decomposition of the Sobolev space $\vec{H}_0^1$ introduced by W. Velte. The third subspace may turn out to be empty; we derive necessary and sufficient conditions that it is present by considering corresponding eigenvalue problems and equations. This leads also to algebraic versions of the Cauchy–Riemann equations and of the Helmholtz–Weyl decomposition. In this way, an algebraic equivalent of classical vector analysis is developed.

This theory is then applied to the finite element solution of the two-dimensional Stokes problem. Here we obtain information about the spectrum of the Schur complement operator and propose iterative methods which proceed in the third, smallest subspace.

Keywords: Velte decomposition, Helmholtz-Weyl decomposition, algebraic vector analysis, Stokes problem, finite elements

Classification: 65N30; 15A22, 31A30, 35Q35

1 Introduction

In [16] an orthogonal decomposition of the Sobolev space (see [1]) $\vec{H}_0^1$ of vector functions into three orthogonal subspaces has been introduced. This decomposition turned out to be of importance for the stability and numerical solution of the first kind Stokes boundary value problem; it becomes interesting from the point of view of linear algebra since after discretization of the original Stokes problem one has to solve a large block system of linear equations the matrices of which have special properties, and, depending on the discretization, several properties of the analytical Velte decomposition may be recovered. Hence, an algebraic Velte decomposition of $\mathbb{R}^n$ can be defined, but not all discretizations of the Stokes problem give rise to such a decomposition. Some classes of discretizations which admit a discrete Velte decomposition have been given in [14] and [5].

In [15] it has been shown that the analytical Velte decomposition can be exploited to optimize iteration methods for the solution of the (undiscretized)

Stokes problem. In reality one is of course confronted with the linear equations of the discretized Stokes problem. Here subspace decompositions are basic for multigrid and other fast solution techniques, and better knowledge of appropriate spectra helps to propose good iteration or smoothing parameters.

In this paper we therefore start from the purely algebraic Velte decomposition introduced in [5] and give a detailed study of related properties.

This leads not only to a better understanding of known properties of the analytical Velte decomposition by illustrating the conditions for their validity but also to new results; further, as anticipated, the investigation has implications for the iterative solution of the discretized Stokes problem. Especially, for discretizations which admit a Velte decomposition we obtain information about the spectrum of the Schur complement operator and propose iterative methods which proceed in the third subspace.

The algebraic vector analysis developed should be of interest also in the numerical solution of other boundary value problems of equations of Mathematical Physics, like for elasticity, biharmonic and Maxwell equations.

An early paper of V.I. Lebedev has a similar direction as ours: in [9] he builds an algebraic Helmholtz decomposition for difference approximations on an advanced form of staggered grids in general orthogonal coordinates and applies it to boundary value problems for the Poisson equation.

Below, we introduce in Section 2 the algebraic Velte decomposition connected with a tripel A, B, C of symmetric matrices, consider in Section 3 related eigenvalue problems and the symmetry of the spectra. Next, in Section 4, to better correspond to the situation of the discrete Stokes problem, we specialize to the case that B and C are given as $\tilde{B}^T\tilde{B}$ and $\tilde{C}^T\tilde{C}$; then, in Section 5, we algebraically model the two-dimensional case. It is here that we consider iterative methods proceeding in the third subspace. Further, in Section 6, we characterize the algebraic Velte decomposition by certain equations (between them, the algebraic Cauchy–Riemann equations) and show in Theorem 4, that connected with an algebraic Velte decomposition, under natural conditions there exist corresponding decompositions of two further accompanying spaces. One of this triad of orthogonal decompositions is an algebraic Helmholtz–Weyl decomposition.

Finally, in Section 7 we turn to conforming finite element discretizations of the two-dimensional Stokes problem and, for Scott–Vogelius elements, show that all important properties of an algebraic Velte decomposition are present.

2 An algebraic Velte decomposition

Let $(\cdot,\cdot)$ denote the Euclidean scalar product in $\mathbb{R}^n$, let $A, B, C \in \mathbb{R}^{n\times n}$ be matrices satisfying

$$A = B + C, \tag{1}$$
$$A = A^T > 0, \tag{2}$$

$$B = B^T \geq 0 \quad , \quad C = C^T \geq 0, \tag{3}$$
$$\dim \ker B, \dim \ker C \quad \geq \quad 1. \tag{4}$$

Then, with a suitable subspace $W \subset I\!\!R^n$ (which may turn out to be empty) and with orthogonality to be understood in the sense of the scalar product $(A\cdot,\cdot)$, there holds

$$I\!\!R^n = \ker B \oplus \ker C \oplus W. \tag{5}$$

In fact, if $u \in \ker B$ and $v \in \ker C$, then from (1)

$$(Au, v) = (Bu, v) + (Cu, v) = (u, Cv) = 0.$$

Hence $\ker B \perp \ker C$ and $W = (\ker B \oplus \ker C)^\perp$. Under the conditions (1)–(4) we shall call (5) an *algebraic Velte decomposition* of $I\!\!R^n$ and speak of a *proper* (algebraic) Velte decomposition in case W is non-empty. This happens provided

$$\dim W = n - \dim \ker B - \dim \ker C > 0 \tag{6}$$

which in the case of a discretized Stokes equation may be less obvious.

Below, we shall denote the correspondence between certain differential operators and matrices by the sign $\sim$; this relation expresses only an analogy and does not necessarily mean some (good) approximation. In this sense we have the following correspondences connected with (1):

$$A \sim -\vec{\Delta}, \;\; B \sim -\operatorname{grad}\operatorname{div}, \;\; C \sim \operatorname{rot}\operatorname{rot}$$

where $\vec{\Delta}$ is the vector Laplace operator, so that (1) corresponds to

$$-\vec{\Delta} = -\operatorname{grad}\operatorname{div} + \operatorname{rot}\operatorname{rot}, \tag{7}$$

a well known identity of vector analysis.

3 Characterization of proper Velte decompositions by eigenvalue problems

In what follows, we characterize Velte decompositions of $I\!\!R^n$ by properties of eigenvalue problems.

