Enhanced Electrical Impedance Tomography via the Mumford-Shah Functional

Luca Rondi* Fadil Santosa†

Abstract

We consider the problem of electrical impedance tomography where conductivity distribution in a domain is to be reconstructed from boundary measurements of voltage and currents. It is well-known that this problem is highly illposed. In this work, we propose the use of the Mumford-Shah functional, developed for segmentation and denoising of images, as a regularization. After establishing existence properties of the resulting variational problem, we proceed by demonstrating the approach in several numerical examples. Our results indicate that this is an effective approach for overcoming the illposedness. Moreover, it has the capability of enhancing the reconstruction while at the same time segmenting the conductivity image.

1 Introduction and formulation of the problem

The purpose of this work is to demonstrate that the Mumford-Shah functional from image processing can be used effectively to regularize the classical problem of electrical impedance tomography. In electrical impedance tomography the objective is to determine the conductivity distribution in a domain from measurements collected at the boundary. Such a problem is often referred to as the inverse conductivity problem. The underlying physical phenomena, that of electrostatics, is modeled by an elliptic partial differential equations. The data available from measurement amounts to (limited) information about the Neumann-to-Dirichlet map.

In this work, we will consider conductivity distribution that is discontinuous. The Mumford-Shah functional, which is used in image processing as a method for segmentation and denoising, can be shown to regularize this otherwise ill-posed inverse problem. Moreover, as we demonstrate in the paper, it allows one not only to obtain a good image of the conductivity distribution, but also to determine the jump set of the conductivity.

*Institut für Industriemathematik/SFB 013, Johannes Kepler Universität Linz, Austria. Present address: School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA.

†School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA.

We begin by giving a precise formulation of the inverse conductivity problem. Consider a bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, with sufficiently smooth boundary, namely we assume $\partial\Omega$ to be Lipschitz. Let $H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega, \mathbb{R}^n)\}$, where ∇u denotes the gradient in the distribution sense. By $H^{1/2}(\partial\Omega)$ we denote the space of traces of $H^1(\Omega)$ on $\partial\Omega$. We recall that $H^{-1/2}(\partial\Omega)$ is the dual space to $H^{1/2}(\partial\Omega)$. With ${}_0H^{1/2}(\partial\Omega)$, ${}_0H^{-1/2}(\partial\Omega)$ and ${}_0L^2(\partial\Omega)$ we denote the corresponding subspaces of elements with zero means. We note that ${}_0H^{1/2}(\partial\Omega)$ and ${}_0H^{-1/2}(\partial\Omega)$ are dual to each other, whereas the dual of ${}_0L^2(\partial\Omega)$ is the space itself. We recall also that if X and Y are two Banach spaces then $B(X,Y)$ will be the space of all bounded linear operators from X to Y, with the usual operator norm. If $X = Y$ we set $B(X,Y) = B(X)$.

Let σ be the conductivity of the medium occupying the region Ω. We make the assumption that σ is a measurable function on Ω satisfying

$$0 < \lambda \leq \sigma(z) \leq \lambda^{-1} \quad \text{for a.e. } z \in \Omega \tag{1.1}$$

where λ is a positive constant less than 1. We let u represent the electrostatic potential in Ω. The potential is created by a current distribution f on the boundary; we assume $f \in {}_0H^{-1/2}(\partial\Omega)$. Then the potential u satisfies the folllowing Neumann type boundary value problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u) = 0 & \text{in } \Omega \\ \sigma\nabla u \cdot \nu = f & \text{on } \partial\Omega \\ u|_{\partial\Omega} \in {}_0H^{1/2}(\partial\Omega) & \end{cases} . \tag{1.2}$$

The boundary value problem (1.2) admits a unique weak solution. That is there exists a unique function $u \in H^1(\Omega)$ whose trace on $\partial\Omega$ has zero mean such that

$$\int_\Omega \sigma\nabla u \cdot \nabla\phi = f[\phi|_{\partial\Omega}] \quad \text{for every } \phi \in H^1(\Omega). \tag{1.2$_w$}$$

We note that if $f \in {}_0L^2(\partial\Omega)$, then $f[\phi|_{\partial\Omega}] = \int_{\partial\Omega} f\phi$ for any $\phi \in H^1(\Omega)$.

For any σ satisfying (1.1) for some λ, $0 < \lambda < 1$, the so-called Neumann-to-Dirichlet map associated to σ, $\Lambda(\sigma,\cdot) : {}_0H^{-1/2}(\partial\Omega) \mapsto {}_0H^{1/2}(\partial\Omega)$, is defined in the following way

$$\Lambda(\sigma, f) = u|_{\partial\Omega} \quad \text{for any } f \in {}_0H^{-1/2}(\partial\Omega)$$

where u is the weak solution to (1.2). We have that $\Lambda(\sigma,\cdot)$ is a bounded linear operator from ${}_0H^{-1/2}(\partial\Omega)$ in ${}_0H^{1/2}(\partial\Omega)$ whose norm depends upon Ω and λ only. In the sequel we shall often consider $\Lambda(\sigma,\cdot)$ as a bounded linear operator from ${}_0L^2(\partial\Omega)$ into itself. We remark that obviously

$$\|\Lambda(\sigma,\cdot)\|_{B({}_0L^2(\partial\Omega))} \leq C\|\Lambda(\sigma,\cdot)\|_{B({}_0H^{-1/2}(\partial\Omega),{}_0H^{1/2}(\partial\Omega))}$$

holds with a constant $C > 0$ depending on Ω only.

The inverse conductivity problem, which has been formulated for the first time by A. P. Calderón in [C], consists in determining if the conductivity σ

is uniquely determined by the associated Neumann-to-Dirichlet map. The pioneering work of R.V. Kohn and M. Vogelius [Ko-V], and later, the work of J. Sylvester and G. Uhlmann [Sy-U], provided the definitive uniqueness results on this problem. For an overview discussion on uniqueness we refer to a recent monograph by V. Isakov, [I].

In this paper we shall deal with the reconstruction issue, that is, we are looking for a procedure which allows us to recover the unknown conductivity σ from the knowledge, possibly partial, of the Neumann-to-Dirichlet map $\Lambda(\sigma,\cdot)$. We are interested in the case when the unknown conductivity σ may present some discontinuities, and our aim is, in particular, to recover the set where such discontinuities occur. In fact we shall suppose to know *a priori* that σ, besides satisfying (1.1) for a fixed constant λ, either is *piecewise* H^1, that is

$$\sigma \in PH^1(\Omega) = \left\{ \sigma \in L^\infty(\Omega) : \begin{array}{c} \sigma \in H^1(\Omega \backslash K) \\ K \text{ closed in } \Omega,\ \mathcal{H}^{n-1}(K) < \infty \end{array} \right\}$$

or is *piecewise constant*, that is

$$\sigma \in PC(\Omega) = \left\{ \sigma \in L^\infty(\Omega) : \begin{array}{c} \sigma \in H^1(\Omega \backslash K),\ \nabla\sigma = 0 \text{ a.e in } \Omega \backslash K \\ K \text{ closed in } \Omega,\ \mathcal{H}^{n-1}(K) < \infty \end{array} \right\}.$$

We recall that $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. We remark that, in an equivalent way, we say that σ belongs to $PC(\Omega)$ if it is constant on any connected component of $\Omega \backslash K$, K being a closed set in Ω with finite $(n-1)$-dimensional Hausdorff measure.

Since the inverse conductivity problem is severely illposed, a regularization procedure is needed to stabilize any reconstruction process. It is known that standard regularization techniques, such as penalizing the H_1-norm stabilizes the problem at the cost of loss of resolution. In this work, we explore the use of techniques from image processing, which stabilizes and enhances the reconstruction. An earlier work of D. Dobson and F. Santosa [D-S] employed minimal total variation (TV) criterion, which is popular in image processing, to stabilize and enhance the reconstruction for electrical impedance tomography.

In the present work, we also borrow another idea from the image processing literature, namely the Mumford-Shah functional, as a regularization procedure. D. Mumford and J. Shah [Mu-Sh1, Mu-Sh2] devised the functional as a way to denoise and segment a black-and-white image. In order to consider the Mumford-Shah functional in the context of electrical impedance tomography, let $F(\sigma)$ be the following L^2 data-fitting functional

$$F(\sigma) = \|\Lambda(\sigma,\cdot) - G\|^2_{B({}_0L^2(\partial\Omega))}. \tag{1.3}$$

The data G being a linear, bounded operator from ${}_0L^2(\partial\Omega)$ into itself representing the measurements of the electrostatic potentials corresponding to any current density belonging to ${}_0L^2(\partial\Omega)$ applied to the conductor Ω. From the point of view of applications, we never have the entire Neumann-to-Dirichlet map but rather N different measurements $g_i \in {}_0L^2(\partial\Omega)$, corresponding to current densities $f_i \in {}_0L^2(\partial\Omega)$, $i = 1, \cdots, N$, and we shall substitute the term described in (1.3) with

$$\sum_{i=1}^{N} \int_{\partial\Omega} |\Lambda(\sigma, f_i) - g_i|^2. \tag{1.4}$$

The variational problem we consider is

$$\min_{(\sigma,K)} \left\{ \alpha \int_{\Omega\backslash K} |\nabla\sigma|^2 + \beta\mathcal{H}^{n-1}(K) + \gamma F(\sigma) \right\} \tag{1.5}$$

where K is closed in Ω and $\sigma \in H^1(\Omega\backslash K)$. Here α, β and γ are positive tuning parameter. Without loss of generality we can always impose $\alpha = 1$. The first term of the functional above is a smoothing term implying that outside the discontinuity set K the image should be smooth. The second is a penalization term on the length of the discontinuity set K, and therefore prevents the creation of spurious discontinuities due to noise. The third term, the fitting term, represents the faithfulness of the reconstruction with respect to the input data.

In the context of image processing, σ would represent a grey-level image, and the third term in (1.5) would be replaced by

$$\int_{\Omega} |\sigma - g|^2$$

where g is the raw input image. Also the so-called *minimal partition* problem has been considered

$$\min_{(\sigma,K)} \left\{ \beta\mathcal{H}^{n-1}(K) + \gamma \int_{\Omega} |\sigma - g|^2 \right\} \tag{1.6}$$

where K is closed in Ω, $\sigma \in H^1(\Omega\backslash K)$ and $\nabla\sigma = 0$ almost everywhere in $\Omega\backslash K$.

