# Engel Expansions of $q$-Series by Computer Algebra

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**Abstract**

The $q$-Engel Expansion is an algorithm that leads to unique series expansions of $q$-series. Various examples related to classical partition theorems, including the Rogers-Ramanujan identities together with the elegant generalization found by Garrett, Ismail and Stanton, have been described recently. The object of this paper is to present the computer algebra package `Engel`, written in Mathematica, that has already played a significant rôle in this work. The package now is made freely available via the web and should help to intensify research in this new branch of $q$-series theory. Among various

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illustrative examples we present a new infinite Rogers-Ramanujan type family that has been discovered by using the package.

# 1 Introduction

Let $\mathbb{F}$ be a field containing the field of rational numbers $\mathbb{Q}$. Let $\mathbb{F}((q))$ denote the field of formal Laurent series over the coefficient field $\mathbb{F}$. In many cases $\mathbb{F}$ will be the field of complex numbers, but in some instances $\mathbb{F}$ will be a field of rational functions in one or several variables. If

$$A = \sum_{n=\nu}^{\infty} c_n q^n \in \mathbb{F}((q)),$$

we call $\nu = \nu(A)$ the *order* of $A$ and define the *norm* of $A$ to be

$$\|A\| = 2^{-\nu(A)}.$$

Note that this norm induces the standard notion of convergence for sequences, infinite series and products of formal Laurent series. In addition, we define the *integral part* $[A] \in \mathbb{F}[q^{-1}]$ of $A$ by

$$[A] = \sum_{\nu \le n \le 0} c_n q^n.$$

As described by Perron [15, sect. 34], Engel originally defined a series expansion for real numbers. Arnold and John Knopfmacher in [12] and [13] extended this concept to formal Laurent series. A special case of their setting which plays a significant rôle for $q$-series can be formulated as follows.

**Definition 1 ("$q$-Engel sequence").** Given a nonnegative integer $\rho$ and $A \in \mathbb{F}((q))$. Set $A_0 = A$, $a_0 = [A]$, and $A_1 = q^\rho(A_0 - a_0)$. Then define recursively for $n \ge 1$:

$$A_{n+1} = q^\rho(a_n A_n - 1)$$

where

$$a_n = \left[\frac{1}{A_n}\right] \quad (n \ge 1).$$

We call $(a_n)_{n\ge 0}$ the *$q$-Engel sequence associated to $A$ and $\rho$*. In the following we will say that $\rho$ is the *extra-exponent* in the Engel expansion.

As a consequence of [13], the following two theorems can be proved.

**Theorem 1 ("Modified $q$-Engel Expansion ($q$-EE)").** *(i) Given a nonnegative integer $\rho$ and $A \in \mathbb{F}((q))$ with associated Engel sequence $(a_n)_{n\ge 0}$. Then*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{q^{-\rho n}}{a_1 \ldots a_n} \tag{1}$$

*holds in $\mathbb{F}((q))$ where the series converges with respect to the above norm. This expansion is finite if and only if $A \in \mathbb{F}((q))$.*
*(ii) For $n \ge 1$:*

$$\nu(a_n) \le -n(\rho+1) \quad \textit{and} \quad \nu(a_{n+1}) \le \nu(a_n) - (\rho+1). \tag{2}$$

The $q$-Engel Expansion turns out to be unique in the following sense:

**Theorem 2 (Uniqueness of $q$-EE).** *Given a nonnegative integer $\rho$ and $A \in \mathbb{F}((q))$. Let $(a_n)_{n\geq 0}$ be a sequence of Laurent polynomials from $\mathbb{F}[q^{-1}]$ with $a_0 = [A]$. If $(a_n)_{n\geq 0}$ satisfies* (1) *and* (2) *then it is the q-Engel sequence associated to $A$ and $\rho$.*

*Proof of Theorem 1 and Theorem 2.* The proofs are implied from the more general setting of Proposition 2 and Theorem 5 in [13]. More precisely, the linkage to the framework of this paper is made up as follows. In section 2 of [13] one first has to set $s_n = a_n$ and $r_n = q^{-\rho}$. Then, if $\rho = 0$, we meet exactly the same situation as described by Theorem 1 and 2 above. For the case $\rho \geq 1$ let us rename the $A(= A_0)$ and $a_0(= [A])$ of Proposition 2 and Theorem 5 in [13] by $A^{(K)}(= A_0^{(K)})$ and $a_0^{(K)}(= [A^{(K)}])$, respectively. After doing so, we set $A^{(K)} = a_0 + q^\rho(A - a_0)$ where $A$ and $a_0$ are taken as in Theorem 1 and 2 above. Since $\rho \geq 1$ we have that $a_0^{(K)} = a_0$. Furthermore, following the machinery of [13] the next value $A_1^{(K)}$ in the $(A_n^{(K)})$-sequence is defined as $A_1^{(K)} = A_0^{(K)} - a_0^{(K)}$; but this equals $A_1 = q^\rho(A_0 - a_0)$, i.e., $A_1^{(K)} = A_1$ and therefore also $a_1^{(K)} = a_1$. For index $n \geq 1$ we have $A_{n+1} = q^\rho(a_n A_n - 1)$ as well as $A_{n+1}^{(K)} = q^\rho(a_n^{(K)} A_n^{(K)} - 1)$. In other words, we have $A_0^{(K)} \neq A_0$, but $A_n^{(K)} = A_n$ $(n \geq 1)$ and $a_n^{(K)} = a_n$ $(n \geq 0)$, and all the statements of Proposition 2 and Theorem 5 in [13] can be carried over accordingly to the situation of Theorem 1 and 2 above. □

We remark that Theorems 1 and 2 in [9] are presented for the special case $\rho = 0$. We also note that in previous papers we have used the term "Extended Engel Expansion" instead of $q$-Engel Expansion.

In [7] and [8] various classical $q$-series identities are shown to be examples of $q$-Engel expansions. For instance, in [7] one finds a detailed proof that the celebrated Rogers-Ramanujan identities [3] fit exactly into this pattern. In [9] we have shown that they form the basis of an *infinite* collection of $q$-Engel expansions, an approach which provides two completely different and alternative proofs of a remarkable result found by T. Garrett, M. Ismail, and D. Stanton [10, (3.5)].

The basis of the investigations in [7], [8], and [9] was a prototype implementation of `Engel` written by the second author of this article in Mathematica. Our object now is to present an updated version of this package to all potential users. It is our hope that the package will help to intensify research in this promising new branch of $q$-series theory.

In Section 2 we will present various introductory examples that should illustrate the scope of the method and the way the `Engel` package is used.

In Section 3 the functionality of the `Engel` package is described in full detail; in addition, comments on the input syntax and the performance are provided.

The `Engel` package serves a tool for discovering or rediscovering $q$-series identities. But also in the case that the "conjectured" identity is well-known, there still remains the task of proving that the identity indeed arises in the context of the given Engel setting. More precisely, in view of Theorem 1 one has to show that the identity is indeed generated as an expansion with respect to a certain associated $q$-Engel sequence. In Section 4 such proofs are given for some of the examples presented in Section 2.

Section 5 illustrates the potential of `Engel` for discovering new identities. We present a new, infinite Rogers-Ramanujan type family — being of similar shape than that one [10, (3.5)] mentioned above — which has been discovered by using the package. Once the identity has been "suggested" to us by `Engel`, we were able to prove it via a finite polynomial version of it.

In Section 6 we conclude the article by raising some open questions.