As is well known, A and B can be diagonalized by the same non-singular transformation, say by X:

$$XAX^T = I, \;\; XBX^T = D_B = \operatorname{diag}(b_i), \text{ and } A^{-1} = X^T X. \tag{8}$$

Then, from (1) we have that C is diagonalized, too:

$$XCX^T = D_C = \operatorname{diag}(c_i),$$

and, since B and C were assumed to be semidefinite, it follows that

$$0 \leq b_i = 1 - c_i \leq 1 \text{ for all } i. \tag{9}$$

The sense of the b_i and c_i is clear: these are the eigenvalues of

$$\mu A u = B u \tag{10}$$

and

$$\nu A u = C u, \tag{11}$$

correspondingly, since we have

$$\mu u = D_B u \;\;\Leftrightarrow\;\; \mu X A X^T u = X B X^T u \;\;\Leftrightarrow\;\; \mu A v = B v,\;\; v := X^T u.$$

As is immediately seen, a system of eigensolutions of (10) is given by $\{\mu_i, u_i\}_{i=1}^n$ where $\mu_i = b_i,\;\; u_i := X^T e_i$ and e_i is the i-th coordinate unit vector. Moreover, if (μ, u) is eigensolution of (10) for any $0 \le \mu \le 1$, then $(1-\mu, u)$ is eigensolution of (11). Therefore, from here on, we mostly omit statements about the pencil A, C. In $(\ker B)^\perp = \ker C \oplus W$ it makes sense to consider also the problem

$$\lambda B u = C u. \tag{12}$$

For $\mu \neq 0$, any solution (μ, u) of (10) gives rise to a solution $(\frac{1-\mu}{\mu}, u)$ of (12). In the analytical case, (12) leads to an eigenvalue problem of Friedrichs for complex functions, see [17].

The eigenvectors of (10) form an A-orthogonal basis of the Velte decomposition (see 5): the eigenvectors to $\mu = 0$ span $\ker B$, those to $\mu = 1$ span $\ker C$ and those to $0 < \mu < 1$ span W. Correspondingly, the numbers of $b_i = 0$, $b_i = 1$ and $b_i \in (0,1)$ are $\dim\ker B$, $\dim\ker C$ and $\dim W$.

In case the b_i take on only the values 0 and 1, D_B and D_C are projectors. Observe that this happens if and only if $B = BA^{-1}B$ since

$$D_B^2 = XB(X^TX)BX^T = XBA^{-1}BX^T = D_B = XBX^T$$

holds just for $B = BA^{-1}B$. We summarize the above results as follows.

Theorem 1. For matrices A, B, C satisfying (1)–(4), a proper algebraic Velte decomposition exists if and only if one of the following conditions holds:

1. (6);
2. there is an eigenvalue $\mu \in (0,1)$ of (10);
3. $B \neq BA^{-1}B$. •

Later, we will give further conditions for a proper Velte decomposition to exist.

We shall call the set of the b_i-s the *spectrum of the algebraic Velte decomposition* (or shorter: the Velte spectrum) and speak about its *symmetry* in case that together with b_i also $1 - b_i$ belongs to the spectrum, including multiplicities.

This symmetry of the spectrum with respect to $\frac{1}{2}$ is a property of the analytical version of the eigenvalue problem (10) which is known from [17] to hold in the two-dimensional case independently of the problem domain (but in general not in the three-dimensional case, [18]), see also [3].

Lemma 1. A necessary condition for the spectrum of the Velte decomposition to be symmetric is that the spectrum of $G := B - C$ is symmetric with respect to 0.

Proof. We have $A = 2B - G$, and with X from (8), it follows $I = 2XBX^T - XGX^T = 2\,\mathrm{diag}(b_i) - XGX^T$. Hence $XGX^T = \mathrm{diag}(2b_i - 1)$. Now Sylvester's theorem (see, e.g., [6] p. 60) gives the result. •

Corollary. A sufficient condition for a Velte decomposition to be proper is that G is singular. In that case the spectrum contains $\frac{1}{2}$. •

Remarks 1. In the analytical case it is well known that the value $\frac{1}{2}$ is not only of infinite multiplicity but also the only limit point of the discrete part of the spectrum as follows from the connection between the Velte and the Cosserat spectrum, see [16]-[18] and [12].

2. It seems to be unknown that in the analytical case, in two and in three dimensions, eigenproblems for G are connected to biharmonic eigenproblems since $G^2 = A^2$.

3. In the situation described in the Corollary, the conditions (1)–(4) of existence of a Velte decompositions are of course assumed to hold and exclude the possibility $\ker B \cap \ker C \neq \{0\}$ which would mean singularity of A (and G). •

4 The algebraic Stokes problem

To come nearer to (7), to the Stokes equation and its discretized versions, we now assume that, with some integers $1 \leq d, r \leq n$,

$$B = \tilde{B}^T \tilde{B}, \;\; \tilde{B} \in I\!R^{d \times n}, \quad C = \tilde{C}^T \tilde{C}, \;\; \tilde{C} \in I\!R^{r \times n}. \tag{13}$$

The correspondence to the analytical case is here:

$$\tilde{B}^T \sim \mathrm{grad}, \;\; \tilde{B} \sim -\mathrm{div}, \;\; \tilde{C} \sim \mathrm{rot}\,. \tag{14}$$

We then have $\dim\ker B = \dim\ker \tilde{B}$ and $\dim\ker C = \dim\ker \tilde{C}$, hence

$$\dim W = n - \dim\ker \tilde{B} - \dim\ker \tilde{C},$$

and in case

$$n - \dim\ker \tilde{B} - \dim\ker \tilde{C} > 0 \tag{15}$$

the third subspace in (5), i.e. in

$$I\!R^n = \ker \tilde{B} \oplus \ker \tilde{C} \oplus W, \tag{16}$$

will be non-empty. We might call the vectors in $\ker \tilde{B}$ and in $\ker \tilde{C}$ "divergence-free" and "rotation-free", correspondingly.