The existence of a solution to the Mumford-Shah minimization problem has been obtained in the framework of free-discontinuity problem by [DG-Ca-L] introducing a relaxed functional in a space of functions of bounded variation. By the direct method in the Calculus of Variations a minimum of such a relaxed functional does exist and, by a regularity theory argument, it produces also a minimum for the original functional. Moreover they showed that if (σ, K) solves the variational problem, then σ is indeed piecewise C^1, that is $\sigma \in C^1(\Omega\backslash K)$. In a similar framework in [Co-Ta] the existence of a solution has been proved also for the minimal partition problem (1.6).

The computational problem associated with these functionals is an active area of research. A main difficulty is presented by the penalty term involving the Hausdorff measure of the set K. One approach is to approximate these functionals in the sense of Γ-convergence by functionals defined on spaces of smooth functions. We refer to the monograph of A. Braides, [Br], for a survey of such results. In this work, we employ the method of Γ-convergence approximation of the Mumford-Shah functional by elliptic, although non-convex, functionals defined on Sobolev spaces due to L. Ambrosio and V.M. Tortorelli, [A-T1, A-T2].

The plan of the paper is as follows. In Section 2 we shall study the properties of the Neumann-to-Dirichlet map with respect to the first argument, that is the conductivity. We shall prove continuity and differentiability with respect to suitable L^p norms trying to keep at a minimum the regularity assumptions of the conductivities involved, given our interest in the recovery of discontinuous functions. The minimization associated with the inverse conductivity problem is considered in Section 3. The main result pertaining to this problem is stated in Theorem 3.7. In addition to the problem stated in (1.5), we also consider the linearized inverse problem, where the Neumann-to-Dirichlet map is linearized about a given conductivity σ_0. Our result for this problem is given in Theorem 3.8. We also treat a more general linear inverse problem where the fitting term in (1.5) is replaced by

$$F(\sigma) = \int_\Omega |A[\sigma] - g|^2 \tag{1.7}$$

where A is a bounded linear operator from $L^p(\Omega)$ into itself for any p, $1 \leq p \leq \infty$. We shall focus our attention upon the case in which A satisfies some compactness properties, for instance when A is a blurring convolution. Existence results for this case are stated in Theorem 3.9. Section 3 ends with a review of the L. Ambrosio and V.M. Tortorelli approximation of the Mumford-Shah functional where we shall note that it may be easily applied to our problems. In Section 4 some numerical examples will be presented in order to show the potential of the method. We shall limit ourselves to the linearized inverse conductivity problem and, concerning the other inverse problems, to the case when the operator A is a convolution. A short discussion section ends the paper.

2 Regularity of the Neumann-to-Dirichlet map

Some regularity results on the Neumann-to-Dirichlet map as a function of the conductivity σ will be needed in the sequel to establish existence results for the minimization problems stated above. We follow some of the techniques developed in [D] where it is shown that higher integrability properties of the solutions to problem (1.2) are decisive for proving differentiability properties of the Neumann-to-Dirichlet map. In order to lower as much as possible the regularity of the conductivities involved, we shall make use of the following regularity results for elliptic equations in divergence form.

We look for conditions upon which weak solutions to elliptic equations in divergence form in a domain Ω belong to $H^{1,q}_{loc}(\Omega)$ with $q > 2$. Let us consider the following definition.

Definition 2.1 Let Ω be a bounded domain contained in $\mathbb{R}^n$, $n \geq 2$, and let $\sigma \in L^\infty(\Omega)$ satisfy (1.1) for a fixed λ, $0 < \lambda < 1$. We say that σ satisfies the *Q-property*, $2 < Q \leq \infty$, if for any $2 < q < Q$ the following holds.

If $f \in L^q(\Omega, \mathbb{R}^n)$, $h \in L^q(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution to

$$\mathrm{div}(\sigma \nabla u) = \mathrm{div}(f) + h \quad \text{in } \Omega$$

then $u \in H^{1,q}_{loc}(\Omega)$ and for any $\Omega_1 \subset\subset \Omega$ the following estimate holds

$$\|u\|_{H^{1,q}(\Omega_1)} \leq C \left(\|u\|_{H^1(\Omega)} + \|f\|_{L^q(\Omega)} + \|h\|_{L^q(\Omega)} \right) \tag{2.1}$$

where the constant C depends on λ, n, q, Ω_1, Ω and σ.

The following result, due to N.G. Meyers, [M], states that any σ satisfying (1.1) with a constant λ, $0 < \lambda < 1$, verifies the Q-property for a constant $Q > 2$ depending on λ and n only.

Theorem 2.2 (Meyers) *Let Ω be a bounded Lipschitz domain contained in $\mathbb{R}^n$, $n \geq 2$. Fixed λ, $0 < \lambda < 1$, there exists a constant Q, $2 < Q < \infty$, depending on λ and on n only, $Q \to 2$ as $\lambda \to 0$ and $Q \to \infty$ as $\lambda \to 1$, such that any $\sigma \in L^\infty(\Omega)$ satisfying* (1.1) *with constant λ verifies the Q-property.*

Moreover the constant C in (2.1) *depends on λ, n, q, Ω_1 and Ω only.*

We note that in the previous theorem no regularity has been assumed on σ. Omitting the dependence upon n, we shall denote by $Q(\lambda)$ the number Q defined in Meyers's Theorem. For any σ satisfying (1.1), we shall denote by $Q(\sigma)$ the supremum of all the numbers Q so that σ verifies the Q-property. The previous theorem ensures that $2 < Q(\lambda) \leq Q(\sigma) \leq \infty$. Some regularity properties of σ may imply that $Q(\sigma)$ is strictly greater than $Q(\lambda)$. In this case the constant in (2.1) will depend upon σ not only through the ellipticity constant λ but also from these regularity properties. Let us recall the following result, [Tr, Theorem 3.7], stating that if σ is Hölder continuous then $Q(\sigma) = \infty$.

Theorem 2.3 *Let Ω be a bounded Lipschitz domain contained in $\mathbb{R}^n$, $n \geq 2$, and let $\sigma \in L^\infty(\Omega)$ satisfy* (1.1) *for a given positive constant λ, $0 < \lambda < 1$. Moreover let us assume that $\sigma \in C^{0,\delta}(\overline{\Omega})$, $0 < \delta < 1$.*

Then σ satisfy the ∞-property and the constant C in (2.1) *depends on λ, δ, $\|\sigma\|_{C^{0,\delta}(\overline{\Omega})}$, n, q, Ω_1 and Ω.*

It might be interesting to find a characterization of $Q(\sigma)$ for σ belonging to the class of functions which are used in this papers, namely piecewise H^1, piecewise constant or SBV functions (which we shall introduce later on in Section 3). In fact these functions have a richer structure than L^∞ ones, although they might be discontinuous, thus preventing the application of Theorem 2.3. With a somewhat different motivation, some studies in this directions have been developed recently in [Bo-V, Li-V].

We recall that, given a bounded and Lipschitz domain Ω, for any σ satisfying (1.1) in Ω for a given positive constant λ we have defined $\Lambda(\sigma, \cdot) : {}_0H^{-1/2}(\partial\Omega) \mapsto {}_0H^{1/2}(\partial\Omega)$ as a bounded linear operator given by

$$\Lambda(\sigma, f) = u|_{\partial\Omega}$$

where u is the solution to (1.2) and f is any element of ${}_0H^{-1/2}(\partial\Omega)$. First of all we notice that there exists a constant C depending on λ and Ω only such that if u solves (1.2) then

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{{}_0H^{-1/2}(\partial\Omega)}. \tag{2.2}$$

Therefore, as we have already noticed in Section 1, the norm of $\Lambda(\sigma,\cdot)$ either in $B({}_0H^{-1/2}(\partial\Omega),{}_0H^{1/2}(\partial\Omega))$ or in $B({}_0L^2(\partial\Omega))$ depends on λ and Ω only.

Let us consider the regularity properties of Λ with respect to σ on the following set of admissible conductivities. We shall fix a constant $\lambda > 0$ and set

$$\mathcal{D} = \{\sigma \in L^\infty(\Omega) : \sigma \text{ verifies (1.1) and } \operatorname{supp}(\sigma - \tau_0) \subset \Omega'\} \tag{2.3}$$

where Ω' is a smooth subset compactly contained in Ω and τ_0 is a given conductivity satisfying (1.1) as well. We shall endow $\mathcal{D}$ with an L^p-norm, $1 \leq p \leq \infty$, usually with $p = 2$.

Let us call $\mathcal{F} : \mathcal{D} \mapsto B({}_0L^2(\partial\Omega))$ the function so defined

$$\mathcal{F}(\sigma) = \Lambda(\sigma,\cdot) \quad \text{for every } \sigma \in \mathcal{D}.$$

First of all we show the uniform continuity of $\mathcal{F}$ in $\mathcal{D}$ with respect to the L^p-norm, for any $1 \leq p \leq \infty$.

In fact let σ_0 and σ_1 be two conductivities belonging to $\mathcal{D}$. Let us fix $f \in {}_0L^2(\partial\Omega)$ and let u_0 and u_1 be the weak solutions to (1.2) where σ is replaced by σ_0 and σ_1 respectively. By the weak formulation of (1.2) we infer that for any $\phi \in H^1(\Omega)$

$$\int_\Omega \sigma_0 \nabla u_0 \cdot \nabla\phi = \int_\Omega \sigma_1 \nabla u_1 \cdot \nabla\phi$$

and so we have that, still for any $\phi \in H^1(\Omega)$,

$$\int_\Omega \sigma_0 \nabla(u_0 - u_1) \cdot \nabla\phi = \int_\Omega (\sigma_1 - \sigma_0)\nabla u_1 \cdot \nabla\phi.$$

If $w \in H^1(\Omega)$ is the weak solution to

$$\begin{cases} \operatorname{div}(\sigma_0 \nabla w) = 0 & \text{in } \Omega \\ \sigma_0 \nabla w \cdot \nu = (u_0 - u_1)|_{\partial\Omega} & \text{on } \partial\Omega \\ w|_{\partial\Omega} \in {}_0H^{1/2}(\partial\Omega) & \end{cases} \tag{2.4}$$

we have that

$$\int_{\partial\Omega} |u_0 - u_1|^2 = \int_\Omega \sigma_0 \nabla w \cdot \nabla(u_0 - u_1) = \int_\Omega (\sigma_1 - \sigma_0)\nabla u_1 \cdot \nabla w.$$

