A General Approach to Algebraic Multigrid Methods *

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Abstract

In this report a general approach to algebraic multigrid methods for problems arising from the finite element discretization of a second order, self-adjoint, elliptic partial differential equation is proposed. Special attention is paid to the coarsening process and the transfer operators. In order to construct a more flexible method an auxiliary matrix is introduced which represents a virtual finite element mesh. In addition this auxiliary matrix is related to the degrees of freedom of the system matrix. The coarsening is performed on the auxiliary matrix, and after defining appropriate transfer operators for the system and the auxiliary matrix, a coarse system can be constructed by Galerkin's method. Moreover, a necessary condition imposed on the corresponding transfer operators is given such that the properties of the fine level system hands over to a coarse level system. It turns out that this approach is a generalization of many existing algebraic multigrid methods. Numerical examples are given which show the efficiency and flexibility of the proposed method.

Keywords algebraic multigrid, coarsening technique, transfer operators, preconditioned conjugate gradient method

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1 Introduction

A finite element (FE) discretization of a second order self-adjoint, elliptic boundary value problem leads usually to a linear (or linearized) system of equations

$$K_h \underline{u}_h = \underline{f}_h \,, \tag{1}$$

where $K_h \in \mathbb{R}^{N_h \times N_h}$ is a symmetric positive definite (SPD) sparse system matrix, $\underline{f}_h \in \mathbb{R}^{N_h}$ the right hand side and $\underline{u}_h \in \mathbb{R}^{N_h}$ the solution vector. The number N_h of unknowns in (1) behaves asymptotically like $N_h = O(h^{-d})$, with $d = 1, 2, 3$ the spatial dimension and h the mesh-size parameter. Therefore the linear system is usually very large. The condition number of the system matrix K_h is typically of order $\kappa(K_h) = O(h^{-2})$, which is the reason for the slow convergence of classical iterative solvers. Consequently, the efficient and fast solution of (1) is an important aspect of the FE-method and therefore optimal solvers, i.e., CPU-time and memory consumption are proportional to N_h, are of interest. Krylov subspace methods together with multigrid methods fulfill these requirements (see [16, 17]).

Algebraic multigrid (AMG) methods are of special interest if geometric multigrid can not be applied. There are at least two reasons for using AMG:

- The discretization provides no hierarchy of FE-meshes.
- The coarsest grid of a geometric multigrid method is too large to be solved efficiently by a direct or classical iterative solver.

In contrast to geometric multigrid methods, where a grid hierarchy is required explicitly, AMG is able to construct a matrix hierarchy and the corresponding transfer operators by only knowing the system matrix. A lot of numerical studies have shown that the convergence rate of AMG is comparable with geometric multigrid methods although it can not be proved in general. Applications of AMG methods in various practical (engineering) areas are given in [12, 13, 18, 19, 20].

The first serious approach to AMG was made 1982 by Brandt, McCormick and Ruge in [5] and an improved version of it can be found in [6]. This method is mainly concerned with SPD matrices K_h, which are additionally M-matrices. In this approach the smoother is usually fixed (e.g. Gauss-Seidel point relaxation) and the prolongation operator is constructed such that the error which is not affected by the smoother is in the range of the prolongation operator. This objective can be well realized for M-matrices [32, 30, 4, 31] but it turned out that it is hard to fulfill for general SPD matrices. A review of this AMG method is made in [33, 34] including also a proposal for the remedy of the M-matrix property. In spite of the fact that this method works well for M-matrices, the setup time (i.e., the construction of the matrix hierarchy with the corresponding transfer operators) and the application as a preconditioner requires plenty of CPU-time for practical problems.

To overcome these drawbacks Braess [3] suggested a quit simple AMG method, where the preconditioner can be constructed and applied very fast. The method benefits from the piecewise constant interpolation and consequently there is less fill in on the coarser levels which in turn implies a fast application. On the other hand this method fails at hand, if anisotropic structures are considered due to the poor prolongation. A related work is given in [21] by Kickinger, where an improved prolongation was proposed. In a subsequent work the agglomeration technique was combined with a block-smoother (see [22]).

Vanek, Mandel and Brezina developed and analyzed a new technique in [35, 23] and [38]. The idea is to construct a 'tentative' prolongation operator, which already takes care of the kernel of K_h (without essential boundary conditions) and improve it by some smoothing steps (usually one damped Jacobi step is used). The smoothing step provides to pull energy out of the basis function. This approach is called 'smoothed aggregation'.

A complete new idea was realized by Jones et al. at the Lawrence Livermore National Laboratory, called AMGe. This method basically assumes access to the element stiffness matrices and is able to construct a measure for the 'algebraic smooth error'. In addition an improved prolongation operator can be constructed, which relies not on the M-matrix property. For the construction and analysis of AMGe see [7, 15]. An element stiffness matrix free version, i.e., working with the assembled system matrix, is given in [14].

In recent time Wagner [37] develops an AMG method that is applicable also for non-symmetric problems. The key point in this approach is to find the two neighbors to a given node, which produce the best possible prolongation. The prolongation weights are computed by local minimization problems and a coarsening is done in a similar way. This approach provides a parallel AMG method in a natural way and it can be used additionally for scalar as well as for block system of equations.

Another approach to overcome the M-matrix property is depicted in [26, 27] which is called 'element preconditioning'. This approach assumes access to the element stiffness matrices, and constructs for every element stiffness matrix a spectral equivalent element stiffness matrix with the right sign condition. If the element stiffness matrices are assembled, we end up with a spectrally equivalent M-matrix B_h with respect to K_h. Applying AMG of Ruge and Stüben to B_h results in a preconditioner for K_h.

The above methods are closely related to an $H^1(\Omega)$-elliptic problems or problems with additional properties (e.g. M-matrix). The edge element FE-discretization in magnetic field computations yield a special matrix which has to be treated very carefully. AMG methods for this kind of problems are given in [2, 28].