For the sake of completeness we recall a few standard notions from $q$-series theory. First, the $q$-shifted factorials are defined as

$$(a;q)_k = \begin{cases} (1-a)(1-aq)\cdots(1-aq^{k-1}), & \text{if } k>0, \\ 1, & \text{if } k=0, \\ 1/((1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{k})), & \text{if } k<0, \end{cases}$$

and

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The Gaussian polynomials are defined as usual as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0\le k\le n, \\ 0, & \text{otherwise.} \end{cases}$$

# 2 Illustrative Examples

The following introductory examples should serve to illustrate the aim and the scope of the method and the way the `Engel` package is used.

## 2.1 The rank function

In 1944, F.J. Dyson defined the *rank* of a partition $\lambda$, $\mathrm{rank}(\lambda)$, as the largest part of $\lambda$ minus the number of its (positive) parts; see, e.g., [3]. Let $r(m,n)$ be the number of partitions $\lambda$ of $n$ into distinct parts such that $\mathrm{rank}(\lambda)=m$. By elementary combinatorial reasoning we obtain for the bivariate generating function

$$\sum_{m,n\ge 0} r(m,n)t^m q^n = 1+\sum_{k=1}^{\infty} t^{k-1}q^k(-q/t;q)_{k-1}. \tag{3}$$

Now let us consider what happens if a truncated version of the right hand side of (3) is taken as input $A(=A_0)$ for $q$-Engel Expansion with extra-exponent $\rho=0$:

$In[1] :=$ **A0Truncated[NN_] :=**
**1 + Sum[t^(k − 1) q^k qfac[−q/t, q, k − 1], {k, 1, NN}];**
$In[2] :=$ **A0Series = Simplify[Series[A0Truncated[20], {q, 0, 12}]];**
$In[3] :=$ **Engel[A0Series]**

$$Out[3] = 1+\frac{q}{1-q\,t}+\frac{q^3}{(1-q\,t)\,(1-q^2\,t)}+\frac{q^6}{(1-q\,t)\,(1-q^2\,t)\,(1-q^3\,t)}+O[q]^{10}$$

By setting the option `EngelSeriesOutput` to `True` we get the explicit values for $a_0=[A_0]=1$ and the first $a_n$ for $n\ge 1$, namely $a_1=(1-qt)/q$, $a_2=(1-q^2t)/q^2$, and $a_3=(1-q^3t)/q^3$:

$In[4] :=$ **Engel[A0Series, EngelSeriesOutput->True]**

$$Out[4] = \mathtt{EngelSeries}\left[q, \left\{1, \left\{\frac{1-q\,t}{q}, \frac{1-q^2\,t}{q^2}, \frac{1-q^3\,t}{q^3}\right\}\right\}, 10\right]$$

In view of Theorem 1 the output suggests the following $q$-Engel Expansion for the rank generating function:

$$\sum_{m,n\geq 0} r(m,n)t^m q^n = 1 + \frac{q}{1-qt} + \frac{q^{1+2}}{(1-qt)(1-q^2t)} + \frac{q^{1+2+3}}{(1-qt)(1-q^2t)(1-q^3t)} + \dots$$
$$= \sum_{k=0}^{\infty} \frac{q^{\binom{k+1}{2}}}{(qt;q)_k}. \tag{4}$$

Once the representation (4) has been suggested, its proof is easy by applying elementary combinatorial reasoning similar to that in the case of equality (3). However, since the `Engel` package has suggested the equality (e.g., [5])

$$1 + \sum_{k=1}^{\infty} t^{k-1} q^k (-q/t;q)_{k-1} = \sum_{k=0}^{\infty} \frac{q^{\binom{k+1}{2}}}{(qt;q)_k}, \tag{5}$$

one might also desire a proof that proceeds by verification of the underlying $q$-EE machinery. Such a proof is given in Section 4. We want to conclude this example by mentioning that the refinement (e.g., [1, Corollary 1])

$$1 + x\sum_{k=1}^{\infty} t^{k-1} q^k (-qx/t;q)_{k-1} = \sum_{k=0}^{\infty} \frac{x^k q^{\binom{k+1}{2}}}{(qt;q)_k}, \tag{6}$$

which also has a straight-forward combinatorial interpretation, again fits perfectly into the $q$-EE scheme.

### 2.2 An identity of Euler

Let us consider one of the classic identities due to Euler (e.g., [3]), namely

$$1 + \sum_{k=1}^{\infty} \frac{z^k q^{\binom{k+1}{2}}}{(1-q)(1-q^2)\dots(1-q^k)} = \prod_{k=1}^{\infty}(1+zq^k). \tag{7}$$

Let us take a truncated version of the right hand side of (7) as input $A(= A_0)$ for $q$-Engel Expansion with extra-exponent $\rho = 0$. Also the first $a_n$ for $n \geq 0$ are delivered by the `Engel` package similar to the example above as follows:

*In*[5] := **A0Series = Simplify[Series[qfac[−z q, q], {q, 0, 10}]];**

*In*[6] := **Engel[A0Series]**

$$Out[6] = 1 + \frac{q\ z}{1-q} + \frac{q^3\ z^2}{(1-q)\ (1-q^2)} + \frac{q^6\ z^3}{(1-q)\ (1-q^2)\ (1-q^3)} + 0[q]^{10}$$

*In*[7] := **Engel[A0Series, EngelSeriesOutput->True]**

$$Out[7] = \mathtt{EngelSeries}\left[q, \left\{1, \left\{\frac{1-q}{q\ z}, \frac{1-q^2}{q^2\ z}, \frac{1-q^3}{q^3\ z}\right\}\right\}, 10\right]$$

In other words, the result of the $q$-EE computation "suggests" identity (7); a proof that $a_0 = 1$ and $a_n = z^{-1}(q^{-n} - 1)$ indeed constitutes the associated $q$-Engel sequence with respect to $A = A_0 = (-zq;q)_\infty$ and extra-exponent $\rho = 0$ is given in [7].

Now let us consider what happens if the special case $z = 1$ of $(-zq;q)_\infty$, namely $A = A_0 = (-q;q)_\infty$, is taken as input for $q$-Engel Expansion with respect to the extra-exponent $\rho = 1$:

*In*[8] := **A0Series = Simplify[Series[qfac[− q, q], {q, 0, 20}]];**

*In*[9] := **Engel[A0Series, ExtraExponent->1]**

$$Out[9] = 1 + \frac{q}{1-q-q^2} - \frac{q^4}{(1-q-q^2)\,(1-q-q^3)} + \frac{q^9}{(1-q)\,(1-q-q^2)\,(1-q-q^3)\,(1-q^3-2\,q^5)} + O[q]^{17}$$

*In*[10] := **Engel[A0Series, EngelSeriesOutput->True, ExtraExponent->1]**

$$Out[10] = \mathtt{EngelSeries}\Big[q, \{1, \{\frac{1-q-q^2}{q^2}, -\frac{1-q-q^3}{q^4}, -\frac{(1-q)\,(1-q^3-2\,q^5)}{q^6}\}\}, 17, \mathtt{ExtraExponent} \to 1\Big]$$

¿From the output we can conclude that in this case the $q$-Engel Expansion is "non-regular".