We take q so that $2 < q < Q(\lambda)$ and $p = q/(q-2)$. Then by Hölder's inequality and by the definition of $\mathcal{D}$

$$\|u_0 - u_1\|^2_{L^2(\partial\Omega)} \leq \|\sigma_1 - \sigma_0\|_{L^p(\Omega')}\|\nabla u_1\|_{L^q(\Omega')}\|\nabla w\|_{L^q(\Omega')}.$$

By Meyers's Theorem and (2.2) there exist constants C and C_1 depending on λ, n, q, Ω' and Ω only such that

$$\|\nabla u_1\|_{L^q(\Omega')} \leq C\|u_1\|_{H^1(\Omega)} \leq C_1\|f\|_{L^2(\partial\Omega)}.$$

By the same reasoning we obtain that

$$\|\nabla w\|_{L^q(\Omega')} \leq C_1\|u_0 - u_1\|_{L^2(\partial\Omega)}.$$

By collecting these last equations we obtain for any p, $Q(\lambda)/(Q(\lambda)-2) < p \leq \infty$,

$$\|\mathcal{F}(\sigma_0) - \mathcal{F}(\sigma_1)\|_{B({}_0L^2(\partial\Omega))} \leq C\|\sigma_1 - \sigma_0\|_{L^p(\Omega')}$$

where C depends on λ, n, p, Ω' and Ω only. In other words $\mathcal{F}$ is Lipschitz continuous in $\mathcal{D}$ with respect to the L^p-norm, $Q(\lambda)/(Q(\lambda)-2) < p \leq \infty$, with a Lipschitz constant depending on λ, n, p, Ω' and Ω only. Since we have $\|\sigma_1 - \sigma_0\|_{L^\infty(\Omega)} \leq \lambda^{-1}$, we deduce by interpolation that for any p, $1 \leq p \leq Q(\lambda)/(Q(\lambda)-2)$, $\mathcal{F}$ is Hölder continuous in $\mathcal{D}$ with respect to the L^p-norm with constants depending on λ, n, p, Ω' and Ω only.

Let us proceed now to the differentiability properties of $\mathcal{F}$. We fix $\sigma \in \mathcal{D}$. For any $f \in {}_0L^2(\partial\Omega)$ we call u the solution to (1.2). Let $\delta\sigma$ be a perturbation to σ belonging to $L^\infty(\Omega')$ and extended to zero outside Ω'. We define $\delta u \in H^1(\Omega)$ as the weak solution to the following linearized problem

$$\begin{cases} \operatorname{div}(\sigma\nabla\delta u) = -\operatorname{div}(\delta\sigma\nabla u) & \text{in } \Omega \\ \sigma\nabla\delta u \cdot \nu = 0 & \text{on } \partial\Omega \\ \delta u|_{\partial\Omega} \in {}_0H^{1/2}(\partial\Omega) \end{cases} \tag{2.5}$$

and we shall call $D\mathcal{F}(\sigma) : L^\infty(\Omega') \mapsto B({}_0L^2(\partial\Omega))$ the map so defined

$$D\mathcal{F}(\sigma)[\delta\sigma][f] = \delta u|_{\partial\Omega}$$

for any $\delta\sigma \in L^\infty(\Omega')$ where $f \in {}_0L^2(\partial\Omega)$ and δu solves (2.5). It is immediate to show that, for each $\delta\sigma \in L^\infty(\Omega')$, $D\mathcal{F}(\sigma)[\delta\sigma]$ is a well defined bounded linear operator from ${}_0L^2(\partial\Omega)$ into itself and that the following estimate holds

$$\|D\mathcal{F}(\sigma)[\delta\sigma]\|_{B({}_0L^2(\partial\Omega))} \leq C\|\delta\sigma\|_{L^\infty(\Omega')}$$

where the constant C depends on λ and Ω only. Since it is clear from the definition that $D\mathcal{F}(\sigma)$ is linear with respect to $\delta\sigma$ we immediately infer that $D\mathcal{F}(\sigma)$ is a bounded linear operator from $L^\infty(\Omega')$ in $B({}_0L^2(\partial\Omega))$ with norm depending on λ and Ω only.

We claim that for any p, $Q(\sigma)/(Q(\sigma)-2) < p < \infty$, we may extend $D\mathcal{F}(\sigma)$ to a bounded linear operator from $L^p(\Omega')$ in $B({}_0L^2(\partial\Omega))$. We shall still denote this operator by $D\mathcal{F}(\sigma)$.

We fix $\delta\sigma \in L^\infty(\Omega')$ and $f \in {}_0L^2(\partial\Omega)$. Let $\delta u = D\mathcal{F}(\sigma)[\delta\sigma][f]$. As before we introduce $w \in H^1(\Omega)$ as the weak solution to

$$\begin{cases} \operatorname{div}(\sigma\nabla w) = 0 & \text{in } \Omega \\ \sigma\nabla w \cdot \nu = \delta u|_{\partial\Omega} & \text{on } \partial\Omega \\ w|_{\partial\Omega} \in {}_0H^{1/2}(\partial\Omega) \end{cases} \tag{2.6}$$

and we observe that

$$\int_{\partial\Omega} |\delta u|^2 = \int_\Omega \sigma \nabla w \cdot \nabla \delta u = -\int_\Omega \delta\sigma \nabla u \cdot \nabla w.$$

Take q so that $2 < q < Q(\sigma)$ and $p = q/(q-2)$. Then by Hölder's inequality

$$\|\delta u\|^2_{L^2(\partial\Omega)} \leq \|\delta\sigma\|_{L^p(\Omega')} \|\nabla u\|_{L^q(\Omega')} \|\nabla w\|_{L^q(\Omega')}.$$

Then since σ satisfies the $Q(\sigma)$-property we have a constant C depending on λ, n, p, Ω', Ω and σ such that

$$\|D\mathcal{F}(\sigma)[\delta\sigma][f]\|_{L^2(\partial\Omega)} \leq C \|\delta\sigma\|_{L^p(\Omega')} \|f\|_{L^2(\partial\Omega)}$$

for any $\delta\sigma \in L^\infty(\Omega')$ and $f \in {}_0L^2(\partial\Omega)$. So we may consider $D\mathcal{F}(\sigma)$ as a bounded linear operator from $L^p(\Omega')$, $Q(\sigma)/(Q(\sigma)-2) < p \leq \infty$, in $B({}_0L^2(\partial\Omega))$ whose norm depends on λ, n, p, Ω', Ω and σ.

The linear operator $D\mathcal{F}(\sigma)$ represents the differential of $\mathcal{F}$ at the point σ in the sense of the following definition. Let a and b be two constants, $-\infty < a < b < +\infty$, and an open set Ω we denote by $X_a^b(\Omega)$ the set of measurable functions σ on Ω such that $a \leq \sigma \leq b$ almost everywhere in Ω. We say that a functional $F : X_a^b(\Omega) \mapsto X$, X being a Banach space, is *differentiable in* $X_a^b(\Omega)$ *at the point* $\sigma_0 \in X_a^b(\Omega)$ *with respect to the* L^p*-norm*, $1 \leq p \leq \infty$, if there exists a linear and bounded operator $DF(\sigma_0) : L^p(\Omega) \mapsto X$ so that

$$\frac{\|F(\sigma) - F(\sigma_0) - DF(\sigma_0)[\sigma - \sigma_0]\|_X}{\|\sigma - \sigma_0\|_{L^p(\Omega)}} \to 0$$

as $\sigma \in X_a^b(\Omega)$ converges to σ_0 in $L^p(\Omega)$.

Let σ and $\sigma + \delta\sigma$ belong to $\mathcal{D}$. If $f \in {}_0L^2(\partial\Omega)$ and $\delta u = D\mathcal{F}(\sigma)[\delta\sigma][f]$ we denote by R the weak solution to

$$\begin{cases} \operatorname{div}((\sigma + \delta\sigma)\nabla R) = -\operatorname{div}(\delta\sigma \nabla \delta u) & \text{in } \Omega \\ (\sigma + \delta\sigma)\nabla R \cdot \nu = 0 & \text{on } \partial\Omega \\ R|_{\partial\Omega} \in {}_0H^{1/2}(\partial\Omega) & \end{cases}$$

then, by taking linear combinations, we notice that

$$R|_{\partial\Omega} = \mathcal{F}(\sigma + \delta\sigma)[f] - \mathcal{F}(\sigma)[f] - D\mathcal{F}(\sigma)[\delta\sigma][f].$$

In order to evaluate $\|R\|_{L^2(\partial\Omega)}$ we introduce, as usual, the auxiliary function w, weak solution to

$$\begin{cases} \operatorname{div}((\sigma + \delta\sigma)\nabla w) = 0 & \text{in } \Omega \\ (\sigma + \delta\sigma)\nabla w \cdot \nu = R|_{\partial\Omega} & \text{on } \partial\Omega \\ w|_{\partial\Omega} \in {}_0H^{1/2}(\partial\Omega) & \end{cases}$$

and we infer that

$$\int_{\partial\Omega} |R|^2 = \int_\Omega (\sigma + \delta\sigma)\nabla w \cdot \nabla R = -\int_\Omega \delta\sigma \nabla \delta u \cdot \nabla w.$$

We take p, q and r so that $2 < r < Q(\lambda)$, $2 < q < Q(\sigma)$ and $p^{-1}+q^{-1}+r^{-1} = 1$. Again by Hölder's inequality

$$\|R\|^2_{L^2(\partial\Omega)} \leq \|\delta\sigma\|_{L^p(\Omega')}\|\nabla\delta u\|_{L^q(\Omega')}\|\nabla w\|_{L^r(\Omega')}.$$

First of all, by Meyers's Theorem, we notice that there exists a constant C depending on λ, n, r, Ω' and Ω such that

$$\|\nabla w\|_{L^r(\Omega')} \leq C\|R\|_{L^2(\partial\Omega)}.$$

By the $Q(\sigma)$-property of σ and since $\delta\sigma$ is zero outside Ω', we obtain, for a constant C_1 depending on λ, n, q, Ω', Ω and σ,

$$\|\nabla\delta u\|_{L^q(\Omega')} \leq C_1\|\delta\sigma\nabla u\|_{L^q(\Omega')}.$$

We choose $\varepsilon > 0$ such that $q+\varepsilon < Q(\sigma)$ and we have, again by the Q-property,

$$\|\delta\sigma\nabla u\|_{L^q(\Omega')} \leq C_2\|\delta\sigma\|_{L^{q(q+\varepsilon)/\varepsilon}(\Omega')}\|f\|_{L^2(\partial\Omega)}$$

where C_2 depends on λ, n, q, ε, Ω', Ω and σ.