The AMG techniques are partly able to deal with matrices K_h stemming from FE-discretization of a system of partial differential equations (see [35, 14, 21, 33]) and to handle non-symmetric matrices K_h arising from convection diffusion problems (see [21, 25, 33, 10, 29, 13, 37]). Other interesting AMG approaches can be found in [11, 24, 36, 9] and references therein.

All presented methods need inherently the efficient interplay of smoothing and coarse grid correction, which are the key ingredients for multigrid methods. One crucial point in the construction of AMG methods is the numerical effort of the coarsening process and the construction of appropriate transfer operators. The challenge is to construct an AMG method with a good convergence rate but rather low costs in the setup and in the application. Therefore we concentrate on the coarsening and the transfer operators in this report. For the other two main ingredients (smoother and coarse grid operator) the standard approaches are used, i.e., point-, block Gauss-Seidel and Galerkin's method. For the rest of this report a two grid method is described and therefore the subscripts h and H are related to fine and coarse grid quantities, respectively. Furthermore the superscripts sys, ker and aux stands for 'system', 'kernel' and 'auxiliary' quantities, respectively. The sub- and superscripts are suppressed whenever this is possible without confusion. In order to construct a more flexible and efficient coarsening strategy, we introduce an auxiliary matrix B_h which represents a virtual FE-mesh. For instance, B_h is a nodal distance matrix which also reflects the underlying partial differential equation. In this way B_h reflects anisotropies in the FE-mesh and in the operator. Obviously, this information has to be provided but this additional information is usually available in standard FE-codes.

After computing B_h on the given grid, a standard coarsening is performed on B_h and additionally appropriate transfer operators are defined. Then a coarse grid auxiliary matrix B_H is computed via the Galerkin method which is interpreted as a virtual FE-mesh on the coarse level. We have to mention, that the auxiliary matrix B_h is constructed such that the degrees of freedom of the system matrix are properly related to the entries of the auxiliary matrix and consequently the coarsening of B_h can be taken for K_h. Finally, transfer and smoothing operators for K_h can be constructed and a coarse grid matrix K_H is computed with Galerkin's method. Once the matrices are defined on a coarser level, the setup process is applied recursively.

Another key point for AMG methods is the prolongation operator. Therefore, we assume to know which underlying variational form (without essential boundary conditions) is under consideration. This is equivalent to know the null space of the main part of the partial differential equation. By this knowledge, a necessary condition on the prolongation operator is posed and prolongation operators for different variational forms are presented.

The report is organized as follows: In Sec. 2 the considered problem classes and notations are introduced. Further we discuss the components of an AMG method in Sec. 3 followed by Sec. 4 presenting some numerical studies. Finally, further remarks are given and conclusions are drawn.

2 Problem Formulation and Notations

Let us consider the variational form:

$$\text{Find } u \in \mathbb{V}: \quad a(u,v) \;=\; \langle F, v\rangle \quad \forall v \in \mathbb{V}, \tag{2}$$

with $a(\cdot,\cdot) : \mathbb{V} \times \mathbb{V} \mapsto \mathbb{R}$ is a symmetric, non-negative bilinear form in a Sobolev space $\mathbb{V}$. In addition, $F \in \mathbb{V}^*$ is a linear functional with the dual space of $\mathbb{V}$ denoted by $\mathbb{V}^*$ and the duality product $\langle\cdot,\cdot\rangle : \mathbb{V}^* \times \mathbb{V} \mapsto \mathbb{R}$. The computational domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is assumed to be a bounded domain with polygonal Lipschitz boundary $\partial\Omega$. If $a(\cdot,\cdot)$ is coercive, then existence and uniqueness is obtained by the Lax-Milgram Lemma (see e.g. [8]). If this is not the case, then the kernel of $a(\cdot,\cdot)$, i.e.,

$$\mathbb{V}_0 = \{u \in \mathbb{V} \,|\, a(u,v) = 0 \;\forall v \in \mathbb{V}\}\,, \tag{3}$$

can be expressed as

$$\mathbb{V}_0 = \Lambda\mathbb{Q} = \{\Lambda u \,|\, u \in \mathbb{Q}\}\,,$$

by introducing a bounded, linear operator $\Lambda : \mathbb{Q} \subset \mathbb{V} \mapsto \mathbb{V}$ applied to the basis $\mathbb{Q}$. If in addition

$$\langle F, v\rangle = 0 \quad \forall v \in \mathbb{V}_0$$

is assumed then the uniqueness of (2) is achieved. Notice, the kernel of the bilinear form is of special interest for the construction of an AMG method.

During this report three different variational forms $a(\cdot,\cdot)$ are considered, which stem from a self-adjoint, linear, elliptic partial differential equation of second order. In particular they are related to the *Electrostatic Equation*, to the *Linearized Elasticity Equations* and to the *Magnetostatic Equations*, subsequently.

Example 2.1. *Let us consider the variational formulation (Electrostatic Equation)*

$$a(u,v) = \int_\Omega \operatorname{grad} v^T D(x) \operatorname{grad} u \, dx + \int_\Omega \sigma u v \, dx \tag{4}$$

with $D(x) \in \mathbb{R}^{d\times d}$ *be SPD and* $\sigma \geq 0$. *An appropriate Sobolev space for (4) is given by*

$$\mathbb{V} = H^1(\Omega) = \{u \in L^2(\Omega) \mid \operatorname{grad} u \in L^2(\Omega)\}\,.$$

Further, let Λ *be the identity operator and* $\mathbb{Q} = \operatorname{span}\{\mathbf{1}\}$. *Then the kernel is given by*

$$\mathbb{V}_0 = \begin{cases} \Lambda\mathbb{Q} = \operatorname{span}\{\mathbf{1}\} & \sigma = 0 \\ \emptyset & \sigma > 0\,. \end{cases}$$