We consider now the following eigenvalue problem:

$$\begin{pmatrix} A & \tilde{B}^T \\ \tilde{B} & \beta I_d \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{17}$$

where I_d is the unit matrix in $\mathbb{R}^{d\times d}$, and $u \in \mathbb{R}^n,\ p \in \mathbb{R}^d$. For any solution (u,p) of (17), we have

$$u = -A^{-1}\tilde{B}^T p, \quad \beta p = -\tilde{B}u = Sp, \quad \text{where } S := \tilde{B}A^{-1}\tilde{B}^T \in \mathbb{R}^{d\times d} \tag{18}$$

is the (negative) Schur complement of the (Stokes) matrix

$$\begin{pmatrix} A & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix}.$$

The eigenvalues of (17) are therefore those of S:

$$Sp = \beta p \tag{19}$$

The corresponding formulation for u is (10) as follows from (18).

As usual, the eigenvalues of (19) and (10) coincide up to multiplicity of the zero eigenvalue; $\beta = 0$ results for $0 \neq p \in \ker \tilde{B}^T$ and $\mu = 0$ for $0 \neq u \in \ker \tilde{B}$. So, for an appropriate enumeration of the non-zero eigenvalues, $\beta_i = \mu_i = b_i = 1 - c_i = 1 - \nu_i$ with μ_i, b_i, c_i, ν_i from (8)–(11).

Since the eigenvalue problem for S is of dimension $d \leq n$, with $d \leq \frac{n}{2}$, typically, it may be easier to solve than the problems (10) or (17). Moreover, S is just the iteration matrix of the Uzawa algorithm for the solution of the Stokes problem, and hence better knowledge of its spectrum is decisive for the fast iterative solution of the discretized Stokes problem [15]; also, the eigenvalues of S are connected to the inf-sup property, see [2], [14], and from there they are denoted by β.

From the above results we conclude the following.

Lemma 2. The eigenvalues of the Schur complement $S = BA^{-1}\tilde{B}^T$ are in $[0,1]$. The eigenvalues 0 correspond to eigenvectors $p \in \ker \tilde{B}^T$ (equivalently: to eigenvectors u of (10) in $\ker \tilde{B}$), the eigenvalues 1 correspond to eigenvectors $u \in \ker \tilde{C}$ (equivalently: to eigenvectors p of (19) in $\tilde{B} \ker \tilde{C}$). •

We also can add two more (necessary and sufficient) conditions to those in Theorem 1 (**first continuation of Theorem 1**):

4. (15) is satisfied;

5. the Schur complement matrix S has an eigenvalue $\beta \in (0,1)$. •

Remarks. 1. Concerning the eigenvalue problem for the "Stokes operator"

$$\begin{pmatrix} A & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \lambda \begin{pmatrix} A & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}, \tag{20}$$

we refer to [10] who shows (between others) that to any eigenvalue $\beta_i > 0$ of (19) there are two eigenvalues of (20): $\lambda_{1,2}^{(i)} := \frac{1}{2}(1 \pm \sqrt{1 + 4\beta_i})$.

2. Further related eigenvalue problems can be obtained by replacing $\tilde{B}$ by $\tilde{C}$ in (17), (19). So, the latter leads to

$$Zq = \nu q, \quad Z = \tilde{C} A^{-1} \tilde{C}^T \in I\!R^{r \times r}, \tag{21}$$

and the corresponding formulation for u is (11) as can be seen by taking

$$u = -A^{-1} \tilde{C}^T q, \quad \nu q = -\tilde{C} u = Zq. \quad \bullet \tag{22}$$

5 An equivalent to the two-dimensional analytical case

The symmetry property of the Velte spectrum in the analytical case is connected to the basic property that in two dimensions $-\mathrm{div}$ of a vector field $\vec{v} = (v_1, v_2)^T$ can be obtained by first rotating $\vec{v}$ by $-90°$ and then calculating rot:

$$-\operatorname{div} \vec{v} = \operatorname{rot} Q\vec{v}, \quad Q\vec{v} = Q \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix}. \tag{23}$$

This isomorphism Q exchanges $\ker \tilde{B}$ and $\ker \tilde{C}$ and leaves W invariant.

To get an algebraic equivalent, we put

$$\tilde{C}\tilde{Q} = \tilde{B} \tag{24}$$

for the matrices in (13), with $\tilde{Q}$ being some orthogonal matrix. Of course, relation (24) can hold only in case $r = d$ in (13).

It becomes now interesting to obtain not only the symmetry of the spectrum but also the other important properties of the Velte decomposition in terms of $\tilde{C}$ and $\tilde{Q}$. In what follows, we use the notation $[M, N]$ for the commutator $MN - NM$ of two square matrices M and N; $\|\cdot\|$ denotes the Euclidean norm.

Theorem 2. Assume that for some $0 \neq \tilde{C} \in I\!R^{r \times n}$ with nontrivial nullspace, and, with an orthogonal matrix $\tilde{Q}$, (24) defines $\tilde{B}$ and, with $d = r$, (13) gives C and B.

Then:

1. $\tilde{C}$ and $\tilde{Q}$ define $A = A^T$ by $A = \tilde{Q}^T C \tilde{Q} + C$, and then $A > 0$ holds and therefore a Velte decomposition exists iff

$$\ker \tilde{C} \cap \ker \tilde{C}\tilde{Q} = \{0\}; \tag{25}$$

2. assuming (25), singularity of $G = \tilde{Q}^T[C, \tilde{Q}]$ implies that the Velte decomposition is proper and that its spectrum contains $\frac{1}{2}$;

3. a necessary condition for symmetry of the Velte spectrum is that the eigenvalue problem $[C, \tilde{Q}]u = \lambda \tilde{Q}u$ has as many positive as negative eigenvalues;

4. $[C, \tilde{Q}^2] = 0$ is sufficient for symmetry of the Velte spectrum;

5. $A^2 = G^2$ holds iff $C\tilde{Q}^T C\tilde{Q}$ is skew-symmetric.

