RESIDUAL-BASED A POSTERIORI ERROR ESTIMATE FOR A MIXED REISSNER-MINDLIN PLATE FINITE ELEMENT METHOD

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Abstract. Reliable and efficient residual-based a posteriori error estimates are established for the stabilised locking-free finite element methods for the Reissner-Mindlin plate model. The error is estimated by a computable error estimator from above and below up to multiplicative constants that do neither depend on the mesh-size nor on the plate's thickness and are uniform for a wide range of stabilisation parameter. The error is controlled in norms that are known to converge to zero in a quasi-optimal way.

1. Introduction

The Reissner-Mindlin plate model [B, BS, Ci] concerns the following problem for a plane simply connected domain Ω with polygonal boundary Γ and a parameter $0 < t < 1$: *Given an applied force $f \in L^2(\Omega)$ seek rotations and displacements $(\vartheta, w) \in V := H^1_0(\Omega)^2 \times H^1_0(\Omega)$ such that, for all $(\varphi, v) \in V$,*

$$\int_\Omega \varepsilon(\vartheta) : \mathbb{C}\varepsilon(\varphi)\, dx + t^{-2} \int_\Omega (\vartheta - \nabla w) \cdot (\varphi - \nabla \mu) dx = \int_\Omega f v\, dx. \tag{1.1}$$

The discretisation of (1.1) is based on a regular triangulation $\mathcal{T}$ and finite element spaces for the conforming or nonconforming discretisations of $H^1_0(\Omega)$ which yield a discrete space V_h.

As an alternative to more complicated nonconforming plate elements from [AF, BBF, BFS, C2], the problem (1.1) can be extended by introducing a new shear-variable γ in $Q := L^2(\Omega)^2$ which is then approximated in another discrete space Q_h [AB, B, BL, CS, Lo].

This paper concerns a stabilised version, the continuous *Problem*, that reads: *Given $f \in L^2(\Omega)$, seek $(\vartheta, w, \gamma) \in V \times Q$ (i.e., $(\vartheta, w) \in V$ and $\gamma \in Q := L^2(\Omega)^2$) that satisfies, for all $(\varphi, v, \eta) \in V \times L^2(\Omega)^2$,*

$$\mathcal{B}_\alpha(\vartheta, w, \gamma; \varphi, v, \eta) = \int_\Omega f v\, dx. \tag{1.2}$$

The continuous problem (1.2) is equivalent to (1.1), while their discrete counterparts and discrete solutions $(\vartheta_h, w_h, \gamma_h) \in V_h \times Q_h$, defined by replacing $V \times Q$ by $V_h \times Q_h$ in (1.2), may differ. There is a hidden stabilisation parameter α (that enters the definition of $\mathcal{B}_\alpha$) such that classical conforming schemes are included for $\alpha = 0$.

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Various suggestions on the choice of α as a function of t and the local mesh-size h_T can be found in the literature [AB, B, BL, CS, Lo] with a corresponding stability and a priori error analysis: $\alpha = 1$ was the first approach with P_1 plus cubic bubbles for ϑ_h, P_2 finite elements for w_h, and P_0 for γ_h and linear convergence in energy norms [AB, B]. The latest suggestion reads $\alpha = 1/(h+t)$ (e.g., for the same finite elements) and linear convergence (in the norms from below) and (compared to $\alpha = 1$) with additional better convergence in L^2-norms for $\vartheta - \vartheta_h$ resp. H^1-norms for $w - w_h$ [CS]; for a multigrid analysis cf. [S].

This paper establishes a quite general a posteriori error analysis for the scheme (1.2) and a wide range of possibly mesh-depending stabilisation parameters α. Our main result is a reliable and efficient computable error estimator $\eta := (\sum_{T\in\mathcal{T}} \eta_T^2)^{1/2}$, where, for each element $T \in \mathcal{T}$ with edges $E \subset \partial T$ (and summation $\sum_{E\subset\partial T}$ over all edges of T),

$$
\begin{aligned}
\eta_T^2 \;:=\;& h_T^2 \,\|\, \alpha_T^2(\vartheta_h - \nabla w_h) + \gamma_h - \operatorname{div}\mathbb{C}\varepsilon(\vartheta_h)\,\|_{L^2(T)}^2 \\
&+ h_T^2/\alpha_T^2 \,\|\, f - \operatorname{div}(\alpha_T^2(\vartheta_h - \nabla w_h) + \gamma_h)\,\|_{L^2(T)}^2 \\
&+ (1+\beta_T/h_T)^{-2} \,\|\, \operatorname{curl} r_h \,\|_{L^2(T)}^2 + (\alpha_T^{-1} + \beta_T)^{-2} \,\|\, r_h \,\|_{L^2(T)}^2 \\
&+ \sum_{E\subset\partial T} h_E \,\|\, [\mathbb{C}\varepsilon(\vartheta_h)]\cdot n_E \,\|_{L^2(E\setminus\Gamma)}^2 \\
&+ \sum_{E\subset\partial T} h_E/(\beta_E\,(\beta_E + h_E)) \,\|\, [r_h]\cdot\tau_E \,\|_{L^2(E)}^2 \\
&+ \sum_{E\subset\partial T} h_E/\alpha_E^2 \,\|\, [\alpha_E^2(\vartheta_h - \nabla w_h) + \gamma_h]\cdot n_E \,\|_{L^2(E\setminus\Gamma)}^2 .
\end{aligned}
\tag{1.3}
$$

Here, we abbreviate $r_h := \vartheta_h - \nabla w_h - \beta^2\gamma_h$ and, for an edge E of length h_E, $[\cdot]$ denotes the jump across E, and n_E and τ_E are normal and tangential unit vectors, respectively. The results of this paper imply that η is a lower and upper bound of the error

$$
e_h := \|\,\vartheta - \vartheta_h\,\|_{H^1(\Omega)} + \|\,\alpha(\vartheta - \vartheta_h - \nabla(w - w_h))\,\|_{L^2(\Omega)} + \|\,\gamma - \gamma_h\,\|_Q . \tag{1.4}
$$

The positive constants c_1 and c_2 in the efficiency and reliability estimate

$$
c_1\,\eta - \text{ h.o.t } \le e_h \le c_2\,\eta \tag{1.5}
$$

are uniform in $0 < t < 1$, α, and h_T (resp. h_E) and depend only on the minimal interior angle in the triangulation $\mathcal{T}$ and on Ω. The higher order terms (h.o.t.) in (1.5) (i.e., efficiency of η) are computable terms.

A posteriori error estimates on other finite element schemes for (1.1) are presented in [C2, Li]. As the estimates therein, our a posteriori error bound η_T may serve as a refinement indicator within an adaptive mesh-refining algorithm.

Adaptive Algorithm (A). *(a) Start with coarse mesh $\mathcal{T}_0$.*
(b) Solve discrete problem with respect to $\mathcal{T}_k$.
(c) Compute η_T from (1.3) for all $T \in \mathcal{T}_k$.
(d) Compute error bound $\left(\sum_{T\in\mathcal{T}_k} \eta_T^2\right)^{1/2}$ and terminate or go to (e).
(e) Mark element T red iff $\eta_T \ge \frac{1}{2}\max_{T'\in\mathcal{T}_k} \eta_{T'}$.
(f) Red-green-blue-refinement to avoid hanging nodes, update mesh $\mathcal{T}_k$ and goto (b).

We refer to [EEHJ, V1] for details on red-green-blue refinement procedures and corresponding data handling and, e.g., to [BR, EEHJ, V1] for corresponding details on the Laplace equation.

The remaining part of the paper is organised as follows. Section 2 describes the mixed formulation and its discretisation $\mathcal{B}_\alpha$ which involves parameters α and β. The main results on reliable and efficient a posteriori error estimates are stated and necessary notation provided in Section 3. The proofs are divided in three main sections. Equivalence of the α-depending error norms and two residuals in V^* and Q^* is established in Section 4 while their estimation is performed in Section 5 and Section 6, respectively, where efficiency and reliability of the two residuals to their estimates is proven separately.

Throughout the paper, $L^2(\Omega)$ and $H^1(\Omega)$ denote the usual Lebesgue and Sobolev spaces [BS, LM] and $H^1_0(\Omega)$ is the subspace of all functions with zero boundary values with a dual space $H^{-1}(\Omega)$. Scalar products in (any power of) $L^2(\Omega)$ are denoted by $(\cdot;\cdot)_{L^2(\Omega)}$ while its extension to the $(H^1_0(\Omega), H^{-1}(\Omega))$-duality is denoted by $\langle\cdot;\cdot\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)}$ which differs from the scalar-product $(\cdot;\cdot)_{H^1(\Omega)}$ in (any power of) $H^1(\Omega)$.

2. Mixed formulation and finite element discretisation

The weak form of the Reißner-Mindlin plate model is rewritten with bilinear forms

$$a(\vartheta, w; \varphi, v) := \int_\Omega \varepsilon(\vartheta) : \mathbb{C}\varepsilon(\varphi)\,dx + \int_\Omega \alpha^2(\vartheta - \nabla w)\cdot(\varphi - \nabla v)\,dx, \tag{2.1}$$

$$b(\vartheta, w; \eta) := \int_\Omega (\vartheta - \nabla w)\cdot \eta\,dx, \tag{2.2}$$

$$c(\gamma; \eta) := \int_\Omega \beta^2\,\gamma\cdot\eta\,dx, \tag{2.3}$$

where $(\vartheta, w) = (\vartheta_1, \vartheta_2, w)$ and $(\varphi, v) \in V := H^1_0(\Omega)^3$, and $\gamma, \eta \in L^2(\Omega)^2$. The linear Green strain $\varepsilon(\vartheta) := \mathrm{sym}D\vartheta = (\frac{1}{2}(\partial\vartheta_j/\partial x_k + \partial\vartheta_k/\partial x_j))_{j,k=1,2}$ is the symmetric gradient and the elasticity operator $\mathbb{C}$ is defined by

$$\mathbb{C}\tau = \frac{E}{12(1-\nu^2)}\Big((1-\nu)\,\tau + \nu\ \mathrm{tr}(\tau)\,\mathbb{I}\Big),$$

where $\mathrm{tr}(\tau)$ denotes the trace of $\tau \in \mathbb{R}^{2\times 2}$, $\mathbb{I}$ is the 2×2-unit matrix, and E is Young's modulus and ν is the Poisson ratio of the elastic plate. On the product space $V \times L^2(\Omega)^2$ we define the bilinear form

$$\mathcal{B}_\alpha(\vartheta, w, \gamma; \varphi, v, \eta) := a(\vartheta, w; \varphi, v) + b(\vartheta, w; \eta) + b(\varphi, v; \gamma) - c(\gamma; \eta).$$

The critical parameter is the small thickness $t > 0$ of the plate which enters (2.1)-(2.3) through $\beta^2 := 1/(t^{-2} - \alpha^2) \in L^2(\Omega)$ where $\alpha \in L^\infty(\Omega)$ is a parameter with $0 < \alpha < 1/t$ to stabilise the finite element discretisation of (2.1)-(2.3) that employs discrete subspaces $V_h \times Q_h$ of $V \times L^2(\Omega)^2$. The *Discrete Problem* reads: *Seek* $(\vartheta_h, w_h, \gamma_h) \in V_h \times Q_h$ *that satisfies, for all* $(\varphi_h, v_h, \eta_h) \in V_h \times Q_h$,

$$\mathcal{B}_\alpha(\vartheta_h, w_h, \gamma_h; \varphi_h, v_h, \eta_h) = \int_\Omega f v_h\,dx. \tag{2.4}$$

The discrete spaces $V_h \times Q_h$ are $\mathcal{T}$-piecewise polynomials (the index h may refer to the mesh-size of $\mathcal{T}$ but we neglect further sub-indices such as $\mathcal{T}_h$, α_h etc.) based on a regular triangulation $\mathcal{T}$ of Ω in the sense of Ciarlet [Ci, BS], i.e., $\mathcal{T}$ is a finite partition of Ω in closed triangles or parallelograms; two distinct elements T_1 and T_2 in $\mathcal{T}$ are either disjoint, or $T_1 \cap T_2$ is a complete edge E or a common node of both T_1 and T_2. The triangulation satisfies a minimum angle condition, i.e., the angles in the triangles or parallelograms are assumed to belong to the interval $(c_\theta, \pi - c_\theta)$ for some positive constant c_θ and so are bounded uniformly away from 0 and π; in addition let c_θ be also a lower bound for the aspect ratios of parallelograms in $\mathcal{T}$.