Example 2.2. *The next variational form is related to Linearized Elasticity Equations), which reads as*

$$a(u,v) = \int_\Omega \epsilon^T(v) E(x) \epsilon(u)\, dx \tag{5}$$

with $E(x) \in \mathbb{R}^{9\times 9}$ *be SPD,* $\epsilon(u) = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{21}, \epsilon_{23}, \epsilon_{32}, \epsilon_{13}, \epsilon_{31})^T$ *and* $\epsilon_{ij}(u) = \frac{1}{2} \cdot \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. *The Sobolev space is given by* $\mathbb{V} = (H^1(\Omega))^p$, $p = 3$ *(The 2D case is given in an analogous way). Again, let* Λ *be the identity operator and*

$$\mathbb{Q} = \operatorname{span} \left\{ \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{z} \\ -\mathbf{y} \end{pmatrix}, \begin{pmatrix} -\mathbf{z} \\ \mathbf{0} \\ \mathbf{x} \end{pmatrix}, \begin{pmatrix} \mathbf{y} \\ -\mathbf{x} \\ \mathbf{0} \end{pmatrix} \right\},$$

with $(\mathbf{x}, \mathbf{y}, \mathbf{z})^T \in \mathbb{R}^{N_h}$, $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{z} \in \mathbb{R}^{M_h}$ *are the* $x-$, $y-$ *and* z*-coordinates of the grid points, respectively. Then the kernel is given by*

$$\mathbb{V}_0 = \Lambda \mathbb{Q}.$$

Further we assume an operator $D(x) \in \mathbb{R}^{d\times d}$ *that provides approximately the principle stress (or strain) in a point* x.

Example 2.3. *An other very important variational form is given by*

$$a(u,v) = \int_\Omega \operatorname{curl} v^T D(x) \operatorname{curl} u\, dx + \int_\Omega \sigma u v\, dx \tag{6}$$

with $D(x) \in \mathbb{R}^{d\times d}$ *be SPD and* $\sigma \geq 0$. *Equation (6) is closely related to Magnetostatic Equations. An appropriate Sobolev space is given by*

$$\mathbb{V} = H_0(\operatorname{curl}, \Omega) = \{u \in (L^2(\Omega))^d \mid \operatorname{curl} u \in (L^2(\Omega))^d\,, u \times \vec{n} = 0 \text{ on } \partial\Omega\}\,,$$

with $\vec{n}$ *the unit outward vector. For* $\sigma = 0$ *the kernel is non-trivial and consists of all gradient fields if the domain* Ω *is simply connected. By defining* $\Lambda = \operatorname{grad}$ *and* $\mathbb{Q} = H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u(x) = 0\,, x \in \partial\Omega\}$ *this kernel can be written as*

$$\mathbb{V}_0 = \begin{cases} \Lambda \mathbb{Q} & \sigma = 0 \\ \emptyset & \sigma > 0\,. \end{cases}$$

The conforming FE-discretization is based on the regular partitioning τ_h of Ω into finite elements [8]. The FE-bases Φ has local support and span the FE-space $\mathbb{V}_h = \operatorname{span} \Phi \subseteq \mathbb{V}$ and additionally we a assume $\mathbb{Q}_h = \operatorname{span} \Psi \subseteq \mathbb{Q}$ for a given bases Ψ. Therefore, (2) changes into

$$\text{Find } u_h \in \mathbb{V}_h \subseteq \mathbb{V}: \quad a(u_h, v_h) = \langle F, v_h \rangle \quad \forall v_h \in \mathbb{V}_h\,,$$

which is equivalent to the linear equation (1) by the FE-isomorphism

$$G^h_{sys} : V_h \mapsto \mathbb{V}_h \ ,$$

with $V_h = \mathbb{R}^{N_h}$. For $\mathbb{Q}_h \subseteq \mathbb{Q}$ we define in an analogous way an isomorphism

$$G^h_{ker} : Q_h \mapsto \mathbb{Q}_h \ ,$$

with Q_h being an appropriate parameter space. After the definition of FE-spaces the discrete mapping $\Lambda_h : Q_h \mapsto V_h$, which maps the discrete kernel into the discrete space, is defined by

$$\Lambda_h \underline{q}_h = (G^h_{sys})^{-1} \, \Lambda \, G^h_{ker} \underline{q}_h \quad \forall \underline{q}_h \in Q_h$$

and thus the discrete kernel reads as

$$\begin{aligned} V_{0h} &= \{\underline{u}_h \in V_h \,|\, a(G^h_{sys}\underline{u}_h, G^h_{sys}\underline{v}_h) = 0 \ \forall \underline{v}_h \in V_h\} \\ &= \{\underline{q}_h \in Q_h \,|\, a(\Lambda G^h_{ker}\underline{q}_h, G^h_{sys}\underline{v}_h) = 0 \ \forall \underline{v}_h \in V_h\} \ = \ \Lambda_h Q_h \ . \end{aligned}$$

By defining coarser spaces $\mathbb{V}_H \subset \mathbb{V}_h$ and $\mathbb{Q}_H \subset \mathbb{Q}_h$ and the appropriate FE-isomorphisms G^H_{sys} and G^H_{ker}, $\Lambda_H : Q_H \mapsto V_H$ is defined by

$$\Lambda_H \underline{q}_H = (G^H_{sys})^{-1} \, \Lambda \, G^H_{ker} \underline{q}_H \quad \forall \underline{q}_H \in Q_H \, .$$

Next, we assume transfer operators $P^{sys}_h : V_H \mapsto V_h$ and $P^{ker}_h : Q_H \mapsto Q_h$ to be given with full rank, then the kernel of the coarse space is given by

$$V_{0H} = \{\underline{v}_H \,|\, P^{sys}_h \underline{v}_H \in V_{0h}\} \, , \tag{7}$$

in the case of using Galerkin's method for the coarse grid operator, i.e.,

$$K_H = (P^{sys}_h)^T \, K_h \, P^{sys}_h \, .$$

Remark 2.4. *The challenging task is to ensure $\Lambda_H Q_H = V_{0H}$ under the assumption that $\Lambda_h Q_h = V_{0h}$. This property is closely related to the transfer operators P^{sys}_h and P^{ker}_h.*