Proof. We consider only (25) and property 4 since the other assertions follow by direct computation and from earlier results in this paper.

When (25) holds, then

$$\|u\|_* := \left(\|\tilde{C}\tilde{Q}u\|^2 + \|\tilde{C}u\|^2\right)^{1/2}$$

defines a norm on $I\!R^n$. But

$$\|\tilde{C}\tilde{Q}u\|^2 + \|\tilde{C}u\|^2 = (\tilde{Q}^T C\tilde{Q}u, u) + (Cu, u) = (Au, u).$$

Hence, A is positive definite. Moreover, it is clear that $\ker B = \ker \tilde{C}\tilde{Q}$ is nontrivial together with $\ker \tilde{C}$; therefore all conditions (1)–(4) are satisfied.

To verify symmetry, let (μ, u) be an eigenpair of (10); then $1-\mu$ is an eigenvalue of (11). Now observe that $[A, \tilde{Q}] = \tilde{Q}^T[C, \tilde{Q}^2]$. Therefore, if condition 4 holds, then $\tilde{Q}A = A\tilde{Q}$ or $\tilde{Q}A\tilde{Q}^T = A$. Further, $\tilde{Q}B\tilde{Q}^T = C$. Hence multiplying (10) from the left by $\tilde{Q}$ and from the right by $\tilde{Q}^T$ (i.e. setting $\tilde{Q}u =: v$), we find that (11) has the eigenvalue μ and thus (10) has the eigenvalue $1-\mu$. •

Remarks 1. $[C, \tilde{Q}] \neq 0$ follows from property 1 of this theorem, since otherwise (using $G = \tilde{Q}^T[C, \tilde{Q}]$)

$$0 = G = \tilde{Q}^T \tilde{C}^T \tilde{C}\tilde{Q} - \tilde{C}^T \tilde{C}$$

which implies $\|u\|_* = 0$ for all $u \in \ker \tilde{C}$.

2. If G is singular then $\frac{1}{2}$ is one of the eigenvalues of (11) (and (10) and (19)); its multiplicity is given by $\dim \ker G$. Observe that $\dim W \geq \dim \ker G$.

Observe also that the interest in property 5 is explained by Remark 2 to Lemma 1.

3. The sufficient condition for the symmetry of the spectrum of (10) is satisfied in the analytical case since the Laplace operator is known to be invariant under any rotation.

4. In the case of (19), the multiplicity of the eigenvalues 0 and 1 may be different even if for (10) and (11) it holds. Let therefore $S_{|W}$ be the restriction of S to W. Then symmetry of the spectrum of (19) can be defined as $\sigma(I - 2S_{|W}) = -\sigma(I - 2S_{|W})$. (We remark also that in the analytical case, the operator $I - 2S_{|W}$ has been investigated in detail in [3].)

Corollary 1. If the assumptions of Theorem 2 and property 4 hold and n is odd then $W \neq \{\emptyset\}$, and then $\frac{1}{2}$ is one of the eigenvalues of (10) and (19).

Proof. By (24), $\dim\ker\tilde{B} = \dim\ker\tilde{C} =: k$, hence, together with n, $\dim W = n - 2k$ is odd. This is also the number of eigenvalues of (10) different from 0 and 1. Due to Theorem 2 and (9), eigenvalues $b_i(=\beta_i) \in (0,1)$ appear in pairs (and are of equal multiplicity) with $1 - b_i$. Therefore, subtracting from $n - 2k$ the number of such eigenvalues different from $\frac{1}{2}$, we conclude that the multiplicity of $\frac{1}{2}$ is odd. •

As a first application, consider now the system of (the "Stokes") equations

$$\begin{pmatrix} A & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{26}$$

and assume that the solvability condition $(g,v) = 0$ for all $v \in \ker\tilde{B}^T$ holds.

Corollary 2. If the assumptions of Theorem 2 and conditions 1 and 4 are satisfied then $\omega = 2$ is the optimal iteration parameter for solving (26) by the following Uzawa algorithm [4], [15] (in which systems of the kind $Au = \varphi$ can be solved, e.g., by a conjugate gradient iteration or some multi-level process):

$$u_0 = A^{-1}f, \;\; p_1 = \tilde{B}u_0 - g, \tag{27}$$

$$u_1 = A^{-1}(f - \tilde{B}^T p_1), \;\; p_2 = p_1 - g + \tilde{B}u_1, \tag{28}$$

for k=2,3,...:

$$u_k = A^{-1}(f - \tilde{B}^T p_k), \;\; p_{k+1} = p_k - \omega(g - \tilde{B}u_k). \tag{29}$$

•

We remark that due to the steps (27)–(28), the iteration (29) proceeds in the subspace W in the sense that the errors of u_k are contained in W, and the errors of p_k in the subspace of $\mathbb{R}^d$ spanned by the eigenvectors to eigenvalues of S in $(0,1)$. The conclusion of the corollary refers to the two-dimensional case to be sure of the optimal parameter. However, (27)–(28) work as well in the three-dimensional case, and instead of the Uzawa iteration also an Arrow-Hurwitz (see [15]) or a conjugate gradient method can be employed: these then also proceed in W and are, in general, even more effective.

6 Characterization of proper Velte decompositions by solutions of equations

In this section, it will be necessary to distinguish orthogonality in the A-scalar product (A-orthogonality, denoted as earlier by $\perp$) and e-orthogonality (orthogonality in the Euclidean scalar product, denoted by $\perp_e$).

Now, to get further information about W, for given $v \in \mathbb{R}^d$ we consider the equation

$$Au = \tilde{B}^T v. \tag{30}$$

Then of course $u = u(v) = A^{-1}\tilde{B}^T v$ is uniquely defined and contained in $(\ker \tilde{B})^\perp$:

$$(Au, y) = (v, \tilde{B}y) = 0 \quad \text{for all } y \in \ker \tilde{B}.$$

Hence $A^{-1}\tilde{B}^T \mathbb{R}^d \subseteq (\ker \tilde{B})^\perp = \ker \tilde{C} \oplus W$. However, in this relation equality is true since the dimensions of the first and second space are equal: using that $\tilde{B}^T \mathbb{R}^d$ is a subspace of $\mathbb{R}^n$ the dimension of which is not changed by A^{-1}, we get from

$$\operatorname{rank} \tilde{B}^T = \operatorname{rank} \tilde{B} = n - \dim \ker \tilde{B} = \dim (\ker \tilde{B})^\perp \tag{31}$$

that indeed $\dim A^{-1}\tilde{B}^T \mathbb{R}^d = \dim (\ker \tilde{B})^\perp$.

