Optimal exploitation of renewable resource stocks: Necessary conditions

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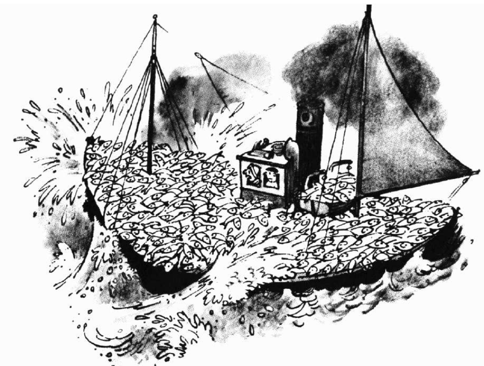
Abstract

We study a model for the exploitation of renewable stocks developed in [CC]. In this particular control problem, the control law contains a measurable and an impulsive control component. We formulate Pontryagin's maximum principle for this kind of control problems, proving first order necessary conditions of optimality. Manipulating the correspondent Lagrange multipliers we are able to define two special switch functions, that allow us to completely describe the optimal trajectories and control policies, for all possible initial conditions in the phase plane.



"I'm worried, Charlie! A catch like this could knock the bottom out of the market."

—Reprinted from *Punch*

1 Introduction

1.1 Description of the model

Consider a bio-economic model of the commercial fishery under sole ownership. The model is governed by the following quantities:

t	: time;
$x(t)$	: population biomass at time t;
$h(t)$	: harvest rate at time t;
$E(t)$	: fishing effort at time t;
$K(t)$	: amount of capital invested in the fishery at time t;
$I(t)$	: investment rate at time t.

The model is based on the following assumptions:

- $h(t) = qE(t)x(t)$; q is a catch coefficient;
- $x'(t) = F(x(t)) - qE(t)x(t)$, $t \geq 0$, $x(0) = x_0$; F is the natural growth function;

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- $K'(t) = -\gamma K(t) + I(t)\,, t \geq 0,\ K(0) = K_0\,;\ \gamma \geq 0$ is the rate of decpreciation;
- constraints: $0 \leq x(t), K(t), E(t)\,;\ E(t) \leq K(t)\,;$
- non-malleability: $0 \leq I(t) \leq \infty, t \geq 0\,;$
- existence of two biological equilibrium points: $F(0) = F(\bar{x}) = 0\,,\ \bar{x} > 0\,;$
- properties of the production function:

$$F \in C^2[0,\infty)\,,\ F(x) > 0\,,\ 0 < x < \bar{x}\,,\ F''(x) < 0\,,\ 0 \leq x \leq \bar{x}\,;$$

- objective function (discounted cash flow): $\int\limits_0^\infty e^{-\delta t}\{ph(t) - cE(t) - rI(t)\}dt\,;$
  $\delta > 0$ is the instanteous rate of discount, $p \geq 0$ is the price of landed fish, $c \geq 0$ is the operating cost per unit effort, $r \geq 0$ is the price of capital.

A concrete production function satisfying the assumption above is given by the logistic mapping $F(x) := ax(1 - \frac{x}{k})\ \ (a > 0, k > 0)$. In our figures we use this production function.

1.2 The optimal control problem

We set $E = u\,K$; u is now a second control variable. Without loss of generality we use $q = 1$. After this manipulation the problem we want to consider is the following optimal control problem:

$$Q(x_0, K_0) \qquad \begin{cases} \text{Minimize } J(x_0, K_0; I, u) := \int\limits_0^\infty e^{-\delta t}\{rI(t) + cu(t)K(t) - pu(t)K(t)x(t)\}dt \\ \text{subject to} \\ \quad x' = F(x) - u(t)K(t)x\ , t \geq 0,\ x(0) = x_0\,, \\ \quad K' = -\gamma K + I(t)\ , t \geq 0,\ K(0) = K_0\,, \\ \quad 0 \leq x(t), K(t)\,,\ 0 \leq u(t) \leq 1\,,\ I(t) \geq 0\,,\ t \in [0,\infty)\,. \end{cases}$$

This problem is considered in [CC] along the “royal road” of Caratheodory. But the analysis is not rigorous in the details (see Section 1.3). In [Mu] the problem is given as an illustration for the problem of the type considered in the paper but the results are not applicable (for proving existence).
In [VP] there is also a hint to this problem and finally, we find the problem in [C2] considered as an example for the application of the maximum principle, but nothing has been made rigorous. For the background in fishery managment see [Fl], [An] and [SS].