The system matrix is stored in such a way, that the p unknowns related to a node (edge) are stored in a sub-matrix k_{ij}, i.e.,

$$K_h = (k_{ij})_{i,j=1,\ldots,M_h} \, , \quad k_{ij} \in \mathbb{R}^{p \times p}$$

with M_h the number of nodes (edges). Thus the number of unknowns is given by $N_h = M_h \cdot p$. Further we call matrices A and $B \in \mathbb{R}^{m \times m}$ spectral equivalent if there exists lower and upper constants $\underline{\gamma}$, $\bar{\gamma} \in \mathbb{R}^+$, respectively, such that

$$\underline{\gamma} \cdot \langle B\underline{u}, \underline{u}\rangle \le \langle A\underline{u}, \underline{u}\rangle \le \bar{\gamma} \cdot \langle B\underline{u}, \underline{u}\rangle \quad \forall \underline{u} \in \mathbb{R}^m$$

with $\langle \cdot, \cdot \rangle$ the Euclidean inner product. This abbreviated by $\underline{\gamma} \cdot B \le A \le \bar{\gamma} \cdot B$. For SPD matrices A we can define the energy norm $\|\underline{u}\|_A^2 = \langle A\underline{u}, \underline{u} \rangle$. The transposed of a matrix A is denoted by A^T. The condition number of an SPD matrix is $\kappa(A)$ and we use the fact that

$$\kappa(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \le \frac{\bar{\gamma}}{\underline{\gamma}},$$

with $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ are the maximal and minimal eigenvalue of A, respectively. Finally, we define the set $Z_m = \{A \in \mathbb{R}^{m \times m} \mid a_{ii} \ge 0\,, a_{ij} \le 0 \ \ \forall i \ne j\}$.

3 Components of an AMG Method

3.1 The Auxiliary Matrix

Let us assume a system matrix K_h stemming from an FE-discretization with FE-mesh $\omega_h = (\omega_h^n, \omega_h^e)$, with ω_h^n the set of nodes and ω_h^e the set of edges in the FE-mesh. In the following we construct an auxiliary matrix $B_h \in \mathbb{R}^{M_h \times M_h}$ with the following properties:

$$(B_h)_{ij} = \begin{cases} b_{ij} \le 0 & i \ne j \\ -\sum_{l \ne i} b_{il} \ge 0 & i = j \end{cases} \qquad \forall i, j \in \omega_h^n$$

and the entries of B_h should be defined in a way such that

1. the distance between two geometric grid points is reflected and
2. the operator $D(x)$ from (4)-(6) is reflected.

Remark 3.1.

1. *$B_h \in Z_{M_h}$ by construction.*
2. *The matrix pattern of B_h can be constructed via several objectives:*
   (a) *B_h reflects the geometric FE-mesh, i.e., $|b_{ij}| \ne 0 \Leftrightarrow (i,j)$ is an edge in the FE-mesh.*
   (b) *B_h has the same pattern as K_h, i.e., $\|k_{ij}\| \ne 0 \Leftrightarrow |b_{ij}| \ne 0$, with $\|\cdot\|$ be an arbitrary matrix norm.*

Example 3.2. *Let $D(x) \in \mathbb{R}^{d \times d}$ $(d = 1, 2, 3)$, the coefficient matrix of the variational form (4), (5) or (6), be SPD. Further let $a_{ij} \in \mathbb{R}^d$ be the geometric vector that connects node i with node j for $i \ne j$, i.e., a_{ij} is a geometric edge in the FE-mesh (see Fig. 1). Note, that $\|a\|_D^2$ represents the length of a_{ij} with respect to the $\|\cdot\|_D$-norm. By defining $b_{ij} = -\frac{1}{\|a\|_D^2}$ for $i \ne j$ results in an appropriate auxiliary matrix (see Example 3.3).*

Example 3.3. *Let us consider the variational form (4) with*

$$D(x) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} ,$$

$\sigma = 0$ and a quadratic finite element with side length $h = 1$ (see Fig. 1) with bilinear FE-functions.

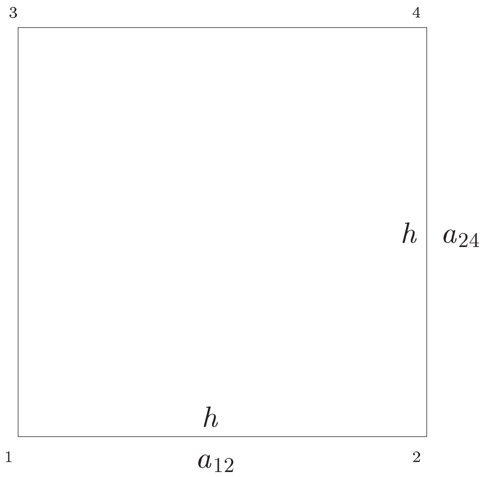

Figure 1: Quadratic finite element.

The element stiffness matrix K_h^r is given by

$$K_h^r = \frac{1}{\epsilon} \cdot \begin{pmatrix} 2+2\cdot\epsilon^2 & 1-2\cdot\epsilon^2 & -2+\epsilon^2 & -1-\epsilon^2 \\ 1-2\cdot\epsilon^2 & 2+2\cdot\epsilon^2 & -1-\epsilon^2 & -2+\epsilon^2 \\ -2+\epsilon^2 & -1-\epsilon^2 & 2+2\cdot\epsilon^2 & 1-2\cdot\epsilon^2 \\ -1-\epsilon^2 & -2+\epsilon^2 & 1-2\cdot\epsilon^2 & 2+2\cdot\epsilon^2 \end{pmatrix}$$

and it can be seen, that for $\epsilon \ll 1$ positive off diagonal entries appear which may cause difficulties in classical AMG methods. With an auxiliary matrix, presenting the same non-zero pattern as the system matrix, the problem is better reflected. The element matrix B_h^r becomes (using the method of Example 3.2)

$$B_h^r = \frac{1}{\epsilon^2+\epsilon} \cdot \begin{pmatrix} \epsilon^2+\epsilon+1 & -\epsilon^2-\epsilon & -\epsilon-1 & -\epsilon \\ -\epsilon^2-\epsilon & \epsilon^2+\epsilon+1 & -\epsilon & -\epsilon-1 \\ -\epsilon-1 & -\epsilon & \epsilon^2+\epsilon+1 & -\epsilon^2-\epsilon \\ -\epsilon & -\epsilon-1 & -\epsilon^2-\epsilon & \epsilon^2+\epsilon+1 \end{pmatrix} .$$

Now, the FE-mesh and the operator anisotropy are represented in the right way, as one would expect. In this case, a standard coarsening finds at hand the 'strong directions'.