Collecting all these estimates and by recalling that $\sigma+\delta\sigma \in \mathcal{D}$ and hence $\|\delta\sigma\|_{L^\infty(\Omega')} \leq \lambda^{-1}$, we deduce that for every p such that $p > \frac{Q(\lambda)Q(\sigma)}{Q(\lambda)Q(\sigma)-Q(\sigma)-Q(\lambda)}$ we may find s, $0 < s \leq 1$, depending on p, $Q(\sigma)$ and $Q(\lambda)$ only and a constant C depending on λ, n, Ω', Ω and σ only such that

$$\|\mathcal{F}(\sigma+\delta\sigma)-\mathcal{F}(\sigma)-D\mathcal{F}(\sigma)[\delta\sigma]\|_{B({}_0L^2(\partial\Omega))} \leq C\|\delta\sigma\|^{1+s}_{L^p(\Omega')} \tag{2.7}$$

for any σ and $\sigma+\delta\sigma$ belonging to $\mathcal{D}$ and this clearly implies that $\mathcal{F}$ is differentiable in $X_\lambda(\Omega) = X_\lambda^{\lambda^{-1}}(\Omega)$ at the point $\sigma \in X_\lambda(\Omega)$ with respect to the L^p-norm for any $p > \frac{Q(\lambda)Q(\sigma)}{Q(\lambda)Q(\sigma)-Q(\sigma)-Q(\lambda)}$. We remark also that, since $Q(\lambda) \leq Q(\sigma)$ we have $\frac{Q(\lambda)Q(\sigma)}{Q(\lambda)Q(\sigma)-Q(\sigma)-Q(\lambda)} \geq Q(\sigma)/(Q(\sigma)-2)$.

3 Existence results and Γ-convergence approximation

In this section we shall prove existence results for the solution of the minimization problems we have outlined in Section 1. We shall introduce a relaxed functional defined on a suitable space of functions of bounded variation. We shall show the existence of a minimum for the relaxed functional and, through a regularity argument, that such a minimum provides also a minimum for the original problem.

At the end of the section we shall recall an approximation, in the sense of Γ-convergence, of the relaxed functional by elliptic, even if not convex, functionals defined on spaces of smooth functions, for instance Sobolev spaces. We have used such an approximation, which has been developed by L. Ambrosio and V.M. Tortorelli in [A-T2], to perform some numerical simulations whose results will be the content of the next section.

We begin by briefly recalling some basic notations and properties of functions of bounded variation. For a more comprehensive treatment of the subject see, for instance, [E-G] and [Gi].

Given a Borel function $\sigma : \Omega \mapsto \mathbb{R}$, Ω being an open and bounded subset of $\mathbb{R}^n$, $n \geq 2$, and a point $x \in \Omega$ we define $\sigma^+(x)$, the *approximate upper limit* of σ at x, as follows

$$\sigma^+(x) = \text{ap-}\limsup_{y \to x} \sigma(y) = \inf \left\{ t \in \mathbb{R} : \lim_{\rho \to 0^+} \frac{|\{y \in B_\rho \cap \Omega : \sigma(y) > t\}|}{\rho^n} = 0 \right\}.$$

In the same fashion we define $\sigma^-(x)$, the *approximate lower limit* of σ at x, as

$$\sigma^-(x) = \text{ap-}\liminf_{y \to x} \sigma(y) = \sup \left\{ t \in \mathbb{R} : \lim_{\rho \to 0^+} \frac{|\{y \in B_\rho \cap \Omega : \sigma(y) < t\}|}{\rho^n} = 0 \right\}.$$

If $\sigma^+(x) = \sigma^-(x)$ then the common value will be called the *approximate limit* of σ at x and will be denoted by $\tilde{\sigma}(x) = \text{ap-}\lim_{y \to x} \sigma(y)$. If the approximate limit of σ at x does exist we say that σ is *approximately continuous* at x. We define the *jump set* of σ as the subset of Ω where the approximate limit of σ does not exist. We denote the jump set of σ by S_σ and we notice that S_σ is a Borel set whose Lebesque measure is zero.

Given an open bounded set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, we denote by $BV(\Omega)$ the Banach space of *functions of bounded variation.* We recall that $\sigma \in BV(\Omega)$ if and only if $\sigma \in L^1(\Omega)$ and its distributional derivative $D\sigma$ is a bounded vector measure. We endow $BV(\Omega)$ with the standard norm as follows. Given $\sigma \in BV(\Omega)$, we denote by $|D\sigma|$ the total variation of its distributional derivative and we set $\|\sigma\|_{BV(\Omega)} = \|\sigma\|_{L^1(\Omega)} + |D\sigma|(\Omega)$.

For any $\sigma \in BV(\Omega)$, by the Lebesque decomposition, we have $D\sigma = D^a\sigma + D^s\sigma$ where $D^a\sigma$ is absolutely continuous with respect to the Lebesque measure whereas $D^s\sigma$ is singular with respect to the Lebesque measure.

We characterize the absolutely continuous part of $D\sigma$ as follows. We denote by $\nabla\sigma$ the density of $D^a\sigma$ with respect to the Lebesque measure and we recall that, for almost every $x \in \Omega$, $\nabla\sigma(x)$ coincides with the *approximate gradient* of σ at x, that is we have

$$\text{ap-}\lim_{y \to x} \frac{\sigma(y) - \tilde{\sigma}(x) - \nabla\sigma \cdot (y - x)}{|y - x|} = 0.$$

If $\sigma \in BV(\Omega)$ then S_σ is *countably* $(\mathcal{H}^{n-1}, n-1)$*-rectifiable* that is there exists a countable family $\{\Gamma_i\}_{i=1}^\infty$ of compact sets each of them contained in a C^1 hypersurface so that $\mathcal{H}^{n-1}(S_\sigma \setminus \cup_{i=1}^\infty \Gamma_i) = 0$. We characterize the singular part of $D\sigma$ as follows. The restriction of $D^s\sigma$ to S_σ will be called the *jump part* of $D\sigma$ and will be denoted by $D^j\sigma$. The remaining part of $D^s\sigma$, that is the restriction of $D^s\sigma$ to $\Omega \setminus S_\sigma$, will be called the *Cantor part* of $D\sigma$ and will be denoted by $D^c\sigma$. Summing up we have that $D\sigma = D^a\sigma + D^s\sigma = D^a\sigma + D^j\sigma + D^c\sigma$ where $D^a\sigma$ is absolutely continuous with respect to the Lebesque measure, $D^s\sigma$, $D^j\sigma$ and $D^c\sigma$ are singular with respect to the Lebesque measure and $D^j\sigma$ is concentrated on S_σ whereas $D^c\sigma$ is concentrated on $\Omega \setminus S_\sigma$.

We may characterize the jump part as follows

$$D^j\sigma = (\sigma^+ - \sigma^-)\nu_\sigma \mathcal{H}^{n-1}|_{S_\sigma}$$

where σ^+ and σ^- are the approximate upper and lower limit of σ respectively and ν_σ is given by $D^j\sigma = \nu_\sigma|D^j\sigma|$, therefore $D^s\sigma$ restricted to S_σ is absolutely continuous with respect to the $(n-1)$-dimensional Hausdorff measure.

We denote by $SBV(\Omega)$ the space of *special functions of bounded variation* that is the space of functions $\sigma \in BV(\Omega)$ whose singular part of $D\sigma$ is concentrated on S_σ, S_σ being the jump set of σ. Equivalently we say that $\sigma \in BV$ belongs to $SBV(\Omega)$ if and only if the Cantor part of $D\sigma$ is zero.

The special functions of bounded variation verify the following compactness and semicontinuity theorem due to L. Ambrosio, [A1, A2].

Theorem 3.1 (SBV Compactness and Semicontinuity) *Letting $p > 1$, if $\{\sigma_h\}_{h=1}^{\infty}$ is a sequence of functions belonging to $SBV(\Omega)$ satisfying for a given constant $C > 0$*

$$\|\sigma_h\|_{BV(\Omega)} \leq C \quad \textit{for any } h$$

and

$$\int_\Omega |\nabla \sigma_h|^p + \mathcal{H}^{n-1}(S_{\sigma_h}) \leq C \quad \textit{for any } h$$

then we may extract a subsequence, which we relabel $\{\sigma_k\}_{k=1}^{\infty}$, such that σ_k converges in $L^1(\Omega)$ to a function $\sigma \in SBV(\Omega)$ and the following lower semicontinuity properties holds

$$\mathcal{H}^{n-1}(S_\sigma) \leq \liminf_k \mathcal{H}^{n-1}(S_{\sigma_k}); \quad \int_\Omega |\nabla\sigma|^p \leq \liminf_k \int_\Omega |\nabla \sigma_k|^p.$$

The following remarks will be useful. Let $\sigma \in BV(\Omega)$. If a, b are two real numbers so that $a < b$ and we denote $\tau = (\sigma \wedge b) \vee a$ then $\tau \in BV$ and

$$|\nabla\tau| \leq |\nabla\sigma| \quad \text{a.e. in } \Omega; \quad \mathcal{H}^{n-1}(S_\tau \backslash S_\sigma) = 0; \quad |D\tau|(\Omega) \leq |D\sigma|(\Omega).$$

Note that if $\sigma \in SBV(\Omega)$ then also $\tau \in SBV(\Omega)$.

We notice also that if $\sigma \in PH^1(\Omega)$, that is $\sigma \in L^\infty(\Omega)$ so that $\sigma \in H^1(\Omega\backslash K)$ for some K closed in Ω, $\mathcal{H}^{n-1}(K) < \infty$, then $\sigma \in SBV(\Omega)$ and $\mathcal{H}^{n-1}(S_\sigma\backslash K) = 0$.

On the other hand we shall need some conditions in order to ensure that an $SBV(\Omega)$ function belongs to $PH^1(\Omega)$. For any $\sigma \in SBV(\Omega)$, any $A \subset \Omega$ and any constant $\beta > 0$ we define

$$MS(\sigma, \beta, A) = \int_A |\nabla\sigma|^2 + \beta\mathcal{H}^{n-1}(S_\sigma \cap A).$$

If $A = \Omega$ we shall simply write $MS(\sigma,\beta,\Omega) = MS(\sigma,\beta)$. Also the dependence upon the constant β will be suppressed when its role is clear from the context.

Let us assume that $\sigma \in SBV(\Omega) \cap L^\infty(\Omega)$. Moreover let a, b be such that $a \leq \sigma \leq b$ almost everywhere in Ω. We also assume that $MS(\sigma, \beta, \Omega)$ is finite.