The set of all edges in $\mathcal{T}$ is denoted as $\mathcal{E}$ and $\cup\mathcal{E}$ is the union of all edges, i.e., the skeleton of all boundaries of elements in $\mathcal{T}$.

For an element $T \in \mathcal{T}$, let $\mathcal{P}_k(T)$ denotes the vector space of algebraic polynomials of (total resp. partial) degree $\leq k$ (if T is a triangle resp. a parallelogram) regarded as a mapping on the domain $T \subset \mathbb{R}^2$. Then,

$$\mathcal{L}^k(\mathcal{T}) := \{p \in L^2(\Omega) : \forall T \in \mathcal{T},\ p|_T \in \mathcal{P}_k(T)\} \quad \text{and} \quad \mathcal{S}^1_0(\mathcal{T}) := \mathcal{L}^1(\mathcal{T}) \cap H^1_0(\Omega). \tag{2.5}$$

Various choices of $V_h \times Q_h$ and mesh-depending parameters $\alpha = \alpha_\mathcal{T} \in L^\infty(\Omega)$ can be found in [AB, B, BL, CS, Lo]. Those results cover a stability and a priori error analysis while this paper establishes an a posteriori error analysis.

3. A posteriori error bound and adaptive algorithm

For the regular triangulation $\mathcal{T}$ of Ω in (closed) triangles or parallelograms let $\mathcal{N}$ be the (finite) set of all vertices and let $\mathcal{K} := \mathcal{N} \cap \Omega$ be the set of interior ones. For simplicity, we assume that the triangulation matches the domain exactly, i.e., $\cup\mathcal{T} = \overline{\Omega}$ and there are no hanging nodes. The set of edges $E = \operatorname{conv}\{x, y\}$ for two distinct $x, y \in \mathcal{N}$ is denoted as $\mathcal{E}$. Their union $\cup\mathcal{E}$ is the skeleton of edges, i.e., the set of all points in $\overline{\Omega}$ which belong to some edge. With each edge, we associate a unit normal vector n_E and a perpendicular tangential unit vector τ_E.

For a $\mathcal{T}$-piecewise uniformly continuous function, the square brackets $[\cdot]$ is defined as the jump over the edges: If $E = T_+ \cap T_-$ is a common edge of two distinct T_+ and T_- in $\mathcal{T}$ then, for $x \in E$, the jump $[G](x)$ is the limit of $G(x + \delta\, n_E) - G(x - \delta\, n_E)$ as $\delta \to 0^+$. (The limit exists if $x \notin \mathcal{K}$ since $x \pm \delta\, n_E \in T_\pm$ and G is uniformly continuous on each $T_\pm$.) In this way, $[\cdot]$ is defined on the skeleton $\cup\mathcal{E} \setminus \partial\Omega$ of all inner boundaries of elements.

The diameter of T is denoted as h_T and the length of E is h_E. For compact notation, let $h_\mathcal{T} \in L^\infty(\Omega)$ and $h_\mathcal{E} \in L^\infty(\cup\mathcal{E})$ be given as $\mathcal{T}$- resp. $\mathcal{E}$-piecewise constant weights

$$h_\mathcal{T}|_T := h_T \quad \text{and} \quad h_\mathcal{E}|_E := h_E \qquad \text{for } T \in \mathcal{T} \text{ and } E \in \mathcal{E}. \tag{3.1}$$

The discrete problem is supposed to generate discrete solutions $(\vartheta_h, w_h, \gamma_h) \in V \times L^2(\Omega)^2$ which are neither expected to be uniquely determined nor to belong to a discrete space $V_h \times Q_h$. We merely suppose that $(\vartheta_h, w_h, \gamma_h)$ is $\mathcal{T}$-piecewise smooth (such that all the derivatives in (3.3) and related traces and jumps on the edges exists in the classical sense and are integrable), i.e., we suppose

$$(\vartheta_h, w_h, \gamma_h) \in H^2(\mathcal{T})^2 \times H^2(\mathcal{T}) \times H^2(\mathcal{T})^2 \tag{3.2}$$

where, for $k = 1, 2$, $H^k(\mathcal{T}) := \{\eta \in L^2(\Omega) :\ \forall T \in \mathcal{T},\ \eta|_T \in H^k(T)\}$. As a minimal condition, we suppose that $(\vartheta_h, w_h, \gamma_h)$ satisfies (2.4) for all $(\varphi_h, v_h, \eta_h) \in \mathcal{S}^1_0(\mathcal{T})^3 \times \mathcal{L}^0(\mathcal{T})^2$.

For each element $T \in \mathcal{T}$, with $\alpha|_T$ constant equal to α_T, $\beta_T := (t^{-2} - \alpha_T^2)^{-1/2}$, and for all $E \in \mathcal{E}$ with $\alpha_E = \min\{\alpha_T : E \subset T \in \mathcal{T}\}$, $\beta_E := \max\{\beta_T : E \subset T \in \mathcal{T}\}$, we define indicators η_T and η_E by and $r_h := \vartheta_h - \nabla w_h - \beta^2\gamma_h$

$$\begin{aligned}
\eta_T^2 \;:=\; & h_T^2 \,\|\, \alpha_T^2(\vartheta_h - \nabla w_h) + \gamma_h - \operatorname{div}\mathbb{C}\varepsilon(\vartheta_h) \,\|_{L^2(T)}^2 \\
& + h_T^2/\alpha_T^2 \,\|\, f - \operatorname{div}(\alpha_T^2(\vartheta_h - \nabla w_h) + \gamma_h) \,\|_{L^2(T)}^2 \\
& + (1 + \beta_T/h_T)^{-2} \,\|\, \operatorname{curl} r_h \,\|_{L^2(T)}^2 + (\alpha_T^{-1} + \beta_T)^{-2} \,\|\, r_h \,\|_{L^2(T)}^2,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\eta_E^2 \;:=\; & h_E \,\|\, [\mathbb{C}\varepsilon(\vartheta_h)] \cdot n_E \,\|_{L^2(E\setminus\Gamma)}^2 + h_E/(\beta_E\,(\beta_E + h_E)) \,\|\, [r_h] \cdot \tau_E \,\|_{L^2(E)}^2 \\
& + h_E/\alpha_E^2 \,\|\, [\alpha_E^2(\vartheta_h - \nabla w_h) + \gamma_h] \cdot n_E \,\|_{L^2(E\setminus\Gamma)}^2.
\end{aligned} \tag{3.4}$$

($[r_h] \cdot \tau_E := 0 - r_h|_\Gamma$ for $E \subset \Gamma$.)

This paper establishes the reliability and efficiency of the a posteriori error bound η,

$$\eta^2 = \sum_{T\in\mathcal{T}} \eta_T^2 + \sum_{E\in\mathcal{E}} \eta_E^2, \tag{3.5}$$

and the error norms in V and Q defined, for $(\varphi, v) \in V$ and $\eta \in Q$, by

$$\|\,(\varphi, v)\,\|_V \;:=\; \left\{\|\,\varphi\,\|_{H^1(\Omega)}^2 + \|\,\alpha(\varphi - \nabla v)\,\|_{L^2(\Omega)}^2\right\}^{1/2}, \tag{3.6}$$

$$\|\,\eta\,\|_{Q,0} \;:=\; \inf_{\eta = p+q,\ \operatorname{div}\, p = 0} \left\{\|\,p\,\|_{H^{-1}(\Omega)}^2 + \|\,q/\alpha\,\|_{L^2(\Omega)}^2\right\}^{1/2}, \tag{3.7}$$

$$\|\,\eta\,\|_Q \;:=\; \left\{\|\,\eta\,\|_{Q,0}^2 + \|\,\beta\eta\,\|_{L^2(\Omega)}^2\right\}^{1/2}. \tag{3.8}$$

In the infimum of (3.7), $p \in L^2(\Omega)$ is divergence free, written div $p = 0$, in the sense that it is $L^2(\Omega)$-orthogonal to $\nabla H_0^1(\Omega)$, the gradients of functions in $H_0^1(\Omega)$, and $q \in L^2(\Omega)$.

Remark 3.1. The set Q equals $L^2(\Omega)$ and their norms are equivalent. However, the constants in the equivalence inequalities are t-dependent and so we are led to specify $(Q, \|\cdot\|_Q)$.

The main contribution of this paper is the proof of reliability and efficiency of the error in the aforementioned norms and the error estimator η. As α is usually chosen as a function of t and h_T and since the elements are shape-regular (whence $h_{T_1}/h_{T_2} \lesssim 1$ for $T_1 \cap T_2 \neq \emptyset$) it is not a too restrictive assumption that the quotient of two α_T is bounded for two neighbour elements; for future reference to those bounds, we introduce

$$\kappa(\alpha, \mathcal{T}) := \max_{T_1, T_2 \in \mathcal{T},\, T_1 \cap T_2 \neq \emptyset} \alpha_{T_1}/\alpha_{T_2}. \tag{3.9}$$

Let $(\vartheta, w, \gamma) \in H_0^1(\Omega)^2 \times H_0^1(\Omega) \times L^2(\Omega)^2$ solve (1.2) and suppose that $(\vartheta_h, w_h, \gamma_h)$ as in (3.2) satisfies (2.4) for all $(\varphi_h, v_h, \eta_h) \in \mathcal{S}_0^1(\mathcal{T})^3 \times \mathcal{L}^0(\mathcal{T})^2$.

Theorem 3.1. *Suppose that α satisfies $0 < \alpha < 1/t$ and $\alpha \leq c_3/h$. Then, there exists a positive $(h_\mathcal{T}, h_\mathcal{E}, t)$-independent constant c_4 which depends only on Ω, c_3, c_Θ, and on $\kappa(\alpha, \mathcal{T})$, such that*

$$\|\,(\vartheta - \vartheta_h, w - w_h)\,\|_V + \|\,\gamma - \gamma_h\,\|_Q \leq c_4\,\eta. \tag{3.10}$$

Theorem 3.2. *Suppose that α satisfies $0 < \alpha < 1/t$ and $\alpha \le c_3/h$. Then, there exists positive $(h_\mathcal{T}, h_\mathcal{E}, t)$-independent constants c_5 and c_6 which depend only on Ω, c_3, c_Θ, and on $\kappa(\alpha, \mathcal{T})$, such that*

$$c_5\,\eta \le \|\,(\vartheta - \vartheta_h, w - w_h)\,\|_V + \|\,\gamma - \gamma_h\,\|_Q + c_6 \inf_{f_h \in \mathcal{L}^1(\mathcal{T})} \|\,h_\mathcal{T}/\alpha(f - f_h)\,\|_{L^2(\Omega)}. \tag{3.11}$$

The remaining sections are devoted to the proof of both theorems.

