A CASCADIC ALGORITHM FOR BOUNDED VARIATION REGULARIZATION*

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Abstract. Bounded variation regularization is a commonly used technique for denoising and deblurring images which simultaneously reduces noise and recovers discontinuities in the image. There are *two* mayor difficulties associated with the numerical realization of bounded variation regularization. The functional to be minimized is *non-differentiable* over a *non-reflexive* Banach space. In numerical realizations these difficulties are usually overcome by considering a *smoothed* functional over a *reflexive* subspace of the non-reflexive Banach space. This approach leads to smooth approximations and consequently discontinuities (which are most important features of an image) are blurred. In this paper we approximate bounded variation regularization by a class of differentiable optimization problems and analyze a *cascadic* algorithm for bounded variation regularization. Moreover, we present some numerical examples for denoising ultrasound data.

Key words. Bounded variation regularization, image denoising, cascadic methods

1. Introduction. *Denoising* and *deblurring* of images are challenging tasks in image processing. In mathematical terms the objective is to find a function u, representing the grey values of an image $u^\dagger$ from a noisy, degraded function

$$z = Ku^\dagger + \varepsilon \,.$$

Here ε is noise and K typically is a convolution operator, which models for instance the blur of an optical device. If $K = I$, then this problem is referred as *denoising*; for $K \neq I$, this problem is called *deblurring*.

In this paper we restrict our attention to the problem of denoising. Any approach for effective denoising has to take into account *a–priori information* on the image to be recovered. Here we assume that the image to be recovered is piecewise smooth.

There are at least two different types of algorithms for denoising images. One relies on *diffusion filtering techniques*. For some recent references on this subject we refer to [21, 17, 3, 15, 1]). The second type of denosing techniques is motivated from regularization techniques. For some recent references on this subject we refer to [20, 5, 2, 22, 23, 14, 9, 10, 4, 3]. Close relation between diffusion filtering techniques and regularization techniques have been pointed out in [18, 19, 11].

In this paper we restrict our attention to the method of *bounded variation regularization*. Other regularization techniques for recovering piecewise smooth images have been analyzed in [19, 11, 16]. In bounded variation regularization an approximation of $u^\dagger$ is given by the minimizer of the functional

$$f_{\mathrm{BV}}(u) = \frac{1}{2}\|u - z\|^2_{L^2(\Omega)} + \alpha |u|_{BV(\Omega)} \,, \tag{1.1}$$

over the space of functions of *bounded variation*; in this functional α is a positive parameter, $|u|_{BV(\Omega)}$ denotes the *bounded variation semi-norm*, and Ω is a regular bounded domain with piecewise smooth boundary. For the definition of the space of

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functions of bounded variation and the bounded variation semi-norm we refer to [12]. Two important properties of the function space $BV(\Omega)$ are that it contains piecewise smooth functions and that for any function $u \in W^{1,1}(\Omega)$

$$|u|_{BV(\Omega)} = \int_\Omega |\nabla u| \, .$$

In this paper we are concerned with numerical aspects of minimizing the functional f_{BV}. Clearly this functional is not differentiable! To overcome this difficulty several authors (see e.g. [20, 2] and the references therein) proposed to consider approximations of the bounded variation functional, such as

$$f_{\mathrm{BV}_\beta}(u) = \frac{1}{2}\|u - z\|^2_{L^2(\Omega)} + \alpha \int_\Omega \sqrt{|\nabla u|^2 + \beta^2} \, ,$$

which is again considered over the space of functions of bounded variation. We note that this approach does not overcome numerical difficulties which result from minimizing a functional over a non-reflexive Banach space. However, at least formally, the first optimality condition can be stated and solved numerically [20, 2, 4]. It is intuitively clear that this modification is counter productive to the original motivation of bounded variation regularization of reconstructing discontinuities in images; in fact the parameter β^2 has some smoothing effect.

To overcome the difficulty of minimizing the functional f_{BV_β} over the nonreflexive Banach space, in this paper, we introduce a family of bounded operators

$$L_h : L^2(\Omega) \to (L^2(\Omega))^d \, ,$$

such that for any smooth function u

$$L_h u \to \nabla u \text{ with respect to the } L^1 - \text{norm for } h \to 0 \, .$$

Replacing ∇u in the functional f_{BV_β} by $L_h u$ results in the optimization problem of minimizing the functional

$$f_{\beta h}(u) = \frac{1}{2}\|u - z\|^2_{L^2(\Omega)} + \alpha \int_\Omega \sqrt{|L_h u|^2 + \beta^2} \, . \tag{1.2}$$

Note that the functional $f_{\beta h}$ is differentiable on the Hilbert space $L^2(\Omega)$.

There are various choices for the family of operators L_h which are practically relevant. A reasonable choice for L_h is a finite difference approximation, where the parameter h denotes the discretization size: for example, if Ω is a bounded interval, then

$$L_h u(x) = \frac{u(x+h) - u(x-h)}{2h}$$

is a reasonable choice. Other choices for L_h are projection operators on finite element subspaces. The problem of minimizing this functional has first been studied in [6], but there the parameters α, β and h are considered to be fixed. In this paper we analyze the interplay of the parameters α, β and h more carefully and update them iteratively in a cascadic algorithm. Many papers which are concerned with the minimization of the functional f_{BV} consider approximate minimization over a subspace of a reflexive Banach space which is contained in $BV(\Omega)$, which in turn excludes discontinuous solutions. In our approach we minimize over the set $L^2(\Omega)$, which does not exclude *a–priori* discontinuous solutions. Finally we note that the idea of or approach is similar to an approach by Catté, Coll, Lions and Morel [13]. They study the approximation of the nonlinear diffusion filtering technique

$$\frac{\partial u}{\partial t} = div\left(\frac{\nabla u}{|\nabla u|}\right)$$

by

$$\frac{\partial u}{\partial t} = div\left(\frac{\nabla u}{|\nabla (L_h u)|}\right) ,$$

where L_h is a convolution operator with a Gaussian kernel and variance h.

The paper is organized as follows: In Section 2 we modify a fixed point iteration introduced by Vogel and Oman [2] and prove convergence of this scheme for minimizing the functional $f_{\beta h}$ - these results are for fixed parameters α, β, h. In Section 3 we study stability and error estimates for the fixed point iteration. In Section 4 we prove convergence rates of the fixed point iteration and study convergence of the cascadic algorithm. Finally we present several numerical experiments (Section 6).

2. Convergence of the Fixed Point Iteration. In this section we study a method for the numerical minimization of the functional $f_{\beta h}$. All along this section the parameters $\alpha > 0$, $\beta > 0$ and the operator $L_h \in \mathcal{L}(L^2(\Omega), \left(L^2(\Omega)\right)^d)$ occurring in the definition of the functional $f_{\beta h}$ are assumed to be fixed.

In [6] it was proven that the functional $f_{\beta h}$ has a unique minimizer $u_{\alpha\beta h}$ in $L^2(\Omega)$. Since the functional $f_{\beta h}$ is differentiable over a reflexive Banach space, the minimizer of $f_{\beta h}$ satisfies the first order optimality condition, i.e.,

$$(I + L_{\alpha\beta h}(u_{\alpha\beta h}))u_{\alpha\beta h} = z \,, \tag{2.1}$$

with

$$L_{\alpha\beta h}(u)v := \alpha L_h^\star\left(\frac{L_h v}{\sqrt{|L_h u|^2 + \beta^2}}\right) ,$$

where $L_h^\star$ is the L^2–adjoint of L_h.

We consider the following fixed point iteration for the solution of (2.1)

$$u_{n+1} = (I + L_{\alpha\beta h}(u_n))^{-1} z \tag{2.2}$$

The fixed point iteration for minimizing $f_{\beta h}$ has been studied in [6]. The algorithm is based on a fixed point iteration introduced by Vogel and Oman [2] for minimizing the functional $f_{\beta h}$.

In the following we show that the fixed point iteration is unconditionally convergent for any parameter setting α, β and for any bounded operator L_h. This generalizes results in [6] where only conditional convergence is proven. Dobson and Vogel [4] proved unconditional convergence of a fixed point iteration of f_{BV_β} under the assumption that $u^\dagger \in W^{1,\infty}$.

The following considerations are taken from [6]. They are summarized here for the readers convenience, since they will be used later on to prove unconditional convergence of the fixed point iteration (2.2).

The sequence $\{u_n\}$ in (2.2) is uniformly bounded, i.e.,

$$\|u_n\|_{L^2(\Omega)} \leq \|z\|_{L^2(\Omega)} \text{ , for all } n \in \mathbb{N} \text{ .} \tag{2.3}$$

Since for any $u \in L^2(\Omega)$, the operator $f'_{\beta h}(u)$ is bounded, by the Riesz representation theorem there exists a generating element $g(u)$, i.e.,

$$\langle g(u), h \rangle_{L^2(\Omega)} = f'_{\beta h}(u)h \quad \text{for any } h \in L^2(\Omega) \text{ .}$$

The generating element is

$$g(u) := (I + L_{\alpha\beta h}(u))(u) - z \text{ .} \tag{2.4}$$

For notational purposes we identify the function $g(u)$ and the operator $f'_{\beta h}(u)$. Below we summarize some properties of the operator $f'_{\beta h}$ which are used in the course of this paper:

$$\langle f'_{\beta h}(u_n), u_{n+1} - u_n \rangle_{L^2(\Omega)} \leq -\|u_{n+1} - u_n\|^2_{L^2(\Omega)} \text{ .} \tag{2.5}$$

Moreover, we have the following identity

$$\left\langle f''_{\beta h}(u)v, v \right\rangle = \int_\Omega v^2 + \frac{\alpha}{\beta} \int_\Omega \frac{|L_h v|^2}{(|L_h u|^2/\beta^2 + 1)^{3/2}} \text{ .} \tag{2.6}$$

In particular we have that the second derivative is positive definite and satisfies

$$\left\langle f''_{\beta h}(u)v, v \right\rangle \geq \|v\|_{L^2(\Omega)} \text{ .}$$

In the following we establish monotonicity of function values during the fixed point iteration.

LEMMA 2.1. *For any pair of parameters α, β and for any bounded operator $L_h : L^2(\Omega) \to L^2(\Omega)^d$ we have*

$$f_{\beta h}(u_{n+1}) - f_{\beta h}(u_n) \leq 0 \quad \textit{for all } n \in \mathbb{N}.$$

Moreover, the sequence $\{f_{\beta h}(u_n)\}_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}$.