Example 3.4. *Let us consider a stiffness matrix arising from linear elasticity, thus* $(K_h)_{ij} = k_{ij} \in \mathbb{R}^{p\times p}$ $(p = 2, 3)$*. Then* $b_{ij} = -\|k_{ij}\|$ *for* $i \neq j$ *is an appropriate auxiliary matrix.*

Example 3.5. *The last example is related to Maxwell's equations and an edge element discretization. In this case the FE-mesh has to be represented by the auxiliary matrix. Using the method of Example 3.2 and* $D(x)$ *of Example 3.3 we get the following element matrix*

$$B_h^r = \frac{1}{\epsilon} \cdot \begin{pmatrix} \epsilon+1 & -\epsilon & -1 & 0 \\ -\epsilon & \epsilon+1 & 0 & -1 \\ -1 & 0 & \epsilon+1 & -\epsilon \\ 0 & -1 & -\epsilon & \epsilon+1 \end{pmatrix}.$$

The entries $(B_h^r)_{14}$*,* $(B_h^r)_{23}$*,* $(B_h^r)_{41}$ *and* $(B_h^r)_{32}$ *are zero, i.e., there is no diagonal edges in the virtual FE-mesh related to Fig. 1 (see [28]).*

Remark 3.6.

1. *We can think of* B_h *as related to elements, such that an element agglomeration is performed (see [15]). This can be realized by defining a distance between two elements (e.g. one over the geometric distance of the barycenter, for elements in a neighborhood).*

2. *The geometric information is required on the finest grid only. By Galerkin's method we get a coarse auxiliary matrix, if the transfer operators for the auxiliary matrix are defined properly.*

3.2 The Coarsening Process

The auxiliary matrix may look artificial but a closer look shows that B_h represents a virtual FE-mesh in the following sense: If a point j is 'far' away from i, then the auxiliary matrix has a small negative entry. On the other hand if a point j is 'close' to i, then the auxiliary matrix has a large negative entry. As it was mentioned above the matrix $B_h \in Z_{M_h}$ and therefore the setup for B_h is straightforward. The crucial point is the identification of the entries of B_h (virtual FE-mesh) for more than one degree of freedom per node (edge).

Let us remember the necessary steps for a matrix $B_h \in Z_{M_h}$. We know that such a matrix represents a virtual FE-mesh $\omega_h = (\omega_h^n, \omega_h^e)$. Such an FE-mesh can be split into two disjoint sets of nodes, i.e.,

$$\omega_h^n = \omega_C^n \cup \omega_F^n, \ \ \omega_C^n \cap \omega_F^n = \emptyset$$

with sets of coarse grid nodes ω_C^n and fine grid nodes ω_F^n. The splitting will be usually performed such that no coarse grid nodes are connected and there should be as much coarse grid nodes as possible. This can be achieved by using the following sets

$$\begin{aligned} N_h^i &= \{j \in \omega_h^n \,|\, |b_{ij}| \neq 0\,, i \neq j\} \\ S_h^i &= \{j \in N_h^i \,|\, |b_{ij}| > \mathrm{coarse}(B_h)\,, i \neq j\} \end{aligned}$$

and by taking one specific cut-off (coarsening) function

$$\mathrm{coarse}(B_h) = \begin{cases} \theta \cdot \sqrt{|b_{ii}||b_{jj}|} & \text{see [35]} \\ \theta \cdot \max_{l \neq i} |b_{lj}| & \text{see [31]} \\ \theta & \text{see [21]} \end{cases}$$

with an appropriate $\theta \in [0, 1]$. Further we define some local sets

$$\omega_C^i = \omega_C^n \cap N_h^i, \quad \omega_F^i = \omega_F^n \cap N_h^i$$

and we assume a 'disjoint' splitting $(I_h^i)_{i=1}^{M_H}$ $(M_H = |\omega_C^n|)$ with

$$I_h^i \cap I_h^j = \emptyset \quad \text{and} \quad \bigcup_{i=1}^{M_H} I_h^i = \omega_h^n\,.$$

By defining an appropriate prolongation P_h^B for B_h a coarse auxiliary matrix is computed by

$$B_H = (P_h^B)^T B_h P_h^B$$

and B_H represents a virtual FE-mesh $\omega_H = (\omega_H^n, \omega_H^e)$, with $\omega_H^n = \omega_C^n$.

Remark 3.7.