In order to ensure that $\sigma \in PH^1(\Omega)$ we only need S_σ to be *essentially closed*, that is $\mathcal{H}^{n-1}((\overline{S_\sigma} \cap \Omega) \backslash S_\sigma) = 0$. In this case, setting $K = \overline{S_\sigma} \cap \Omega$, we have that $\sigma \in PH^1(\Omega)$.

We have the following result which is an immediate consequence of Proposition 4.12 in [DG-Ca-L].

Proposition 3.2 *Let $\sigma \in SBV(\Omega)$ be such that $a \leq \sigma \leq b$ almost everywhere in Ω, where a, b are two real numbers so that $a < b$. We assume that, for a constant $\beta > 0$, $MS(\sigma, \beta, \Omega)$ is finite.*

If for every compact set $A \subset \Omega$ we have a constant $C > 0$ and a constant p, $1 \leq p < n/(n-1)$, so that

$$MS(\sigma, \beta, A) \leq MS(\tau, \beta, A) + C\|\sigma - \tau\|_{L^p(A)}$$

for any $\tau \in SBV(\Omega)$ satisfying $a \leq \tau \leq b$ almost everywhere in Ω and such that $\tau = \sigma$ outside A, then S_σ is essentially closed.

Let us now consider the case of piecewise constant functions. Clearly a piecewise constant function σ, that is a function $\sigma \in L^\infty(\Omega)$ such that $\sigma \in H^1(\Omega \backslash K)$ with $\nabla \sigma = 0$ almost everywhere in $\Omega \backslash K$, K being closed in Ω, $\mathcal{H}^{n-1}(K) < \infty$, is an $SBV(\Omega)$ function such that $\mathcal{H}^{n-1}(S_\sigma \backslash K) = 0$ and $\nabla \sigma = 0$ almost everywhere in Ω.

Let us look at some properties of $SBV(\Omega)$ functions whose approximate gradient $\nabla \sigma$ is zero almost everywhere. If σ satisfies these assumptions and we have also that $\mathcal{H}^{n-1}(S_\sigma) < \infty$ then, see [Co-Ta, Lemma 1.11], there exists a Borel partition $\{U_i\}_{i=1}^\infty$ of Ω and a sequence $\{t_i\}_{i=1}^\infty$ in $\mathbb{R}$ with $t_i \neq t_j$ if $i \neq j$ so that

$$\sigma = \sum_{i=1}^\infty t_i \chi_{U_i} \quad \text{a.e. in } \Omega$$

and

$$\sum_{i=1}^\infty P(U_i, \Omega) < \infty$$

where with $P(U_i, \Omega)$ we denote the *perimeter* of U_i in Ω that is $|D\chi_{U_i}|(\Omega)$.

Again if $\sigma \in SBV(\Omega) \cap L^\infty(\Omega)$ is such that $\nabla \sigma = 0$ almost everywhere in Ω and $\mathcal{H}^{n-1}(S_\sigma) < \infty$ we have that σ is piecewise constant if S_σ is essentially closed. In fact in this case, by taking $K = \overline{S_\sigma} \cap \Omega$ we have that u verifies the definition of piecewise constant functions.

A result analogous to Proposition 3.2 may be obtained also for this case. Here we refer to [Co-Ta, Theorem 2.4].

Proposition 3.3 *Let $\sigma \in SBV(\Omega)$ be such that $a \leq \sigma \leq b$ almost everywhere in Ω, where a, b are two real numbers so that $a < b$. Let us assume that $\nabla \sigma$ is zero almost everywhere in Ω and let T be a countable subset of $[a, b]$ so that $\sigma(x) \in T$ for almost every $x \in \Omega$. We also assume that $\mathcal{H}^{n-1}(S_\sigma) < \infty$.*

If for every compact set $A \subset \Omega$ we have a constant $C > 0$ and a constant p, $1 \leq p < n/(n-1)$, so that

$$\mathcal{H}^{n-1}(S_\sigma \cap A) \leq \mathcal{H}^{n-1}(S_\tau \cap A) + C\|\sigma - \tau\|_{L^p(A)}$$

for any $\tau \in SBV(\Omega)$ such that $\tau(x) \in T$ almost everywhere in Ω and such that $\tau = \sigma$ outside A, then S_σ is essentially closed.

With these results we can prove the existence of a solution to the following kinds of minimization problems. First we consider

$$\min_{(\sigma,K)} \left\{ \begin{array}{c} G(\sigma, K) = \int_{\Omega\setminus K} |\nabla\sigma|^2 + \beta\mathcal{H}^{n-1}(K) + \gamma F(\sigma) \\ K \text{ closed in } \Omega; \quad \sigma \in H^1(\Omega\setminus K); \quad a \leq \sigma \leq b \text{ a.e. in } \Omega\setminus K \end{array} \right\} \tag{3.1}$$

then we consider also the minimal partition version

$$\min_{(\sigma,K)} \left\{ \begin{array}{c} G_1(\sigma, K) = \beta\mathcal{H}^{n-1}(K) + \gamma F(\sigma) \\ K \text{ closed in } \Omega; \quad \sigma \in H^1(\Omega\setminus K) \\ \nabla\sigma = 0 \text{ a.e. in } \Omega\setminus K; \quad a \leq \sigma \leq b \text{ a.e. in } \Omega\setminus K \end{array} \right\}. \tag{3.2}$$

Here β and γ are positive parameters whereas a and b satisfy $-\infty < a < b < +\infty$. We remark that we may impose,without loss of generality, $\mathcal{H}^{n-1}(K) < \infty$, therefore σ will be defined almost everywhere in Ω.

The following theorem may be proved.

Theorem 3.4 *If the faithfulness term $F(\sigma)$ is Lipschitz continuous from $L^p(\Omega)$ in $\mathbb{R}$ for some p, $1 \leq p < n/(n-1)$, then the minimization problems* (3.1) *and* (3.2) *admit a solution.*

Proof. We begin by proving the existence of a solution to problem (3.1). For any $\sigma \in SBV(\Omega)$ such that $a \leq \sigma \leq b$ almost everywhere in Ω we define the following relaxed functional

$$\tilde{G}(\sigma) = \int_\Omega |\nabla\sigma|^2 + \beta\mathcal{H}^{n-1}(S_\sigma) + \gamma F(\sigma).$$

We notice that on $X_a^b(\Omega)$, the set of measurable functions σ on Ω satisfying $a \leq \sigma \leq b$ almost everywhere in Ω, the faithfulness functional F is continuous with respect to the $L^1(\Omega)$ norm (indeed with respect to any $L^q(\Omega)$ norm, $1 \leq q \leq \infty$).

We remark also that for any admissible couple (σ, K) in (3.1) we have that $\sigma \in SBV(\Omega)$ and

$$G(\sigma, K) = \tilde{G}(\sigma). \tag{3.3}$$

Then we prove existence of a solution for the following minimization problem

$$\min\{\tilde{G}(\sigma) : \sigma \in SBV(\Omega) \text{ and } a \leq \sigma \leq b\}. \tag{3.4}$$

The SBV Compactness and Semicontinuity Theorem 3.1 and the direct method in the Calculus of Variations allow us to prove immediately that such a minimization problem admits a solution, which we denote by σ.

Then for any admissible $\tau \in SBV(\Omega)$ we infer

$$MS(\sigma) + \gamma F(\sigma) \leq MS(\tau) + \gamma F(\tau)$$

and then

$$MS(\sigma) \leq MS(\tau) + \gamma(F(\tau) - F(\sigma)) \leq MS(\tau) + C\|\tau - \sigma\|_{L^p(\Omega)}.$$

So, by applying Proposition 3.2, we obtain that S_σ is essentially closed and this in turn implies that $(\sigma, \overline{S_\sigma} \cap \Omega)$ is an admissible couple in (3.1). By (3.3) we also immediately deduce that $(\sigma, \overline{S_\sigma} \cap \Omega)$ solves (3.1).

With a completely analogous reasoning also the existence of a solution to (3.2) may be proved. We introduce the relaxed functional

$$\tilde{G}_1(\sigma) = \beta \mathcal{H}^{n-1}(S_\sigma) + \gamma F(\sigma)$$

for any $\sigma \in SBV(\Omega)$ such that $a \leq \sigma \leq b$ and $\nabla\sigma = 0$ almost everywhere in Ω. We minimize $\tilde{G}_1$ on this class of functions. The SBV Compactness and Semicontinuity Theorem 3.1 ensures that any minimizing sequence converges in L^1 to a function of the same class which is therefore a minimum for $\tilde{G}_1$. The same regularity argument, using this time Proposition 3.3, shows the existence of a couple which solves (3.2).$\blacksquare$

Remark 3.5 In the classical formulation of the Mumford-Shah problem, that is when $F(\sigma) = \int_\Omega |\sigma - g|^2$ with $g \in L^\infty(\Omega)$, we *a priori* know that a minimum σ of the relaxed functional $\tilde{G}$ (or $\tilde{G}_1$ respectively) has to satisfy $\|\sigma\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}$. Therefore, in that case, no *a priori* bound on the ess-inf$_\Omega\, \sigma$ and on the ess-sup$_\Omega\, \sigma$ is required.

Here, however, we have replaced the usual faithfulness term of the Mumford-Shah with a more general functional F which may have some compactness properties which produce a lack of coercivity for the functionals $\tilde{G}$ and $\tilde{G}_1$. Then we have enforced an *a priori* lower and upper bound on the values of the admissible functions. This allows us to have coercivity and hence the existence results. Moreover we would like to apply this method to develop reconstruction procedures in inverse problems. Usually these inverse problems are illposed and *a priori* bounds on the admissible solutions are needed in order to ensure stability. We may see our bounds on the L^∞ norm of the admissible functions σ as such a kind of *a priori* bound.

Such bounds arise in a natural way when we deal with digitized images, since we have natural bounds on the grey levels of the picture used, and also in the inverse conductivity problem where we have to ensure that σ verifies (1.1) for a positive constant λ so that the Neumann-to-Dirichlet map is well defined.

We may also impose further regularity conditions on the faithfulness term F so that the solution σ to (3.1) outside $K = \overline{S_\sigma} \cap \Omega$ is more regular than simply belonging to H^1, for instance it is C^1. For example, this is the case for the classical formulation of the Mumford-Shah problem, see [DG-Ca-L]. This result would allow us to formulate our minimization problems in a class of more regular functions, for instance piecewise C^1 functions, and, nevertheless, to obtain existence results.