4. Equivalence of error and residual norms

This section is devoted to the proof of equivalence between the error and the residual with respect to the norms in $V \times Q$ and $V^* \times Q^*$, respectively, with emphasis on the independence of the equivalence constants from α, h, and t.

According to Korn's inequality and setting of the parameters $E > 0$ and $0 < \nu < 1$, the energy norm $\|\,\mathbb{C}^{1/2}\varepsilon(\cdot)\,\|_{L^2(\Omega)}$ is equivalent to the Sobolev norm $\|\cdot\|_{H^1_0(\Omega)}$ with the global positive constants c_7 and c_8 which merely depend on Ω and $\mathbb{C}$, i.e., for all $\varphi \in H^1_0(\Omega)^2$,

$$c_7\,\|\,\varphi\,\|^2_{H^1_0(\Omega)} \le \int_\Omega \varepsilon(\varphi) : \mathbb{C}\varepsilon(\varphi)dx \le c_8\,\|\,\varphi\,\|^2_{H^1_0(\Omega)}. \tag{4.1}$$

Thus the bilinear form $a(\cdot;\cdot)$ is elliptic and continuous with constants $c_9 := \min\{1, c_7\}$ and $c_{10} := \max\{1, c_8\}$, i.e., for all $(\varphi, v) \in V$,

$$c_9\|\,(\varphi, v)\,\|^2_V \le a(\varphi, v; \varphi, v) \le c_{10}\|\,(\varphi, v)\,\|^2_V. \tag{4.2}$$

The norm $\|\cdot\|_{Q,0}$ is chosen such that both, stability and continuity of the bilinear form b are satisfied with bound 1.

Lemma 4.1. *The stability and continuity conditions of the bilinear form b with respect to the norms $\|\cdot\|_V$ and $\|\cdot\|_{Q,0}$ are fulfilled with the optimal constant one, i.e.,*

$$\|\,\eta\,\|_{Q,0} = \sup_{(\varphi,v)\in V\setminus\{0\}} \frac{b(\varphi, v; \eta)}{\|\,(\varphi, v)\,\|_V} \qquad \textit{for all } \eta \in Q. \tag{4.3}$$

Proof. Let $(\varphi, v) \in V$, $\eta = p + q \in Q$ with $\operatorname{div} p = 0$ (i.e., $\int_\Omega p \cdot \nabla v\,dx = 0$) and deduce

$$\begin{aligned} b(\varphi, v; \eta) &= \int_\Omega (\varphi - \nabla v)\cdot(p + q)\,dx \\ &= \int_\Omega \varphi \cdot p\,dx + \int_\Omega \alpha(\varphi - \nabla v)\cdot q/\alpha\,dx \\ &\le \|\,\varphi\,\|_{H^1_0(\Omega)}\,\|\,p\,\|_{H^{-1}(\Omega)} + \|\,\alpha(\varphi - \nabla v)\,\|_{L^2(\Omega)}\,\|\,q/\alpha\,\|_{L^2(\Omega)} \\ &\le \|\,(\varphi, v)\,\|_V \left\{\|\,p\,\|^2_{H^{-1}(\Omega)} + \|\,q/\alpha\,\|^2_{L^2(\Omega)}\right\}^{1/2}. \end{aligned}$$

Since the split $\eta = p + q$ was arbitrary, this estimate shows

$$\sup_{(\varphi,v)\in V\setminus\{0\}} \frac{b(\varphi, v; \eta)}{\|\,(\varphi, v)\,\|_V} \le \|\,\eta\,\|_{Q,0} \qquad \text{for all } \eta \in Q. \tag{4.4}$$

An explicit decomposition $\eta = p + q$ will be constructed to show inequality reverse to (4.4). Given $\eta \in Q \setminus \{0\}$, let $(\vartheta, w) \in V$ solve, for all $(\varphi, v) \in V$,

$$(\vartheta; \varphi)_{H^1_0(\Omega)} + (\alpha^2(\vartheta - \nabla w); \varphi - \nabla v)_{L^2(\Omega)} = -b(\varphi, v; \eta). \tag{4.5}$$

(The same arguments that show ellipticity of a prove that of the bilinear form on left-hand side of (4.5) and so imply unique existence of $(\vartheta, w) \in V$.) The choice $(\varphi, v) = (\vartheta, w)$ in (4.5) shows

$$\|(\vartheta, w)\|_V = -\frac{b(\vartheta, w; \eta)}{\|(\vartheta, w)\|_V} \leq \sup_{(\varphi,v)\in V\setminus\{0\}} \frac{b(\varphi, v; \eta)}{\|(\varphi, v)\|_V}. \tag{4.6}$$

Set $q := -\alpha^2(\vartheta - \nabla w)$ and $p := \eta - q$. Because of (4.5), we have, for all $v \in H^1_0(\Omega)$,

$$(p; \nabla v)_{L^2(\Omega)} = (\eta + \alpha^2(\vartheta - \nabla w); \nabla v)_{L^2(\Omega)} = 0$$

and so div $p = 0$. Furthermore, (4.5) (with $v = 0$) shows

$$\begin{aligned}\|p\|_{H^{-1}(\Omega)} &= \sup_{\varphi\in H^1_0(\Omega)\setminus\{0\}} \frac{(\eta + \alpha^2(\vartheta - \nabla w); \varphi)_{L^2(\Omega)}}{\|\varphi\|_{H^1_0(\Omega)}} \\ &= \sup_{\varphi\in H^1_0(\Omega)\setminus\{0\}} \frac{(\vartheta; \varphi)_{H^1_0(\Omega)}}{\|\varphi\|_{H^1_0(\Omega)}} = \|\vartheta\|_{H^1_0(\Omega)}. \qquad (4.7)\end{aligned}$$

The proof of the missing inequality is concluded for $\eta = p + q$ with (4.6)(4.7) and

$$\begin{aligned}\|\eta\|^2_{Q,0} &\leq \|p\|^2_{H^{-1}(\Omega)} + \|q/\alpha\|^2_{L^2(\Omega)} \\ &= \|\vartheta\|^2_{H^1_0(\Omega)} + \|\alpha(\vartheta - \nabla w)\|^2_{L^2(\Omega)} \\ &= \|(\vartheta, w)\|^2_V \leq \sup_{(\varphi,v)\in V} \frac{b(\vartheta, v; \eta)^2}{\|(\varphi, v)\|^2_V}. \qquad \square\end{aligned}$$

Theorem 4.2. *The bilinear form $\mathcal{B}_\alpha(\cdot;\cdot)$ provides an isomorphism between $V \times Q$ and its dual, i.e., for all $(\vartheta, w, \gamma) \in V \times Q$, we have (when in the supremum $(v, \varphi, \eta) \in V \times Q \setminus \{0\}$)*

$$\frac{\min\{c_9, c_{10}^{-1}\}}{\sqrt{13}} \|(w, \vartheta, \gamma)\|_{V\times Q} \leq \sup_{(v,\varphi,\eta)} \frac{\mathcal{B}_\alpha(\vartheta, w, \eta; \varphi, w, \gamma)}{\|(v, \varphi, \eta)\|_{V\times Q}} \leq (1 + c_{10})\|(w, \vartheta, \gamma)\|_{V\times Q}. \tag{4.8}$$

Remark 4.1. The bounds for the global stability and continuity estimates only depend on the bounds of the norm equivalence (4.1). If we use the energy norm $\|\mathbb{C}^{1/2}\varepsilon(\cdot)\|_{L^2(\Omega)}$ instead of the $H^1_0(\Omega)$-norm, then $c_7 = 1 = c_8$ and the constants in Theorem 4.2 are absolute constants $1/\sqrt{13}$ and 2.

Proof of Theorem 4.2. To prove the stability we define operators $A : V \to V$, $B : V \to Q$, $B^* : Q \to V$, and $C : Q \to Q$ by $(A(\vartheta, w); (\varphi, v))_V = a(\vartheta, w; \varphi, v)$, $(B(\vartheta, w); \eta)_Q = (B^*\eta; \vartheta, w)_V = b(\vartheta, w; \eta)$, and $(C\eta; \gamma)_Q = c(\eta; \gamma)$. The operators A and C are selfadjoint (with respect to the scalar product in V and Q), respectively. Since A is elliptic, $A^{\pm 1/2}$ and A^{-1} are well defined. Lemma 4.1 proves the isometry of B^* with respect to the norms $\|\cdot\|_{Q,0}$ and $\|\,.\,\|_V$, i.e., for all $\eta \in Q$,

$$\|\eta\|_{Q,0} = \sup_{(\varphi,v)\in V\setminus\{0\}} \frac{b(\varphi, v; \eta)}{\|(\varphi, v)\|_V} = \sup_{(\varphi,v)\in V\setminus\{0\}} \frac{(B^*\eta; \varphi, v)_V}{\|(\varphi, v)\|_V} = \|B^*\eta\|_V. \tag{4.9}$$

For fixed $(\vartheta, w, \gamma) \in V \times Q$ set

$$\begin{aligned} f &:= A(\vartheta, w) + B^*\gamma \in V, \\ g &:= B(\vartheta, w) - C\gamma \in Q, \end{aligned}$$

and eliminate the primal variable $(w,\vartheta) = A^{-1}(f - B^*\gamma)$. With the Schur complement S,

$$S := BA^{-1}B^* + C : Q \to Q \quad \text{selfadjoint isomorphism,}$$

(since A and C are elliptic) this implies the identity

$$S\gamma = BA^{-1}f - g. \tag{4.10}$$

The continuity and ellipticity bounds on $S^{\pm 1}$ (may depend on α, β, h, t and so) are analysed explicitly in the sequel. For $\eta \in Q$, the definition of S and (4.2) yield

$$\begin{aligned}(S\eta;\eta)_Q &= ((BA^{-1}B^* + C)\eta;\eta)_Q = (A^{-1}B^*\eta; B^*\eta)_V + (C\eta;\eta)_Q \\ &\le c_9^{-1}\| B^*\eta \|_V^2 + c(\eta;\eta) = c_9^{-1}\| \eta \|_{Q,0}^2 + c(\eta;\eta) \le c_9^{-1}\| \eta \|_Q^2,\end{aligned} \tag{4.11}$$

and, similarly, one verifies the reverse estimate to finally obtain

$$c_{10}^{-1}\| \eta \|_Q^2 \le (S\eta;\eta)_Q \le c_9^{-1}\| \eta \|_Q^2 \qquad \text{for all } \eta \in Q. \tag{4.12}$$