However, why not trying again with extra-exponent $\rho = 2$ instead of $\rho = 1$:

*In*[11] := **A0Series = Simplify[Series[qfac[− q, q], {q, 0, 40}]];**

*In*[12] := **Engel[A0Series, ExtraExponent->2]**

$$Out[12] = 1 + \frac{q}{(1-q)\,(1-q^2)} + \frac{q^6}{(1-q)\,(1-q^2)\,(1-q^3)\,(1-q^4)} + \frac{q^{15}}{(1-q)\,(1-q^2)\,(1-q^3)\,(1-q^4)\,(1-q^5)\,(1-q^6)} + O[q]^{28}$$

*In*[13] := **Engel[A0Series, EngelSeriesOutput->True, ExtraExponent->2]**

$$Out[13] = \mathtt{EngelSeries}\Big[q, \{1, \{\frac{(1-q)\,(1-q^2)}{q^3}, \frac{(1-q^3)\,(1-q^4)}{q^7}, \frac{(1-q^5)\,(1-q^6)}{q^{11}}\}\}, 28, \mathtt{ExtraExponent} \to 2\Big]$$

In other words, now the result of the $q$-EE computation "suggests" the identity

$$1 + \sum_{k=1}^{\infty} \frac{q^{k(2k-1)}}{(1-q)(1-q^2)\dots(1-q^{2k})} = \prod_{k=1}^{\infty}(1+q^k), \tag{8}$$

where $a_0 = 1$ and $a_n = (1-q^{2n-1})(1-q^{2n})/q^{4n-1}$ is the associated Engel sequence with respect to $A = A_0 = (-q;q)_\infty$ and extra-exponent $\rho = 2$. The $q$-EE context is entirely new, but identity (8) itself is well-known. In fact, it is entry (85) of Slater's list [17]. We want to mention that running the $q$-EE machinery on a slightly different $A$, for instance on $A = A_0 = (1-q)(-q;q)_\infty$, one can produce a companion to (8):

*In*[14] := **A0Series = Simplify[Series[(1 − q)qfac[− q, q], {q, 0, 40}]];**

*In*[15] := **Engel[A0Series, ExtraExponent->2]**

$$Out[15] = 1 + \frac{q^3}{(1-q^2)\,(1-q^3)} + \frac{q^{10}}{(1-q^2)\,(1-q^3)\,(1-q^4)\,(1-q^5)} + \frac{q^{21}}{(1-q^2)\,(1-q^3)\,(1-q^4)\,(1-q^5)\,(1-q^6)\,(1-q^7)} + O[q]^{36}$$

*In*[16] := **Engel[A0Series, EngelSeriesOutput->True, ExtraExponent->2]**

$Out[16] = \texttt{EngelSeries}[\texttt{q},$
$\{1, \{\frac{(1-q^2)\ (1-q^3)}{q^5}, \frac{(1-q^4)\ (1-q^5)}{q^9}, \frac{(1-q^6)\ (1-q^7)}{q^{13}}\}\}, 36,$
$\texttt{ExtraExponent} \to 2]$

This time the computation "suggests" the identity

$$\sum_{k=0}^{\infty} \frac{q^{k(2k+1)}}{(1-q)(1-q^2)\dots(1-q^{2k+1})} = \prod_{k=1}^{\infty}(1+q^k), \tag{9}$$

where $a_0 = 1$ and $a_n = (1-q^{2n})(1-q^{2n+1})/q^{4n+1}$ is the associated Engel sequence with respect to $A = A_0 = (1-q)(-q;q)_\infty$ and extra-exponent $\rho = 2$. Identity (9) again is well-known. In fact, it is entry (84) (which is the same as entry (9)) of Slater's list [17]. Concerning the combinatorics underlying (8) and (8), see e.g. [2].

## 2.3 An identity of Cauchy

Let us consider another classic identity,

$$\sum_{k=0}^{\infty} \frac{z^k q^{k^2}}{(q;q)_k (zq;q)_k} = \prod_{k=1}^{\infty} \frac{1}{1-zq^k}. \tag{10}$$

which is due to Cauchy; see e.g., [3].

Let us take the right hand side of (10) as input $A(= A_0)$ and $\rho = 1$ for the $q$-Engel Expansion. Again also the first $a_n$ for $n \geq 0$ are delivered by the Engel package in a manner similar to the example above as follows:

$In[17] :=$ **A0Series = Simplify[Series[1/qfac[z q, q], {q, 0, 10}]];**

$In[18] :=$ **Engel[A0Series, ExtraExponent->1]**

$$Out[18] = 1 + \frac{q\ z}{(1-q)\ (1-q\ z)} + \frac{q^4\ z^2}{(1-q)\ (1-q^2)\ (1-q\ z)\ (1-q^2\ z)} + O[q]^9$$

$In[19] :=$ **Engel[A0Series, ExtraExponent->1, EngelSeriesOutput->True]**

$Out[19] = \texttt{EngelSeries}[\texttt{q}, \{1, \{\frac{(1-q)\ (1-q\ z)}{q^2\ z}, \frac{(1-q^2)\ (1-q^2\ z)}{q^4\ z}\}\}, 9,$
$\texttt{ExtraExponent} \to 1]$

This computation confirms the identity (10). Moreover, $a_0 = 1$ and $a_n = z^{-1}q^{-2n}(1-q^n)(1-q^n z)$ is the associated $q$-Engel sequence with respect to $A = A_0 = 1/(zq;q)_\infty$ and extra-exponent $\rho = 1$. The $q$-Engel Expansion for the special case $z = 1$ (the resulting identity is due to Euler) has been proven in [7].

## 2.4 Slater's list

In [17], L.J. Slater has listed 130 identities all of Rogers-Ramanujan type. Some of the $q$-Engel Expansions we have treated in [7] and [8] can be found there. However, not all of these identities fit directly into the $q$-EE machinery, for particular instances one needs to introduce a certain variation. We illustrate this point by considering the identities,

$$\sum_{k=0}^{\infty} \frac{q^{2k^2+2k}}{(1-q)(1-q^2)\dots(1-q^{2k+1})} = \prod_{\substack{k=1 \\ k\equiv \pm1,\pm4,\pm6,\pm7 \pmod{16}}}^{\infty} \frac{1}{1-q^k} \tag{11}$$

and

$$1+\sum_{k=1}^{\infty}\frac{q^{2k^2}}{(1-q)(1-q^2)\dots(1-q^{2k})}=\prod_{\substack{k=1\\ k\equiv\pm2,\pm3,\pm4,\pm5 \pmod{16}}}^{\infty}\frac{1}{1-q^k} \tag{12}$$

which are entries (38) and (39), respectively, in Slater's list [17]. Taking the right hand side of (12) as $A = A_0$ and $\rho = 1$, the `Engel` computation confirms that this identity fits directly into $q$-EE. But trying `Engel` on the first one with $A = A_0$ being the product side and $\rho = 1$, the `Engel` computation results in:

*In*[20] := **A0 = 1/(**
**qfac[q^1, q^16]qfac[q^(16 − 1), q^16]∗**
**qfac[q^4, q^16]qfac[q^(16 − 4), q^16]∗**
**qfac[q^6, q^16]qfac[q^(16 − 6), q^16]∗**
**qfac[q^7, q^16]qfac[q^(16 − 7), q^16]);**

*In*[21] := **A0Series = Series[A0, {q, 0, 15}];**

*In*[22] := **Engel[A0Series, ExtraExponent->1]**

$$Out[22]=1+\frac{q}{1-q}+\frac{q^4}{(1-q)\,(1-q^2-q^3)}-\frac{q^9}{(1-q)\,(1-q^2-q^3)^2\,(1-q^6)}+\frac{q^{12}}{(1-q)\,(1-q^2-q^3)^2\,(1-q^6)}+O[q]^{16}$$

*In*[23] := **Engel[A0Series, ExtraExponent->1, EngelSeriesOutput->True]**

$$Out[23]=\mathtt{EngelSeries}\left[q,\left\{1,\left\{\frac{1-q}{q^2},\frac{1-q^2-q^3}{q^4},-\frac{(1-q^2-q^3)\,(1-q^6)}{q^6\,(1-q^3)}\right\}\right\},16,\mathtt{ExtraExponent}\to 1\right]$$

However, if one varies the problem by taking as $A = A_0$ the right hand side of (11) *multiplied by the factor* $(1-q)$ and again $\rho = 1$, the computation returns a nice result which after dividing by $(1-q)$ is nothing but a truncated version of identity (11):