1. *In this discussion it is always assumed that the coarse grid degrees of freedom are numbered first.*

2. *B_h represents a virtual FE-mesh and therefore it can be related to the degrees of freedom in the original matrix. Consequently it is very important that the prolongation operators P_h^B, P_h^{sys} and P_h^{ker} are consistent, i.e.,*

   (a) *if Lagrange FE-functions are used, then $\|k_{ij}\| \neq 0 \Leftrightarrow |b_{ij}| \neq 0$ for $i \neq j$ on all levels.*

   (b) *if Nédélec FE-functions are used then b_{ij} $i > j$ (or $i < j$) represents an edge in a virtual FE-mesh (see [28]).*

3. *If $K_h \in Z_{N_h}$ stems from a scalar problem then we can take $B_h \equiv K_h$ which results in a classical AMG method, e.g. [31] (small positive off-diagonal entries of K_h are admissible).*

4. *If K_h stems from an FE-discretization of a scalar boundary value problem and we construct a preconditioner $B_h \in Z_{N_h}$ such that $\gamma_1 \cdot B_h \leq K_h \leq \gamma_2 \cdot B_h$, $0 < \gamma_1 \leq \gamma_2$, based on the element stiffness matrices, then the technique is equal to the element preconditioning technique (see [26, 27]).*

3.3 The Prolongation Operator

In most AMG-approaches the kernel of the underlying operator is not, or only implicitly considered. In many AMG approaches the constant functions are preserved, which is closely related to the variational form (4). But this prerequisite is not sufficient for (5) or (6). It is of great importance for multilevel methods that the characteristics of the discretized operator are the same on all levels, e.g., especially, the kernel has to be preserved. Consequently, AMG-methods have to meet this requirement, too. The following theorem provides a necessary condition for the prolongation operator.

Theorem 3.8. *Let V_H, V_h, Q_H, Q_h, Λ_H and Λ_h be defined as in Sec. 2. Moreover, $P_h^{sys} : V_H \mapsto V_h$ and $P_h^{ker} : Q_H \mapsto Q_h$ are matrices with full rank. If the equation*

$$P_h^{sys} \Lambda_H \underline{q}_H = \Lambda_h P_h^{ker} \underline{q}_H \quad \forall \underline{q}_H \in Q_H \tag{8}$$

holds and additionally

$$\forall \underline{q}_h \in Q_h \, \exists \underline{q}_H \in Q_H \, : \underline{q}_h = P_h^{ker} \underline{q}_H$$

is fulfilled, then

$$V_{0H} = \Lambda_H Q_H$$

is valid.

Proof. The proof splits into two parts:

1. We show that $\Lambda_H Q_H \subseteq V_{0H}$. Let us take an arbitrary but fixed $\underline{q}_H \in Q_H$ and recall the definition of V_{0H} (7). Thus we get

$$P_h^{sys} \Lambda_H \underline{q}_H = \Lambda_h P_h^{ker} \underline{q}_H \, ,$$

which is true because of (8). Consequently we obtain $\Lambda_H \underline{q}_H \in V_{0H}$.

2. We show that $V_{0H} \subseteq \Lambda_H Q_H$. We take a $\underline{v}_H \in V_{0H}$ and perform the following calculation

$$P_h^{sys} \underline{v}_H = \Lambda_h \underline{q}_h = \Lambda_h P_h^{ker} \underline{q}_H = P_h^{sys} \Lambda_H \underline{q}_H \, .$$

Because P_h^{sys} is assumed to have full rank we conclude that $\underline{v}_H = \Lambda_H q_H$, which is the desired result.

□

Example 3.9. *The first example is given by a Poisson type problem (4). In our setting we define $P_h^{sys} = P_h^{ker} = P_h^B$ and then one verifies easily the assumptions of Theorem 3.8, if P_h^{sys} is appropriately chosen, e.g*

$$(P_h^{sys})_{ij} = \begin{cases} 1 & i = j \in \omega_C^n \\ -\frac{k_{ij}+cij}{k_{ii}+c_{ii}} & i \in \omega_F^n,\, j \in \omega_C^i \\ 0 & else \end{cases} \tag{9}$$

with

$$c_{ij} = \sum_{p\in\omega_F^i} \frac{k_{ip}\cdot k_{pj}}{\sum_{q\in\omega_C^i} k_{pq} + k_{pi}}\,.$$

An other possibility for this case is a simple prolongation

$$(P_h^{sys})_{ij} = \begin{cases} 1 & i = j \in \omega_C^n \\ \frac{1}{|S_h^i\cap\omega_C^n|} & i \in \omega_F^n,\, j \in S_h^i \cap \omega_C^n \\ 0 & else \end{cases} \tag{10}$$

which again fulfills Theorem 3.8.

Example 3.10. *The preliminaries for Theorem 3.8 can not be shown for linear elasticity . It is known from geometric multigrid, that at least linear functions have to be prolongated exactly, in order to preserve the kernel. For AMG methods this is hardly possible. Anyway, we propose an interpolation which shows a good convergence behavior. We choose again $P_h^{sys} = P_h^{ker} = P_h^B$, with*

$$(P_h^{sys})_{ij} = \begin{cases} I_n^n & i = j \in \omega_C^n \\ -k_{ii}^{-1}\big(k_{ij} + c_{ij}\big) & i \in \omega_F^n,\, j \in \omega_C^i \\ 0 & else \end{cases} \tag{11}$$

with $I_n^n \in \mathbb{R}^{n\times n}$ the identity matrix and

$$c_{ij} = \sum_{p\in\omega_F^i} \Big(\sum_{q\in\omega_C^i} k_{pq}\Big)^{-1} k_{ip}k_{pj}\,.$$

An other possibility for this case is the analogue to prolongation (10)

$$(P_h^{sys})_{ij} = \begin{cases} I_n^n & i = j \in \omega_C^n \\ \frac{1}{|S_h^i\cap\omega_C^n|}\cdot I_n^n & i \in \omega_F^n,\, j \in S_h^i \cap \omega_C^n \\ 0 & else\,. \end{cases} \tag{12}$$

Example 3.11. *In [28] an appropriate setting for the prolongation operators P_{sys} and $P_{ker} = P_B$ is suggested if $\mathbb{V} = H_0(\text{curl}, \Omega)$, $\Lambda = \text{grad}$, $\mathbb{Q} = H_0^1$.*

$$(P_h^{sys})_{ij} = \begin{cases} +1 & i = j \in \omega_C^e,\ i,j \text{ have the same orientation} \\ -1 & i = j \in \omega_C^e,\ i,j \text{ have not the same orientation} \\ 0 & \text{else} \end{cases} \tag{13}$$

and $P_h^{ker} = P_h^B$ is given by

$$(P_h^{ker})_{ij} = \begin{cases} 1 & i = j \in \omega_C^n,\ i,j \in I_h^i \\ 0 & \text{else}. \end{cases}$$

4 Numerical Studies

The AMG techniques are used as a preconditioner for the preconditioned conjugate gradient (PCG) method (see [16, 17]). The iteration was stopped if an error reduction in the preconditioner energy norm by a factor of 10^{-8} has been achieved. We used a V(2,2)-cycle and solved the coarsest grid with a Cholesky factorization (degrees of freedom ≤ 500). All calculations were done on an SGI Octane 300 MHz workstation. In the subsequent sections the following short-cuts are used:

- 'iter': number of iterations in the PCG-method,
- 'setup': CPU-time (seconds) for the construction of the matrix hierarchy,
- 'solver': CPU-time (seconds) for the solution time of the PCG-method.