In other words, choosing $v \in \mathbb{R}^d$ appropriately, any $u \in \ker \tilde{C} \oplus W$ is obtainable as solution of (30) and hence especially, if $W \neq \{0\}$, any $u \in W$, too. In this case we then not only have $u \perp \ker \tilde{B}$ but also $u \perp \ker \tilde{C}$; for fixed $u \in W$ and arbitrary z in $\ker \tilde{C}$ it follows that

$$0 = (Au, z) = (v, \tilde{B}z), \text{ i.e. } v \perp_e \tilde{B} \ker \tilde{C}.$$

Vice versa, for a v with this property it follows that $u \in W$ for the solution of (30); hence for $W \neq \{0\}$ it is necessary that $\dim \tilde{B} \ker \tilde{C} < d$. Observe further that $\dim \tilde{B} \ker \tilde{C} = 0$ or $\dim \tilde{B} \ker \tilde{C} < \dim \ker \tilde{C}$ would mean that there are vectors in $\ker \tilde{C}$ which are annihilated by $\tilde{B}$. This however entails singularity of A. Hence, for any Velte decomposition, $1 \leq \dim \tilde{B} \ker \tilde{C} = \dim \ker \tilde{C}$, and

$$1 \leq \dim \tilde{B} \ker \tilde{C} = \dim \ker \tilde{C} < d \tag{32}$$

for a proper Velte decomposition.

Continuing (31) to the left, we get

$$d - \dim \ker \tilde{B}^T = \operatorname{rank} \tilde{B}^T = n - \dim \ker \tilde{B},$$

for the dimension of the space of right-hand sides of (30). Requiring that these right-hand sides are A-orthogonal to $\ker \tilde{C}$ means $\dim \ker \tilde{C}$ further conditions. Therefore (or referring to (6)), we find the following formula for the dimension of W:

$$\dim W = d - \dim \ker \tilde{B}^T - \dim \ker \tilde{C}. \tag{33}$$

Comparing (32) and (33) we see that from those v which are e-orthogonal to $\tilde{B} \ker \tilde{C}$ one has to subtract yet those which are in $\ker \tilde{B}^T$; only then $0 \neq u(v) \in W$ follows.

Now it pays off to consider, along with (30), for given $w \in \mathbb{R}^r$ also the equation

$$Au = \tilde{C}^T w. \tag{34}$$

First, exchanging in the above approach $\tilde{B}^T$ and $\tilde{C}^T$, we have $u = u(w) = A^{-1}\tilde{C}^T w \in (\ker \tilde{B})^\perp$ in general and $u(w) \in W$ for $w \perp_e \tilde{B} \ker \tilde{C}$, and from

$$r - \dim \ker \tilde{C}^T = \operatorname{rank} \tilde{C}^T = \operatorname{rank} \tilde{C} = n - \dim \ker \tilde{C} = \dim (\ker \tilde{C})^\perp \tag{35}$$

we find

$$\dim W = r - \dim\ker \tilde{C}^T - \dim\ker \tilde{B}. \tag{36}$$

Especially, for a proper Velte decomposition it is necessary that

$$1 \leq \dim \tilde{C}\ker \tilde{B} = \dim\ker \tilde{B} < r. \tag{37}$$

Secondly, and more important, we see from (30) and (34) that

$$\tilde{B}^T v = Au = \tilde{C}^T w$$

for $u \in W$. This gives the possibility to build first A times the third subspace of the Weyl decomposition, $AW = \tilde{B}^T I\!R^d \cap \tilde{C}^T I\!R^r$, by obtaining the general solution $v \in (\ker \tilde{B}^T)^{\perp_e} \subset I\!R^d$, $w \in (\ker \tilde{C}^T)^{\perp_e} \subset I\!R^r$ of

$$\tilde{B}^T v = \tilde{C}^T w \tag{38}$$

and then determining W. We know in advance that (38) has $d - \dim\ker \tilde{B}^T - \dim\ker \tilde{C} = r - \dim\ker \tilde{C}^T - \dim\ker \tilde{B}$ linearly independent solutions with $\tilde{B}^T v = \tilde{C}^T w \neq 0$.

By comparing with (14) we see that in the two-dimensional analytical case (38) is nothing else than the Cauchy–Riemann equations, or, when compared to the three-dimensional case, the Moisil–Teodorescu equations of quaternionic analysis. Therefore, as a by-product we have the following result.

Theorem 3. In the analytical case, the third Velte subspace consists of biharmonic functions with zero boundary values and nonzero Neumann data. •

Remark. The functions in the other two subspaces are connected to solutions of biharmonic equations with right-hand sides $\operatorname{div} \Delta\vec{u}$ and $\operatorname{rot} \Delta\vec{u}$. This situation will be considered in more detail in a separate paper. •

We return now to (30) and give an additional characterization of the $v \in I\!R^d$ assuring $u(v) \in W$. Assume

$$\dim\ker \tilde{B}^T < \dim\ker \tilde{C}\tilde{B}^T \leq d - \dim W \tag{39}$$

and let

$$L := \tilde{B}C\tilde{B}^T = \tilde{B}\tilde{C}^T\tilde{C}\tilde{B}^T = (\tilde{C}\tilde{B}^T)^T\tilde{C}\tilde{B}^T = \tilde{B}(A - \tilde{B}^T\tilde{B})\tilde{B}^T.$$

If now $q \in I\!R^d$ is any element of $\ker \tilde{C}\tilde{B}^T = \ker L$ then $z := \tilde{B}^T q$ is contained in $\ker \tilde{C}$, and for $u \in W$ we obtain

$$0 = (Au, z) = (v, \tilde{B}z) = (v, \tilde{B}\tilde{B}^T q) = (\tilde{B}\tilde{B}^T v, q).$$

This shows that if $u(v) \in W$ then v is "harmonic" in the sense that

$$\tilde{B}\tilde{B}^T v \perp_e \ker L.$$

(Concerning the notation "harmonic", remark that $\tilde{B}\tilde{B}^T \sim -\Delta$ according to (14) and that L is the zero operator in the analytical case.)