*In*[24] := **A0 = (1 − q)/(**
**qfac[q^1, q^16]qfac[q^(16 − 1), q^16]∗**
**qfac[q^4, q^16]qfac[q^(16 − 4), q^16]∗**
**qfac[q^6, q^16]qfac[q^(16 − 6), q^16]∗**
**qfac[q^7, q^16]qfac[q^(16 − 7), q^16]);**

*In*[25] := **A0Series = Series[A0, {q, 0, 40}];**

*In*[26] := **Engel[A0Series, ExtraExponent->1]**

$$Out[26]=1+\frac{q^4}{(1-q^2)\,(1-q^3)}+\frac{q^{12}}{(1-q^2)\,(1-q^3)\,(1-q^4)\,(1-q^5)}+\frac{q^{24}}{(1-q^2)\,(1-q^3)\,(1-q^4)\,(1-q^5)\,(1-q^6)\,(1-q^7)}+O[q]^{40}$$

*In*[27] := **Engel[A0Series, ExtraExponent->1, EngelSeriesOutput->True]**

$$Out[27]=\mathtt{EngelSeries}\left[q,\left\{1,\left\{\frac{(1-q^2)\,(1-q^3)}{q^5},\frac{(1-q^4)\,(1-q^5)}{q^9},\frac{(1-q^6)\,(1-q^7)}{q^{13}}\right\}\right\},40,\mathtt{ExtraExponent}\to 1\right]$$

A complete proof of both identities, (11) and (12), via $q$-Engel Expansion is given in Section 4.

# 3 The `Engel` Package

## 3.1 How to Use the Package

The Mathematica package `Engel.m` and its online manual `EngelManual.nb` are available from the software site of the RISC combinatorics group at

`http://www.risc.uni-linz.ac.at/research/combinat/risc/`

To have a quick start, open `EngelManual.nb` from within Mathematica; it contains all examples of this paper.

In a Mathematica session, load the package `Engel.m` by executing

*In*[28] := << **Engel.m**

```
Engel.m (version of May 2000) by B.Zimmermann@risc.uni-linz.ac.at
 in cooperation with G.E.Andrews, A.Knopfmacher and P.Paule.
Using qNormal.m by Axel.Riese@risc.uni-linz.ac.at.
```

As a first example, let us compute a truncated version of the first Rogers-Ramanujan identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty};$$

in [7] a complete $q$-EE proof is given. We enter its right hand side,

*In*[29] := **A0 = 1/qfac[q, q^5]/qfac[q^4, q^5]**;

expand it into a Laurent series up to an error of $O(q^{11})$,

*In*[30] := **A0Series = Series[A0, {q, 0, 10}]**

*Out*[30] $= 1 + q + q^2 + q^3 + 2\,q^4 + 2\,q^5 + 3\,q^6 + 3\,q^7 + 4\,q^8 + 5\,q^9 + 6\,q^{10} + O[q]^{11}$

and start the `Engel` program by

*In*[31] := **result = Engel[A0Series]**

obtaining as output a truncated version of the left hand side of the First Rogers Ramanujan Identity:

*Out*[31] $= 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)\,(1-q^2)} + O[q]^9$

To check the result, we convert it back to a power series by

*In*[32] := **Series[result, {q, 0, 30}]**

*Out*[32] $= 1 + q + q^2 + q^3 + 2\,q^4 + 2\,q^5 + 3\,q^6 + 3\,q^7 + 4\,q^8 + O[q]^9$

Inspection shows that the series expansion is indeed correct.

The function `Engel` computes as many terms of the $q$-Engel Expansion as possible. If we give the input with higher precision, we get more terms back:

*In*[33] := **A0Series = Series[A0, {q, 0, 20}]**

*In*[34] := **Engel[A0Series]**

*Out*[34] $= 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)\,(1-q^2)} + \frac{q^9}{(1-q)\,(1-q^2)\,(1-q^3)} + O[q]^{16}$

## 3.2 The Input Syntax

The expression `qfac[`$a$`,`$q$`,`$n$`]` denotes the $q$-shifted factorial $(a;q)_n$ and `qfac[`$a$`,`$q$`]` abbreviates `qfac[`$a$`,`$q$`,Infinity]` which is $(a;q)_\infty$

Let $A$ be an expression involving the indeterminate $q$ and let $n$ be a nonnegative integer. The Mathematica command

`Series[`$A$`, {`$q$`, 0, `$n$`}]`

tries to expand $A$ in a truncated Laurent series in $q$ up to an error of $O(q^{n+1})$. Note that the package `Engel.m` extends `Series` to input involving $q$-shifted factorials. For example, we obtain the counting generating function of all partitions, which is case $z = 1$ of (10), as follows:

*In*[35] := **A0 = 1/qfac[q, q]**;

*In*[36] := **A0Series = Series[A0, {q, 0, 15}]**

*Out*[36] $= 1 + q + 2\ q^2 + 3\ q^3 + 5\ q^4 + 7\ q^5 + 11\ q^6 + 15\ q^7 + 22\ q^8 + 30\ q^9 + 42\ q^{10} + 56\ q^{11} + 77\ q^{12} + 101\ q^{13} + 135\ q^{14} + 176\ q^{15} + O[q]^{16}$

Given a nonnegative integer $\rho$ and a (truncated) Laurent series $L$ in $\mathbb{F}((q))$ where $\mathbb{F}$ is a computable extension field of $\mathbb{Q}$, the expression

`Engel[`$L$`, ExtraExponent->`$\rho$`]`

evaluates to a (truncated) $q$-Engel Expansion of $L$ of the form

$$a_0 \quad + \quad \sum_{n=1}^{N} \frac{q^{-n\rho}}{a_1 \ldots a_n} \quad + \quad O(q^m) \tag{13}$$

where the integer $N$ is chosen by the program as big as the precision of $L$ allows, and for $A_0 = A$:

$$\begin{aligned} a_0 &= [A_0], \\ a_n &= [1/A_n] \quad \text{for all} \quad n \geq 1, \\ A_1 &= q^\rho (A_0 - a_0), \\ A_{n+1} &= q^\rho (a_n A_n - 1) \quad \text{for all} \quad n \geq 2, \\ m &= \sum_{1 \leq n \leq N} \nu(\frac{1}{a_n}) + \nu(A_{n+1}) - (N+1)\,\rho. \end{aligned}$$

For example,

*In*[37] := **Engel[A0Series, ExtraExponent->1]**

*Out*[37] $= 1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2\ (1-q^2)^2} + \frac{q^9}{(1-q)^2\ (1-q^2)^2\ (1-q^3)^2} + O[q]^{16}$

To compute (unmodified) $q$-Engel Expansions (in contrast to modified $q$-Engel Expansions), it suffices to omit the option `ExtraExponent`$\rightarrow \rho$; in this case the function `Engel` implicitly assumes `ExtraExponent`$\rightarrow 0$.

To get the $q$-Engel Expansion terms $a_0$, $a_1$, $a_2, \ldots$ explicitly, we use the option

`EngelSeriesOutput->True`

.

*In*[38] := **Engel[A0Series, ExtraExponent->1, EngelSeriesOutput->True]**

$Out[38] = \texttt{EngelSeries}[\texttt{q}, \{1, \{\frac{(1-q)^2}{q^2}, \frac{(1-q^2)^2}{q^4}, \frac{(1-q^3)^2}{q^6}\}\}, 16, \texttt{ExtraExponent} \to 1]$

To understand this output, note that

$$\texttt{EngelSeries}[q, \{a_0, \{a_1, a_2 \ldots a_N\}\}, m, \texttt{ExtraExponent} \to \rho]$$

is the `Engel` package's internal representation of the series expansion (13).