Proposition 3.6 *Assume that the hypotheses of Theorem* 3.4 *are satisfied. We also assume that for any* $\sigma_0 \in X_a^b(\Omega)$ *there exists* p, $1 \leq p < n/(n-1)$, *such that* F *is differentiable in* $X_a^b(\Omega)$ *at the point* σ_0 *with respect to the* L^p*-norm.*

Then if (σ, K) *is a solution to the minimization problem* (3.1), *we have that* $\sigma \in C^1(\Omega \backslash K)$.

Proof. We already know that there exists a solution (σ_0, K) to (3.1). We consider a point $x_0 \in \Omega\backslash K$ and we want to prove that locally in a neighbourhood of x_0 the function σ_0 is C^1. First of all we consider $DF(\sigma_0)$. We may identify it with an $L^q(\Omega)$ function f, with $q > n$.

Our argument is based on regularity theory for variational inequalities and obstacle problems. We refer to [K-St].

We introduce the following notation. If $\sigma \in H^1(\Omega)$, Ω being an open set, we say that $\sigma(x) > 0$ at $x \in \Omega$ (in the sense of $H^1(\Omega)$) if there exist $B_r(x)$ and $\phi \in H_0^{1,\infty}(B_r(x))$, $\phi \geq 0$, $\phi(x) > 0$, so that $u - \phi \geq 0$ on $B_r(x)$ (in H^1). Remark that the set $\{x \in \Omega : \sigma(x) > 0\}$ is open.

Let us assume that x_0 is such that $a < \sigma_0(x_0) < b$. Then we have that locally σ_0 solves the following equation

$$\Delta\sigma_0 = f \quad \text{in } B_r(x_0).$$

Then, since $f \in L^q$ with $q > n$, standard regularity arguments shows that σ_0 is C^1 in a neighbourhood of x_0.

Some care is needed in treating the case when $\sigma_0(x_0)$ is either equal to a or to b. We consider only the first case since the second one may be treated in a completely analogous way.

Without loss of generality we may assume that $x_0 = 0$, that $B_r = B_r(0)$ is contained in Ω and that $\sigma_0 < (a+b)/2$ in B_r. We fix

$$\mathbb{K} = \{\tau \in H^1(B_r) : \tau = \sigma_0 \text{ on } \partial B_r \text{ and } \tau \geq a \text{ in } B_r\}$$

and we notice that if we denote by τ_0 the solution to

$$\min_{\tau \in \mathbb{K}} \frac{1}{2} \int_{B_r} |\nabla\tau|^2 + \int_{B_r} f\tau$$

we have that $\tau_0 \in C^1(B_r)$, see [K-St].

Therefore it will be enough to prove that $\sigma_0 = \tau_0$ to obtain our result. This may be achieved through the following procedure. First of all we notice that, since σ_0 in B_r is a supersolution in the following sense

$$\int_{B_r} \nabla\sigma_0 \cdot \nabla\phi \geq -\int_{B_r} f\phi \quad \text{for any } \phi \in H_0^1(B_r),\ \phi \geq 0$$

and clearly belongs to $\mathbb{K}$ we have that $\tau_0 \leq \sigma_0$ in B_r. We assume, by contradiction, that the open set $D = \{x \in B_r : \sigma_0(x) > \tau_0(x)\}$ is not empty. We notice the following facts. Since we have that $\sigma_0 = \tau_0$ on ∂B_r we infer that $\sigma_0 = \tau_0$ on ∂D. Furthermore we have that $\Delta\sigma_0 = f$ in D and this, in turn, implies that σ_0 coincides with τ_0 on D. This contradiction allows us to conclude the proof. ■

We apply the previously stated results to the inverse conductivity problem, and to other inverse problems as well, in the following way.

For the inverse conductivity problem we consider the following procedure. Let Ω be a bounded and Lipschitz domain contained in $\mathbb{R}^n$. We fix a constant λ, $0 < \lambda < 1$, a smooth domain Ω' compactly contained in Ω and a measurable function σ_0 which satisfies (1.1) in Ω with constant λ.

We shall try to force the minimum to be equal to σ_0 outside Ω' and to this aim for any σ belonging to $X_\lambda(\Omega) = X_\lambda^{\lambda^{-1}}(\Omega)$ (or, equivalently, satisfying (1.1) with constant λ) we set

$$\hat{\sigma} = \begin{cases} \sigma & \text{in } \Omega' \\ \sigma_0 & \text{otherwise} \end{cases}$$

and we shall penalize, in a suitable norm, the term $\sigma - \sigma_0$ on $\Omega \backslash \Omega'$.

We shall consider the following cases. In the first one we assume knowledge of the complete Neumann-to-Dirichlet map, that is we have a linear, bounded operator G from ${}_0L^2(\partial\Omega)$ into itself which corresponds to the measurements on the boundary of the electrostatic potentials determined by applying any current density to the conductor Ω. In this case we write the faithfulness term as follows

$$F(\sigma) = \|\Lambda(\hat{\sigma}, \cdot) - G\|^2_{B({}_0L^2(\partial\Omega))} + \|\sigma - \sigma_0\|^q_{L^p(\Omega\backslash\Omega')} \tag{3.5}$$

for any $\sigma \in X_\lambda(\Omega)$. Here p and q are two real numbers greater than or equal to 1.

On the other hand, in view of applications, we consider N different current densities $f_i \in {}_0L^2(\partial\Omega)$, we measure $g_i \in {}_0L^2(\partial\Omega)$, $i = 1, \ldots, N$, the corresponding potentials on the boundary, and we set

$$F(\sigma) = \sum_{i=1}^{N} \int_{\partial\Omega} |\Lambda(\hat{\sigma}, f_i) - g_i\|^2 + \|\sigma - \sigma_0\|^q_{L^p(\Omega\backslash\Omega')} \tag{3.6}$$

for any $\sigma \in X_\lambda(\Omega)$.

Then we may state the main result.

Theorem 3.7 *Under the previously stated assumptions if we have that* $Q(\lambda) > 2n$ *and* $\frac{nq}{(n-1)p} \geq 1$ *we obtain that* (3.1) *and* (3.2) *admit a solution if we take as* F *the functional defined either in* (3.5) *or in* (3.6) *and we fix* $a = \lambda$ *and* $b = \lambda^{-1}$.

With the same notation as before, as an easy application of Proposition 3.6 and of the regularity results of Section 2, we obtain that the solution to problem

(3.1) with F as in (3.6) is piecewise C^1 if, for instance, the conditions $Q(\lambda) > 2n$ and $p = q = 2$ hold.

In order to avoid the somehow restrictive hypothesis that $Q(\lambda) > 2n$, we shall consider the following linearized version of the problem. We shall substitute, either in (3.5) or (3.6) the operator $\Lambda(\hat{\sigma}, \cdot)$ with its differential $D\mathcal{F}(\sigma_0)[\hat{\sigma} - \sigma_0](\cdot)$ around the reference conductivity σ_0 and the measurements g_i with the perturbation of the reference data related to the reference conductivity, namely $g_i - \Lambda(\sigma_0, f_i)$. If σ_0 is regular enough we may relax the condition on $Q(\lambda)$ and still obtain a similar existence result.

Theorem 3.8 *For any λ, $0 < \lambda < 1$, assume $Q(\sigma_0) > 2n$ and $\frac{nq}{(n-1)p} \geq 1$. Then if we define as F the functional obtained by replacing either in* (3.5) *or in* (3.6) *the operator $\Lambda(\hat{\sigma}, \cdot)$ by $D\mathcal{F}(\sigma_0)[\hat{\sigma} - \sigma_0](\cdot)$ and the measurements g_i by $g_i - \Lambda(\sigma_0, f_i)$ and we fix, as before, $a = \lambda$ and $b = \lambda^{-1}$ we have that* (3.1) *and* (3.2) *admit a solution.*

It is clear that, if we consider F as in (3.6), $Q(\sigma_0) > 2n$ and $p = q = 2$ we have that the solution whose existence is proved in Theorem 3.8 is piecewise C^1.

In order to complete this survey of results and applications to inverse problems of Theorem 3.4 and Proposition 3.6 we consider the case stated in (1.7).

Theorem 3.9 *Let A be a linear bounded operator from $L^p(\Omega)$ into itself for any p, $1 \leq p \leq \infty$, and let $F(\sigma) = \int_\Omega |A[\sigma] - g|^2$ where g is an $L^\infty(\Omega)$ function corresponding to the additional measurement.*

Then, for any constants a, b, $-\infty < a < b < +\infty$, problems (3.1) *and* (3.2) *admit a solution. Furthermore, for what concerns* (3.1), *such solution is piecewise C^1.*

With the existence of the solutions and their regularity properties established for problems (3.1)-(3.2), we will use an approximation procedure to construct the solutions. We recall here just the definition and some basic properties of Γ-convergence. For a more detailed introduction we refer to [DM].

Let (X, d) be a metric space. Then a sequence $F_k : X \mapsto [-\infty, +\infty]$ Γ-converges as $k \to \infty$ to a function $F : X \mapsto [-\infty, +\infty]$ if for every $x \in X$ we have

(i) for every x_k converging to x we have

$$F(x) \leq \liminf_k F_k(x_k); \tag{3.7}$$

(ii) there exists a sequence x_k converging to x so that

$$F(x) = \lim_k F_k(x_k). \tag{3.8}$$

The function F will be called the Γ-limit of F_k as $k \to \infty$ with respect to the metric d and we denote it by $F = \Gamma\text{-}\lim_k F_k$.

The following theorem, usually known as the Fundamental Theorem of Γ-convergence, illustrates the motivations for the definition of such a kind of convergence.

Theorem 3.10 *Let (X, d) be a metric space and let $F_k : X \mapsto [-\infty, +\infty]$ be a sequence of functions defined on X. If there exists a compact set K such that $\inf_K F_k = \inf_X F_k$ for any k and $F = \Gamma\text{-}\lim_k F_k$ then F admits a minimum over X and we have*

$$\min_X F = \lim_k \inf_X F_k.$$

Furthermore if x_k is a sequence of points in X which converges to a point $x \in X$ and satisfies $\lim_k F_k(x_k) = \lim_k \inf_X F_k$ then x is a minimum point for F.

We simply recall the following property of Γ-convergence which will be used in the sequel. If $F = \Gamma\text{-}\lim_k F_k$ and G is a continuous function on X (with respect to the metric d) then

$$F + G = \Gamma\text{-}\lim_k (F_k + G).$$

The definition of Γ-convergence may be extended in a natural way for families depending on a continuous parameter. For instance we say that the family of functions F_ε, defined for every $\varepsilon > 0$, Γ-converges to a function F as $\varepsilon \to 0^+$ if for every sequence of positive ε_k converging to 0 we have $F = \Gamma\text{-}\lim_k F_{\varepsilon_k}$.