Proof. We introduce the following functional defined on $L^2(\Omega)$

$$\begin{aligned} j_{u_n}(u) \quad &:= \quad \tfrac{1}{2}\|u - z\|^2 + \tfrac{\alpha}{2} \int_\Omega \tfrac{|L_h u|^2}{\sqrt{|L_h u_n|^2 + \beta^2}} \\ &- \quad \tfrac{\alpha}{2} \int_\Omega \tfrac{|L_h u_n|^2}{\sqrt{|L_h u_n|^2 + \beta^2}} + \alpha \int_\Omega \sqrt{|L_h u_n|^2 + \beta^2} \text{ .} \end{aligned}$$

The formal derivatives of the functionals $f_{\beta h}$ and j_{u_n} are given by

$$\begin{aligned} f'_{\beta h}(u)v \quad &= \quad \langle u - z, v \rangle_{L^2(\Omega)} + \alpha \left\langle \tfrac{L_h u}{\sqrt{|L_h u|^2 + \beta^2}}, L_h v \right\rangle_{L^2(\Omega)} \text{ ,} \\ j'_{u_n}(u)v \quad &= \quad \langle u - z, v \rangle_{L^2(\Omega)} + \alpha \left\langle \tfrac{L_h u}{\sqrt{|L_h u_n|^2 + \beta^2}}, L_h v \right\rangle_{L^2(\Omega)} \text{ .} \end{aligned}$$

It is easy to see that

$$f_{\beta h}(u_n) = j_{u_n}(u_n) \text{ and } f'_{\beta h}(u_n) = j'_{u_n}(u_n) \; .$$

In particular this shows that the functional j_{u_n} is the quadratic approximation of the functional $f_{\beta h}$ at u_n. Moreover, from the definition of j_{u_n} it follows that for all $v \in L^2(\Omega)$

$$f_{\beta h}(v) \leq j_{u_n}(v) \; . \tag{2.7}$$

Since the functional

$$C(v) := \frac{1}{2}\|v - z\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_\Omega \frac{|L_h v|^2}{\sqrt{|L_h u_n|^2 + \beta^2}}$$

is strictly convex, a minimizer v of C satisfies the first order optimality condition,

$$(I + L_{\alpha\beta h}(u_n))v = z \; ,$$

and vice versa. This shows that u_{n+1} minimizes the functional C, and in particular it is also the minimizing element of the functional j_{u_n}. Since $f_{\beta h}(u_n) = j_{u_n}(u_n)$ it follows from (2.7) that

$$f_{\beta h}(u_{n+1}) \leq j_{u_n}(u_{n+1}) \leq j_n(u_n) = f_{\beta h}(u_n) \; .$$

Thus we have proven that the sequence $\{f_{\beta h}(u_n)\}_{n\in\mathbb{N}}$ is monotonically decreasing. Since $f_{\beta h}(u_n) \geq 0$, the sequence $\{f_{\beta h}(u_n)\}_{n\in\mathbb{N}}$ is convergent. □

Using the above lemmata we are able to prove convergence of the sequence $\{u_n\}_{n\in\mathbb{N}}$.

THEOREM 2.2. *(Convergence) The sequence $\{u_n\}_{n\in\mathbb{N}}$ generated by the fixed-point iteration (2.2) is convergent to the fixed point $u_{\alpha\beta h}$.*

Proof. From (2.3) we know that the sequence $\{u_n\}$ is uniformly bounded. Since L_h is bounded from $L^2(\Omega)$ into $L^2(\Omega)^d$ it follows from Cauchy-Schwarz inequality

$$\int_\Omega \sqrt{|L_h u_n|^2 + \beta^2} \leq \sqrt{\text{meas }(\Omega)}\sqrt{\int_\Omega |L_h u_n|^2 + \beta^2}$$

that $\int_\Omega \sqrt{|L_h u_n|^2 + \beta^2}$ is uniformly bounded. Thus there exists a weakly convergent subsequence $\{u_k\}$ in $L^2(\Omega)$ of $\{u_n\}$ with weak limit $\tilde{z}$ which satisfies

$$\lim_{k\to\infty}\int_\Omega \sqrt{|L_h u_k|^2 + \beta^2} = \liminf_{n\in\mathbb{N}}\int_\Omega \sqrt{|L_h u_n|^2 + \beta^2} \; .$$

From the weak lower semi-continuity of the $L^2(\Omega)$–norm it follows that

$$\frac{1}{2}\|\tilde{z} - z\|^2 \leq \liminf_{k\to\infty}\frac{1}{2}\|u_k - z\|^2.$$

Let

$$\mathcal{V} := \{v \in L^2(\Omega;\mathbb{R}^d) : \text{esssup } |\text{v(x)}| \leq 1\} \; .$$

From the weak convergence of u_k to $\tilde{z}$ and Fenchel duality (see e.g. [8]) it follows that

$$\begin{aligned}
\int_\Omega \sqrt{|L_h\tilde{z}|^2+\beta^2} &\le \sup_{v\in\mathcal{V}} \int_\Omega \left(\tilde{z}L_h^\star v + \beta\sqrt{1-|v|^2}\right) \\
&= \sup_{v\in\mathcal{V}} \lim_{k\to\infty} \int_\Omega \left(u_k L_h^\star v + \beta\sqrt{1-|v|^2}\right) \\
&\le \lim_{k\to\infty} \sqrt{|L_h u_k|^2+\beta^2}\ .
\end{aligned}$$

This shows that

$$f_{\beta h}(\tilde{z}) \le \liminf f_{\beta h}(u_k)\ . \tag{2.8}$$

In the following we show that $\left\langle f'_{\beta h}(u_k), u_k - \tilde{z}\right\rangle_{L^2(\Omega)} \to 0$ and $f_{\beta h}(u_k) - f_{\beta h}(\tilde{z}) \to 0$. In order to do so, we introduce the set

$$L^2_\beta(\Omega) := \left\{ w : w(x) > \frac{1}{2\beta} \text{ almost everywhere} \right\}$$

and the functional $f^\star_{\beta h} : L^2(\Omega) \times L^2_\beta(\Omega) \to \mathbb{R}$,

$$f^\star_{\beta h}(u,w) = \frac{1}{2}\|u-z\|^2 + \frac{\alpha}{2}\int_\Omega \left(w\left(|L_h u|^2+\beta^2\right) + \frac{1}{w}\right)\ .$$

Let $v_1^1, v_1^2, v_2^1, v_2^2 \in L^2_\beta(\Omega)$ and $\tilde{w}_1 = v_1^1 - v_1^2$, $\tilde{w}_2 = v_2^1 - v_2^2$. Then, the first and second derivatives of $f^\star_{\beta h}$ in the directions $\tilde{w}_1$ and $\tilde{w}_2$ are

$$\frac{\partial f^\star_{\beta h}(u,w)}{\partial w}\tilde{w}_1 = \left\langle \frac{\alpha}{2}\left(|L_h u|^2+\beta^2\right) - \frac{\alpha}{2}\frac{1}{w^2}, \tilde{w}_1 \right\rangle_{L^2(\Omega)}$$

and

$$\frac{\partial^2 f^\star_{\beta h}(u,w)}{\partial w^2}(\tilde{w}_1,\tilde{w}_2) = \left\langle \frac{\alpha}{w^3}, \tilde{w}_1\tilde{w}_2 \right\rangle_{L^2(\Omega)}\ . \tag{2.9}$$

This shows that the infimum of $f^\star_{\beta h}(u,w)$ with respect to w is attained at

$$w^\star = \left(|L_h u|^2+\beta^2\right)^{-\frac{1}{2}}\ . \tag{2.10}$$

Consequently

$$\inf_{w\in L^2(\Omega)} f^\star_{\beta h}(u,w) = f_{\beta h}(u) \quad \text{for all } u \in L^2(\Omega)\ . \tag{2.11}$$

Now, we proceed along the lines in [3, 4], and define an iteration process

$$\begin{aligned}
w_{k+1} &= \operatorname{argmin}_{w\in L^2(\Omega)} f^\star_{\beta h}(\rho_k, w)\ , \\
\rho_{k+1} &= \operatorname{argmin}_{u\in L^2(\Omega)} f^\star_{\beta h}(\rho, w_{k+1})\ .
\end{aligned} \tag{2.12}$$

Then, from (2.10) and (2.11) it follows that

$$\begin{aligned}
w_{k+1} &= \left(|L_h\rho_k|^2+\beta^2\right)^{-\frac{1}{2}}\ , \\
\rho_{k+1} &= \operatorname{argmin}_{\rho\in L^2(\Omega)} \left(\tfrac{1}{2}\|u-z\|^2 + \tfrac{\alpha}{2}\int_\Omega \frac{|L_h\rho|^2}{\sqrt{|L_h\rho_k|^2+\beta^2}}\right)\ .
\end{aligned} \tag{2.13}$$

In particular (2.13) shows that $u_k = \rho_k$. Moreover, from (2.13) it follows that

$$\begin{aligned} f_{\beta h}(u_{k-1}) - f_{\beta h}(u_k) = &\left(f^\star_{\beta h}(u_k, w_k) - f^\star_{\beta h}(u_k, w_{k+1})\right) \\ &+ \left(f^\star_{\beta h}(u_{k-1}, w_k) - f^\star_{\beta h}(u_k, w_k)\right) \ . \end{aligned}$$

From (2.12) it follows that both terms on the right hand side are non-negative. Since by Lemma 2.1 the sequence $\{f_{\beta h}(u_k)\}_{n\in\mathbb{N}}$ is convergent, we obtain

$$f^\star_{\beta h}(u_k, w_k) - f^\star_{\beta h}(u_k, w_{k+1}) \to 0 \ .$$

Using that w_{k+1} is the minimizing element of $f^\star_{\beta h}(u_k, w)$, it follows from Taylor series expansion of the functional $f^\star_{\beta h}$ with respect to the variable w that

$$\begin{aligned} &f^\star_{\beta h}(u_k, w_k) - f^\star_{\beta h}(u_k, w_{k+1}) \\ &= \left\langle \tfrac{\partial f^\star_{\beta h}(u_k, w_{k+1})}{\partial w}, w_k - w_{k+1} \right\rangle_{L^2(\Omega)} \\ &\quad + \quad \int_0^1 t \int_0^1 \left\langle \tfrac{\partial^2 f^\star_{\beta h}(u_k, w_{k+1} + t\tau(w_k - w_{k+1}))}{\partial w^2}(w_k - w_{k+1}), w_k - w_{k+1} \right\rangle_{L^2(\Omega)} \\ &= \int_0^1 t \int_0^1 \left\langle \tfrac{\partial^2 f^\star_{\beta h}(u_k, w_{k+1} + t\tau(w_k - w_{k+1}))}{\partial w^2}(w_k - w_{k+1}), w_k - w_{k+1} \right\rangle_{L^2(\Omega)} d\tau dt \ . \end{aligned}$$