It is known (see for instance [Mu]), that control problems may have no solution if the control variable is unbounded and linear in the cost functional and the dynamics. This situation is given here with respect to the control I. In order to avoid non–existence we have to replace the conventional control I by an impulse control; jumps in the state result. Therefore we consider I as a Borel measure and the capital function K as a function of bounded variation. Then the problem $Q(x_0, K_0)$ becomes:

$$P(x_0, K_0) \qquad \begin{cases} \text{Minimze } J(x_0, K_0; \mu, u) := \int\limits_0^\infty e^{-\delta t} r\mu(dt) + \int\limits_0^\infty e^{-\delta t}\{c - pqx(t)\}u(t)K(t)dt \\ \text{subject to} \quad (u,\mu) \in U_{ad} \times C^* \ \text{ and} \\ \quad x' = F(x) - u(t)Kx\ ,\ x(0) = x_0\,, \\ \quad dK = -\gamma K dt + \mu(dt)\,,\ K(0) = K_0\,. \end{cases}$$

Here

$$\begin{aligned} U_{ad} &:= \{v \in L_\infty[0,\infty) | 0 \le v(t) \le 1 \text{ a.e. in } [0,\infty)\}\,, \\ C^* &:= \{\mu | \mu \text{ a non–negative Borel measure on } [0,\infty)\}\,. \end{aligned}$$

Notice that the constraints $0 \le x(t), K(t)$, $t \in [0,\infty)$, are satisfied due to the assumptions above if $x_0 \ge 0, K_0 \ge 0$.

The initial value problem

$$dK = -\gamma K dt + \mu(dt)\,,\; K(0) = K_0 \tag{1}$$

has to be considered as differential equation with a measure: a function $K : [0, t_1) \longrightarrow I\!R$ ($t_1 \in (0,\infty]$) is a solution if

$$K(t) = K_0 - \int_0^t \gamma K(s) ds + \int_{[0,t]} \mu(ds)\,,\; 0 \le t < t_1\,; \tag{2}$$

This implies that K is right continuous in $(0, t_1)$ and $K(0) = K_{0,+}\mu(\{t_0\})$ where $K_{0,+}$ denotes $\lim_{t \downarrow 0} K(t)$.

Remark 1 *Due to the fact that the coefficients in front of the measure control μ does not depend on the state we may use the solution concept as given above, the so called Young solutions (see [Ri]). Otherwise we would have to use the concept of robust solutions considered in [SV], [DR], [MR], [SR], [BL].* □

We set $\kappa := \delta + \gamma$, $r' := r\kappa$, $c_* := c + r'$ and define functions g, ψ, ψ_* on $(0, \bar{x})$ by

$$\begin{aligned} g(x) &:= \delta - F'(x) + \frac{F(x)}{x}\,, \\ \psi(x) &:= (px - c)(\delta - F'(x)) - \frac{cF(x)}{x}\,, \\ \psi_*(x) &:= (px - c_*)(\delta - F'(x)) - \frac{c_* F(x)}{x}\,. \end{aligned}$$

Further we consider the following conditions:

$(V1)$ $F \in C^2[0,\infty) \cap C^3(0,\bar{x})$;
$F(0) = F(\bar{x}) = 0$; $F(x) > 0\,,\ 0 < x < \bar{x}$; $F''(x) < 0\,,\ 0 \le x \le \bar{x}$.

$(V2)$ $\delta > 0$, $r > 0$, $c > 0$, $\gamma > 0$.

$(V3)$ $c_* - p\bar{x} < 0$.

$(V4)$ There exist $\tilde{x}$, $x^* \in (0, \bar{x})$ with

$$\psi(x) < 0\,,\ 0 < x < \tilde{x}\,,\ \psi(\tilde{x}) = 0\,,\ \psi(x) > 0\,,\ \tilde{x} < x < \bar{x}\,,$$

$$\psi_*(x) < 0\,,\ 0 < x < x^*\,,\ \psi_*(x^*) = 0\,,\ \psi_*(x) > 0\,,\ x^* < x < \bar{x}\,.$$

$(V5)$ $\psi'(x) > 0$, $x \in (0, \tilde{x})$, $\psi_*'(x) > 0$, $x \in (\tilde{x}, \bar{x})$.

$(V6)$ $g'(x) > 0,\ x \in (0, \bar{x})$.