The following Γ-convergence results have been proved by L. Ambrosio and V.M. Tortorelli in [A-T2], see also [A-T1] and [Br].

We introduce the functional MS_ε on $L^1(\Omega) \times L^1(\Omega)$ as follows. We define

$$MS_\varepsilon(\sigma, s) = \int_\Omega (s^2 + o_\varepsilon)|\nabla\sigma|^2 + \beta\varepsilon|\nabla s|^2 + \beta\frac{(s-1)^2}{4\varepsilon} \tag{3.9}$$

if $\sigma \in H^1(\Omega) \cap X_a^b(\Omega)$ and $s \in H^1(\Omega) \cap X_0^1(\Omega)$ and we set $MS_\varepsilon(\sigma, s) = \infty$ otherwise.

Then we formally add a second variable to the functional MS in the following way

$$MS(\sigma, s) = \begin{cases} \int_\Omega |\nabla\sigma|^2 + \beta\mathcal{H}^{n-1}(S_\sigma) & \text{if } \begin{cases} \sigma \in SBV(\Omega) \cap X_a^b(\Omega) \\ s \equiv 1 \end{cases} \\ +\infty & \text{otherwise} \end{cases} \tag{3.10}$$

Let F be any functional on $X_a^b(\Omega)$ continuous with respect to the L^1-norm. We extend the functional F onto $L^1(\Omega)$ by setting its value to ∞ outside $X_a^b(\Omega)$.

Theorem 3.11 *If o_ε is a nonnegative infinitesimal of higher order than ε, then $\tilde{G}_\varepsilon = MS_\varepsilon + F$ Γ-converges (as $\varepsilon \to 0^+$) in $L^1(\Omega) \times L^1(\Omega)$ to $\tilde{G} = MS + F$.*

Moreover if $(\sigma_\varepsilon, s_\varepsilon)$ minimizes $\tilde{G}_\varepsilon$, then σ_ε is compact in $L^1(\Omega)$ and any limit point of σ_ε as $\varepsilon \to 0^+$ determines a pair $(\sigma, 1)$ so that σ minimizes $\tilde{G}$.

For a proof see [A-T2]. In the same paper an analogous approximation has been developed for the partition problem by taking the following approximating family. Again we define a functional $(MS_1)_\varepsilon$ on $L^1(\Omega) \times L^1(\Omega)$ as follows

$$(MS_1)_\varepsilon(\sigma, s) = \int_\Omega (M_\varepsilon s^2 + o_\varepsilon)|\nabla\sigma|^2 + \beta\varepsilon|\nabla s|^2 + \beta\frac{(s-1)^2}{4\varepsilon} \tag{3.11}$$

if $\sigma \in H^1(\Omega) \cap X_a^b(\Omega)$ and $s \in H^1(\Omega) \cap X_0^1(\Omega)$. As usual we set $(MS_1)_\varepsilon(\sigma, s) = \infty$ otherwise.

We take a functional F continuous on $X_a^b(\Omega)$ endowed with the L^1-norm. For any $\sigma \in L^1(\Omega) \backslash X_a^b(\Omega)$ we set $F(\sigma) = \infty$. We define the functional $\tilde{G}_1$ on $L^1(\Omega) \times L^1(\Omega)$ as

$$\tilde{G}_1(\sigma, s) = \begin{cases} \beta\mathcal{H}^{n-1}(S_\sigma) + F(\sigma) & \text{if } \begin{cases} \sigma \in SBV(\Omega);\ \nabla\sigma = 0 \text{ a.e.} \\ \sigma \in X_a^b(\Omega);\ s \equiv 1 \end{cases} \\ +\infty & \text{otherwise} \end{cases} \tag{3.12}$$

Then we have the following result, whose proof is contained in [A-T2].

Theorem 3.12 *If o_ε is a nonnegative infinitesimal of higher order than ε and $M_\varepsilon \to \infty$ as $\varepsilon \to 0^+$, then $(\tilde{G}_1)_\varepsilon = (MS_1)_\varepsilon + F$ Γ-converges (as $\varepsilon \to 0^+$) in $L^1(\Omega) \times L^1(\Omega)$ to $\tilde{G}_1$.*

Moreover if $(\sigma_\varepsilon, s_\varepsilon)$ minimizes $(\tilde{G}_1)_\varepsilon$, then σ_ε is compact in $L^1(\Omega)$ and any limit point of σ_ε as $\varepsilon \to 0^+$ determines a pair $(\sigma, 1)$ so that σ minimizes $\tilde{G}_1$.

It is evident that for any of the cases treated in Theorem 3.7, 3.8 and 3.9, the Γ-convergence results stated above hold true.

4 Numerical results

In this section we present some results from a numerical implementation of the method described at the end of the previous section. We begin by showing some examples of reconstructing conductivity in the linearized inverse conductivity problem. Finally we present also an example where the direct problem is described by a linear operator. In our example, we choose A to be a blurring convolution operator; therefore the inverse problem can be viewed as one of deblurring and segmentation.

For the inverse conductivity problem we shall consider the following framework. We shall limit ourselves to the linearized case. However, we will be inverting *nonlinear data* since the data will be generated by solving the true direct problem, not a linearized one. Moreover, a small amount of noise will be added to the data.

For simplicity we choose the domain Ω to be a rectangle of sides L_1, L_2. We shall assume that the unknown conductivity σ to be determined is a perturbation of a contrast conductivity, which we assume to be smooth (at least Hölder continuous). In the numerical tests we actually choose $\sigma_0 \equiv 1$ in Ω. In practice we shall assume also that σ is equal to σ_0 outside a rectangle slightly smaller than Ω.

We fix a number n and m of equally spaced points on each vertical and horizontal side of Ω respectively. We have $2(n+m)$ points and we order them in an anticlockwise order. We identify the $2(n+m)+1$ point with the first one. These points will constitute the electrode locations and also where the measurements of the potential are collected. We assume that the current densities used are given by putting a positive electrode and a negative electrode, both of intensity one, on two adjacent locations. This will give us $2(n+m)$ different current patterns, which we shall denote by $f_1, \ldots, f_{2(n+m)}$. For each f_i we shall measure the corresponding potential u_i on the boundary. The potential u_i is measuread on all the possible electrode sites as a voltage drop between two adjacent electrode locations. Thus we have $2(n+m)$ data for each measurement making a total of $(2(n+m))^2$ data. Since we are in the linearized case, we consider only the perturbation with respect to the potentials corresponding to σ_0. So if u^i_j, $j = 1, \ldots, 2(n+m)$ and U^i_j are the values on the electrode sites of the potentials corresponding to the current density f_i, respectively with conductivities σ and σ_0, the data will be given by $g^i_j = u^i_{j+1} - u^i_j - (U^i_{j+1} - U^i_j)$, $i, j = 1, \ldots, 2(n+m)$. Globally such data will be denoted by the vector G.

We shall divide Ω into a uniform grid of mesh h. Therefore the unknown conductivity will be discretized onto the nodes of this uniform mesh. Let N, M be the number of nodes for the vertical and horizontal side respectively (that is $N = L_1/h$ and $M = L_2/h$).

For each measurement f_i we compute, by a finite differences scheme, the discretization of the forward (linearized) map $D\mathcal{F}(\sigma_0)[\cdot - \sigma_0](f_i)$ and we shall denote it by DF^i which will be a linear map from the space of $N \times M$ matrices onto the $2(n+m)$ vector which represents the measured potentials on the electrode sites.

Globally our operator $A = DF$ will be given by a matrix mapping $N \times M$ onto $(2(n+m))^2$, which is the number of total measured data. The scaled discretized penalization term will be then given by

$$h' \sum (DF(\sigma) - G)^2$$

where h' is the gap between the electrode sites.

The data, for the reference conductivity and the unknown one, have been computed by using a discretization of the (full) operator $\mathcal{F}(\sigma_0)[\cdot]$ and $\mathcal{F}(\sigma)[\cdot]$ over the same mesh and a finite difference scheme. Some noise have been then added to the vector data G.

In our experiments we have used the following values. The domain Ω is a square of size 2, we chose $h = 0.05$, thus making $N = M = 40$. We took 5 electrode locations per side, thus making 20 total measueremnts and 400 data points. We added about 1 percent noise.

We consider the Γ-convergence procedure described above. We fix accordingly the functional parameter and the Γ-convergence parameter ϵ. We discretize the other parts of the functional over the same mesh and we solve the minimiza-

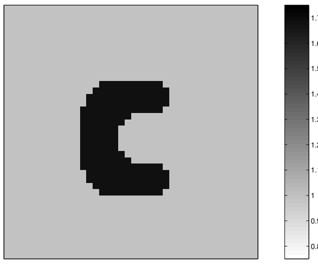

Figure 1-a: The true conductivity distribution.

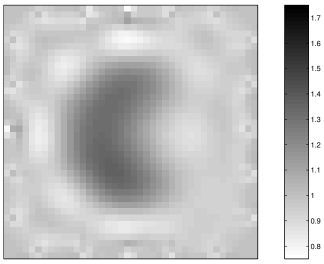

Figure 1-b: SVD-regularized reconstruction.

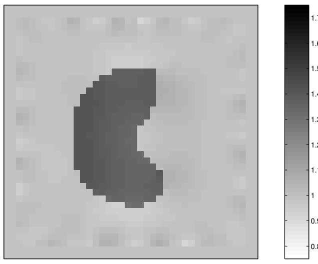

Figure 1-c: Reconstruction using variational approach.

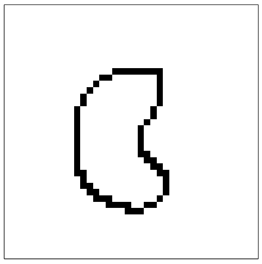

Figure 1-d: Segmentation of the reconstructed conductivity distribution.

tion problem by a conjugate gradient method, using again finite differences. It should be pointed out that the functional to be minimized is not convex and therefore the minimum might be not unique.

To test the method we shall compare the results with the pseudoinverse solution which is obtained through an SVD-regularization of DF and by computing as first guess $M^{\dagger}G$ where $M^{\dagger}$ is the pseudoinverse. We shall not use the pseudoinverse as a starting point for our minimization procedure. We shall use σ_0 instead. This because the method depends on two variables (σ, s) and we have no way to compute an s which might be consistent with the pseudoinverse. We shall therefore assume as initial values $\sigma = \sigma_0$ and $s \equiv 1$, that is no jumps.