From (2.13), (2.10), and (2.9) it follows that the second derivative of $f^\star_{\beta h}$ with respect to w is bounded from below by $\alpha\beta^3$, and consequently

$$\frac{1}{2}\alpha\beta^3 \int_\Omega (w_k - w_{k+1})^2 \leq f^\star_{\beta h}(u_k, w_k) - f^\star_{\beta h}(u_k, w_{k+1}) \to 0 \ .$$

Thus we have shown that

$$\|w_k - w_{k+1}\|_{L^2(\Omega)} \to 0 \ . \tag{2.14}$$

Since the second derivative of $f_{\beta h}$ is positive semidefinite (cf. 2.6) it follows that

$$\begin{aligned} &\limsup_{k\in\mathbb{N}} \left(f_{\beta h}(u_k) - f_{\beta h}(\tilde{z})\right) \\ &= \limsup_{k\in\mathbb{N}} \Bigg(\left\langle f'_{\beta h}(u_k), u_k - \tilde{z} \right\rangle_{L^2(\Omega)} \\ &\qquad - \quad \int_0^1 t \int_0^1 \left\langle f''_{\beta h}(u_k + t\tau(\tilde{z} - u_k))(\tilde{z} - u_k), \tilde{z} - u_k \right\rangle d\tau dt \Bigg) \\ &\leq \limsup_{k\in\mathbb{N}} \left\langle f'_{\beta h}(u_k), u_k - \tilde{z} \right\rangle_{L^2(\Omega)} \ . \end{aligned}$$

Therefore, using (2.13), (2.14) and the fact that $\rho_k = u_k$, it follows that

$$\begin{aligned} &\limsup \left(f_{\beta h}(u_k) - f_{\beta h}(\tilde{z})\right) \\ &\leq \limsup \langle f'_{\beta h}(u_k), u_k - \tilde{z}\rangle \\ &= \limsup \left\langle \left(L_{\alpha\beta h}(u_k) - L_{\alpha\beta h}(u_{k-1}\right) u_k, u_k - \tilde{z}\right\rangle \\ &= \alpha \limsup \int_\Omega \left(\tfrac{L_h u_k}{\sqrt{|L_h u_k|^2 + \beta^2}} - \tfrac{L_h u_k}{\sqrt{|L_h u_{k-1}|^2 + \beta^2}} \right) \cdot \ L_h(u_k - \tilde{z}) \\ &= \alpha \limsup \int_\Omega (w_{k+1} - w_k) L_h u_k L_h(u_k - \tilde{z}) \\ &\leq 2\alpha \limsup \|w_{k+1} - w_k\| \||L_h\|^2 \|z\|^2 \\ &= 0 \ . \end{aligned} \tag{2.15}$$

Therefore, using (2.8), which implies that $\limsup \left(f_{\beta h}(u_k) - f_{\beta h}(\tilde{z})\right) \geq 0$, we obtain that

$$\lim_{k\to\infty} f_{\beta h}(u_k) = f_{\beta h}(\tilde{z}) \ , \tag{2.16}$$

and in turn it follows from (2.15) that

$$\lim_{k\to\infty} \left\langle f'_{\beta h}(u_k), u_k - \tilde{z} \right\rangle = 0 . \tag{2.17}$$

It remains to prove that u_k converges to $\tilde{z}$ strongly. Using (2.6), (2.16), and (2.17), we see that

$$\begin{aligned} 0 &= \liminf_{k\to\infty} (f_{\beta h}(u_k) - f_{\beta h}(\tilde{z})) \\ &\leq \liminf_{k\to\infty} \Big\{ \left\langle f'_{\beta h}(u_k), u_k - \tilde{z} \right\rangle \\ &\qquad - \liminf_{k\to\infty} \int_0^1 t \int_0^1 \left\langle f''_{\beta h}(u_k + t\tau(\tilde{z} - u_k))(\tilde{z} - u_k)(\tilde{z} - u_k) \right\rangle d\tau\, dt \Big\} \\ &\leq -\liminf_{k\to\infty} \|\tilde{z} - u_k\|^2_{L^2(\Omega)} \\ &\leq 0 . \end{aligned} \tag{2.18}$$

Taking $u_{\tilde{k}}$ a subsequence satisfying

$$\lim_{\tilde{k}} \|\tilde{z} - u_{\tilde{k}}\|^2_{L^2(\Omega)} = \liminf_{k\to\infty} \|\tilde{z} - u_k\|^2_{L^2(\Omega)} ,$$

it follows from (2.18) that $u_k \to \tilde{z}$ in $L^2(\Omega)$. Hence, every subsequence of the sequence $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $u_{\tilde{k}}$ and therefore $\{u_n\}_{n\in\mathbb{N}}$ is itself convergent. $\square$

In the above theorem, we have established convergence of the fixed point iteration with respect to the $L^2(\Omega)$–norm. Typical convergence results for functions of bounded variation guarantee that the sequence is convergent in the $L^p(\Omega)$-norm and that the functional $\int_\Omega |Du|$ is convergent; note that $\int_\Omega |Du|$ is defined via a weak formulation and is in general different from $\int_\Omega |\nabla u|$. A similar result also holds in our setting, namely

$$\lim_{n\to\infty} \int_\Omega \sqrt{|L_h u_n|^2 + \beta^2} = \int_\Omega \sqrt{|L_h u_{\alpha\beta h}|^2 + \beta^2} .$$

Lemma 2.3. *Let $\{u_n\}$ be the sequence of (2.2). Then*

$$\lim_{n\to\infty} \int_\Omega \sqrt{|L_h u_n|^2 + \beta^2} = \int_\Omega \sqrt{|L_h u_{\alpha\beta h}|^2 + \beta^2} .$$

Proof. From frequent application of the triangle inequality it follows that

$$\begin{aligned} &\int_\Omega \left| \sqrt{|L_h u_n|^2 + \beta^2} - \sqrt{|L_h \hat{z}|^2 + \beta^2} \right| = \\ &= \int_\Omega \left| \frac{|L_h u_n|^2 - |L_h u_{\alpha\beta h}|^2}{\sqrt{|L_h u_n|^2 + \beta^2} + \sqrt{|L_h u_{\alpha\beta h}|^2 + \beta^2}} \right| \\ &\leq \int_\Omega ||L_h \hat{u}_n| - |L_h u_{\alpha\beta h}|| \\ &\leq \int_\Omega |L_h(u_n - u_{\alpha\beta h})| \\ &\leq \sqrt{\text{meas } |\Omega|} \|L_h\| \|u_n - u_{\alpha\beta h}\|_{L^2(\Omega)} \\ &\to 0 \quad \text{as } n \to \infty \end{aligned}$$

This shows the assertion. $\square$

3. Stability. In this section we analyze the stability of the fixed point iteration (2.2) with respect to perturbations in the data z. Let $z^\delta \in L^2(\Omega)$ denote perturbed data of z satisfying

$$\|z - z^\delta\|_{L^2(\Omega)} \le \delta \; .$$

In this situation, instead of minimizing the functional $f_{\beta h}$ we consider minimization of the functional

$$f_{\beta h}^\delta(u) = \frac{1}{2}\|u - z^\delta\|^2_{L^2(\Omega)} + \alpha \int_\Omega \sqrt{|L_h u|^2 + \beta^2} \; . \tag{3.1}$$

In [6] it was shown that there exists a unique minimizer of the functional $f_{\beta h}^\delta$, which will be denoted by $u_{\alpha\beta h}^\delta$. Moreover, there we proved stability of the minimizer of the functional $f_{\beta h}$. Since the minimizer of the functional is a fixed point of (2.1) and vice versa, the following result is an immediate consequence of the stability result in [6].

THEOREM 3.1. *(Stability) Let $\delta \to 0$. Then*

$$u_{\alpha\beta h}^\delta \to u_{\alpha\beta h} \; .$$

The fixed point iteration using perturbed data z^δ is given by

$$u_{n+1}^\delta = (I + L_{\alpha\beta h}(u_n^\delta))^{-1} z^\delta \; . \tag{3.2}$$

From the general results in Section 2 it follows that this iteration process is unconditionally convergent and the limit is the minimizer of the functional $f_{\beta h}^\delta$.

4. Convergence Rates. In this section we present a convergence rates result for (2.2). A similar result holds for (3.2) but this is omitted to avoid dublications.

First, we will prove a Lemma giving the abstract framework for the estimate of the convergence rate. Then we will apply the lemma to the problem at hand.

LEMMA 4.1. *Let $(V, \|.\|_V)$ be a Hilbert space, $f : V \to \mathbb{R}$ a continuous functional, and $\|.\|_{A_u}$, $u \in V$ a family of Hilbert space norms equivalent to $\|.\|_V$. Assume there holds*

$$\underline{j}_u(v) \le f(v) \le \overline{j}_u(v) \qquad \forall\, u, v \in V \tag{4.1}$$

for the lower and upper quadratic approximations

$$\begin{aligned}\overline{j}_u(v) &= f(u) + \langle f'(u), v - u\rangle + \frac{1}{2}\|u - v\|^2_{A_u},\\ \underline{j}_u(v) &= f(u) + \langle f'(u), v - u\rangle + \frac{\nu}{2}\|u - v\|^2_{A_u}\end{aligned}$$

and some $\nu \in (0, 1]$. Then the iteration

$$u_{n+1} = argmin\, \overline{j}_{u_n}(v) \tag{4.2}$$

applied for some $u_0 \in V$ converges to the unique minimizer u^*. The function values converge monotone and linearly, namely

$$f(u_{n+1}) - f(u^*) \le (1 - \nu)\; (f(u_n) - f(u^*))\, . \tag{4.3}$$

The iterates converge linearly, namely

$$\|u_n - u^*\|_V \leq c(1-\nu)^{k/2} \tag{4.4}$$

with a constant c depending on the norm equivalence constants between $\|.\|_{A_{u^*}}$ and $\|.\|_V$, and the parameter ν.