4.1 Anisotropic Scalar Equation

Let us consider the variational form (4) on $\Omega \subset \mathbb{R}^2$ be the unit square,

$$D = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}$$

and $\sigma = 10^{-4}$. We assume homogeneous Neumann boundary conditions on $\partial\Omega$. Further, the FE-discretization was done with bilinear FE-functions. The arising linear equation is solved on one hand by the new method based on an auxiliary matrix constructed via the method given in Example 3.2 and on the other hand by the classical Ruge/Stüben method. We used in both cases a Block-Gauss-Seidel smoother with maximal block size 3 per node patch. The prolongation for the AMG method based on the auxiliary matrix is given in (10) and the prolongation for the classical AMG method is due to (9).

N_h	ϵ	iter	setup (sec)	solver (sec)
10201		11	4.57	3.31
40401	10^{-1}	12	23.40	19.87
90601		12	54.78	46.78
10201		59	2.00	13.94
40401	10^{-2}	78	9.03	88.53
90601		84	24.12	228.27
10201		91	1.87	21.50
40401	10^{-3}	103	8.44	118.36
90601		167	18.57	441.84

Table 1: Results for the anisotropic 2D problem with classical AMG.

N_h	ϵ	iter	setup (sec)	solver (sec)
10201		15	1.28	2.72
40401	10^{-1}	14	5.12	11.63
90601		24	11.57	44.90
10201		9	1.37	1.75
40401	10^{-2}	12	5.28	10.37
90601		14	12.06	27.61
10201		10	1.36	1.90
40401	10^{-3}	11	5.26	9.50
90601		13	12.12	25.82

Table 2: Results for the anisotropic 2D problem with auxiliary matrix.

The results are depicted in Tab. 2 and Tab. 1 for the new and the classical AMG method, respectively.

It can be seen, that the new method performs much better than the classical method. In addition the new method is robust with respect to the anisotropic parameter, whereas the classical method fails in the sense of required PCG iterations, in the case of $\epsilon \leq 10^{-2}$. The classical AMG method works well, if the anisotropy is moderate ($\epsilon = 10^{-1}$). Let us further mention that the operator complexity, i.e.,

$$\frac{\sum_{i=1}^{\ell} NNE_i}{NNE_1}$$

with NNE_i is the number of non-zero entries on level i of the system matrix and ℓ the number of levels, is considerable less in the new method than in the classical one.

Next, we are concerned with the 3D case of the variational form (4) on Ω be an L-shaped domain,

$$D = \begin{pmatrix} 2+\epsilon & -\epsilon & 2-\epsilon \\ -\epsilon & 2+\epsilon & -2+\epsilon \\ 2-\epsilon & -2+\epsilon & 4+\epsilon \end{pmatrix},$$

$\sigma = 0$ and homogeneous Dirichlet boundary conditions. For an FE-discretization we use linear tetrahedra. The prolongation operator for both methods is given in (10) and a block Gauss-Seidel smoother is used with maximal block size 10. Again, the auxiliary matrix is constructed as mentioned in Example 3.2.

The results for different ϵ are given in Tab. 3 and Tab. 4 for the classical and the new method, respectively.

N_h	ϵ	iter	setup (sec)	solver (sec)
2025		5	0.37	0.70
14161	10^{-3}	9	3.45	15.28
105633		16	32.65	242.19
2025		5	0.35	0.69
14161	10^{0}	8	3.32	12.49
105633		11	28.92	156.09
2025		6	0.41	0.81
14161	10^{+3}	11	3.58	18.77
105633		19	32.99	291.36

Table 3: Results for an anisotropic problem in 3D with classical AMG.

N_h	ϵ	iter	setup (sec)	solver (sec)
2025		5	0.53	0.72
14161	10^{-3}	9	4.09	14.55
105633		14	35.53	195.67
2025		5	0.50	0.75
14161	10^{0}	8	4.05	12.47
105633		10	34.52	137.66
2025		7	0.63	1.29
14161	10^{+3}	13	5.15	24.68
105633		23	43.18	372.02

Table 4: Results for an anisotropic problem in 3D with auxiliary matrix.

Both methods are similar compared to each other. The reason therefore is that they detect approximately the same strong connections, i.e., the nodes which are potential candidates for prolongation. In this case, where the anisotropy is not aligned with the grid, an optimal solver is hard to realize.

4.2 Static Linear Elasticity Equations

The linear elasticity equations of (5) gives rise to block-system of equations. Subsequently, a cantilever beam and a crank shaft are presented. For further discussion the following abbreviations are used: type=1 is related to the new AMG method and prolongation given in (12), type=2 is due to the classical method with prolongation given in (12) and type=3 is related to the classical method with prolongation given in(11).