Since C is elliptic, $(A^{-1}B^*\gamma; B^*\gamma)_V \le (S\gamma;\gamma)_Q = \| S^{1/2}\gamma \|_Q^2$. This and (4.10) lead to

$$\begin{aligned}\| S^{1/2}\gamma \|_Q^2 &= (BA^{-1}f - g;\gamma)_Q \\ &\le \| A^{-1/2}f \|_V \| A^{-1/2}B^*\gamma \|_V + \| S^{-1/2}g \|_Q \| S^{1/2}\gamma \|_Q \\ &\le \big(\| A^{-1/2}f \|_V + \| S^{-1/2}g \|_Q\big)\| S^{1/2}\gamma \|_Q, \quad \text{whence}\end{aligned}$$

$$\| S^{1/2}\gamma \|_Q \le \| A^{-1/2}f \|_V + \| S^{-1/2}g \|_Q. \tag{4.13}$$

Similar arguments for the primal variable $(w,\vartheta) = A^{-1}(f - B^*\gamma)$ show

$$\begin{aligned}\| A^{1/2}(\vartheta,w) \|_V^2 &= (A(\vartheta,w);(\vartheta,w))_V = (f - B^*\gamma;(\vartheta,w))_V \\ &\le \| A^{-1/2}f \|_V \| A^{1/2}(\vartheta,w) \|_V + \| A^{-1/2}B^*\gamma \|_V \| A^{1/2}(\vartheta,w) \|_V \\ &\le \big(\| A^{-1/2}f \|_V + \| S^{1/2}\gamma \|_Q\big)\| A^{1/2}(\vartheta,w) \|_V, \quad \text{whence}\end{aligned}$$

$$\| A^{1/2}(\vartheta,w) \|_V \le \| A^{-1/2}f \|_V + \| S^{1/2}\gamma \|_Q. \tag{4.14}$$

The lower bounds in (4.2) resp. (4.12) in the first inequality, (4.14) in the second, and (4.13) in the third imply

$$\begin{aligned}&\max\{c_9^{-1}, c_{10}\}^{-1/2}\| (\vartheta,w,\gamma) \|_{V\times Q} \le \| A^{1/2}(\vartheta,w) \|_V + \| S^{1/2}\gamma \|_Q \\ &\le \| A^{-1/2}f \|_V + 2\| S^{1/2}\gamma \|_Q \le 3\| A^{-1/2}f \|_V + 2\| S^{-1/2}g \|_Q.\end{aligned}$$

This and the bounds on $A^{-1/2}$ resp. $S^{-1/2}$ in (4.2) resp. (4.12) (with a Cauchy inequality in $\mathbb{R}^2$ at the end) lead to

$$\begin{aligned}&\max\{c_9^{-1}, c_{10}\}^{-1}\| (\vartheta,w,\gamma) \|_{V\times Q} \le 3\| f \|_V + 2\| g \|_Q \le \sqrt{3^2+2^2}\,\| (f,g) \|_{V\times Q} \\ &= \sqrt{13} \sup_{(v,\varphi,\gamma)\in(V\times Q)\setminus\{0\}} \frac{((f,g);(\varphi,v,\gamma))_{V\times Q}}{\|(v,\varphi,\gamma)\|_{V\times Q}} = \sqrt{13} \sup_{(v,\varphi,\gamma)\in(V\times Q)\setminus\{0\}} \frac{\mathcal{B}_\alpha(w,\vartheta,\gamma;v,\varphi,\gamma)}{\|(v,\varphi,\eta)\|_{V\times Q}}\end{aligned}$$

which is the first claimed inequality in the theorem. The proof of the second follows, for (ϑ,w,γ), $(\varphi,v,\eta) \in V\times Q$, from

$$\begin{aligned}\mathcal{B}_\alpha(\vartheta,w,\gamma;\varphi,v,\eta) &= a(\vartheta,w;\varphi,v) + b(\vartheta,w;\eta) + b(\varphi,v;\gamma) - c(\gamma,\eta) \\ &\le \big(a(\vartheta,w;\vartheta,w) + \| \vartheta,w \|_V^2 + \| \gamma \|_{Q,0}^2 + c(\gamma,\gamma)\big)^{1/2} \\ &\quad \times \big(a(\varphi,v;\varphi,v) + \| \eta \|_{Q,0}^2 + \| (\varphi,v) \|_V^2 + c(\eta,\eta)\big)^{1/2} \\ &\le (1 + c_{10})\| (\vartheta,w,\eta) \|_{V\times Q}\| (\varphi,v,\gamma) \|_{V\times Q}. \qquad \square\end{aligned}$$

The theorem relates the error to the two residuals in V^* and Q^* estimated in Section 5 and 6, respectively.

Corollary 4.3. *The discretisation error is equivalent to the sum of residuals, i.e., there holds (with equivalence constants which depend only on c_7 and c_8)*

$$\begin{aligned}\|\,(\vartheta-\vartheta_h, w-w_h, \gamma-\gamma_h)\,\|_{V\times Q}\\ \approx\ \|\,f-a(\vartheta_h,w_h;\cdot)-b(\cdot;\gamma_h)\,\|_{V^*}+\|\,b(\vartheta_h,w_h;\cdot)-c(\cdot;\gamma_h)\,\|_{Q^*}.\end{aligned}\tag{4.15}$$

Proof. Theorem 4.2 relates the error (ϑ, w, η) to the dual norm of the residuals, i.e.,

$$\begin{aligned}\|(\vartheta-\vartheta_h, w-w_h, \gamma-\gamma_h)\|_{V\times Q} &\approx \sup_{(\varphi,v,\eta)\in(V\times Q)\setminus\{0\}} \frac{\mathcal{B}_\alpha(\vartheta-\vartheta_h, w-w_h, \gamma-\gamma_h;\varphi,v,\eta)}{\|(v,\varphi,\eta)\|_{V\times Q}}\\ &= \sup_{(\varphi,v,\eta)\in(V\times Q)\setminus\{0\}} \frac{f(v)-a(\vartheta_h,w_h;\varphi,v)-b(v,\varphi;\gamma_h)-b(\vartheta_h,w_h;\eta)+c(\eta,\gamma_h)}{\|(v,\varphi,\eta)\|_{V\times Q}}\\ &\approx \sup_{(\varphi,v)\in V\setminus\{0\}} \frac{f(v)-a(\vartheta_h,w_h;\varphi,v)-b(v,\varphi;\gamma_h)}{\|(v,\varphi)\|_V} + \sup_{\eta\in Q\setminus\{0\}}\frac{-b(\vartheta_h,w_h;\eta)+c(\eta;\gamma_h)}{\|\eta\|_Q}. \qquad \square\end{aligned}$$

5. Reliable and Efficient Computable Estimates for the residual in V^*

This section is devoted to the proof of equivalence of the dual norm $\|\,r_V\,\|_{V^*}$ of the primal residual $r_V := f - a(\vartheta_h, w_h;\cdot) - b(\cdot;\gamma_h)$ and the computable error estimator η_V,

$$\begin{aligned}\eta_V^2 := \sum_{T\in\mathcal{T}}\Big(h_T^2\|\,\alpha_T^2(\vartheta_h-\nabla w_h)+\gamma_h-\operatorname{div}\mathbb{C}\varepsilon(\vartheta_h)\,\|_{L^2(T)}^2\\ +h_T^2/\alpha_T^2\,\|\,f-\operatorname{div}(\alpha_T^2(\vartheta_h-\nabla w_h)+\gamma_h)\,\|_{L^2(T)}^2\Big)\\ +\sum_{E\in\mathcal{E}}\Big(h_E\|\,[\mathbb{C}\varepsilon(\vartheta_h)]\cdot n_E\,\|_{L^2(E\setminus\Gamma)}^2+h_E/\alpha_E^2\,\|\,[\gamma_h+\alpha^2(\vartheta_h-\nabla w_h)]\cdot n_E\,\|_{L^2(E\setminus\Gamma)}^2\Big).\end{aligned}\tag{5.1}$$

Recall $\alpha_E := \min\{\alpha_{T_1}, \alpha_{T_2}\}$ if $E = T_1\cap T_2$ is the joint edge of the distinct elements $T_1, T_2 \in \mathcal{T}$ and $\alpha_T := \alpha|_T$ is supposed to be constant on each $T\in\mathcal{T}$.

Theorem 5.1. *There exists an $(h_\mathcal{T}, t)$-independent constant c_{11}, which depends on α and $\mathcal{T}$ only through $\kappa(\alpha,\mathcal{T})$, $c_3 = \|\,\alpha\, h_\mathcal{T}\,\|_{L^\infty(\Omega)}$, and c_Θ, such that*

$$\|r_V\|_{V^*} \le c_{11}\,\eta_V.\tag{5.2}$$

The proof is based on a refined approximation property of the Clément-interpolant (or any other weak approximation operator which is locally exact for affine functions) of Lemma 5.2 in which the upper bound $\|(\varphi,v)\|_V$ is more involved than $\|\,(D\varphi,\alpha\,\nabla v)\,\|_{L^2(\Omega)}$.

Lemma 5.2. *There exists an $(h_\mathcal{T}, t)$-independent constant c_{12}, which depends on α and $\mathcal{T}$ only through $\kappa(\alpha,\mathcal{T})$, $c_3 = \|\,\alpha\, h_\mathcal{T}\,\|_{L^\infty(\Omega)}$, and c_Θ, such that, given $(\varphi, v)\in V$, there exists $(\varphi_h, v_h)\in S_0^1(\mathcal{T})^3$ which satisfies*

$$\begin{aligned}\sum_{T\in\mathcal{T}} h_T^{-2}\|\varphi-\varphi_h\|_{L^2(T)}^2+\sum_{T\in\mathcal{T}}\alpha_T^2/h_T^2\|v-v_h\|_{L^2(T)}^2\\ +\sum_{E\in\mathcal{E}} h_E^{-1}\|\varphi-\varphi_h\|_{L^2(E)}^2+\sum_{E\in\mathcal{E}}\alpha_E^2/h_E\|v-v_h\|_{L^2(E)}^2 \le c_{12}\,\|(\varphi,v)\|_V^2.\end{aligned}\tag{5.3}$$

Proof. The Clément approximation operator for a scalar v defines a $\mathcal{T}$-piecewise affine v_h by their nodal values. For a node z at the boundary, $v_h(z) = 0$, and at an interior node z with a patch $\omega_z := \text{interior}(\cup\{T \in \mathcal{T} : z \in T\})$ we let $v_h(z) = p(z)$ where p is the $L^2(\omega)$-best-approximation of $v_h|_{\omega_z}$ in $\mathcal{P}_1(\omega_z)$. The two components of φ_h are defined by the same procedure applied on the two components of φ. Well-established approximation estimates [BS, Cl, Ci, V1] prove that the terms on the left-hand side of (5.3) which contain $\varphi - \varphi_h$ are bounded by a constant times $\| D\varphi \|_{L^2(\Omega)} \leq \|(\varphi, v)\|_V$. It therefore remains to prove

$$\sum_{T\in\mathcal{T}} \alpha_T^2/h_T^2 \|v - v_h\|^2_{L^2(T)} + \sum_{E\in\mathcal{E}} \alpha_E^2/h_E \|v - v_h\|^2_{L^2(E)} \lesssim \|(\varphi, v)\|_V^2, \tag{5.4}$$

which is completely standard up to the weights $\alpha \lesssim h_{\mathcal{T}}^{-1}$. Indeed, from the proof of first order convergence and stability of the Clément operator we know

$$h_T^{-2}\|v - v_h\|^2_{L^2(T)} + h_E^{-1}\|v - v_h\|^2_{L^2(E)} + \|\nabla v_h\|^2_{L^2(T)} \lesssim \|\nabla v\|^2_{L^2(\omega_T)} \tag{5.5}$$

for each element $T \in \mathcal{T}$ with an edge $E \subset \partial T$ and its patch $\omega_T := \text{interior}(\cup\{K \in \mathcal{T} : K \cap T \neq \emptyset\})$. Suppose in the first case that one vertex of T belongs to the boundary $\partial\Omega$. Then, the intersection of $\partial\omega_T$ with $\partial\Omega$ contains at least one edge and so (after $\alpha \lesssim h_{\mathcal{T}}^{-1}$) a Friedrichs inequality shows

$$\| \alpha\, \varphi \|_{L^2(\omega_T)} \lesssim \| h_{\mathcal{T}}^{-1}\, \varphi \|_{L^2(\omega_T)} \lesssim \| D\varphi \|_{L^2(\omega_T)}. \tag{5.6}$$