Remark 2 *Notice that the conditions* $(V1), \ldots, (V6)$ *are satisfied for the logistic production function if the constants are chosen appropriately. Notice too that* $(V4), (V5)$ *contain redundant information.* □

Remark 3 *In the following analysis of the problem it is very important that* $x = \bar{x}$ *is an attracting and* $x = 0$ *is an unstable equilibrium point.* □

1.3 The verification approach

As already mentioned the problem $P(x_0, K_0)$ is considered in [CC]. By using a Hamilton–Jacobi–Bellman equation on $(0, \bar{x}) \times (0, \infty)$ a candidate for an optimal control policy is defined for each $(x_0, K_0) \in (0, \bar{x}) \times (0, \infty)$. This results in the definition of a function

$$S : (0, \bar{x}) \times (0, \infty) \longrightarrow I\!R\,,$$

such that for all $(x, K) \in (0, \bar{x}) \times (0, \infty)$, for all $u \in [0, 1]$ and for all $I \geq 0$ – controls with jumps are avoided by considering them as "limits" of regular controls –

$$\begin{aligned} \delta S(x, K) + F(x) S_x(x, K) - \gamma K S_K(x, K) \\ \geq I(S_K(x, K) + r) + uK\{qx S_x(x, K) - pqx + c\} \end{aligned} \tag{3}$$

holds. Then it is stated that S is the value function V where the value function V is given here by

$$V(x_0, K_0) := \inf\{J(x_0, K_0; I, u) | (I, u) \text{ admissible}\}\,,\ (x_0, K_0) \in (0, \bar{x}) \times (0, \infty).$$

Implicitly there are only used controls which result in states x, K such that

$$\lim_{t \to \infty} e^{-\delta t} S(x(t), K(t)) = 0.$$

One can follow the analysis in [CC] partly but for some steps the assumptions are not sufficient and some arguments are not complete. Since S is not differentiable everywhere they use the argument that each problem $P(x_0, K_0)$ may be approximated by the problem $Q(x_0, K_0)$. This density argument is a very deep topological argument and no results to make this argument rigorous are available from the literature. It is open whether on this road the verification of optimal controls is possible. Thus, the verification of the optimal policy in [CC] has to be considered as an open problem.

Two different ways may be considered in order to circumvent these difficulties:

1. Extension of the so–called Hamilton–Jacobi–Bellman equation such that jumps are allowed.

2. Proof of the closure property inherent in the density argument.

The verification of the optimal policy in [CC] has to be considered as an open problem.

1.4 Main results

In Section 2 we study the necessary conditions, furnished by a special version of Pontryagin's maximum principle (see Appendix). Through manipulation of the Lagrange multipliers we manage to define two special switch functions. The first switch helps to determine the bang–bang behavior of the measurable component of the control policy, while the second one is a jump switch, which gives us a necessary condition for discontinuities in the state variables.

In Section 3 we use the necessary conditions of optimality, rewritten for the switch functions, in order to detect both optimal and non optimal behaviour of the admissible processes. Following optimal trajectories backwards in time and observing the evolution of the switches, we are able to construct some auxiliary curves in the phase plane (x, K), that are very useful to determine optimal behaviour. Particularly we are able to detect jump curves in the phase plane. This shows that the application of the Pontryagin maximum principle can be used in order to construct the extremals of the problem.

In Section 4 we put all arguments together and resume the obtained results in the form of Theorem 27. The next two theorems, 28 and 29, treat some special initial conditions, which may occur. However, the argumentation follow the spirit of Theorem 27.

In [Si] it is shown that an optimal solution for problem $P(x_0, K_0)$ exists. It is worth to mention that our results are in agreement with the suggestions in [CC]. In a forthcoming paper we will take up the problem of sufficient conditions; see [BL].

2 Necessary conditions

In this section we use the maximum principle to derive first order necessary conditions for problem $P(x_0, K_0)$ and define, with the aid of the Lagrange multipliers, two auxiliary functions (switches) that play a key rule in the analysis of the optimal trajectories. We start defining the Hamiltonfunction $\widetilde{H}$ by

$$\widetilde{H}(t,\tilde{x},\widetilde{K},w,\widetilde{\lambda}_1,\widetilde{\lambda}_2,\eta) := \widetilde{\lambda}_1(F(\tilde{x}) - w\widetilde{K}\tilde{x}) - \widetilde{\lambda}_2\gamma\widetilde{K} - \eta e^{-\delta t}(c - p\tilde{x})w\widetilde{K}\,.$$