### 3.3 Performance

How many Engel Expansion terms can be computed in reasonable time by `Engel.m`? To give a rough estimate, we list the time* needed to compute the first Rogers-Ramanujan Identity

$$\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} = \sum_{n=0}^{N-1} \frac{q^{n^2}}{(q;q)_n} + O(q^{N^2}).$$

up to an error of $O(q^{N^2})$.

| N | seconds |
|---|---|
| 0 | 0.01 |
| 10 | 0.89 |
| 20 | 20.06 |
| 30 | 197.94 |

These timings are typical for input whose Engel Expansion exhibits a nice pattern. On the other hand, computing Engel Expansions for random input is usually much slower. For example, let us try to expand

$$\frac{1}{(q;q^5)_\infty(q^3;q^5)_\infty}$$

into an Engel series:

$In[39] :=$ **Engel[Series[1/qfac[q, q^5]/qfac[q^3, q^5], {q, 0, 100}]]**

$Out[39] =$ `$Aborted`

After waiting for some time without getting any result, we are forced to interrupt the computer. Retrying at a lower precision

$In[40] :=$ **Engel[Series[1/qfac[q, q^5]/qfac[q^3, q^5], {q, 0, 10}]]**

is successful:

$$Out[40] = 1 + \frac{q}{1-q} + \frac{q^3}{1-q} + \frac{4\,q^6}{(1-q)\,(2-q^2-2\,q^3)} + O[q]^{10}$$

In practice, a useful strategy is to raise the precision stepwise in a loop:

$In[41] :=$ **Do[**
**Print[ Engel[Series[1/qfac[q, q^5]/qfac[q^3, q^5], {q, 0, NN}]]],**
**{NN, 0, 100, 2}];**

*Timings were measured in Mathematica 4.0 under Linux on a Pentium III running at 450 MHz with 196MB RAM.

# 4 Proving Identities via $q$-Engel Expansion

As demonstrated in Section 2 the `Engel` package serves a tool for discovering or rediscovering $q$-series identities. But also in the case that the "conjectured" identity is well-known, there still remains the task of proving that the identity indeed arises in the context of the given Engel setting. More precisely, in view of Theorem 1 one has to show that the identity is generated as an expansion with respect to a certain associated $q$-Engel sequence. In this section we present such proofs for some of the examples from Section 2.

## 4.1 The rank identity

In order to give an $q$-EE-proof of (5), we will prove the following statement:

**Theorem 3.** *Let $A$ be the left hand side of* (5)*. Then the sequence $(a_n)_{n\geq 0}$ defined as*

$$a_n = \begin{cases} 1, & \text{if } n = 0, \\ \frac{1-q^n t}{q^n}, & \text{if } n \geq 1, \end{cases}$$

*is the q-Engel sequence associated to $A$ and $\rho = 0$.*

This proves identity (5) since (1) then implies

$$\begin{aligned} A = a_0 + \sum_{k\geq 1} \frac{1}{a_1 \dots a_k} &= 1 + \sum_{k\geq 1} \frac{q^{1+2+\dots+k}}{(qt;q)_k} \\ &= 1 + \sum_{k\geq 1} \frac{q^{\binom{k+1}{2}}}{(qt;q)_k}. \end{aligned}$$

*Proof of Theorem 3.* Define $A$ to be the left hand side of (5) and set $A_0 = A$. For $n \geq 1$ set

$$A_n = \sum_{k=1}^{\infty} q^{nk} t^{k-1} (-q/t;q)_{k-1}. \tag{14}$$

Given $(a_n)_{n\geq 0}$ as in the statement of Theorem 3 and $\rho = 0$, the proof according to Definition 1 and Theorem 1 splits into two parts: (i) verifying the relations

$$A_1 = A_0 - a_0 \quad \text{and} \quad A_{n+1} = a_n A_n - 1 \quad (n \geq 1), \tag{15}$$

and (ii) showing that

$$a_0 = [A] \quad \text{and} \quad a_n = \left[\frac{1}{A_n}\right] \quad (n \geq 1). \tag{16}$$

*Part (i):* The case $n = 0$, i.e., $A_1 = A_0 - a_0 = A - 1$ is obvious. For $n \geq 1$ we compute

$$\begin{aligned} A_{n+1} &= \frac{1-q^n t}{q^n} A_n - 1 \\ &= \sum_{k\geq 1} q^{n(k-1)} t^{k-1} (-q/t;q)_{k-1} - \sum_{k\geq 1} q^{nk} t^k (-q/t;q)_{k-1} - 1 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\geq 1} q^{nk} t^k (-q/t;q)_k - \sum_{k\geq 1} q^{nk} t^k (-q/t;q)_{k-1} \\
&= \sum_{k\geq 1} q^{nk} t^k (1 + q^k/t - 1)(-q/t;q)_{k-1} \\
&= \sum_{k\geq 1} q^{(n+1)k} t^{k-1} (-q/t;q)_{k-1}.
\end{aligned}
$$

*Part (ii):* The case $n=0$, i.e., $a_0 = [A] = 1$ is again obvious. In order to prove the case $n \geq 1$, before extracting the integral part of $A_n$, we first derive a suitable asymptotic representation of $A_n$.

$$
\begin{aligned}
A_n &= \sum_{k=1}^{\infty} q^{nk} t^{k-1} (-q/t;q)_{k-1} \\
&= q^n + q^{2n} t (-q/t;q)_1 + O(q^{3n}) \\
&= q^n \left(1 + q^n t + q^{n+1} + O(q^{2n})\right) \\
&= q^n \left(1 + q^n t + O(q^{n+1})\right).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\left[\frac{1}{A_n}\right] &= \left[\frac{1}{q^n}\left(1 - q^n t + O(q^{n+1})\right)\right] \\
&= \frac{1}{q^n} - t = \frac{1-q^n t}{q^n},
\end{aligned}
$$

which completes the proof of Theorem 3. □

We conclude this section by mentioning that the $q$-EE proof of the refined identity (6) works entirely analogous.

## 4.2 Identities (38) and (39) from Slater's list

In order to prove the identities (11) and (12) via $q$-Engel Expansion, we begin with two polynomial sequences originally defined by Santos [16], namely

$$
S_N = \sum_{j=-\infty}^{\infty} q^{4j^2-j} \begin{bmatrix} N \\ \lfloor \frac{N+1-4j}{2} \rfloor \end{bmatrix} \tag{17}
$$

and

$$
T_N = \sum_{j=-\infty}^{\infty} q^{4j^2-3j} \begin{bmatrix} N \\ \lfloor \frac{N+2-4j}{2} \rfloor \end{bmatrix}. \tag{18}
$$

These polynomials were subsequently studied in [6] where it was noted that

$$
\lim_{N\to\infty} S_N = \prod_{\substack{n=1 \\ n \equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}}}^{\infty} \frac{1}{1-q^n} \tag{19}
$$

and

$$\lim_{N\to\infty} T_N = \prod_{\substack{n=1 \\ n\equiv\pm1,\pm4,\pm6,\pm7 \pmod{16}}}^{\infty} \frac{1}{1-q^n} \tag{20}$$

Although it is not shown explicitly in [6], it is easy to verify that these polynomials satisfy the following defining recurrences

$$\begin{aligned} S_N - S_{N-1} &= q^N\, T_{N-1}, \\ T_N - T_{N-1} &= q^{N-1}\, S_{N-1}\,, \end{aligned}$$

together with the initial values $S_0 = 1$, $T_0 = 0$.