A multiplication of (5.5) with α_T, a triangle inequality, and (5.6) yield

$$\begin{aligned} \alpha_T\, h_T^{-1}\|v - v_h\|_{L^2(T)} + \alpha_E \kappa(\alpha, \mathcal{T})^{-1} h_E^{-1/2}\|v - v_h\|_{L^2(E)} &\lesssim \kappa(\alpha, \mathcal{T})\|\alpha\nabla v\|_{L^2(\omega_T)} \\ &\lesssim \kappa(\alpha, \mathcal{T})\|\alpha(\varphi - \nabla v)\|_{L^2(\omega_T)} + \kappa(\alpha, \mathcal{T})\|D\varphi\|_{L^2(\omega_T)}. \end{aligned} \tag{5.7}$$

In the second case, the vertices of T are interior nodes and so $(v - v_h)|_T$ remains the same if we change v to $v - z$ for an affine function z with on ω_T when we change v_h accordingly (cf. the above mentioned construction for details and a proof); the Clément approximation operator locally preserves affine functions. We choose the constant vector $A := \nabla z$ as the integral mean of φ on ω_T. As a consequence, (5.5) can be recast as

$$h_T^{-2}\|v - v_h\|^2_{L^2(T)} + h_E^{-1}\|v - v_h\|^2_{L^2(E)} + \|\nabla v_h - A\|^2_{L^2(T)} \lesssim \|\nabla v - A\|^2_{L^2(\omega_T)}. \tag{5.8}$$

Hence (after $\alpha \lesssim h_{\mathcal{T}}^{-1} \lesssim 1/\text{diam}(\omega_T)$ on ω_T) a Poincaré inequality shows

$$\| \alpha(\varphi - A) \|_{L^2(\omega_T)} \lesssim \| h_{\mathcal{T}}^{-1}(\varphi - A) \|_{L^2(\omega_T)} \lesssim \| D\varphi \|_{L^2(\omega_T)}. \tag{5.9}$$

A multiplication of (5.8) with α_T, a combination with (5.9), and the above arguments yield

$$\begin{aligned} \alpha_T\, h_T^{-1}\|v - v_h\|_{L^2(T)} + \alpha_E \kappa(\alpha, \mathcal{T})^{-1} h_E^{-1/2}\|v - v_h\|_{L^2(E)} &\lesssim \kappa(\alpha, \mathcal{T})\|\alpha(\nabla v - A)\|_{L^2(\omega_T)} \\ &\lesssim \kappa(\alpha, \mathcal{T})\|\alpha(\varphi - \nabla v)\|_{L^2(\omega_T)} + \kappa(\alpha, \mathcal{T})\|D\varphi\|_{L^2(\omega_T)}. \end{aligned} \tag{5.10}$$

A Summation of (5.7) resp. (5.10) over all $T \in \mathcal{T}$ and $E \in \mathcal{E}$ concludes the proof of (5.3). ☐

Proof of Theorem 5.1. Given $(\varphi, v) \in V$, $\mathcal{T}$-piecewise integrations by parts shows

$$\begin{aligned} r_V(\varphi, v) &= f(v) - a(\vartheta_h, w_h; \varphi, v) - b(v, \varphi; \gamma_h) \\ &= \int_\Omega f v\, dx - \int_\Omega \varepsilon(\vartheta_h) : \mathbb{C}\varepsilon(\varphi)\, dx - \int_\Omega (\alpha^2(\vartheta_h - \nabla w_h) + \gamma_h) \cdot (\varphi - \nabla v)\, dx \\ &= \sum_{T\in\mathcal{T}} \int_T (r^{T,w}\, v + r^{T,\vartheta} \cdot \varphi) dx + \sum_{E\in\mathcal{E}} \int_E (r^{E,w}\, v + r^{E,\vartheta} \cdot \varphi) ds \end{aligned} \tag{5.11}$$

with the element and edge residual terms

$$
\begin{aligned}
r^{T,\vartheta} &= \operatorname{div}\mathbb{C}\varepsilon(\vartheta_h) - \alpha_T^2(\vartheta_h - \nabla w_h) - \gamma_h, & r^{E,\vartheta} &= [\mathbb{C}\varepsilon(\vartheta_h)]\cdot n_E,\\
r^{T,w} &= f - \operatorname{div}(\alpha_T^2(\vartheta_h - \nabla w_h) + \gamma_h), & r^{E,w} &= -[\gamma_h + \alpha^2(\vartheta_h - \nabla w_h)]\cdot n_E.
\end{aligned}
\tag{5.12}
$$

The Galerkin orthogonality allows the substitution of (φ, v) by $(\varphi - \varphi_h, v - v_h)$ in (5.11) for (φ_h, v_h) as in Lemma 5.2. From this and the lemma, we infer with Cauchy inequalities

$$
\begin{aligned}
r_V(v,\varphi) &\leq \sum_{T\in\mathcal{T}} h_T \,\|\,(r^{T,\vartheta}, r^{T,w}/\alpha_T)\,\|_{L^2(T)}\, h_T^{-1}\|(\varphi-\varphi_h, \alpha_T(v-v_h))\|_{L^2(T)}\\
&+\sum_{E\in\mathcal{E}} h_E^{1/2}\,\|\,(r^{E,\vartheta}, r^{E,w}/\alpha_E)\,\|_{L^2(E)}\, h_E^{-1/2}\|(\varphi-\varphi_h, \alpha_E(v-v_h))\|_{L^2(E)}\\
&\leq \sqrt{c_{12}}\,\eta_V\,\|(\varphi, v)\|_V. \qquad \square
\end{aligned}
\tag{5.13}
$$

The estimator η_V is efficient: The converse inequality of (5.2) holds even in a more local form than stated (cf. the proof of (5.14) below).

Theorem 5.3. *There exists an $(h_{\mathcal{T}}, t)$-independent constant c_{13}, which depends on α and $\mathcal{T}$ only through $\kappa(\alpha,\mathcal{T})$, $c_3 = \|\,\alpha\, h_{\mathcal{T}}\,\|_{L^\infty(\Omega)}$, and c_Θ, such that*

$$
\eta_V(r_V) \leq c_{13}\left(\|r_V\|_{V^*} + \inf_{f_h\in\mathcal{L}^1(\mathcal{T})}\|\,h_{\mathcal{T}}/\alpha(f-f_h)\,\|_{L^2(\Omega)}\right). \tag{5.14}
$$

Proof. For each triangle T adopt notation from (5.12) and let b_T be the cubic bubble-function (i.e., 27 times the product of all barycentric coordinates on T) which satisfies $\operatorname{supp}(b_T) = T$, $0 \leq b_T \leq \max b_T = 1$. Let f_h denote the $L^2(T)$-best-approximation of f in $\mathcal{P}_1(T)$ and consider $\varphi := b_T\, r^{T,\vartheta}$ and $v := b_T\, \bar r^{T,w}/\alpha_T^2$, $\bar r^{T,w} := r^{T,w} - f + f_h$ in (5.11) to observe

$$
\|\, b_T^{1/2}\, r^{T,\vartheta}\,\|_{L^2(T)}^2 + \|\, b_T^{1/2}\,\bar r^{T,w}\,\|_{L^2(T)}^2/\alpha_T^2 + \int_T b_T(f-f_h)\bar r^{T,w}\,dx/\alpha_T^2 = r_V(v,\varphi). \tag{5.15}
$$

Equivalence of the norms $\|\, b_T^{1/2}\cdot\,\|_{L^2(T)}$ and $\|\cdot\|_{L^2(T)}$ on a polynomial space (the components of $r^{T,\vartheta}$ and $\bar r^{T,w}$ belong to) and Young's inequality yield with $V_T := H_0^1(T) \hookrightarrow V$ (and so $V_T^* = H^{-1}(T)$) in (5.15)

$$
\begin{aligned}
\|\, r^{T,\vartheta}\,\|_{L^2(T)}^2 + \frac{1}{2\,\alpha_T^2}\|\,\bar r^{T,w}\,\|_{L^2(T)}^2 &\lesssim \|\, b_T^{1/2}\, r^{T,\vartheta}\,\|_{L^2(T)}^2 + \frac{1}{2\,\alpha_T^2}\|\, b_T^{1/2}\,\bar r^{T,w}\,\|_{L^2(T)}^2\\
&= -\frac{1}{2\,\alpha_T^2}\|\, b_T^{1/2}\,\bar r^{T,w}\,\|_{L^2(T)}^2 - \int_T b_T(f-f_h)\bar r^{T,w}\,dx/\alpha_T^2 + r_V(v,\varphi)\\
&\leq \frac{1}{2\,\alpha_T^2}\|\, b_T^{1/2}\,(f-f_h)\,\|_{L^2(T)}^2 + \|r_V\|_{V_T^*}\|\,(v,\varphi)\,\|_{V_T}.
\end{aligned}
\tag{5.16}
$$

Inverse estimates for the polynomials $b_T\, r^{T,\vartheta}$ and $b_T\,\bar r^{T,w}$ and $\alpha_T \lesssim 1/h_T$ guarantee

$$
\begin{aligned}
h_T\,\|\,(v,\varphi)\,\|_V &\lesssim \|\,\varphi\,\|_{L^2(T)} + h_T\|\,\alpha_T(\varphi-\nabla v)\,\|_{L^2(T)}\\
&\leq \|\,\varphi\,\|_{L^2(T)} + h_T\,\alpha_T\,\|\,\varphi\,\|_{L^2(T)} + h_T\alpha_T\,\|\,\nabla v\,\|_{L^2(T)}\\
&\lesssim \|\, r^{T,\vartheta}\,\|_{L^2(T)} + \alpha_T\|\,v\,\|_{L^2(T)} \leq \|\, r^{T,\vartheta}\,\|_{L^2(T)} + \|\,\bar r^{T,w}\,\|_{L^2(T)}/\alpha_T.
\end{aligned}
\tag{5.17}
$$

A multiplication of (5.16) with h_T^2 and using (5.17) to absorb $h_T^2 \| (v,\varphi) \|_V$ we obtain

$$\| h_T \, r^{T,\vartheta} \|_{L^2(T)} + \| h_T/\alpha_T \, \bar{r}^{T,w} \|_{L^2(T)} \lesssim h_T/\alpha_T \| f - f_h \|_{L^2(T)} + \|r_V\|_{V_T^*}. \tag{5.18}$$

This and a triangle inequality $\| r^{T,w} \|_{L^2(T)} \leq \| \bar{r}^{T,w} \|_{L^2(T)} + \| f - f_h \|_{L^2(T)}$ prove

$$\| h_T \, r^{T,\vartheta} \|_{L^2(T)} + \| h_T/\alpha_T \, r^{T,w} \|_{L^2(T)} \lesssim \|r_V\|_{V_T^*} + h_T/\alpha_T \| f - f_h \|_{L^2(T)}. \tag{5.19}$$

The proof is the same for a parallelogram T with a different b_T.