Proof. Let $A_u(.,.)$ be the inner product associated to the Hilbert space norm $\|.\|_{A_u}$. Then the first order optimality condition of (4.2) reads as

$$0 = \left\langle \overline{\jmath}'_{u_n}(u_{n+1}), v \right\rangle = \langle f'(u_n), v\rangle + A_{u_n}(u_{n+1} - u_n, v) = 0 \qquad \forall\, v \in V.$$

This identity applied for $v = u_{n+1} - u_n$ gives the decrease in $\overline{\jmath}_{u_n}(.)$ as

$$\begin{aligned} \overline{\jmath}_{u_n}(u_{n+1}) - \overline{\jmath}_{u_n}(u_n) &= \langle f'(u_n), u_{n+1} - u_n\rangle + \frac{1}{2}\|u_{n+1} - u_n\|^2_{A_{u_n}} \\ &= -\frac{1}{2}\|u_{n+1} - u_n\|^2_{A_{u_n}}. \end{aligned}$$

Analogue, the minimization of the lower bound $\underline{j}_{u_n}(.)$ gives

$$\min_v \underline{j}_{u_n}(v) - \underline{j}_{u_n}(u_n) = -\frac{1}{2\nu}\|u_{n+1} - u_n\|^2_{A_{u_n}}.$$

Summing up, we get

$$\begin{aligned} f(u_{n+1}) - f(u^*) &\leq \overline{\jmath}_{u_n}(u_{n+1}) - f(u^*) \\ &= f(u_n) - 1/2\|u_{n+1} - u_n\|^2_{A_{u_n}} - f(u^*) \\ &= f(u_n) + \nu\left(\min_v \underline{j}_{u_n}(v) - \underline{j}_{u_n}(u_n)\right) - f(u^*) \\ &\leq f(u_n) + \nu\left(\min_v f(v) - f(u_n)\right) - f(u^*) \\ &= (1-\nu)\left(f(u_n) - f(u^*)\right). \end{aligned}$$

Using the equivalence of $\|.\|_V$ and $\|.\|_{A_{u^*}}$ we get the convergence of the iterates

$$\begin{aligned} \|u_n - u^*\|^2_V &\leq c\,\|u_n - u^*\|^2_{A_u^*} \\ &= \frac{2c}{\nu}\left(\underline{j}_{u^*}(u_n) - \underline{j}_{u^*}(u^*)\right) \\ &\leq \frac{2c}{\nu}\left(f(u_n) - f(u^*)\right) \\ &\leq (1-\nu)^k \frac{2c}{\nu}\left(f(u_0) - f(u^*)\right), \end{aligned}$$

and the proof is complete. ☐

We recall the function of interest

$$f_{\beta h}(v) = \frac{1}{2}\|v - z\|^2_{L_2} + \alpha \int_\Omega \sqrt{|L_h v|^2 + \beta^2}.$$

We use the family of Hilbert space norms defined as

$$\|v\|^2_{A_u} := \|v\|^2_{L_2} + \alpha \int_\Omega \frac{|L_h u|^2}{\sqrt{|L_h u|^2 + \beta^2}}.$$

This norm and its inner product $A_u(.,.)$ is related to the operator (2.1) of the fixed point iteration as

$$A_u(v,w) = ((I + L_{\alpha\beta h}(u))v, w)_{L_2} \qquad \forall\, u, v, w, \in V.$$

Thus it is easy to see that the iterates obtained from the fixed point iteration and iteration (4.2) are identic.

LEMMA 4.2. *The convergence rate of the iteration (4.2) applied for the functional $f_{\beta h}$ with the definition of $\|.\|_{A_u}$ from above is bounded by*

$$1 - \nu = \frac{\frac{\alpha}{\beta}\|L_h\|^2}{1 + \frac{\alpha}{\beta}\|L_h\|^2} < 1. \tag{4.5}$$

Proof. We have to verify the estimates (4.1). The upper bound has been proved in Lemma 2.1. Since the functional $\int \sqrt{|L_h v|^2 + \beta^2}$ is convex, the functional $f_{\beta h}$ is strongly convex and fulfills

$$f(v) \geq f(u) + \langle f'(u), v - u\rangle + \frac{1}{2}\|u - v\|_{L_2}^2 \qquad \forall\, u, v \in V.$$

If we set ν such that

$$\nu\|v\|_{A_u}^2 \leq \|v\|_{L_2}^2 \qquad \forall\, u, v \in V \tag{4.6}$$

holds, then the lower bound of (4.1) is fulfilled. Indeed,

$$\begin{aligned}\|v\|_{A_u}^2 &= \|v\|_{L_2}^2 + \alpha \int \frac{|L_h v|^2}{\sqrt{|L_h u|^2 + \beta^2}} \\ &\leq \|u\|_{L_2}^2 + \frac{\alpha}{\beta}\|L_h u\|_{L_2}^2 \\ &\leq \left(1 + \frac{\alpha}{\beta}\|L_h\|^2\right)\|u\|_{L_2}^2\end{aligned}$$

leads to

$$\nu := \left(1 + \frac{\alpha}{\beta}\|L_h\|^2\right)^{-1}$$

and to the convergence rate (4.5). □

Although the fixed point iteration (2.2) is convergent for any parameter setting $\alpha > 0$, $\beta > 0$ and $h > 0$, the convergence rate deteriors for small values β and h.

The following theorem proves linear and monotone convergence of the iterates $\{u_n\}_{n\in\mathbb{N}}$ if we impose additional restrictions on the parameters α, β and h:

THEOREM 4.3. *For $\frac{\alpha}{\beta}\|L_h\|^2 \leq 2$ we have linear convergence:*

$$\|u_{n+1} - u_{\alpha\beta h}\| \leq \sqrt{2\,(1-\nu)\left(\frac{1}{2} + \frac{\alpha}{\beta}\|L_h\|^2\right)}\,\|u_n - u_{\alpha\beta h}\|$$

with ν as in Lemma 4.2.

Proof. Using the previous lemma and Taylor-series expansion we get

$$\begin{aligned}
\frac{1}{2}\|u_{n+1} - u_{\alpha\beta h}\|^2 &\leq f_{\beta h}(u_{n+1}) - f_{\beta h}(u_{\alpha\beta h}) \\
&\leq (1-\nu)\left(f_{\beta h}(u_n) - f_{\beta h}(u_{\alpha\beta h})\right) \\
&\leq (1-\nu)\left(\langle f'_{\beta h}(u_n), u_n - u_{\alpha\beta h}\rangle - \frac{1}{2}\|u_n - u_{\alpha\beta h}\|^2\right) \\
&\leq (1-\nu)\left(\frac{1}{2} + \frac{\alpha}{\beta}\|L_h\|^2\right)\|u_n - u_{\alpha\beta h}\|^2
\end{aligned}$$

□

In particular it follows from Theorem 4.3 that linear convergence of (3.2) holds for a parameter setting where α is sufficiently small, the discretization is relatively coarse, i.e., that $\|L_h\|$ is relatively small, or if β is relatively large. These conditions have to be chosen compatiblely with the general asymptotical convergence results in [6], which is possible.

5. Cascadic Algorithm. In the following we analyze a cascadic algorithm for minimizing the functional $f_{\beta h}$.

For describing a cascadic algorithm we state the dependence of u_n^δ on the parameters α, β and on the operator L_h explicitly, i.e., $u_{n,\alpha,\beta,h}^\delta = u_n^\delta$.

For bounded variation minimization analyze the following cascadic algorithm:

1. Initialization ($N = 0$): Choose a parameter setting α_0, β_0, and an operator L_{h_0} (note that in typical applications the operator L_h is a discretization operator). Set
$$u_{0,\alpha_0,\beta_0,L_{h_0}}^\delta = 0\,,$$
and use the fixed point iteration (2.2) to calculate a minimizer of $f_{\alpha_0,\beta_0,L_{h_0}}$.
2. Iterative Refinement ($N \to N+1$): Let $u_{\alpha_N,\beta_N,L_{h_N}}$ be the minimizer of $f_{\alpha_N,\beta_N,L_{h_N}}$. Set
$$u_{0,\alpha_{N+1},\beta_{N+1},L_{h_{N+1}}}^\delta = u_{\alpha_N,\beta_N,L_{h_N}}^\delta\,, \tag{5.1}$$
and use the fixed point iteration (2.2) to calculate a minimizer of the functional $f_{\alpha_{N+1},\beta_{N+1},L_{h_{N+1}}}$.

The cascadic algorithm as defined above is well defined by the general results in Section 2. The two crucial steps in bounded variation minimization with a cascadic algorithm are

1. The minimizer of $f_{\alpha_0,\beta_0,L_{h_0}}$ is not calculated exactly using (2.2). Therefore, we have to implement an appropriate stopping criterion at each iteration level N.
2. The parameters α_N, β_N and the operator L_{h_N} have to be updated appropriately, to guarantee that the cascade algorithm is effective.

The following theorem suggests to use a stopping criterion at fixed level based on the norm of the difference of two successive iterates.

THEOREM 5.1. *(Error Estimate) Let the parameters α, β be positive and fixed. Moreover, let the operator L_h be fixed. Let $u_{\alpha\beta h}^\delta$ be the minimizer of the functional*

$f^\delta_{\alpha,\beta,h}$. *Let* u^δ_n *be the iterates of the fixed point iteration (3.2). Then*

$$\left\| u^\delta_n - u^\delta_{\alpha\beta h} \right\|_{L^2(\Omega)} \leq \left(2 + \tfrac{\alpha}{\beta}\|L_h\|^2\right) \|u^\delta_n - u^\delta_{n-1}\| \,.$$

Proof. From triangle inequality it follows that

$$\begin{aligned} \|u^\delta_n - u^\delta_{\alpha\beta h}\| &\leq \|u^\delta_n - u^\delta_{n-1}\| + \|u^\delta_{n-1} - u^\delta_{\alpha\beta h}\| \\ &\leq \left(2 + \frac{\alpha}{\beta}\|L_h\|^2\right) \|u^\delta_n - u^\delta_{n-1}\| \,. \end{aligned}$$

□

In practical application (cf. Section 6) the norm of two successive iterates of (2.2) is small after at most three iterations, where we can stop the iteration (cf. Figures 6.14 and 6.15).

Let $u^\delta_{n,\alpha_{N+1},\beta_{N+1},L_{h_{N+1}}}$ be the iterates of the fixed point iteration (2.2) for a parameter setting $\alpha_{N+1}, \beta_{N+1}, h_{N+1}$. Then it follows from Theorem 4.3 that

$$\begin{aligned} &\left\| u^\delta_{1,\alpha_{N+1},\beta_{N+1},L_{h_{N+1}}} - u^\delta_{\alpha_{N+1},\beta_{N+1},L_{h_{N+1}}} \right\|_{L^2(\Omega)} \\ &\leq \quad \sqrt{2(1-\nu)\left(\tfrac{1}{2} + \tfrac{\alpha}{\beta}\|L_h\|^2\right)} \left\| u^\delta_{\alpha_N,\beta_N,L_{h_N}} - u^\delta_{\alpha_{N+1},\beta_{N+1},L_{h_{N+1}}} \right\|_{L^2(\Omega)} \,. \end{aligned}$$

This shows that the first iteration of (2.2) at level $N+1$ is a good approximation to the fixed point at level $N+1$, using a parameter choice satisfying that $\frac{\alpha}{\beta}\|L_h\|^2$ is small and if the fixed point at level N and $N+1$ are close. In particular this shows that for small variations of the parameters α, β, h only a few iterations at each level have to be performed. This is confirmed by our numerical experiments (cf. Figure 6.16, 6.17, 6.18, 6.19, 6.20 and 6.21). Moreover, we note that the parameter choice for linear convergence of the fixed point iteration has to be compatible with the general convergence results in [6]. Combination of convergence results nicely results from regularization theory with the numerical analysis.