In the next step one may reduce these recurrences to separate recurrences for each sequence to obtain

$$S_N - S_{N-1} - q\, S_{N-1} + q\, S_{N-2} = q^{2N-2}\, S_{N-2}\,, \quad \text{for } N > 1, \tag{21}$$

and

$$T_N - T_{N-1} - q\, T_{N-1} + q\, T_{N-2} = q^{2N-2}\, T_{N-2}\,, \text{ for } N > 1. \tag{22}$$

We remark that with $q$-WZ theory, for instance with the computer algebra package `qZeil` [14], one can produce these recurrences in purely automatic fashion with the original sum representations (17) and (18) as input.

To prove (12), which is (39) in [17], we define

$$\begin{aligned} A_n &= \sum_{j=1}^{\infty} q^{(2n-1)j+1} T_{j-1} \text{ for } n > 0\,, \\ A_0 &= \lim_{N\to\infty} S_N\,, \\ a_n &= q^{-4n+1}\,(1-q^{2n})(1-q^{2n-1}) \text{ for } n > 0, \\ a_0 &= 1\,, \end{aligned} \tag{23}$$

and we apply the $q$-Engel Expansion with extra-exponent $\rho = 1$.

First of all,

$$\begin{aligned} q(a_0A_0 - 1) = q\left(\lim_{N\to\infty}(S_N - S_0)\right) &= q \lim_{N\to\infty} \sum_{j=0}^{N-1} (S_{j+1} - S_j) \\ &= q\sum_{j=0}^{\infty} q^{j+1} T_j = \sum_{j=1}^{\infty} q^{j+1} T_{j-1} = A_1\,. \end{aligned}$$

Next

$$
\begin{aligned}
&q(a_nA_n - 1)\\
&= q\left((1-q^{-2n}-q^{-2n+1}+q^{-4n+1})\sum_{j=1}^{\infty} q^{(2n-1)j+1}T_{j-1} - 1\right)\\
&= \sum_{j=1}^{\infty} q^{(2n-1)j+2}T_{j-1} - \sum_{j=0}^{\infty} q^{(2n-1)j+1}T_j - \sum_{j=0}^{\infty} q^{(2n-1)j+2}T_j\\
&\qquad\qquad + \sum_{j=-1}^{\infty} q^{(2n-1)j+1}T_{j+1} - q\\
&= \sum_{j=1}^{\infty} q^{(2n-1)j+1}(T_{j+1}-T_j-q\,T_j+q\,T_{j-1})\\
&= \sum_{j=1}^{\infty} q^{(2n-1)j+1}q^{2j}T_{j-1}\\
&= A_{n+1}\,.
\end{aligned}
$$

Finally

$$
\begin{aligned}
\left[\frac{1}{A_n}\right] &= \left[\frac{1}{q^{4n-1}+(1+q)q^{6n-2}+q^{8n-3}(1+q+q^2+q^4)+\cdots}\right]\\
&= \left[q^{-4n+1}(1+q^{2n-1}+q^{2n}+q^{4n-2}+q^{4n-1}+\cdots)^{-1}\right]\\
&= \left[q^{-4n+1}(1-(q^{2n-1}+q^{2n}+q^{4n-2}+q^{4n-1})\right.\\
&\qquad\qquad\qquad\qquad \left. +q^{4n-2}+2q^{4n-1}+O(q^{4n}))\right]\\
&= [q^{-4n+1}(1-q^{2n-1}-q^{2n}+q^{4n-1}+O(q^{4n}))]\\
&= q^{-4n+1}(1-q^{2n-1})(1-q^{2n})\\
&= a_n
\end{aligned}
$$

Hence according to Theorem 1 we may conclude that

$$
\begin{aligned}
A_0 &= 1+\sum_{n=1}^{\infty}\frac{q^{-n}}{a_1a_2\cdots a_n}\\
&= 1+\sum_{n=1}^{\infty}\frac{q^{2n^2}}{(1-q)(1-q^2)\cdots(1-q^{2n})}
\end{aligned}
$$

which combined with (19) and (23) proves identity (12).

To prove (11), we define

$$U_n = T_n - q\,T_{n-1}\,.$$

So $U_1 = 1$, $U_0 = 0$, and by (22)

$$U_n - U_{n-1} = q^{2n-2}\,T_{n-2}\,.$$

Now by (20)

$$\lim_{n\to\infty} U_n = (1-q) \prod_{\substack{n=1 \\ n\equiv\pm1,\pm4,\pm6,\pm7 \pmod{16}}}^{\infty} \frac{1}{1-q^n}\,. \tag{24}$$

We now define

$$\begin{aligned} A_n &= \sum_{j=1}^{\infty} q^{2nj+1} T_{j-1} \text{ for } n>0 \\ A_0 &= \lim_{n\to\infty} U_n, \\ a_n &= q^{-4n-1}(1-q^{2n})(1-q^{2n+1}) \quad \text{ for } n>0, \\ a_0 &= 1\,, \end{aligned} \tag{25}$$

and again we apply the $q$-Engel Expansion with extra-exponent $\rho = 1$.

Initially then

$$\begin{aligned} q(a_0A_0-1) &= q\left(\lim_{n\to\infty} U_n - 1\right) = q \lim_{n\to\infty} \sum_{j=2}^{n} (U_j - U_{j-1}) \\ &= \sum_{j=2}^{\infty} q^{2j-1} T_{j-2} = \sum_{j=1}^{\infty} q^{2j+1} T_{j-1} = A_1\,. \end{aligned}$$

Next

$$\begin{aligned} &q(a_nA_n - 1) \\ =&q\left((1-q^{-2n}-q^{-2n-1}+q^{-4n-1})\sum_{j=1}^{\infty} q^{2nj+1}T_{j-1} - 1\right) \\ =&\sum_{j=1}^{\infty} q^{2nj+2}T_{j-1} - \sum_{j=0}^{\infty} q^{2nj+2}T_j - \sum_{j=0}^{\infty} q^{2nj+1}T_j \\ &\qquad\qquad + \sum_{j=-1}^{\infty} q^{2nj+1}T_{j+1} - q \\ =&\sum_{j=1}^{\infty} q^{2nj+1}(T_{j+1} - T_j - q\,T_j - q\,T_{j-1}) \\ =&\sum_{j=1}^{\infty} q^{2nj+1} q^{2j} T_{j-1} \\ =&A_{n+1}\,. \end{aligned}$$

Finally

$$\left[\frac{1}{A_n}\right] = \left[\frac{1}{q^{4n+1}+q^{6n+1}(1+q)+q^{8n+1}(1+q+q^2+q^4)+\cdots}\right]$$

$$= \left[\frac{1}{q^{4n+1}(1+q^{2n}+q^{2n+1}+q^{4n}+q^{4n+1}+O(q^{4n+2}))}\right]$$

$$= \big[q^{-4n-1}(1-(q^{2n}+q^{2n+1}+q^{4n}+q^{4n+1})$$

$$+q^{4n}+2q^{4n+1}+O(q^{4n+2}))\big]$$

$$= q^{-4n-1}(1-q^{2n}-q^{2n+1}+q^{4n+1})$$
$$= a_n.$$

Hence from Theorem 1 we may conclude that

$$\begin{aligned} A_0 &= 1+\sum_{n=1}^{\infty}\frac{q^{-n}}{a_1a_2\cdots a_n} \\ &= 1+\sum_{n=1}^{\infty}\frac{q^{2n^2+2n}}{(1-q^2)(1-q^3)\cdots(1-q^{2n+1})}\ , \end{aligned}$$

which combined with (24) and (25) proves identity (11).

# 5 A New Infinite Family of Rogers-Ramanujan Type

In this section we want to demonstrate the potential of the `Engel` package with respect to the discovery of new identities. We remark that initially our search that led to the Theorems 4 and 5 below was inspired by the Garrett-Ismail-Stanton result [10, (3.5)]; see also [9]. Nevertheless, we also note that the way they derived and proved their result is completely different to the $q$-Engel approach.