In the second part of this proof, we consider an interior edge E with patch $\omega_E :=$ interior$(\cup\{T \in \mathcal{T} : E \subset \partial T\})$ and construct functions $b_E^k \in H_0^1(\omega_E)$ for non-negative integer k. On each of the two neighbouring elements T_1 and T_2 in $\overline{\omega}_E = T_1 \cup T_2$ the function b_E^k equals $p_k \, b_E - q_k^j \, b_{T_j}$ on T_j where b_E is the product of the two barycentric coordinates in T_j such that $b_E(s) = s/h_E(1 - s/h_E)$ is quadratic in the arc-length parameter $0 < s < h_E$ along E. The (one-dimensional) monomial $p_k(s) = s^k$ for the parameter $s := t_E \cdot (x - x_1)$ (where t_E is the unit tangential vector along E and $x_1 \in E$ the first vertex of T) defines $p_k(x)$. The polynomial q_k (of degree $\leq K$) is chosen such that b_E^k is $L^2(T_j)$-orthogonal to $\mathcal{P}_K(T_j)$ for $j = 1, 2$; the parameter K is the highest degree of the polynomials $r^{T_j,\vartheta}$, $\bar{r}^{T_j,w}$, $r^{E,\vartheta}$, and $r^{E,\vartheta}$. As a consequence, for $k = 0, \ldots, K$,

$$\int_{\omega_E} r^{T,\vartheta} b_E^k \, dx = 0 \quad \text{and} \quad \int_{\omega_E} \bar{r}^{T,w} b_E^k \, dx = 0. \tag{5.20}$$

Let $r^{E,\vartheta} =: \sum_{k=0}^K \alpha_k \, p_k|_E$ (resp. $r^{E,w} =: \sum_{k=0}^K \beta_k \, p_k|_E$) define real coefficients $\alpha_0, \ldots, \alpha_K \in \mathbb{R}^2$ (resp. $\beta_0, \ldots, \beta_K \in \mathbb{R}$) and then set $\varphi := \sum_{k=0}^K \alpha_k \, b_E^k \in H_0^1(\omega_E)^2$ and $v := \sum_{k=0}^K \beta_k \, b_E^k/\alpha_E^2 \in H_0^1(\omega_E)$. The equivalence of norms $\| b_E^{1/2} \cdot \|_{L^2(E)}$ and $\| \cdot \|_{L^2(E)}$ (on a polynomial space the components of $r^{E,\vartheta}$ and $r^{E,w}$ belong to) and (5.20) show (with (5.12) at the end)

$$\begin{aligned} \| r^{E,\vartheta} \|_{L^2(E)}^2 + \| r^{E,w} \|_{L^2(E)}^2/\alpha_E^2 &\lesssim \int_E (r^{E,\vartheta} \cdot \varphi + r^{E,w} \, v) ds \\ &= \sum_{j=1,2} \int_{T_j} (r^{T_j,\vartheta} \cdot \varphi + \bar{r}^{T_j,w} \, v) dx + \int_E (r^{E,\vartheta} \cdot \varphi + r^{E,w} \, v) ds \\ &= r_V(v,\varphi) + \sum_{j=1,2} \int_{T_j} (f_h - f) \, v \, dx. \end{aligned} \tag{5.21}$$

Since $f - f_h$ has the integral mean zero we have with the integral mean $\bar{v}$ of v and a Poincaré inequality on T_j that, with $V_E := H_0^1(\omega_E) \hookrightarrow V$, $V_E^* = H^{-1}(\omega_E))$, (5.21) leads to

$$\begin{aligned} &\| r^{E,\vartheta} \|_{L^2(E)}^2 + \| r^{E,w} \|_{L^2(E)}^2/\alpha_E^2 \\ &\qquad \lesssim \| r_V \|_{V_E^*} \| (\varphi, v) \|_V + h_E \| f - f_h \|_{L^2(\omega_E)} \| \nabla v \|_{L^2(\omega_E)}. \end{aligned} \tag{5.22}$$

The arguments in (5.17) apply to the present $\| (\varphi, v) \|_V$ as well and yield

$$h_E \| r^{E,\vartheta} \|_{L^2(E)} + h_E/\alpha_E \| r^{E,w} \|_{L^2(T)} \lesssim \|r_V\|_{V_E^*} + h_E/\alpha_E \| f - f_h \|_{L^2(\omega_E)}. \tag{5.23}$$

A summation of the estimates (5.19) and (5.23) for all T and E concludes the proof since,

$$\sum_{T \in \mathcal{T}} \|r_V\|_{V_T^*}^2 + \sum_{E \in \mathcal{E}} \|r_V\|_{V_E^*}^2 \lesssim \|r_V\|_{V^*}^2. \tag{5.24}$$

Choose $\varphi_E \in H^1_0(\omega_E)$ (extended by zero) with $\|r_V\|^2_{V_E^*} = \|\varphi_E\|^2_{H^1_0(\omega_E)} = \int_\Omega r_V(\varphi_E)dx$ and set $\varphi_j = \sum_{E\in\mathcal{E}_j} \varphi_E$ where $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_J$ is a partition such that $(\omega_E : E \in \mathcal{E}_j)$ are pairwise disjoint. It is not hard to see that J is bounded. Then,

$$\sum_{E\in\mathcal{E}_j} \|r_V\|^2_{V_E^*} = \sum_{E\in\mathcal{E}_j} \int_\Omega r_V(\varphi_E)dx = \int_\Omega r_V(\varphi_j)dx \le \|r_V\|_{V^*}\|\varphi_j\|_V. \tag{5.25}$$

By construction and since $(\omega_E : E \in \mathcal{E}_j)$ are pairwise disjoint,

$$\|\varphi_j\|^2_V = \sum_{E\in\mathcal{E}_j} \|\varphi_E\|^2_{H^1_0(\omega_E)} = \sum_{E\in\mathcal{E}_j} \|r_V\|^2_{V_E^*}. \tag{5.26}$$

A combination of (5.25)-(5.26) shows (the main part of) (5.24). The proof is finished. □

6. Reliable and Efficient Computable Estimates for the residual in Q^*

This section is devoted to the reliable and efficient estimation of $r_Q \in Q^*$ with the $L^2(\Omega)$-representation $r_h := \vartheta_h - \nabla w_h - \beta^2\gamma_h$, which is L^2-orthogonal to Q_h owing to the Galerkin property, in the norm $\|\cdot\|_{Q^*}$ by η_Q,

$$\begin{aligned}\eta_Q^2 \;:=\; & \sum_{T\in\mathcal{T}} \frac{h_T^2}{(h_T+\beta_T)^2}\| \operatorname{curl} r_h \|^2_{L^2(T)} + \sum_{T\in\mathcal{T}} \| r_h/(1/\alpha+\beta) \|^2_{L^2(T)} \\ & + \sum_{E\in\mathcal{E}} \frac{h_E}{\beta_E(h_E+\beta_E)} \| [r_h]\cdot\tau_E \|^2_{L^2(E)}.\end{aligned} \tag{6.1}$$

The estimator is a lower and upper bound of the residual $\| r_Q \|_{Q^*}$. The reliability and efficiency proof are based on the following lemma.

Lemma 6.1. *If Ω is simply connected, we have*

$$\sup_{z\in H^1(\Omega)\setminus\{0\}} \frac{(r_h;\operatorname{Curl} z)_{L^2(\Omega)}}{\|z\|_{L^2(\Omega)} + \|\beta\nabla z\|_{L^2(\Omega)}} \lesssim \| r_Q \|_{Q^*}, \tag{6.2}$$

$$\| r_h/(1/\alpha+\beta) \|_{L^2(\Omega)} \lesssim \| r_Q \|_{Q^*}, \tag{6.3}$$

$$\| r_Q \|_{Q^*} \lesssim \| \alpha\, r_h \|_{L^2(\Omega)} + \sup_{z\in H^1(\Omega)\setminus\{0\}} \frac{(r_h;\operatorname{Curl} z)_{L^2(\Omega)}}{\|z\|_{L^2(\Omega)} + \|t\,\nabla z\|_{L^2(\Omega)}}, \tag{6.4}$$

$$\| r_Q \|_{Q^*} \le \|r_h/\beta\|_{L^2(\Omega)}. \tag{6.5}$$

Proof. The definition of $\| \eta \|_Q$ (where $p, q \in L^2(\Omega)^2$, $p+q \ne 0$, and $\operatorname{div} p = 0$) reads

$$\| r_Q \|_{Q^*} = \sup_{p,q} \frac{(r_h; p+q)_{L^2(\Omega)}}{\left\{\| p \|^2_{H^{-1}(\Omega)} + \| q/\alpha \|^2_{L^2(\Omega)} + \| \beta(p+q) \|_{L^2(\Omega)}\right\}^{1/2}}. \tag{6.6}$$

Since Ω is simply connected and $p \in L^2(\Omega)$ is divergence free, we have $p = \operatorname{Curl} z := (-\partial z/\partial x_2, \partial z/\partial x_1)$ for some $z \in H^1(\Omega)$ [GR]. Adding a constant to z, if necessary, we obtain $\int_\Omega z\,dx = 0$ and infer from the existence of solutions to the Stokes equations that $z = \operatorname{div}\eta$ for some $\eta \in H^1_0(\Omega)^2$ [GR]; furthermore, writing $\psi = (-\eta_2, \eta_1) \in H^1_0(\Omega)^2$,

$$z = \operatorname{rot}\psi \quad \text{and} \quad \| \psi \|_{H^1_0(\Omega)} \le c_{14} \| z \|_{L^2(\Omega)}, \tag{6.7}$$

where c_{14} depends only on Ω. Using this and an integration by parts, we deduce

$$\begin{aligned}\|z\|_{L^2(\Omega)} = (z;\operatorname{rot}\psi)_{L^2(\Omega)} = \langle \operatorname{Curl} z;\psi\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} = \langle p;\psi\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} \\ \le \|p\|_{H^{-1}(\Omega)}\,\|\psi\|_{H^1_0(\Omega)} \le c_{14}\,\|p\|_{H^{-1}(\Omega)}\,\|z\|_{L^2(\Omega)}\end{aligned}$$

and (by duality and integration by parts) $\|p\|_{H^{-1}(\Omega)} = \|\operatorname{Curl} z\|_{H^{-1}(\Omega)} \le \|z\|_{L^2(\Omega)}$, whence

$$\|p\|_{H^{-1}(\Omega)} \le \|z\|_{L^2(\Omega)} \le c_{14}\,\|p\|_{H^{-1}(\Omega)}. \tag{6.8}$$