For the cantilever beam we assume a Poisson ratio 0.3 and an FE-discretization with bilinear FE-functions on a rectangular grid with ratio $1 : \epsilon$. Additionally, homogeneous Dirichlet boundary conditions are assumed on one side of the beam and free boundary conditions on the rest of the boundary, see Fig 2. First,

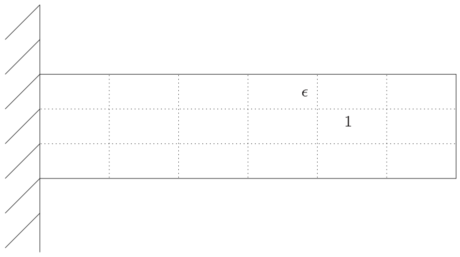

Figure 2: Cantilever Beam in 2D.

the results for $\epsilon = 1$ are presented in Tab. 5. The classical AMG method with harmonic extension operator (type 3) performs best, because the kernel of the operator (without essential boundary conditions) is represented well, which is not true for the other type of prolongation operators.

The results for the cantilever beam with $\epsilon = 10^{-1}$ are listed in Tab. 6. In this case the classical method is much better than the new AMG method. The reason therefore is again the better prolongation operator, as it was mentioned above.

To finish this subsection we present a 3D case which is related to a crankshaft. The geometry is given in Fig. 3 and the input data was the same as for the cantilever beam. The results are depicted in Tab. 7 and show that both methods perform comparable well. As for the cantilever beam the classical method with the harmonic extension for the prolongation operator is the best method in that comparison.

N_h	type	iter	setup (sec)	solver (sec)
20402		15	1.79	3.67
80802	1	21	7.21	21.57
181202		22	16.29	51.29
20402		12	1.11	3.00
80802	2	14	4.48	14.61
181202		15	9.96	35.40
20402		11	1.28	2.77
80802	3	11	5.11	11.64
181202		12	11.44	28.70

Table 5: Results for the Cantilever Beam with $\epsilon = 1$.

N_h	type	iter	setup (sec)	solver (sec)
20402		113	7.00	54.74
80802	1	109	30.80	224.27
181202		140	71.57	654.36
20402		50	1.58	15.68
80802	3	65	6.26	87.71
181202		81	13.57	244.43

Table 6: Results for the Cantilever Beam with $\epsilon = 10^{-1}$.

N_h	type	iter	setup (sec)	solver (sec)
3039		8	2.45	0.89
17769	1	16	12.98	10.63
118359		23	110.0	118.86
3039		8	0.63	0.82
17769	2	16	6.18	10.63
118359		23	13.52	118.75
3039		7	0.67	0.73
17769	3	11	2.18	7.38
118359		19	17.53	96.83

Table 7: Results for the Crank Shaft in 3D.

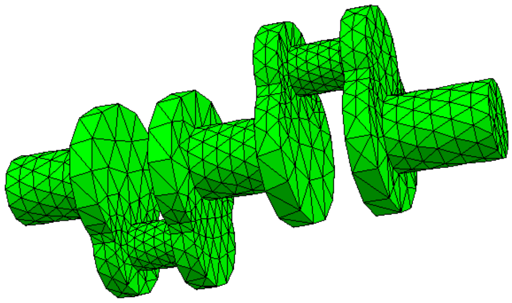

Figure 3: Crank Shaft in 3D.

4.3 Static Linear Maxwell Equations

The last numerical example is due to the variational form (6). In this case the proposed AMG method needs inherently the auxiliary matrix in order to construct an appropriate prolongation (13) and smoothing operator [1]. For more details see [28]. We consider a static linear magnetic field problem in 3D with geometry $\Omega \subset \mathbb{R}^3$ given in Fig. 4. Further we assume homogeneous Dirichlet boundary conditions,

$$D = \begin{cases} 10^0 \cdot I_3^3 & \text{in air} \\ 10^{-3} \cdot I_3^3 & \text{in ferromagnetics} \end{cases}$$

and $\sigma = 10^{-4}$. The results are given in Tab. 8. In this case we detect a stronger dependency on the mesh parameter h, but this is due to the non optimal prolongation operator. Nevertheless, this method is much better than standard preconditioners, i.e., incomplete Cholesky Factorization or even classical AMG which does not work for this example.

N_h	iter	setup (sec)	solver (sec)
8714	14	2.49	0.53
65219	28	44.01	4.17
504246	63	792.49	34.49

Table 8: Results for the magnetic valve with new AMG method.

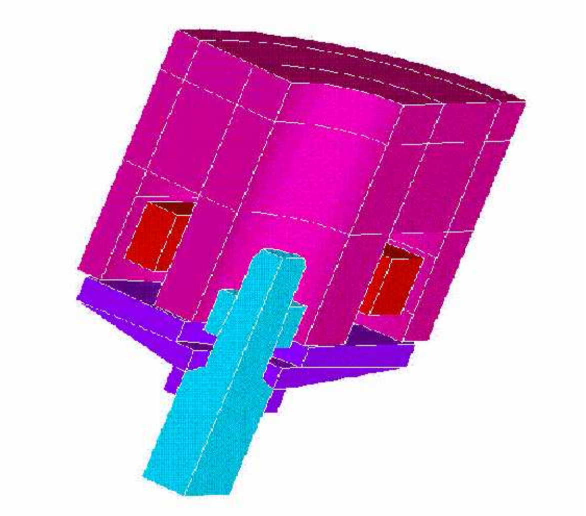

Figure 4: Magnetic valve in the linear static case.

5 Conclusions and Further Remarks

In this report a general AMG method for SPD problems, which arise from an FE-discretization of second order, self-adjoint, elliptic partial differential equations, was presented. We concentrated on the coarsening process and the prolongation operator. For the coarsening process an auxiliary matrix was introduced which is related to a virtual FE-mesh. A hierarchy of virtual FE-meshes (auxiliary matrices) was constructed and the degrees of freedom of the original problem are related properly to the auxiliary matrix. We suggested a necessary condition for the prolongation operator in order to preserve the properties of the underlying operator (i.e., especially the kernel). This condition is not sufficient to obtain an optimal solver. Nevertheless, this condition can be used as a starting point for non-standard problems, as it was done for edge element FE-discretizations. The proposed approach is general and can be easily implemented in standard FE-codes. In addition it has a great potential for practical applications, because optimal solvers can be constructed rather simple for a wide range of problems.

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