6. Numerical Results. In this section we present some numerical experiment for bounded variation regularization using (2.2). The numerical simulations where carried out within the software package DIFFPACK [7] using as test data an ultrasound image (cf. Figure 6.1)

Figure 6.2 shows the results for different values of the regularization parameter α, once calculated without, once with grid refinement. There is no visible difference whether one uses grid refinement or not, but the CPU requirements decrease significantly when using refinement, since only the last iterations have to be carried out with high resolution.

We first show results for different fixed parameter settings, varying the smoothing parameter β (Figure 6.4), the discretization, i.e. the parameter h (Figure 6.3). The numerical simulations show that the function values $f_{\beta h}(u_n)$ are monotonely decreasing. This confirms the theoretical results of Lemma 2.1.

FIG. 6.1. *test image of the size* 112×194 *pixels.*

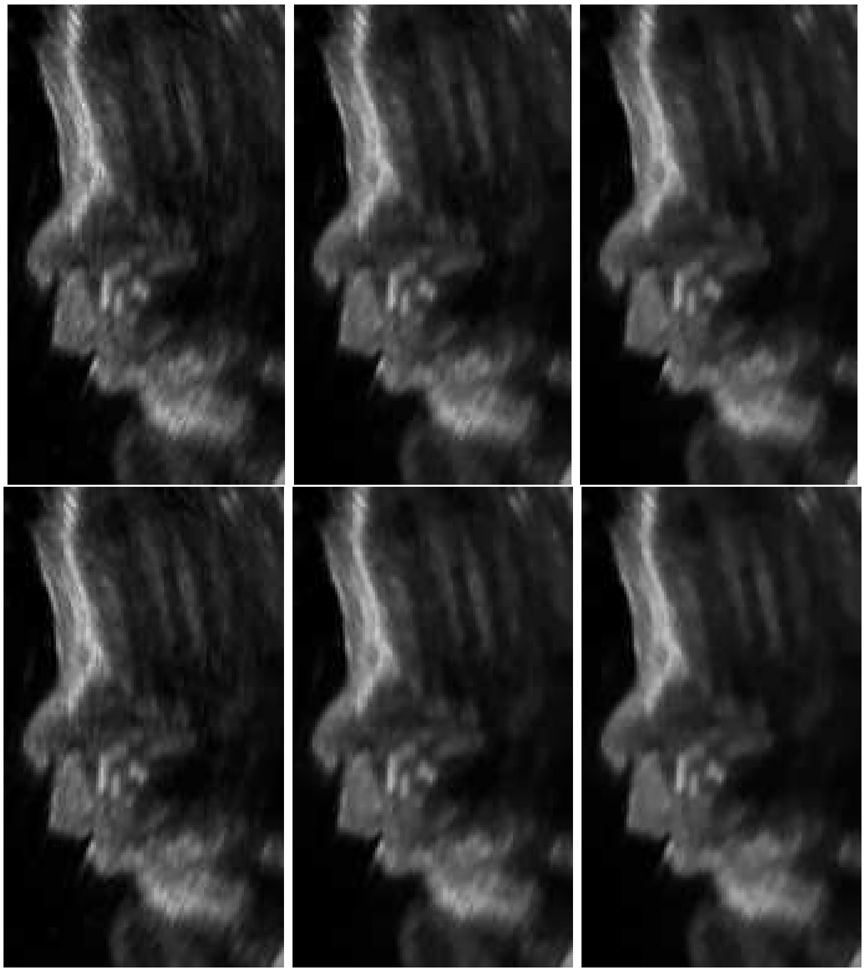

FIG. 6.2. *denoised versions of the original dataset after* 8 *iterations; the parameter settings are* $\beta = 0.001$ *and* $\alpha = 0.5, 2.0, 4.0$ *(from left to right); on top the results without refinement, the bottom line shows the results calculated with grid refinement.*

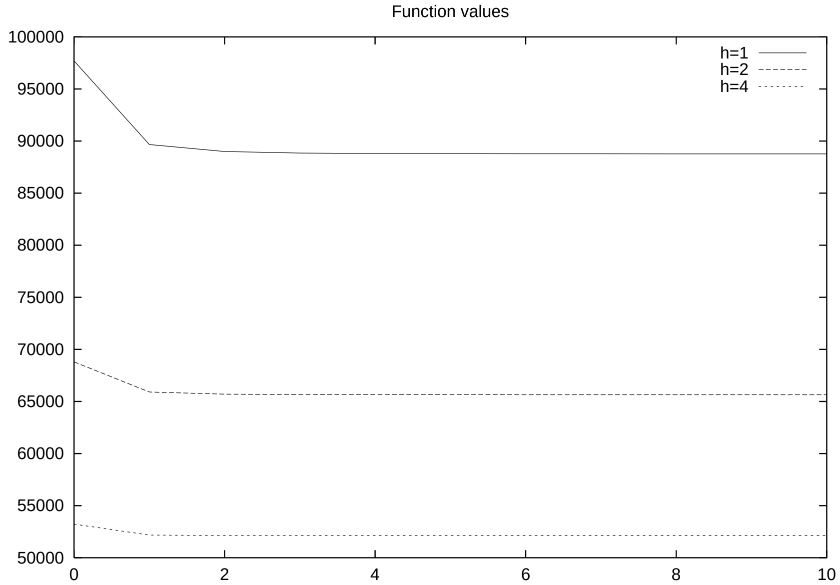

FIG. 6.3. *the development of the function values $f_{\beta h}$ for increasing iteration number with mesh size $h = 1, 2, 4$ respectively; $\alpha = 0.5$, $\beta = 10^{-3}$*

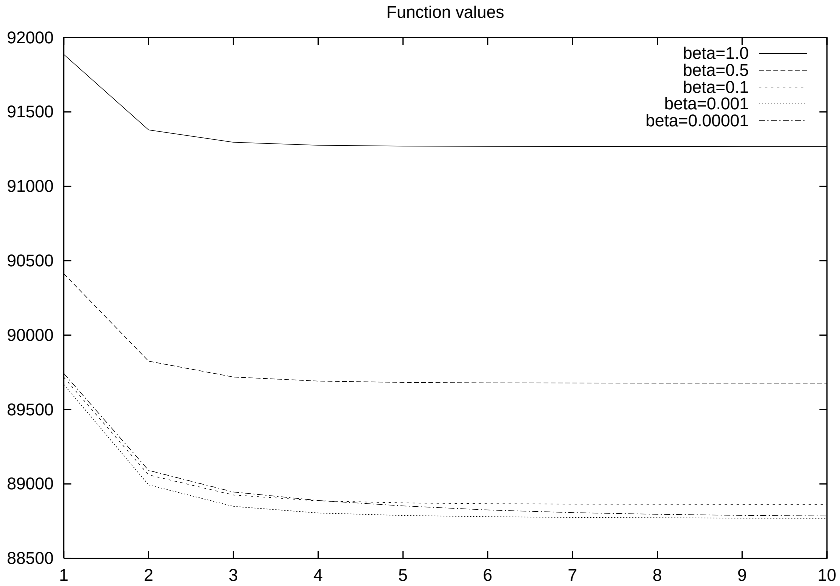

FIG. 6.4. *the development of the function values $f_{\beta h}$ for increasing iteration number; $\alpha = 0.5$, $h = 1$*

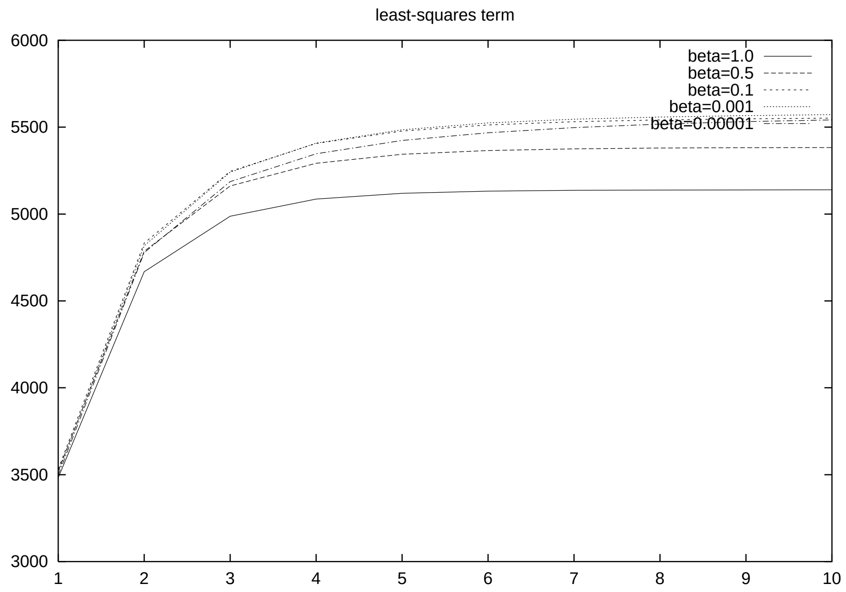

FIG. 6.5. *the development of the least-squares term of the optimization functional $f_{\beta h}$ for increasing iteration number; $\alpha = 0.5$, $h = 1$*

Moreover, in all numerical examples it turned out that the least-squares term of the optimization functional is increasing with the iteration number (Figures 6.5, 6.7), whereas the regularization term is decreasing (Figures 6.6 and 6.8).