Therefore, a substitution of $p = \operatorname{Curl} z$ and $\|p\|_{H^{-1}(\Omega)}$ by $\|z\|_{L^2(\Omega)}$ in (6.6) shows

$$\|r_h\|_{Q^*} \approx \sup_{q\in L^2(\Omega),\, z\in H^1_0(\Omega)} \frac{(r_h;q+\operatorname{Curl} z)_{L^2(\Omega)}}{\|z\|_{L^2(\Omega)} + \|q/\alpha\|_{L^2(\Omega)} + \|\beta(q+\operatorname{Curl} z)\|_{L^2(\Omega)}}. \tag{6.9}$$

The right-hand side of (6.9), equivalent to $\|r_Q\|_{Q^*}$, allows immediate proofs of (6.2) and (6.3): For $q=0$, we obtain (6.2); for $z=0$ we deduce (6.3) for a proper $q\in L^2(\Omega)^2$ from

$$\frac{(r_h/(1/\alpha+\beta);q(1/\alpha+\beta))_{L^2(\Omega)}}{\|q(1/\alpha+\beta)\|_{L^2(\Omega)}} = \frac{(r_h;q)_{L^2(\Omega)}}{\|q(1/\alpha+\beta)\|_{L^2(\Omega)}} \le \frac{\sqrt{2}(r_h;q)_{L^2(\Omega)}}{\|q/\alpha\|_{L^2(\Omega)} + \|\beta q\|_{L^2(\Omega)}}.$$

In the verification of (6.4) show a triangle inequality,

$$\|t\nabla z\|_{L^2(\Omega)} = \|t\operatorname{Curl} z\|_{L^2(\Omega)} \le \|t(q+\operatorname{Curl} z)\|_{L^2(\Omega)} + \|t\,q\|_{L^2(\Omega)},$$

and $t<\alpha^{-1}$ resp. $t<\beta$ that

$$\begin{aligned}\|z\|_{L^2(\Omega)} + \|t\nabla z\|_{L^2(\Omega)} + \|q/\alpha\|_{L^2(\Omega)} \\ \le \|z\|_{L^2(\Omega)} + 2\|q/\alpha\|_{L^2(\Omega)} + \|\beta(q+\operatorname{Curl} z)\|_{L^2(\Omega)}.\end{aligned} \tag{6.10}$$

A substitution of the lower bound $\|z\|_{L^2(\Omega)} + \|t\nabla z\|_{L^2(\Omega)}$ resp. $\|q/\alpha\|_{L^2(\Omega)}$ for (6.10) in the terms $(r_h;\operatorname{Curl} z)_{L^2(\Omega)}$ resp. $(r_h;q)_{L^2(\Omega)}$ of the right-hand side of (6.9) shows (6.4).

The estimate (6.5) follows immediately from (6.6). □

The error estimator η_Q is a (global) reliable upper bound.

Theorem 6.2. *Suppose that Ω is simply connected and that $\mathcal{T}$ consists of triangles. There exists an $(h_{\mathcal{T}},t)$-independent constant c_{15}, which depends on α and $\mathcal{T}$ only through $\kappa(\alpha,\mathcal{T})$, and c_Θ, such that*

$$\|r_Q\|_{Q^*} \le c_{15}\,\eta_Q. \tag{6.11}$$

Proof. The (closure of the) domain is split into two (essentially) disjoint (closed) sets A_1 and A_2 according to the value of β on neighbouring elements by

$$\begin{aligned}A_1 &:= \cup\{T\in\mathcal{T} : 2t<\beta_K \text{ for some } K\in\mathcal{T} \text{ with } K\cap T\neq\emptyset\},\\ A_2 &:= \cup\{T\in\mathcal{T} : 2t\ge\beta_K \text{ for all } K\in\mathcal{T} \text{ with } K\cap T\neq\emptyset\}.\end{aligned}$$

(A_1 is $\{T\in\mathcal{T} : 2t<\beta_T\}$ enlarged by neighbour elements.) Define $r_j := r_h$ on A_j and $r_j := 0$ on $\omega\setminus A_j$ for each $j=1,2$ so that we have $r_h = r_1 + r_2$. The estimates (6.4) and (6.5) can be separately applied to r_1 and r_2 (see the proof of Lemma 6.1) and show, with $|||z|||_\omega := \|z\|_{L^2(\omega)} + t\,\|\nabla z\|_{L^2(\omega)}$ for $\omega\subseteq\Omega$, that

$$\begin{aligned}\|r_Q\|_{Q^*} &\le \|r_1\|_{Q^*} + \|r_2\|_{Q^*}\\ &\le \|r_1/\beta\|_{L^2(A_1)} + \|\alpha\,r_2\|_{L^2(A_2)} + \sup_{z\in H^1(\Omega)\setminus\{0\}} \frac{(r_2,\operatorname{Curl} z)_{L^2(\Omega)}}{|||z|||_\Omega}.\end{aligned} \tag{6.12}$$

We claim $1/t \lesssim \alpha$ on A_1. For a proof consider (not necessarily distinct) $T, K \in \mathcal{T}$ with $K \cap T \neq \emptyset$ and $2t < \beta_K$. As a consequence, the definition of β_K yields $1/t \leq 2\alpha_K/\sqrt{3}$ and with $\alpha_K \leq \kappa(\alpha, \mathcal{T})\, \alpha_T$ the assertion $1/t \lesssim \alpha$. This estimate yields

$$1/\beta \lesssim 1/(1/\alpha + \beta) \quad \text{on } A_1, \qquad \text{whence } \| r_1/\beta \|_{L^2(A_1)} \lesssim \eta_Q. \tag{6.13}$$

Because of $\alpha < 1/t \leq 2/\beta$ on A_2, we have

$$\| \alpha\, r_2 \|_{L^2(A_2)} \leq 2 \| \min\{\alpha, 1/\beta\}\, r_2 \|_{L^2(A_2)} \leq 2\, \eta_Q. \tag{6.14}$$

Given $z \in H^1(\Omega) \setminus \{0\}$, it remains to check $(r_h; \operatorname{Curl} z)/|||\, z\, |||_\Omega \leq \eta_Q$. The bound of $(r_2, \operatorname{Curl} z)_{L^2(\Omega)} = (r_2; \operatorname{Curl}(z - z_h))_{L^2(\Omega)}$ utilises the observation that r_h and so r_2 is $L^2(\Omega)$-orthogonal onto $\mathcal{T}$-piecewise constants such as $\operatorname{Curl} z_h$ for the Clément approximation z_h to z in $\mathcal{S}^1_0(\mathcal{T})$. Besides $\|z - z_h\|_{L^2(T)} \lesssim \|z\|_{L^2(\omega_T)}$, we have

$$h_T^{-2} \|z - z_h\|^2_{L^2(T)} + h_E^{-1} \|z - z_h\|^2_{L^2(E)} + \|\nabla(z - z_h)\|^2_{L^2(T)} \lesssim \|\nabla z\|^2_{L^2(\omega_T)} \tag{6.15}$$

as in (5.5). With $\|z - z_h\|_{L^2(T)} \lesssim \min\{\|z\|_{L^2(\omega_T)}, h_T \|\nabla z\|_{L^2(\omega_T)}\}$ and $\beta_T \leq 2t$ if $T \subset A_2$,

$$(1 + \beta_T/h_T)\, \|z - z_h\|_{L^2(T)} \lesssim \|z\|_{L^2(\omega_T)} + t\, \|\nabla z\|_{L^2(\omega_T)} \lesssim |||\, z\, |||_{\omega_T}. \tag{6.16}$$

For each edge $E \subset A_2 \cap \partial T$ with neighbour $T \in \mathcal{T}$, we have $\beta_E := \max\{\beta_K : E \subset K \in \mathcal{T}\} \leq 2t$ and a trace inequality [Cl], [BS, p. 35],

$$\|z - z_h\|_{L^2(E)} \lesssim \|z - z_h\|^{1/2}_{L^2(T)} \Big(\|z - z_h\|^{1/2}_{L^2(T)} + \|\nabla(z - z_h)\|^{1/2}_{L^2(T)} \Big). \tag{6.17}$$

The definition of the norm $|||\, z\, |||_{\omega_T}$ and (6.16)-(6.17) show eventually that

$$\Big(\beta_E (1 + \beta_E/h_E) \Big)^{1/2} \|z - z_h\|_{L^2(E)} \lesssim |||\, z\, |||^{1/2}_{\omega_T}\, \beta_E^{1/2}\, \|\nabla z\|^{1/2}_{L^2(\omega_T)} \leq |||\, z\, |||_{\omega_T}. \tag{6.18}$$

A $\mathcal{T}$-piecewise integration by parts, Cauchy inequalities, (6.16), and (6.18) yield

$$\begin{aligned}
(r_2; \operatorname{Curl} z)_{L^2(\Omega)} &= (r_2; \operatorname{Curl}(z - z_h))_{L^2(\Omega)} = (\operatorname{curl}_{\mathcal{T}} r_2; z - z_h)_{L^2(\Omega)} + ([r_2] \cdot \tau_{\mathcal{E}}; z - z_h)_{L^2(\cup \mathcal{E})} \\
&\leq \sum_{E \in \mathcal{E}} \sqrt{\beta_E (1 + \beta_E/h_E)}\, \|z - z_h\|_{L^2(E)}\, \| [r_2] \cdot \tau_E / \sqrt{\beta_E (1 + \beta_E/h_E)}\, \|_{L^2(E)} \\
&\quad + \sum_{T \in \mathcal{T}} (1 + \beta_T/h_T)\, \|z - z_h\|_{L^2(T)}\, \| \operatorname{curl} r_h / (1 + \beta_T/h_T) \|_{L^2(T)} \\
&\lesssim |||\, z\, |||_\Omega \big(\eta_Q^2 + \sum_{E \subset \partial A_2} \| [r_2] \cdot \tau_E / \sqrt{\beta_E (1 + \beta_E/h_E)}\, \|^2_{L^2(E)} \big)^{1/2} \qquad (6.19) \\
&\lesssim |||\, z\, |||_\Omega \big(\eta_Q^2 + \sum_{E \subset \partial A_2} \| [r_1] \cdot \tau_E / \sqrt{\beta_E (1 + \beta_E/h_E)}\, \|^2_{L^2(E)} \big)^{1/2}.
\end{aligned}$$

The summation in the last sums in (6.19) is over all edges E which do not belong to the boundary Ω but to the boundary of A_2. While the jumps of $[r_h] \cdot \tau_E$ do contribute to η_Q the remaining jumps of $[r_2] \cdot \tau_E = [r_h] \cdot \tau_E - [r_1] \cdot \tau_E$ do not; we employed a triangle inequality in the last step and focus on the estimate of $\| [r_1] \cdot \tau_E / \sqrt{\beta_E (1 + \beta_E/h_E)}\, \|_{L^2(E)}$ for an edge $E = T_1 \cap T_2$ with $T_j \subset A_j$, $T_j \in \mathcal{T}$, for $j = 1, 2$. Therefore, the definition of A_2 leads to $\beta_{T_j} \leq 2t$ for $j = 1, 2$ while there exists a $K \in \mathcal{T}$ with $K \cap T_1 \neq \emptyset$ and $2t < \beta_K$. Thus

$$4/t^2 - 4\alpha_K^2 = 4/\beta_K^2 < 1/t^2, \quad \text{whence } 3/t^2 < 4\alpha_K^2. \tag{6.20}$$