Conversely, if for a given $v \in I\!R^d$ there holds $(\tilde{B}\tilde{B}^T v, q) = 0$ for all $q \in \ker L$, then let $z := \tilde{B}^T q$ for a fixed such q. It follows that $z \in \ker \tilde{C}$ and

$$(Au, z) = (v, \tilde{B}z) = 0,$$

and to conclude that $u \in W$ we need $\ker \tilde{C} \subseteq \tilde{B}^T \ker L$. This means that $\tilde{B}^T q = z$ $(\in \ker \tilde{C})$ must be solvable for any such z; in other words: "every rotation-free vector must be representable as a gradient".

It is clear that $\tilde{B}^T \ker L \subseteq \ker \tilde{C}$, moreover we have $\dim \tilde{B}^T \ker L \geq \dim \ker L - \dim \ker \tilde{B}^T$. Therefore, if

$$\dim \ker \tilde{B}^T + \dim \ker \tilde{C} \leq \dim \ker L =: \ell, \tag{40}$$

then $\ker \tilde{C} = \tilde{B}^T \ker L$ follows. We observe that one easily may construct examples with $\tilde{B}^T \ker L \neq \ker \tilde{C}$ such that at the same time $A = \tilde{B}^T \tilde{B} + \tilde{C}^T \tilde{C}$ is positive definite.

Concerning condition (39) we remark that whereas the lower bound is needed to make L interesting, the upper bound is necessary. In fact if $\dim L = \dim \ker \tilde{C}\tilde{B}^T = \ell > d - \dim W$ then, by (33), $\ell > \dim \ker \tilde{B}^T + \dim \ker \tilde{C}$. Hence, because of $\tilde{B}^T \ker L \subseteq \ker \tilde{C}$ and $\dim \tilde{B}^T \ker L \geq \ell - \dim \ker \tilde{B}^T$, it follows that

$$\dim \ker \tilde{C} \geq \dim \tilde{B}^T \ker L \geq \ell - \dim \ker \tilde{B}^T > \dim \ker \tilde{C},$$

a contradiction.

Summarizing, we have the following necessary and sufficient conditions for a proper Velte decomposition to exist (**second continuation of Theorem 1**):

6. there holds $\dim W > 0$ with $\dim W$ given by (33) or (36);

7. (38) has a solution (v, w) with $\tilde{B}^T v = \tilde{C}^T w \neq 0$;

moreover, for the same conclusion to be true, the conditions (32) or (37) are necessary, and, if (39) holds, then it is sufficient that one rotation-free vector is representable as a gradient. $\bullet$

Finally, we have under natural conditions also algebraic Helmholtz–Weyl decompositions. Now we shall use $\oplus_e$ to denote the direct sum of subspaces with orthogonality in the Euclidean scalar product.

Theorem 4. Assume that $\ker \tilde{B}^T > 0$ and $\ker \tilde{C}^T > 0$ hold for the matrices in (13). Then, along with a proper Velte decomposition of $I\!R^n$, there are also orthogonal decompositions of $I\!R^r$ and $I\!R^d$ into three nontrivial subspaces, and these three decompositions can be written and are interconnected by one-to-one, orthogonality conserving mappings as follows:

$$I\!R^r \quad = \quad \underbrace{\tilde{C} \ker \tilde{B} \quad \oplus_e \quad \tilde{C}W}_{} \quad \oplus_e \quad \ker \tilde{C}^T \tag{41}$$

$$
\begin{array}{rcccccl}
 & & A^{-1}\tilde{C}^T \Downarrow & \Uparrow & \tilde{C} & & \\
\mathbb{R}^n & = & \ker \tilde{B} & \oplus & W & \oplus & \ker \tilde{C}, \\
 & & & & \tilde{B} & \Downarrow \quad \Uparrow & A^{-1}\tilde{B}^T \\
\mathbb{R}^d & = & \ker \tilde{B}^T & \oplus_e & \overbrace{\tilde{B}W \quad \oplus_e \quad \tilde{B}\ker \tilde{C}}, & &
\end{array}
\qquad \begin{array}{r} (42) \\ \\ (43) \end{array}
$$

Proof. Let us consider (41). Clearly $\tilde{C}\ker\tilde{B}$, $\ker\tilde{C}^T$ and $\tilde{C}W$ are contained in $\mathbb{R}^r$, moreover, if $p_1 \in \ker\tilde{C}^T$ and $p_2 = \tilde{C}u$ with $u \in \ker\tilde{B}$, then $(p_1, p_2) = (\tilde{C}^T p_1, u) = 0$. Similarly $p_1 \perp_e p_3$ for $p_3 = \tilde{C}w \in \tilde{C}W$. Finally, $(p_2, p_3) = (\tilde{C}u, \tilde{C}w) = (\tilde{C}^T\tilde{C}u, w) = (Au, w) = 0$ since $Bu = 0$ and $u \perp w \in W$. In other words, $\tilde{C}$ performs a mapping of $\ker\tilde{B} \oplus W$ into $\tilde{C}(\ker\tilde{B} \oplus W) = \tilde{C}\ker\tilde{B} \oplus_e \tilde{C}W$ and this mapping conserves orthogonality. Further, this mapping is "onto": no element of $\ker\tilde{B} \oplus W$ is annihilated due to (36) and (37). (43) is proved similarly.