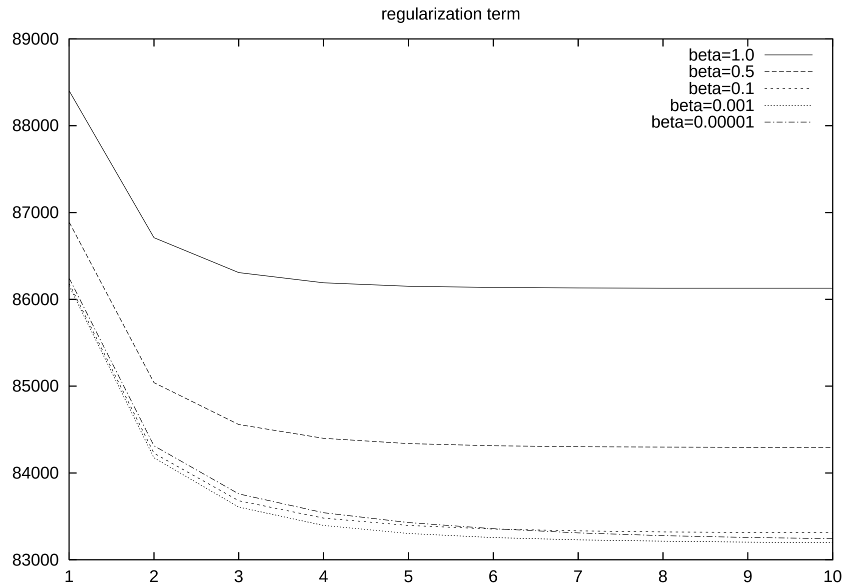

FIG. 6.6. *the development of the regularization term of the optimization functional $f_{\beta h}$ for increasing iteration number;* $\alpha = 0.5,\ h = 1$

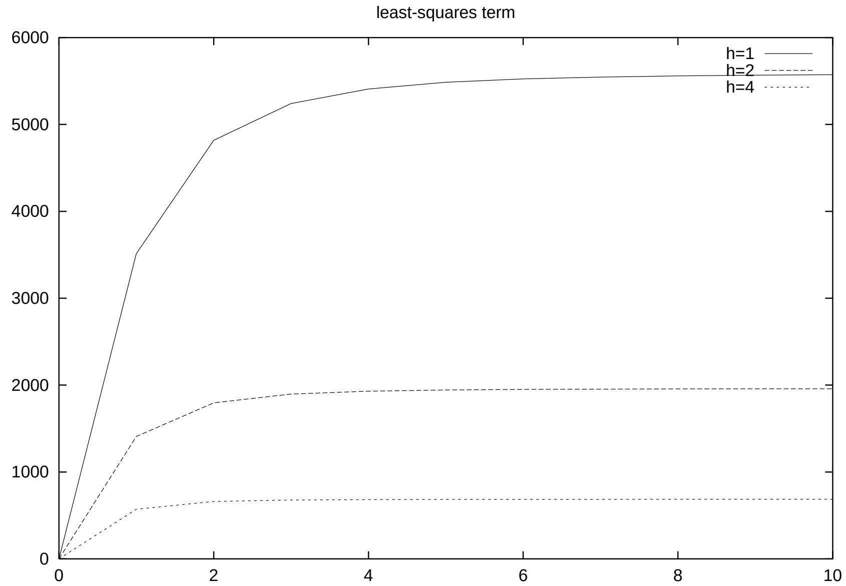

FIG. 6.7. *the development of the least-squares term of the optimization functional $f_{\beta h}$ for increasing iteration number;* $\alpha = 0.5,\ \beta = 0.001$

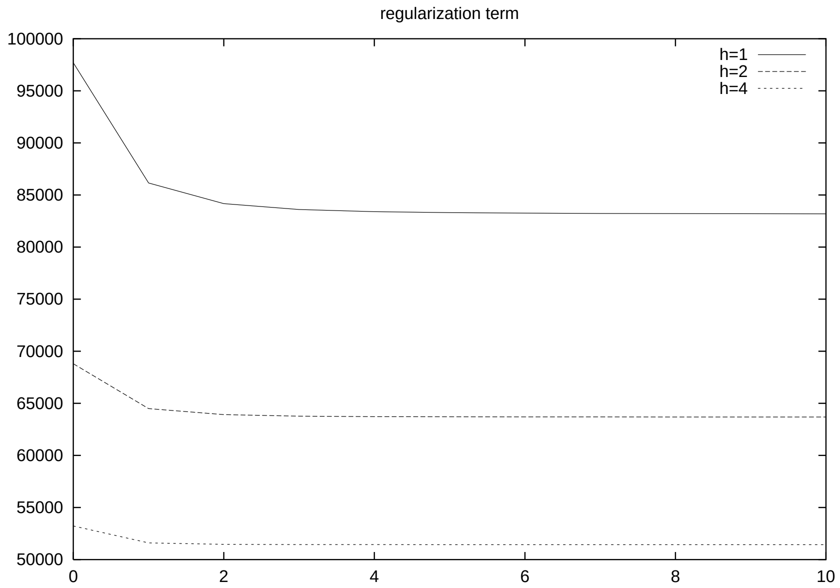

FIG. 6.8. *the development of the regularization term of the optimization functional $f_{\beta h}$ for increasing iteration number;* $\alpha = 0.5$, $\beta = 0.001$

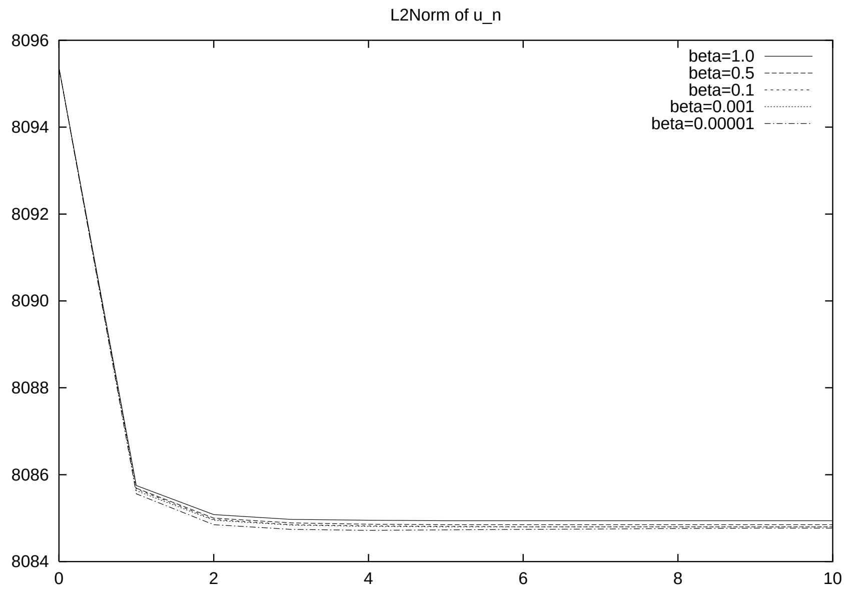

FIG. 6.9. *the development of the L^2-norm of the iterates u_n for increasing iteration number n;* $\alpha = 0.5$, $h = 1$

From 2.5 and Theorem 2.2 it follows that the $\|u_n\|_{L^2\Omega}$ is decreasing. This can also be observed in numerical experiments (cf. Figure 6.9).

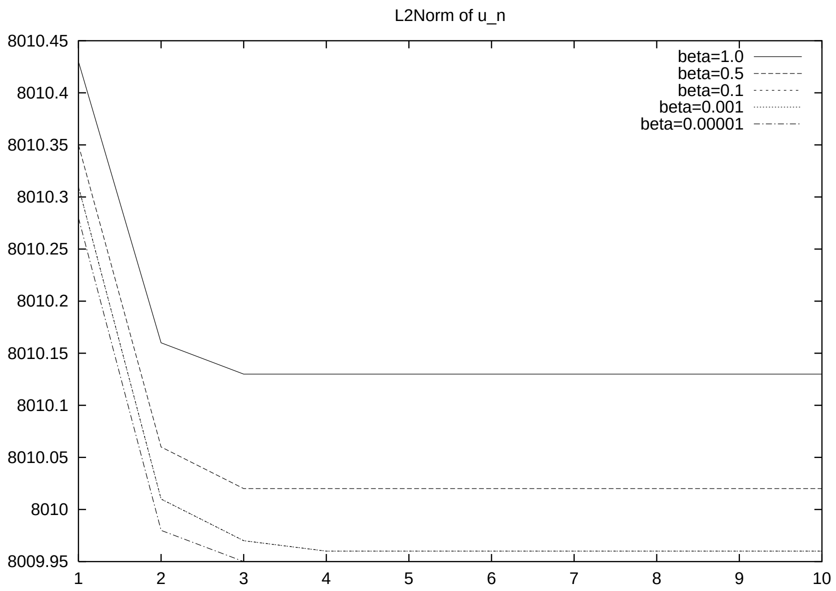

FIG. 6.10. $\|u_n\|_{L^2(\Omega)}$ *as a function of* n *is decreasing. We used the parameters* $\alpha = 0.5$, $h = 2$

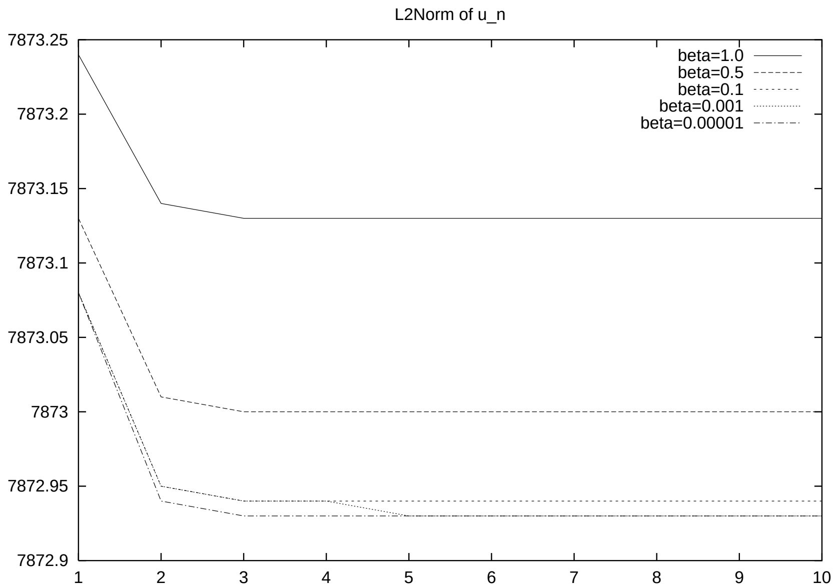

FIG. 6.11. *the development of the* L^2*-norm of the iterates* u_n *for increasing iteration number* n*;* $\alpha = 0.5$, $h = 4$

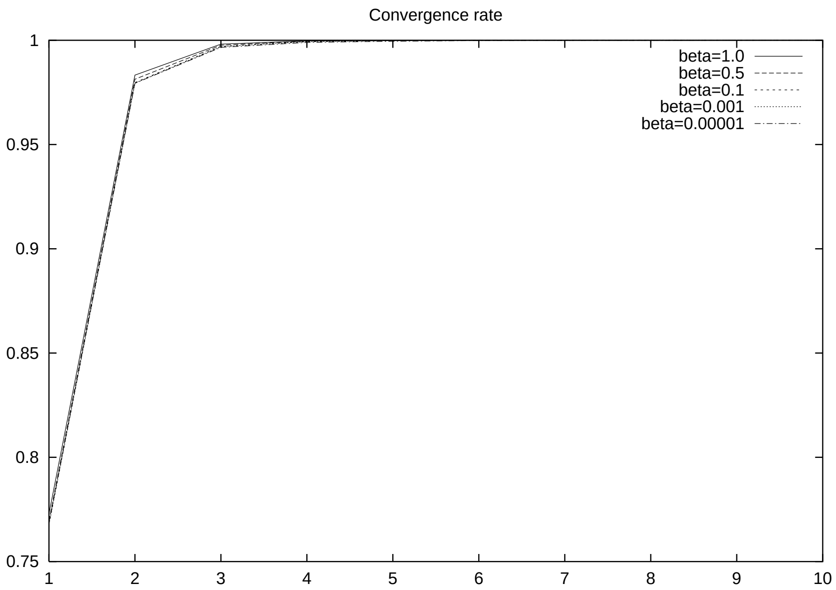

FIG. 6.12. *the convergence rate for different values of β, i.e.* $\frac{\|u_n - u_{\alpha\beta h}\|}{\|u_{n-1} - u_{\alpha\beta h}\|}$; $\alpha = 0.5$, $h = 2$