The reverse arguments show $4\alpha_{T_1}^2 \le 3/t^2$ and so, with $\alpha_K \le \kappa(\alpha, \mathcal{T})\alpha_{T_1}$ and (6.20),

$$1/t \lesssim \alpha_K \lesssim \alpha_{T_1} \lesssim 1/t, \quad \text{whence } \alpha_{T_1} \approx 1/t. \tag{6.21}$$

This, $0 < \alpha$, and the definition of β show

$$1/\beta_{T_1}^2 = 1/t^2 - \alpha_{T_1}^2 \le 1/t^2 \lesssim \alpha_{T_1}^2, \quad \text{whence } 1/\beta_{T_1} \lesssim \min\{1/\beta_{T_1}, \alpha_{T_1}\}. \tag{6.22}$$

On the other hand, r_1 vanishes on T_2 and equals r_h on T_1, hence, the jump $[r_1] \cdot \tau_E$ equals the trace $(r_h \cdot \tau_E|_{T_1})|_E$ of $r_h \cdot \tau_E$ on T_1. A trace inequality for r_h on $E \subset \partial T_1$ and the inverse estimate $h_E \| \nabla r_h \|_{L^2(T_1)} \lesssim \| r_h \|_{L^2(T_1)}$ for the polynomial $r_h|_{T_1}$ show

$$\| [r_1] \cdot \tau_E \|_{L^2(E)} \le \| r_h|_{T_1} \|_{L^2(E)} \lesssim h_E^{-1/2} \| r_h \|_{L^2(T_1)} + h_E^{1/2} \| \nabla r_h \|_{L^2(T_1)} \lesssim h_E^{-1/2} \| r_h \|_{L^2(T_1)}. \tag{6.23}$$

With $\beta_{T_1} \le \beta_E$ and (6.22)-(6.23) we deduce

$$\| [r_1] \cdot \tau_E / \sqrt{\beta_E(1 + \beta_E/h_E)} \|_{L^2(E)} \lesssim \| r_h \|_{L^2(T_1)}/\beta_E \lesssim \| \min\{\alpha, 1/\beta\}\, r_h \|_{L^2(T_1)}. \tag{6.24}$$

The evaluation of (6.24) in (6.19) concludes the proof. $\square$

Remark 6.1. The assumption on triangles can be weakened to the hypothesis on parallelograms $T \in \mathcal{T}$ that $\int_T p\, r_h\, dx$ vanishes for all $p \in P_1(T)$. (The only condition is that $\int_\Omega \operatorname{curl} z_h\, r_h\, dx = 0$ which is then satisfied for $z_h|_T$ in $P_1(T)$ resp. $Q_1(T)$.)

The error estimator η_Q is efficient: The converse inequality of (6.11) holds even in a more local form than stated (cf. the proof of (6.25) below).

Theorem 6.3. *Suppose that Ω is simply connected. Then there exists an $(h_\mathcal{T}, t)$-independent constant c_{16}, which depends on α and $\mathcal{T}$ only through $\kappa(\alpha, \mathcal{T})$, and c_Θ, such that*

$$\eta_Q \le c_{16} \| r_Q \|_{Q^*}. \tag{6.25}$$

Proof. For each triangle T let b_T be a bubble-function as in the first part of the proof of Theorem 5.3; set $z|_T := b_T\ \operatorname{curl} r_h/(1 + \beta_T/h_T) \in H_0^1(T)$ and define $|\!|\!|\cdot|\!|\!|_\omega := \| \cdot \|_{L^2(\omega)} + \| \beta \nabla \cdot \|_{L^2(\omega)}$ for $\omega \subseteq \Omega$. Then, equivalence of the norms $\| b_T^{1/2} \cdot \|_{L^2(T)}$ and $\| \cdot \|_{L^2(T)}$ for polynomials, an $\mathcal{T}$-piecewise integration by parts, and (6.2) of Lemma 6.1 show

$$\begin{aligned} \| \operatorname{curl}_\mathcal{T} r_h \|_{L^2(\Omega)}^2 &\lesssim ((1 + \beta/h_\mathcal{T})z; \operatorname{curl}_\mathcal{T} r_h)_{L^2(\Omega)} \\ &= (\operatorname{Curl}_\mathcal{T} z(1 + \beta/h_\mathcal{T}); r_h)_{L^2(\Omega)} \le \| r_Q \|_{Q^*} |\!|\!| (1 + \beta/h_\mathcal{T})z |\!|\!|_\Omega. \end{aligned} \tag{6.26}$$

By $\| \nabla b_T \|_{L^\infty(T)} \lesssim 1/h_T$ and (the inverse estimate) $h_T \| \nabla \operatorname{curl} r_h \|_{L^2(T)} \lesssim \| \operatorname{curl} r_h \|_{L^2(T)}$,

$$\begin{aligned} |\!|\!| (1 + \beta_T/h_T)z |\!|\!|_T &= \| b_T \operatorname{curl} r_h \|_{L^2(T)} + \beta_T \| \nabla(b_T\ \operatorname{curl} r_h) \|_{L^2(T)} \lesssim \| \operatorname{curl} r_h \|_{L^2(T)} \\ &\quad + \beta_T \Big(\| \nabla b_T\ \operatorname{curl} r_h \|_{L^2(T)} + \| \nabla \operatorname{curl} r_h \|_{L^2(T)} \Big) \lesssim (1 + \beta_T/h_T) \| \operatorname{curl} r_h \|_{L^2(T)}. \end{aligned} \tag{6.27}$$

Combining (6.26)-(6.27) we deduce the asserted estimate of the volume contributions

$$\| \operatorname{curl}_\mathcal{T} r_h/(1 + \beta_\mathcal{T}/h_\mathcal{T}) \|_{L^2(\Omega)} \lesssim \| r_Q \|_{Q^*}. \tag{6.28}$$

The related proof of the estimate on the edge contributions is more involved. For each $E \in \mathcal{E}$, say $E = \operatorname{conv}\{a, b\}$ for end-points $a, b \in \mathcal{N}$ and with patch ω_E consider $\sigma_E := \{x \in \omega_E : 0 < \tau_E \cdot (x - a) < h_E\}$, which might be strictly smaller than ω_E if some inner angles are larger than $\pi/2$. The reduced patch σ_E consists of (at most) two (neighbouring) elements T_1 and T_2

on which we define $b_E^k \in H_0^1(\sigma_E) \subseteq H_0^1(\omega_E)$ as in the second part of the proof of Theorem 5.3. Given $[r_h]\cdot\tau_E$ set $\phi_E := \sum_{k=0}^K \alpha_k\, b_E^k \in H_0^1(\sigma_E) \subseteq H_0^1(\omega_E)$ with $(\phi_E)|_E = b_E\,[r_h]\cdot\tau_E$ and so

$$h_E^{-1/2}\|\,\phi_E\,\|_{L^2(\omega_E)} + h_E^{1/2}\|\,\nabla\phi_E\,\|_{L^2(\omega_E)} \lesssim \|\,[r_h]\cdot\tau_E\,\|_{L^2(E)}. \tag{6.29}$$

To cover the situation of very small β_E/h_E, we employ an idea of Verfürth [V2] and consider, for $0<\delta_E := \min\{1,\beta_E/h_E\}$, the affine bijection $\Phi : \sigma_E \to \omega_E^\delta$ defined by

$$\Phi(x) := a + s\,\tau_E + \delta_E\, t\, n_E \quad \text{for } x = a + s\,\tau_E + t\, n_E, \tag{6.30}$$

onto a smaller domain ω_E^δ ($s := \tau_E\cdot(x-a)$, $t := n_E\cdot(x-a)$); Φ describes a stretch in the direction n_E by a factor δ_E. We define $\Psi_E := \Phi^{-1} : \omega_E^\delta \to \sigma_E$ with constant derivative $B := \tau_E\otimes\tau_E + \delta^{-1}\; n_E\otimes n_E$ and its determinant $\det B = 1/\delta_E$; let $z_E := \phi_E\circ\Psi_E \in H_0^1(\omega_E^\delta) \subseteq H_0^1(\omega_E)$ and set $\rho_E^2 := \beta_E(1+\beta_E/h_E)$ for each $E\in\mathcal{E}$. The family $(z_E : E\in\mathcal{E})$ (regarded as functions in $H_0^1(\Omega)$) has finite overlap and so $z := \sum_{E\in\mathcal{E}} z_E/\rho_E^2$ is well-defined in $H_0^1(\Omega)$. Since $z|_E = (\phi_E)|_E$, an integration by parts shows (after equivalence $\|\,b_E^{1/2}\cdot\,\|_{L^2(E)}$ and $\|\cdot\|_{L^2(E)}$ for polynomials)

$$\begin{aligned}\|\,[r_h]\cdot\tau_{\mathcal{E}}/\rho_{\mathcal{E}}\,\|^2_{L^2(\cup\mathcal{E})} &\lesssim \sum_{E\in\mathcal{E}}\rho_E^{-2}\int_E [r_h]\cdot\tau_E\, z_E\, ds\\ &= \sum_{E\in\mathcal{E}}\rho_E^{-2}\int_{\omega_E^\delta} r_h\cdot\operatorname{Curl} z_E\,dx = (r_h;\operatorname{Curl} z)_{L^2(\Omega)} \le \|\,r_Q\,\|_{Q^*}\,|||\,z\,|||_\Omega,\end{aligned} \tag{6.31}$$

where we used the transformed analogy $\int_{\omega_E^\delta}\operatorname{curl} r_h\, z_E\,dx = 0$ of (5.20). Since $z_E = \phi_E\circ\Psi$, we infer from a transformation formula that

$$\|\,z_E\,\|_{L^2(\omega_E^\delta)} \le \delta_E^{1/2}\,\|\,\phi_E\,\|_{L^2(\omega_E)} \quad\text{and}\quad \|\,\nabla z_E\,\|_{L^2(\omega_E^\delta)} \le \sqrt{\delta_E + 1/\delta_E}\,\|\,\nabla\phi_E\,\|_{L^2(\omega_E)}. \tag{6.32}$$

Since the ω_E have finite overlap, $\beta_E := \|\,\beta\,\|_{L^\infty(\Omega_E)}$, a combination of (6.29) and (6.32) show

$$\begin{aligned}|||\,z\,|||^2_\Omega &\lesssim \sum_{E\in\mathcal{E}}(\|\,z_E\,\|^2_{L^2(\omega_E^\delta)} + \beta_E^2\,\|\,\nabla z_E\,\|^2_{L^2(\omega_E^\delta)})/\rho_E^4\\ &\lesssim \sum_{E\in\mathcal{E}}(\delta h_E + \beta_E^2/(\delta h_E))\|\,[r_h]\cdot\tau_E\,\|^2_{L^2(E)}/\rho_E^4 \le \sum_{E\in\mathcal{E}}\|\,[r_h]\cdot\tau_E/\rho_E\,\|^2_{L^2(E)}\end{aligned} \tag{6.33}$$

because, in any case, $(\delta h_E + \beta_E^2/(\delta h_E)) \le \rho_E^2$. A combination of (6.31) and (6.33) proves $\|\,[r_h]\cdot\tau_{\mathcal{E}}/\rho_{\mathcal{E}}\,\|^2_{L^2(\cup\mathcal{E})} \lesssim \|\,r_Q\,\|_{Q^*}$. The remaining assertion (6.3) is already verified. □

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