The mappings in the direction to (42) are given by (22) and (18), and orthogonal bases for all the spaces are provided by the eigenvectors of (21), (10) and (19), correspondingly, see Lemma 2. Finally, $A^{-1}\tilde{C}^T$ conserves orthogonality: If $u \in \tilde{C}\ker\tilde{B}$ and $v \in \tilde{C}W$, i.e. $v \perp_e u = \tilde{C}q$ with $\tilde{B}q = 0$, and $v = \tilde{C}w,\ w \in W$, then $A^{-1}\tilde{C}^T u \perp A^{-1}\tilde{C}^T v$:

$$
\begin{aligned}
(A(A^{-1}\tilde{C}^T u), A^{-1}\tilde{C}^T v) &= (\tilde{C}^T\tilde{C}q, A^{-1}\tilde{C}^T\tilde{C}w) = (Aq, A^{-1}\tilde{C}^T\tilde{C}w) \\
&= (q, \tilde{C}^T\tilde{C}w) = (\tilde{C}q, \tilde{C}w) = (u, v) = 0. \quad \bullet
\end{aligned}
$$

Remarks. (41) is an algebraic Helmholtz–Weyl decomposition, and (43) is a decomposition of the algebraic "pressure" space which parallels the Velte decomposition, see [14] for the analytical case.

The triad of orthogonal decompositions (41)–(43) underlines the central role of the Velte decomposition (42) where, compared to (16), it was now more appropriate to write W as second subspace. $\bullet$

7 Application to the finite element solution of the Stokes problem

We consider now finite element methods for the solution of the two-dimensional first kind Stokes problem in an open, bounded, simply connected polygonal domain Ω.

Let $\vec{V} := (H_0^1(\Omega))^2$ be the Sobolev space (see [1]) of vector functions $\vec{u}(x) = (u_1(x), u_2(x))^T$ defined for $x = (x_1, x_2) \in \Omega$, with generalized derivatives in $(L_2(\Omega))^{2\times 2}$ and with zero boundary values in the sense of traces on the boundary $\partial\Omega$ of Ω. Denoting by $L_{2,0}(\Omega)$ the subspace of $L_2(\Omega)$ of square integrable functions with zero integral over Ω, for a given $\vec{f} = (f_1, f_2)^T \in (L_2(\Omega))^2$ consider the variational problem of finding a velocity vector $\vec{u} \in \vec{V}$ and a pressure

$p \in L_{2,0}(\Omega)$ such that

$$\begin{aligned} a(\vec{u},\vec{v}) + b(\vec{v},p) &= (\vec{f},\vec{v}) \quad \text{for all } \vec{v} \in \vec{V}, && (44)\\ b(\vec{u},q) &= 0, \quad \text{for all } q \in L_{2,0}(\Omega), && (45) \end{aligned}$$

where

$$\begin{aligned} (\vec{f},\vec{v}) &= \sum_{i=1}^{2}(f_i,v_i) = \int_\Omega \sum_{i=1}^{2} f_i(x)v_i(x)\,dx, \\ a(\vec{u},\vec{v}) &= \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,dx, && (46)\\ b(\vec{u},p) &= -(\operatorname{div}\vec{u},p). && (47) \end{aligned}$$

This problem is solvable, see [11], [7].

For its numerical solution (see [7] and [2] for important results on this topic), let a basis of the conforming velocity finite element space $\vec{V}_h$ be given as

$$\begin{aligned} \vec{V}_h &:= \operatorname{span}\{\vec{w}_i\}_{i=1}^{2n_h}, && (48)\\ \vec{w}_i &= \begin{pmatrix} w_i \\ 0 \end{pmatrix}, \quad \vec{w}_{i+n_h} = \begin{pmatrix} 0 \\ w_i \end{pmatrix}, \quad 1 \le i \le n_h, \end{aligned}$$

where w_i are scalar piecewise polynomial finite element basis functions defined on a regular triangulation of Ω. Moreover, let a finite element pressure space P_h be given,

$$P_h := \operatorname{span}\{q_j\}_{j=1}^{m_h}, \tag{49}$$

with piecewise polynomial basis functions q_j defined on the same triangulation as w_i.

Compared to our earlier notations, we have now $n = 2n_h$ and $d = m_h$ as dimensions of the discrete velocity and pressure space, respectively.

The finite element velocities and pressures are then given by

$$\vec{u}_h = \sum_{i=1}^{2n_h} u_i\vec{w}_i, \quad p_h = \sum_{j=1}^{m_h} p_j q_j,$$

with nodal values u_i and p_j which will be collected into vectors u and p. The mass matrix M_h of the pressure basis and the stiffness matrix of the velocity basis are then given by

$$\begin{aligned} M_h = (m_{ij}) \in \mathbb{R}^{m_h \times m_h}, \ m_{ij} = (q_j,q_i), \\ A_h = (a_{ij}) \in \mathbb{R}^{2n_h \times 2n_h}, \ a_{ij} = a(\vec{w}_j,\vec{w}_i), \end{aligned}$$

and are positive definite. Here A_h corresponds to A in the algebraic part of this paper. Defining the discrete divergence operator $\operatorname{div}_h : \vec{V}_h \to P_h$ as usual by projection into the pressure space,

$$(\operatorname{div}_h \vec{u}_h, q) = (\operatorname{div}\vec{u}_h, q) \text{ for all } q \in P_h, \tag{50}$$

one obtains $\|\operatorname{div}_h \vec{u}_h\|^2 = (B_h^T M_h^{-1} B_h u, u)$, where u is the nodal vector of $\vec{u}_h$ and $B_h := (b_{ij}),\ \ b_{ij} := b(\vec{w}_j, q_i)$, with b given by (47), see [14]. Hence $B = B_h^T M_h^{-1} B_h$ and

$$\tilde{B} = M_h^{-1/2} B_h \tag{51}$$

when comparing with (1) and (13).