A plot of the convergence rate $\frac{\|u_n - u_{\alpha\beta h}\|}{\|u_{n-1} - u_{\alpha\beta h}\|}$ shows that only the first iterations yield an important amelioration of the result, especially for small values of β (Figures 6.12 and 6.13):

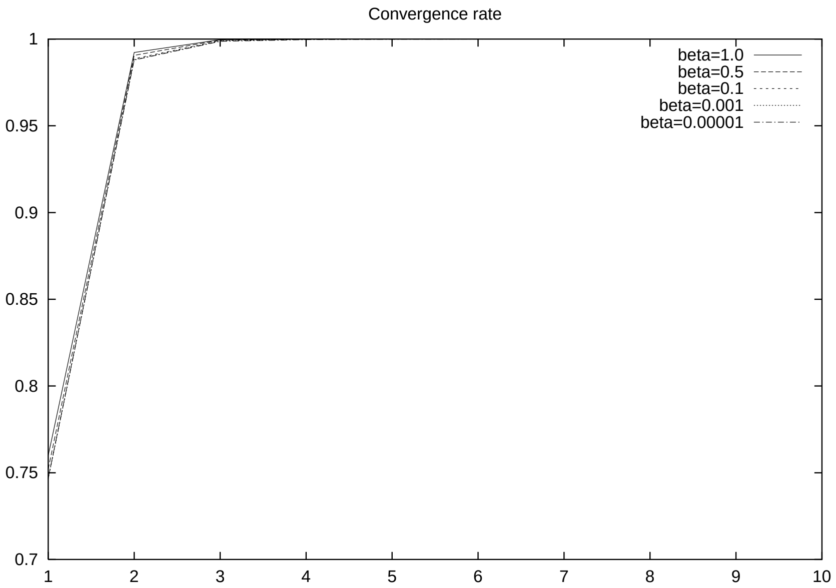

FIG. 6.13. *the convergence rate for different values of* β*, i.e.* $\frac{\|u_n - u_{\alpha\beta h}\|}{\|u_{n-1} - u_{\alpha\beta h}\|}$; $\alpha = 0.5$, $h = 4$

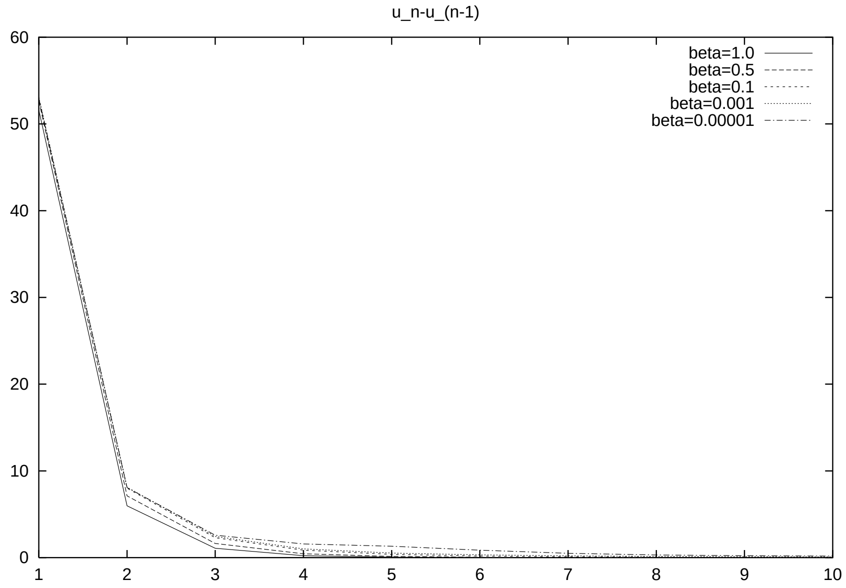

FIG. 6.14. $\|u_n - u_{n-1}\|$ *for different* β *and* $\alpha = 0.5$, $h = 2$

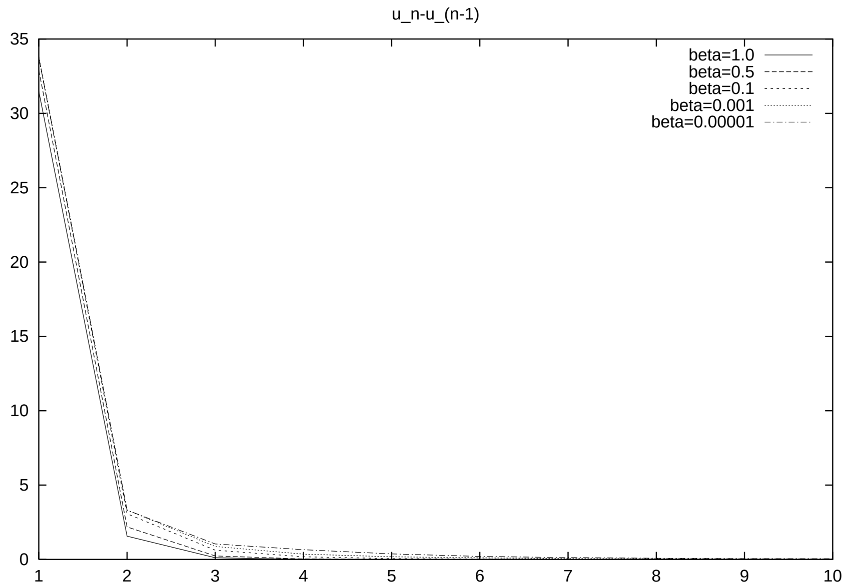

FIG. 6.15. $\|u_n - u_{n-1}\|$ *for different* β *and* $\alpha = 0.5$, $h = 4$

These results suggest, especially Figures 6.14, 6.15, 6.12 and 6.13 that a multi-level algorithm should perform only very few steps on each level:

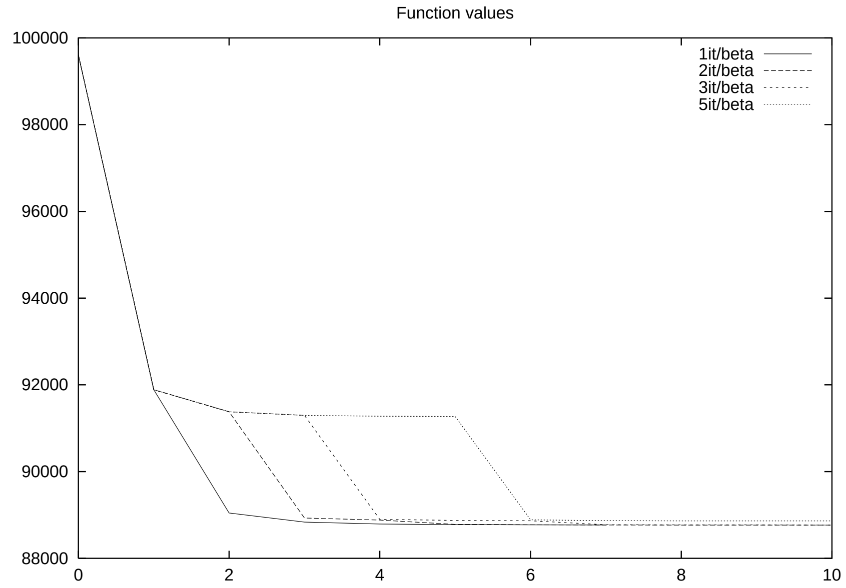

FIG. 6.16. *the development of the function values $f_{\beta h}$ for increasing iteration number n; $\alpha = 0.5$, $h = 1$; the start value for β was choosen 1.0, on every new level β was refined by a factor of 0.1; the different curves are for a changement of level every 1, 2, 3, 5 iterations respectively.*

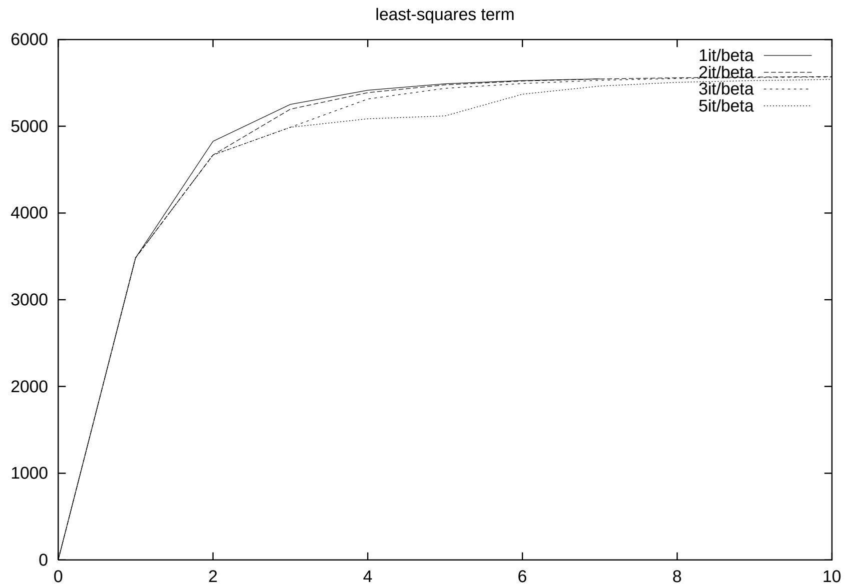

FIG. 6.17. *the development of the least-squares term of $f_{\beta h}$ for increasing iteration number n; $\alpha = 0.5$, $h = 1$; the start value for β was choosen 1.0, on every new level β was refined by a factor of 0.1; the different curves are for a changement of level every 1, 2, 3, 5 iterations respectively.*