For the sake of simplicity let us abbreviate the product sides of (19) and (20) by $S_\infty$ and $T_\infty$, respectively. Our goal is to combine both products in such a way that gives a generalization of (11) or (12). In view of the Garrett-Ismail-Stanton result [10, (3.5)] one can expect that some variation of an ansatz like

$$S_n\cdot T_\infty - T_n\cdot S_\infty, \tag{26}$$

$S_n$ and $T_n$ being the Santos polynomials we needed in Section 4.2, could possibly lead to a result in this direction.

Before calling the `Engel` procedure, we need to define the polynomials $S_n$ and $T_n$, which is done recursively by

```
In[42] := S[0] = 1;
          S[1] = 1;
          S[n_] := S[n] = Simplify[(1 + q) S[n − 1] − q(1 − q^(2n − 3)) S[n − 2]];
In[43] := T[0] = 0;
          T[1] = 1;
          T[n_] := T[n] = Simplify[(1 + q) T[n − 1] − q(1 − q^(2n − 3)) T[n − 2]];
```

and the infinite products $S_\infty$ and $T_\infty$:

$In[44] :=$ **SInfinity = 1/(qfac[q^2, q^16] * qfac[q^3, q^16]***
**qfac[q^4, q^16] * qfac[q^5, q^16] * qfac[q^11, q^16]***
**qfac[q^12, q^16] * qfac[q^13, q^16] * qfac[q^14, q^16]);**

$In[45] :=$ **TInfinity =**
**1/(qfac[q, q^16] * qfac[q^4, q^16] * qfac[q^6, q^16]***
**qfac[q^7, q^16] * qfac[q^9, q^16] * qfac[q^10, q^16]***
**qfac[q^12, q^16] * qfac[q^15, q^16]);**

First we try the ansatz (26) for $n = 1$:

$In[46] :=$ **Mixed1 = (S[1] TInfinity − T[1] SInfinity);**
**A = Series[Mixed1, {q, 0, 45}];**
**Engel[A, ExtraExponent->1]**

$$Out[46] = q + \frac{q^7}{(1-q^2)\,(1-q^3)} + \frac{q^{17}}{(1-q^2)\,(1-q^3)\,(1-q^4)\,(1-q^5)} + O[q]^{31}$$

This looks already promising; nevertheless, the output suggests to normalize via division by $q$:

$In[47] :=$ **Mixed1 = (S[1] TInfinity − T[1] SInfinity)/q;**
**A = Series[Mixed1, {q, 0, 45}];**
**Engel[A, ExtraExponent− > 1]**

$$Out[47] = 1 + \frac{q^6}{(1-q^2)\,(1-q^3)} + \frac{q^{16}}{(1-q^2)\,(1-q^3)\,(1-q^4)\,(1-q^5)} + \frac{q^{30}}{(1-q^2)\,(1-q^3)\,(1-q^4)\,(1-q^5)\,(1-q^6)\,(1-q^7)} + O[q]^{46}$$

The denominator pattern is obvious, the pattern for the numerator exponents e.g. can be found by interpolation:

$In[48] :=$ **Expand[InterpolatingPolynomial[{{1, 6}, {2, 16}, {3, 30}}, k]]**

$Out[48] = 4\ k + 2\ k^2$

The last line confirms that (after division by $1-q$) we have indeed found an expansion of the form $\sum_{k\geq 0} q^{2k^2+4k}/(q;q)_{2k+1}$ instead of $\sum_{k\geq 0} q^{2k^2+2k}/(q;q)_{2k+1}$, the latter being Slater's (11) which is the special case $n = 0$ of (26).

Let us proceed with $n = 2$ where we now divide the ansatz (26) by $q^2$ instead of $q$:

$In[49] :=$ **Mixed2 = (S[2] TInfinity − T[2] SInfinity)/q^2;**
**A = Series[Mixed2, {q, 0, 45}];**
**Engel[A, ExtraExponent->1]**

$$Out[49] = 1 - q^3 + \frac{q^8}{1-q^2} + \frac{q^{20}}{(1-q^2)\,(1-q^4)\,(1-q^5)} + O[q]^{36}$$

This suggests that we should divide by $(1-q^3)$:

$In[50] :=$ **Mixed2 = (S[2] TInfinity − T[2] SInfinity)/(q^2(1 − q^3));**
**A = Series[Mixed2, {q, 0, 55}];**
**Engel[A, ExtraExponent− > 1]**

$$Out[50] = 1 + \frac{q^8}{(1-q^2)\,(1-q^3)} + \frac{q^{20}}{(1-q^2)\,(1-q^3)\,(1-q^4)\,(1-q^5)} + \frac{q^{36}}{(1-q^2)\,(1-q^3)\,(1-q^4)\,(1-q^5)\,(1-q^6)\,(1-q^7)} + O[q]^{56}$$

$In[51] :=$ **Expand[InterpolatingPolynomial[{{1, 8}, {2, 20}, {3, 36}}, k]]**

$Out[51] = 6\ k + 2\ k^2$

The last line confirms that this way we have found a variation of the ansatz (26) that results, after division by $(1-q)$, in a representation of the form $\sum_{k\geq 0} q^{2k^2+6k}/(q;q)_{2k+1}$.

Finally let us check the case $n=3$. So far the pattern suggests to divide in this case by $q^3$ and by $(1-q^5)$, instead of $q^2$ and $(1-q^3)$ in the case $n=2$:

```
In[52] := Mixed3 = (S[3] TInfinity − T[3] SInfinity)/(q^3 (1 − q^5));
          A = Series[Mixed3, {q, 0, 45}];
          Engel[A, ExtraExponent->1]
```

$$Out[52] = 1 - q^3 + \frac{q^{10}}{1-q^2} + \frac{q^{24}}{(1-q^2)\ (1-q^4)\ (1-q^5)} + O[q]^{42}$$

This output again suggests a division, namely by the factor $(1-q^3)$, and we end up with

```
In[53] := Mixed3 = (S[3] TInfinity − T[3] SInfinity)/(q^3 (1 − q^3)(1 − q^5));
          A = Series[Mixed3, {q, 0, 65}];
          Engel[A, ExtraExponent->1]
```

$$Out[53] = 1 + \frac{q^{10}}{(1-q^2)\ (1-q^3)} + \frac{q^{24}}{(1-q^2)\ (1-q^3)\ (1-q^4)\ (1-q^5)} + \frac{q^{42}}{(1-q^2)\ (1-q^3)\ (1-q^4)\ (1-q^5)\ (1-q^6)\ (1-q^7)} + O[q]^{64}$$

$In[54] :=$ **Expand[InterpolatingPolynomial[{{1, 10}, {2, 24}, {3, 42}}, k]]**

$Out[54] = 8\ k + 2\ k^2$

This confirms that we succeeded again. Moreover, now the pattern is obvious and points to the general case which we can be stated as follows.

**Theorem 4.** *Let $S_n$ and $T_n$ be the polynomials defined in* (17) *and* (18), *and recall that*

$$S_\infty = \prod_{\substack{n=1 \\ n\equiv \pm 2,\pm 3,\pm 4,\pm 5 \pmod{16}}}^{\infty} \frac{1}{1-q^n} \quad \text{and} \quad T_\infty = \prod_{\substack{n=1 \\ n\equiv \pm 1,\pm 4,\pm 6,\pm 7 \pmod{16}}}^{\infty} \frac{1}{1-q^n}. \tag{27}$$

*Then for all nonnegative integers $n$, we have*

$$S_n \cdot T_\infty - T_n \cdot S_\infty = q^n (q;q^2)_n \sum_{k=0}^{\infty} \frac{q^{2k^2+2(n+1)k}}{(q;q)_{2k+1}}. \tag{28}$$

We remark that M. Ismail, H. Prodinger, and D. Stanton [11] have found a different generalization of (11) and (12).