Similarly, the discrete rotation operator is defined by

$$(\operatorname{rot}_h \vec{u}_h, q) = (\operatorname{rot} \vec{u}_h, q) \text{ for all } q \in P_h, \tag{52}$$

giving rise to $C = C_h^T M_h^{-1} C_h$ and

$$\tilde{C} = M_h^{-1/2} C_h, \tag{53}$$

where $C_h := (c_{ij}),\ \ c_{ij} := (\operatorname{rot} \vec{w}_j, q_i)$, and $\|\operatorname{rot}_h \vec{u}_h\|^2 = (C_h^T M_h^{-1} C_h u, u)$. Observe that both $\tilde{B}$ and $\tilde{C}$ are of dimension $d \times n = m_h \times 2n_h$.

For the above finite element matrices A, B, C, in [5] it has been proved that $A - B - C$ is positive semidefinite for general conforming velocity spaces, and $A = B + C$ holds in case that, additionally,

$$\operatorname{div} \vec{V}_h \subseteq P_h. \tag{54}$$

Then, an algebraic Velte decomposition exists, if $\ker B$ and $\ker C$ are nontrivial, see (1)–(4).

We show now first that the symmetry conditions of Theorem 2 hold for general finite element families.

Lemma 3. For the finite element spaces (48)–(49), relation (24) and property 4 of Theorem 2 hold with

$$\tilde{Q} = \tilde{Q}_h = \begin{pmatrix} 0 & I_{n_h} \\ -I_{n_h} & 0 \end{pmatrix}.$$

Proof. (23) carries over to the discrete operators: $-\operatorname{div}_h \vec{u}_h = \operatorname{rot}_h(Q\vec{u}_h)$, as follows from (50) and (52). The matrix corresponding to $-\operatorname{div}_h \vec{u}_h$ is $\tilde{B} = M_h^{-1/2} B_h$ by (51). In the space of nodal vectors u, the $\tilde{Q}_h$ given above is the matrix form of Q. Therefore the matrix form of $\operatorname{rot}_h(Q\vec{u}_h)$ is $M_h^{-1/2} C_h \tilde{Q}_h = \tilde{C}\tilde{Q}_h$, compare with (53), and so (24) follows.

Concerning property 4, observe that $\tilde{Q}_h^2 = -I_n$ is commuting with any matrix. •

For the Scott–Vogelius finite element family (see [13], [19] and [2]) with conforming polynomial velocity spaces of degree $p+1$, and with discontinuous pressure spaces of degree p, the above theory applies.

First we consider the case that the triangulation may contain singular vertices as explained in [13], or [2], p. 227.

Theorem 5. For the Scott-Vogelius elements with $p \geq 3$, and on a regular triangulation, an algebraic Velte decomposition exists, it is proper, and its spectrum is symmetric.

Proof. In our case, the velocity space is conforming, (54) is satisfied and $\ker B$ is nontrivial in general, its dimension being given in [13], and according to the above construction of $\tilde{B}$ and $\tilde{C}$, $\dim\ker\tilde{B} = \dim\ker\tilde{C}$. Symmetry is then obtained from Lemma 3 and Theorem 2. That the decomposition is proper follows from (15) by using the results of [13]: the number of velocity degrees of freedom n is given by

$$n = (p-1)pT + 2pE_0 + 2V_0$$

with p the polynomial degree (≥ 3), T the number of triangles, E_0 the number of inner edges, V_0 the number of inner vertices. The relation $\operatorname{div}\vec{V}_h \subseteq P_h$ means that in $\vec{V}_h$ there holds $\operatorname{div} = \operatorname{div}_h$, hence the dimension of $\ker\operatorname{div}$ given in [13] equals $\dim\ker\tilde{B}$ and $\dim\ker\tilde{C}$,

$$\dim\ker\tilde{B} = \frac{1}{2}p(p-5)T + (2p-1)E_0 + 3V_0 + \sigma - 1.$$

Here σ gives the number of singular vertices.

Then, applying the Euler formula $T = E + 1 - V = E_0 + 1 - V_0$, (15) gives

$$\dim W = n - 2\dim\ker\tilde{B} = 2(p+1)(E_0 + 2 - 2V_0) - 2(\sigma+1).$$

Using induction on T and the assumption that our triangulation covers a simply connected domain, we can prove that $E_0 + 2 - 2V_0 = V - 1$. Hence

$$\dim W = 2\left((p+1)(V-1) - \sigma - 1\right).$$

In the worst case all vertices are singular: $V \geq \sigma$, and therefore, since $V, p \geq 3$,

$$\dim W \geq 8. \quad \bullet$$

Observe that only regularity of the triangulation has been assumed; the symmetry of the spectrum has nothing to do with any symmetry properties of the triangulation.

Turning now to the case that the triangulation does not contain singular vertices, we have the following further results.

In the absence of singular vertices, according to the results of [13], instead of (54) there holds $\operatorname{div}\vec{V}_h = P_h$ meaning $\beta_h \geq \beta$ for the discrete (β_h) and the continuous (β) inf-sup constants (see [2], [7] for the inf-sup condition; see also [14] for specific values of these constants). Then, the lower bound $\underline{\beta} := \sin\min(\omega/2)$ for β (which goes back to [8], see [15], and which only requires to measure a certain angle ω along the boundary of Ω) is useful and, due to $\beta_h \geq \beta$ and the symmetry of both spectra with respect to $\frac{1}{2}$, provides the inclusion

$$I_\sigma(S_{h_{|W_h}}) \subseteq I_\sigma(S_{|W}) \subseteq [\underline{\beta}^2, 1 - \underline{\beta}^2]$$

for the intervals I_σ containing the spectra of $S_{|W}$ and of the corresponding discretized operator $S_{h_{|W_h}}$.

Using then (27)–(29) for the solution of (26), the convergence rate becomes $\cos\min(\omega)$. But when, after applying the two Uzawa steps (27), (28), we use an Arrow-Hurwitz, or Chebyshev, or conjugate gradient method instead of (29), the convergence rate becomes $\cos\min(\omega)/(1+\sin\min(\omega))$, see [15], where these convergence rates referred only to the undiscretized case.

Acknowledgement. The author has been supported by Austrian Science Foundation FWF under the grant SFB F013 and by the Hungarian Ministry for Culture and Education under grant number FKFP 0174/1997.

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