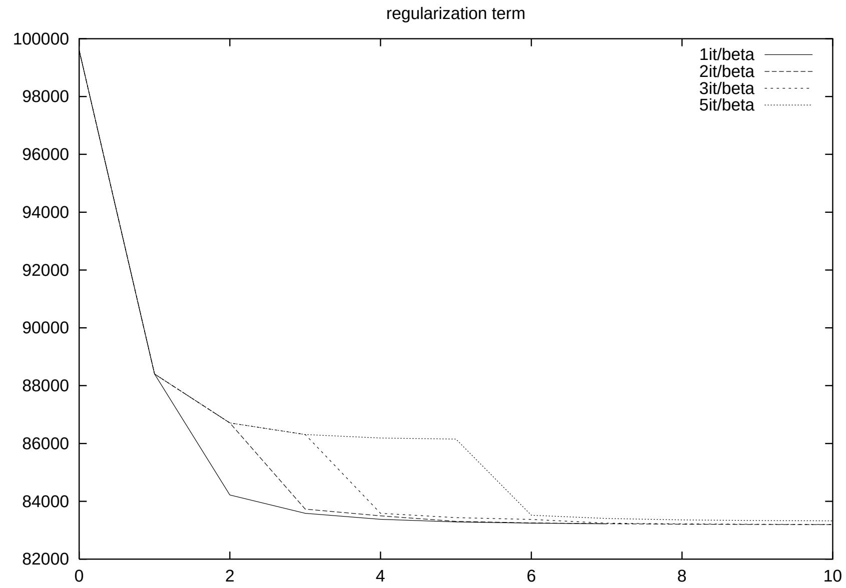

FIG. 6.18. *the development of the regularization term of* $f_{\beta h}$ *for increasing iteration number* n*;* $\alpha = 0.5$*,* $h = 1$*; the start value for* β *was choosen* 1.0*, on every new level* β *was refined by a factor of* 0.1*; the different curves are for a changement of level every* 1*,* 2*,* 3*,* 5 *iterations respectively.*

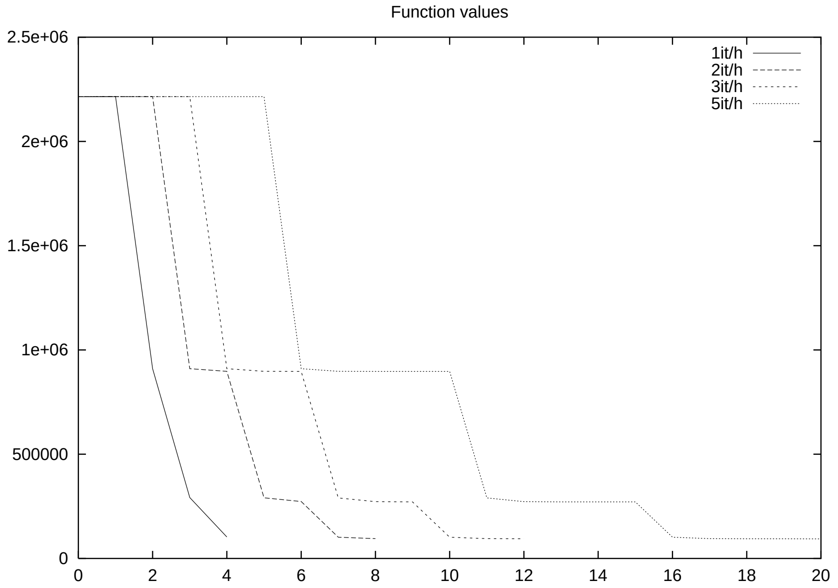

FIG. 6.19. *the development of the function values $f_{\beta h}$ for increasing iteration number n; $\alpha = 0.5$, $\beta = 0.001$; the starting grid was only 14×24, on every new level the grid was refined by a factor 2 in both directions; the different curves are for a changement of level every 1, 2, 3, 5 iterations respectively.*

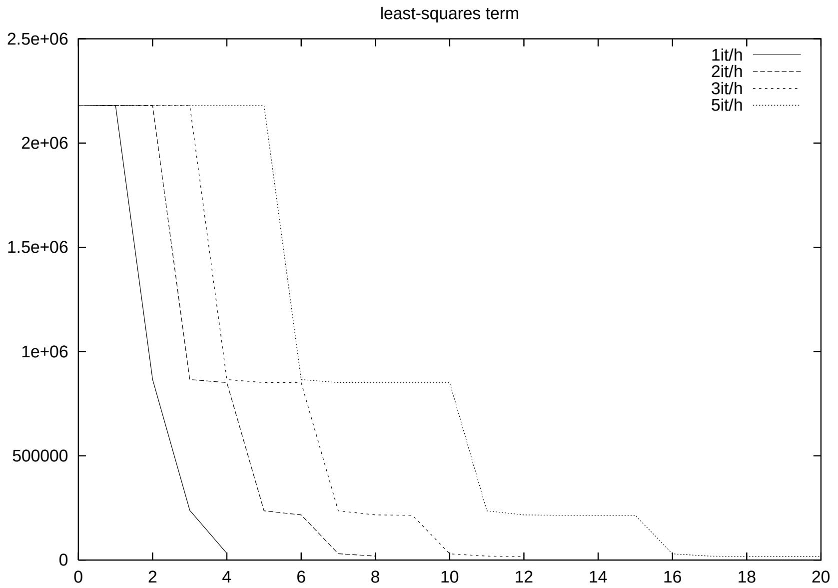

FIG. 6.20. *the development of the least-squares term of $f_{\beta h}$ for increasing iteration number n; $\alpha = 0.5$, $\beta = 0.001$; the starting grid was only 14×24, on every new level the grid was refined by a factor 2 in both directions; the different curves are for a changement of level every 1, 2, 3, 5 iterations respectively.*

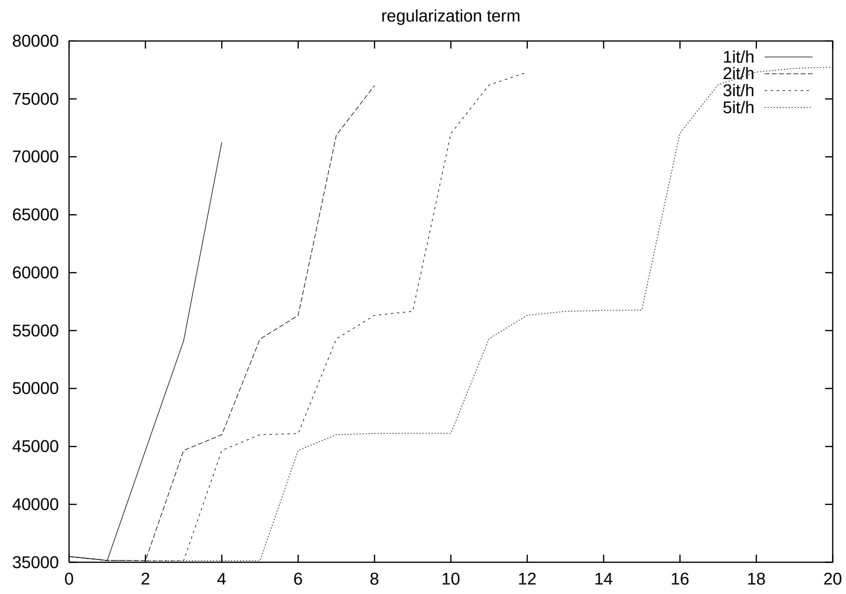

FIG. 6.21. *the development of the regularization term of $f_{\beta h}$ for increasing iteration number n; $\alpha = 0.5$, $\beta = 0.001$; the starting grid was only 14×24, on every new level the grid was refined by a factor 2 in both directions; the different curves are for a changement of level every 1, 2, 3, 5 iterations respectively.*

REFERENCES

[1] A. Chambolle and P.-J. Lions. Image recovery via total variation minimization and related problems. *Numerische Mathematik*, 76(2):167–188, 1997.

[2] C. R. Vogel and M. E. Oman. Iterative methods for total variation denoising. *SIAM Journal of Scientific Computing*, 17:227–238, 1996.

[3] Pierre Charbonnier, Laure Blanc-Féraud, Gilles Aubert, and Michel Barlaud. Deterministic edge-preserving regularization in computed imaging. *IEEE Transactions on Image Processing*, 5(12):298–311, 1997.

[4] D. C. Dobson and C. R. Vogel. Convergence of an iterative method for total variation denoising. *SIAM Journal of Numerical Analysis*, 34(5):1779–1791, October 1997.

[5] D. C. Dobson and F. Santosa. Recovery of blocky images from noisy and blurred data. *SIAM Journal of Applied Mathematics*, 56(4):1181–1198, August 1996.

[6] D. C. Dobson and O. Scherzer. Analysis of regularized total variation penalty methods for denoising. *Inverse Problems*, 12:601–617, 1996.

[7] Morten Dæhlen and Aslak Tveito. *Numerical Methods and Software Tools in Industrial Mathematics*. Birkhäuser, Boston, 1997.

[8] Klaus Deimling. *Nonlinear Functional Analysis*. Springer, Berlin, 1985.

[9] E. Casas, K. Kunisch, and C. Pola. Regularization by functions of bounded variation and applications to image enhancement. *Applied Mathematics and Optimization*, to appear.

[10] E. Casas, K. Kunisch, and C. Pola. Some applications of bv functions in optimal control and calculus of variations. *Applied Mathematics and Optimization*, to appear.

[11] E. Radmoser, O. Scherzer, and J. Weickert. Scale-space properties of regularization methods. Technical report, Proceedings of Scale-Space 1999, 1999. to appear.

[12] Lawrence C. Evans and Ronald F. Gariepy. *Measure Theory and Fine Properties of Functions*. CRC Press, Boca Raton, Florida, 1992.

[13] F. Catte, P.-L. Lions, J.-M. Morel, and T. Coll. Image selective smoothing and edge detection by nonlinear diffusion. *SIAM J. Numer. Anal.*, 29(1):182–193, 1992.

[14] G. Chavent and K. Kunisch. Regularization of linear least squares problems by total bounded variation. *Control, Optimisation and Calculus of Variations*, 2:359–376, December 1997.

[15] Joachim Weickert. *Anisotropic Diffusion in Image Processing*. B. G. Teubner, stuttgart, ecmi edition, 1998.

[16] K. Ito and K. Kunisch. Efg-images. Technical report, Karl-Franzens-University of Graz, 1999. to appear.

[17] L. I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.

[18] O. Scherzer. Stable evaluation of differential operators and linear and nonlinear multi-scale filtering. *Electronic Journal of Differential Equations*, 15:1–12, 1997.

[19] O. Scherzer and J. Weickert. On regularization and diffusion filtering. *Journal of Mathematics in Imaging and Vision*, 1999. to appear.

[20] R. Acar and C. R. Vogel. Analysis of bounded variation penalty methods for ill-posed problems. *Inverse Problems*, 10:1217–1229, 1994.

[21] S. Osher and J. A. Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on hamilton-jacobi formulations. *J. Comput. Phys.*, 79:12–49, 1988.

[22] T. F. Chan, H. M. Zhou, and R. H. Chan. Continuation method for total variation denoising problems. UCLA CAM Report 18, Department of Mathematics, University of California, Los Angeles, April 1995.

[23] Tony. F. Chan, Gene H. Golub, and Pep Mulet. A nonlinear primal-dual method for total variation-based image restoration. UCLA CAM Report 45, Department of Mathematics, University of California, Los Angeles, September 1995.