A careful choice of the coefficients has to be made. For the usual Mumford-Shah functional a detailed study of the meaning of the coefficients, and how to choose them, has been carried over, see for instance [B-Z]. Such a study has never been done for our model problem. The choice of the parameter is not easy, given also the great number of them involved. However this fact may be exploited to recover particular kinds of features of the unknown conductivity. See for instance the three different reconstruction of Example 2.

Example 1

The true conductivity distribution, along with its grey level values, are given in Figure 1a. The reconstruction by the SVD-regularization is given in Figure 1b. Note that a large amount of blurring has taken place. The reconstruction using the approximate Mumford-Shah variational is given in Figure 1c, where we have used $\epsilon = 0.008$, $\alpha = 0.02$, $\beta = 0.002$ and $\gamma = 2 \times 10^6$. These values were arrived at by experimentation. Clearly, this is a weakness of this approach. However, we emphasize that when good values of these parameters are used, the reconstruction is highly effective. The convergence of the conjugate gradient method is also quite slow; 12000 iterations were needed to obtain the result shown. The auxilliary function s, when thresholded to 0.2, gives the segmentation given in Figure 1d.

Example 2

In this example, we illustrate the behavior of the reconstruction when the parameters are altered. First we display the true conductivity distribution and its reconstruction using the SVD regularization in Figures 2a and 2b. In the subsequent three pairs of figures, we display the reconstruction and its segmentation by the variational method for three different choices of ϵ and β as indicated. The other parameters are set at $\gamma = 2 \times 10^6$ and $\alpha = 0.1$. We display the results at the end of 6000 iterations of the conjugate gradient method.

Example 3

In examples 1 and 2, we have made no assumptions on the blocky character of the unknown conductivities. We now present a case where the true conductivity consists of a blocky part and a 'ramp' part. In principle this method deals with a much wider class of admissible conductivities. On the other hand the reconstruction of blocky conductivities might be less precise than with BV regularization as in [D-S]. The calculations were carried out with $\epsilon = 0.025$, $\alpha = 0.1$, $\beta = 0.021$ and $\gamma = 2 \times 10^6$. Again, 6000 conjugate gradient iterations were performed to get the results shown in Figure 3. Note that the segmentation is able to make out the larger jump towards the top of the ramp, but misses the jump towards the bottom. The reconstruction faithfully images the jump near the top.

Example 4

We consider now the case where the forward map is a blurring operator. The point spread function, which is assumed to be known, is in the form of a pyramid as displayed in Figure 4a. The true image is shown in Figure 4b, while the blurred and noisy version (5% noise) is shown in Figure 4c. We show two calculations. In the first, $\epsilon = 0.002$ and $\alpha = 0.05$, whereas in the second, $\epsilon = 0.001$ and $\alpha = 0.01$. In both cases, $\beta = 0.005$ and $\gamma = 10^4$, and 1000 conjugate gradient iterations were taken.

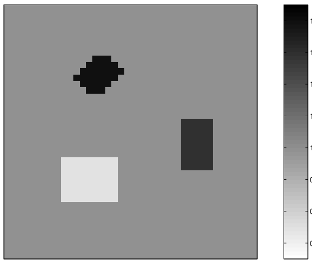

Figure 2-a: The true conductivity distribution.

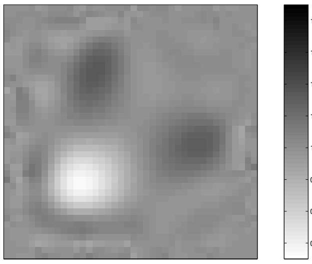

Figure 2-b: SVD-regularized reconstruction.

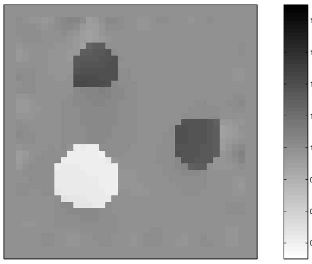

Figure 2-c: Reconstruction with $\epsilon = 0.019$ and $\beta = 0.019$.

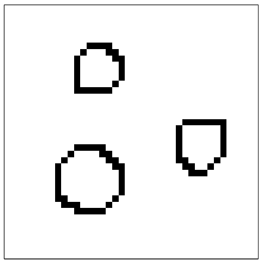

Figure 2-d: Segmentation.

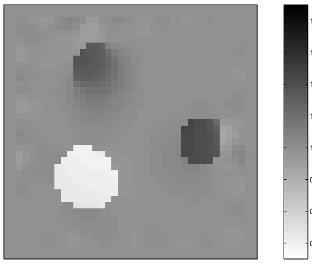

Figure 2-e: Reconstruction with $\epsilon = 0.02$ and $\beta = 0.025$.

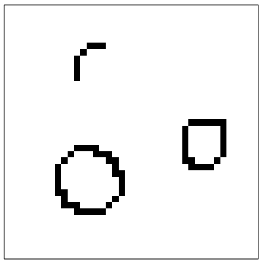

Figure 2-f: Segmentation.

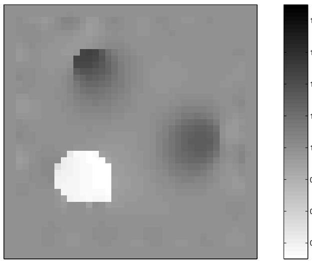

Figure 2-g: Reconstruction with $\epsilon = 0.035$ and $\beta = 0.042$.

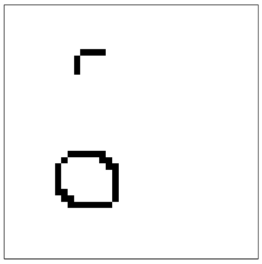

Figure 2-h: Segmentation.

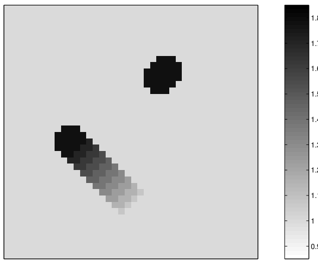

Figure 3-a: The true conductivity distribution.

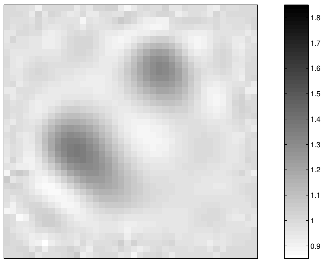

Figure 3-b: SVD-regularized reconstruction.

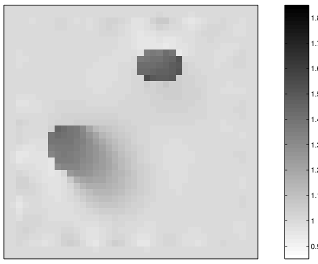

Figure 3-c: Reconstruction using variational approach.

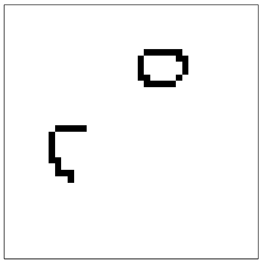

Figure 3-d: Segmentation of the reconstructed conductivity distribution.

What is remarkable about the calculations is that, in both cases, we are able to reconstruct the unknown image rather well. However, in the first case, we were able to find a segmentation, while in the second, no segmentation was found. This demonstration illustrates the sensitivity of the segmentation process to the parameters. It can be seen that the reconstruction is visibly better when a segmentation is found.

5 Discussion

We have considered an approach for stabilizing and enhancing the reconstruction of conductivity distribution in electrical impedance tomography. The approach uses the Mumford-Shah functional which has been introduced as a method to denoise and segment grey-level images. We have shown that the resulting variational problem admits a solution. The efficacy of the method is studied in several numerical experiments. We found the approach to be quite promising. It has the capability to resolve conductivity distributions with jumps with good accuracy even in the presence of small amount of data noise, and by only considering the linearized problem. However, we also found that aside from the computational complexity involved, the method requires tuning of several parameters, some of which can be very sensitive. Further work to understand how the parameters

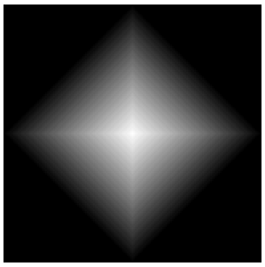

Figure 4-a: The point spread function of the blurring operator.

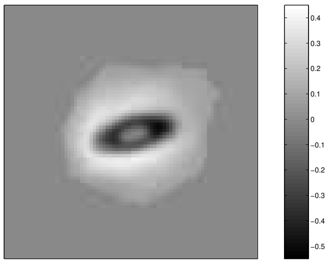

Figure 4-b: The true image.

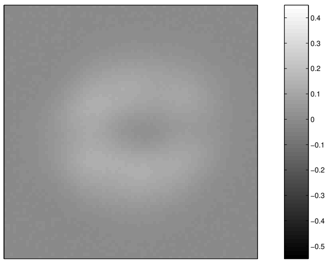

Figure 4-c: The blurred image.

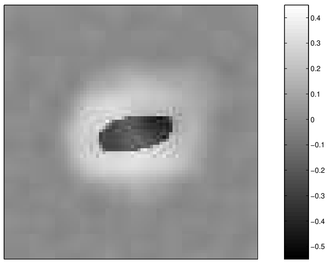

Figure 4-d: Reconstruction with $\epsilon = 0.002$ and $\alpha = 0.05$.

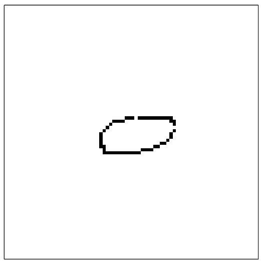

Figure 4-e: Segmentation.

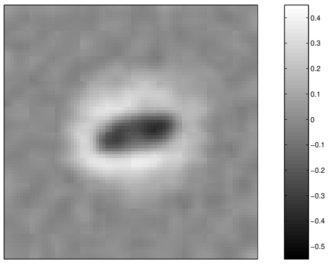

Figure 4-f: Reconstruction with $\epsilon = 0.001$ and $\alpha = 0.01$.

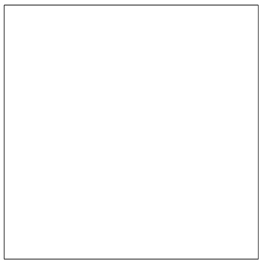

Figure 4-g: Segmentation.

should be set and how to accelerate the computation is needed. Nevertheless, we believe the results we have obtain are encouraging and warrant further research.

Acknowledgments

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