Theorem 4 now can be proved by applying Theorem 1; i.e., by verifying the corresponding $q$-Engel relations. But such a proof would follow essentially the same steps as spelled out in the proofs of Slater's (11) and (12) presented in Section 4.2. Therefore we find it more instructive to give an alternative verification, namely by proving the following finite, polynomial version of it.

**Theorem 5.** *Let $S_n$ and $T_n$ be the polynomials defined in* (17) *and* (18). *Then for nonnegative integers $n$ and $N$ we have*

$$S_n \cdot T_{n+N} - T_n \cdot S_{n+N} = q^n (q;q^2)_n \sum_{k\geq 0} \begin{bmatrix} N \\ 2k+1 \end{bmatrix} q^{2k^2+2(n+1)k}. \tag{29}$$

We observe that sending $N$ to infinity in Theorem 5 implies Theorem 4 immediately.

*Proof of Theorem 5.* For $n+N \geq 2$ each of the sequences $(S_{n+N})$, $(T_{n+N})$, and

$$V_{n+N} := \sum_{k \geq 0} \begin{bmatrix} N \\ 2k+1 \end{bmatrix} q^{2k^2+2(n+1)k}$$

satisfies the recurrence

$$Y_{n+N} = (1+q)Y_{n+N-1} - q(1-q^{2n+2N-3})Y_{n+N-2}. \tag{30}$$

That fact that $(S_{n+N})$ and $(T_{n+N})$ are solutions of (30) has already been stated in (21) and (22) above; the fact that also $V_{n+N}$ is a solution of (30), for instance, can be derived automatically by using the package `qZeil` described in [14]. Now let us consider the expressions $S_{n+N}$, $T_{n+N}$, and $V_{n+N}$ as sequences in $N$ with free nonnegative integer parameter $n$. They all satisfy the recurrence (30), now interpreted as a recurrence solely in $N$ of order 2. Consequently, both sides of (29) satisfy the same recursion in $N$ of order 2. Thus the proof of Theorem 5 is completed, once we have shown that (29) is true for the initial values $N=0$ and $N=1$.

The case $N=0$ is trivial. The case $N=1$ is equivalent to showing

$$S_n \cdot T_{n+1} - T_n \cdot S_{n+1} = q^n (q;q^2)_n \tag{31}$$

for all nonnegative integers $n$. To this end we represent the left hand side of (31) as a determinant and reduce it according to (21) and (22),

$$\begin{vmatrix} S_n & S_{n+1} \\ T_n & T_{n+1} \end{vmatrix} = \begin{vmatrix} S_n & (1+q)S_n - q(1-q^{2n-1})S_{n-1} \\ T_n & (1+q)T_n - q(1-q^{2n-1})T_{n-1} \end{vmatrix} = q(1-q^{2n-1}) \begin{vmatrix} S_{n-1} & S_n \\ T_{n-1} & T_n \end{vmatrix}.$$

Consequently induction with respect to $n$ completes the proof of (29). Hence Theorem 5 is proved. □

# 6 Some Open Problems

The study of $q$-Engel Expansion in connection with $q$-series is only at the very beginning, so we are still faced with a variety of open questions. We want to conclude by stating a few of these explicitly.

(1) How many of the entries of Slater's list can be treated by $q$-EE? In particular, can this set be extended significantly if one uses the $q$-EE machinery from [13] in full generality, i.e., with properly chosen sequences $(r_n)$ and $(s_n)$. — We note that so far all $q$-series applications were carried out with the setting $s_n = a_n$ and $r_n = q^{-\rho}$ for all $n$.

(2) Find a $q$-EE proof of

$$\sum_{k=0}^{\infty} \frac{q^{k(2k+1)} t^k}{(q^2;q^2)_k} = (tq;q)_\infty \sum_{k=0}^{\infty} \frac{q^{\binom{k+1}{2}} t^k}{(q;q)_k (tq;q)_k}, \tag{32}$$

which is an identity of Rogers; see e.g. [4]. This seems to fit perfectly into the $q$-EE context, an observation suggested by applying the `Engel` package on $A$ defined as the right hand side of (32) and $\rho = 0$. However, so far we have not succeeded to find a $q$-EE proof for this.

(3) It seems that finite versions of $q$-series identities, as for instance the $q$-binomial theorem (e.g., [3])

$$\sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix} (-1)^k z^k q^{\binom{k}{2}} = (z;q)_N,$$

do not fit into $q$-EE. Is it possible to extend the $q$-EE machinery in this direction?

(4) Section 5 illustrates the potential of `Engel` for discovering new identities. Besides [10, (3.5)] which has been derived in a completely different manner, using `Engel` we were able to find another parameterized family of Rogers-Ramanujan type. This suggest to use the package in a more systematic search for further families of similar type.

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## References

[1] G.E. Andrews, *On basic hypergeometric series, mock theta functions, and partitions II*, Quaterly J. Math. Oxford (2) **17** (1966), 132–143.

[2] G.E. Andrews, *Enumerative proofs of certain q-identities*, Glasgow Math. J. **8** (1967), 33–40.

[3] G.E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and Its Applications, Vol. 2, G.-C. Rota ed., Addison-Wesley, Reading, 1976. (Reissued: Cambridge University Press, Cambridge, 1985.)

[4] G.E. Andrews and R. Askey, *Enumeration of partitions: the role of Eulerian series and q-orthogonal polynomials*, Higher Combinatorics (M. Aigner, ed.), Reidel Publ. Co., Dordrecht, Holland, 3–26.

[5] G.E. Andrews, *Ramanujan's "lost" notebook V: Euler's partition identity*, Advances in Math. **61** (1986), 156–164.

[6] G.E. Andrews and J.P.O. Santos, *Rogers-Ramanujan type identities for partitions with attached odd parts*, Ramanujan Journal **1** (1997), 91–99.

[7] G.E. Andrews, A. Knopfmacher, and J. Knopfmacher, *Engel expansions and the Rogers-Ramanujan identities*, J. of Number Theory **80** (2000), 273–290.

[8] G.E. Andrews and A. Knopfmacher, *An algorithmic approach to discovering and proving q-series identities*, (to appear).

[9] G.E. Andrews, A. Knopfmacher, and P. Paule, *An infinite family of Engel Expansions of Rogers-Ramanujan type*, (to appear).

[10] T. Garrett, M.E.H. Ismail, and D. Stanton, *Variants of the Rogers-Ramanujan identities*, Adv. Appl. Math. **23** (1999), 274–299.

[11] M.E.H. Ismail, H. Prodinger, and D. Stanton, *Schur's determinants and partition theorems*, Séminaire Lotharingien de Combinatoire, B44a (2000), 10 pp.

[12] A. Knopfmacher and J. Knopfmacher, *Inverse polynomial expansions of Laurent series*, Constructive Approx. **4** (1988), 379–389.

[13] A. Knopfmacher and J. Knopfmacher, *Inverse polynomial expansions of Laurent series, II*, J. Comput. and Appl. Math. **28** (1989), 249–257.

[14] P. Paule and A. Riese, *A Mathematica $q$-Analogue of Zeilberger's Algorithm Based on an Algebraically Motivated Approach to $q$-Hypergeometric Telescoping*, pp. 179–210 in: Fields Institute Communications, Vol. **14**, Amer. Math. Soc., Providence, 1997.

[15] O. Perron, *Irrationalzahlen*, reprinted: Chelsea, New York, 1951.

[16] J.P.O. Santos, *Computer Algebra and Identities of the Rogers-Ramanujan type*, Ph.D. thesis, Pennsylvania State University, 1991.

[17] L.J. Slater, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (2) **54** (1952), 147–167.