Let (u, μ) be an optimal control policy for $P(x_0, K_0)$ and let (x, K) be the associated state. From the maximum principle in the appendix we obtain constants $\widetilde{\lambda}_{1,0}$, $\widetilde{\lambda}_{2,0}$, $\eta \in I\!R$ and adjoint functions $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2 : [0,\infty) \to I\!R$ such that

$$\begin{aligned}
\widetilde{\lambda}_{1,0}^2 + \widetilde{\lambda}_{2,0}^2 + \eta^2 &\neq 0\,,\ \eta \geq 0\,,\\
x' &= F(x) - u(t)Kx\,,\ x(0) = x_0\,,\\
dK &= -\gamma K dt + \mu(dt)\,,\ K(0) = K_0\,,\\
\widetilde{\lambda}_1' &= -\widetilde{\lambda}_1(F'(x) - u(t)K) - \eta e^{-\delta t} p u(t) K\,,\ \widetilde{\lambda}_1(0) = \widetilde{\lambda}_{1,0}\,,\\
\widetilde{\lambda}_2' &= \widetilde{\lambda}_1 x u + \gamma\widetilde{\lambda}_2 + \eta e^{-\delta t}(c - px)u(t)\,,\ \widetilde{\lambda}_2(0) = \widetilde{\lambda}_{2,0}\,,\\
\widetilde{\lambda}_2(t) - \eta e^{-\delta t} r &\leq 0 \text{ for all } t \in [0,\infty)\,,\\
\widetilde{\lambda}_2(t) - \eta e^{-\delta t} r &= 0\ \mu\text{–a.e. in } [0,\infty)\,,\\
\widetilde{H}(t,x(t),K(t),u(t),\widetilde{\lambda}_1(t),\widetilde{\lambda}_2(t),\eta) &= \max_{w\in[0,1]} \widetilde{H}(t,x(t),K(t),w,\widetilde{\lambda}_1(t),\widetilde{\lambda}_2(t),\eta) \text{ a.e. in } [0,\infty)\,.
\end{aligned}$$

Now we set

$$\lambda_1(t) := \widetilde{\lambda}_1(t)e^{\delta t},\ \lambda_2(t) := \widetilde{\lambda}_2(t)e^{\delta t},\ \lambda_{1,0} := \lambda_1(0)\,,\ \lambda_{2,0} := \lambda_2(0)\,,$$

$$H(t,\tilde{x},\widetilde{K},w,\lambda,\eta) := (-\lambda\tilde{x} + \eta(p\tilde{x} - c))\widetilde{K}w\,.$$

Next we define the auxiliary functions $z,\, \lambda : [0,\infty) \to I\!R$

$$z := -\lambda_1 x + \eta(px - c)\,,\ \lambda := \lambda_2\,,\ z_0 := \lambda_{1,0}\,,\ \lambda_0 := \lambda_{2,0}\,.$$

that are used to reinterpret the necessary conditions. We call z and λ switch variables. Note that z defines along the maximum condition the value of $u(t)$, namely $u(t) = 0$ if $z(t) < 0$ and $u(t) = 1$ if $z(t) > 0$. If $z(t)$ vanishes, then the value of $u(t)$ has to be determined by other means. The function λ defines a jump switch, since $\mu(\{t\}) > 0$ for some $t \in [0,\infty)$ implies $\lambda(t) = \eta r$.

Finally we are able to rewrite the necessary optimality conditions in the form

$$\begin{aligned}
z_0^2 + \lambda_0^2 + \eta^2 &\neq 0\,,\ \eta \geq 0\,,\\
x' &= F(x) - u(t)Kx\,,\ x(0) = x_0\,,\\
dK &= -\gamma K dt + \mu(dt)\,,\ K(0) = K_0\,,\\
z' &= zg(x) - \eta\psi(x)\,,\ z(0) = z_0\,,\\
\lambda' &= \kappa\lambda - zu(t)\,,\ \lambda(0) = \lambda_0\\
\lambda(t) - \eta r &\leq 0 \text{ for all } t \in [0,\infty)\,,\\
\lambda(t) - \eta r &= 0\ \mu\text{–a.e. in } [0,\infty)\,,\\
z(t)K(t)u(t) &= \max_{w\in[0,1]} z(t)K(t)w \ \text{ a.e. in } [0,\infty)\,.
\end{aligned}$$

We want to exclude the irregular case $\eta = 0$. This is prepared by

Lemma 4 *A policy (u,μ) such that there exists $\tau > 0$ with $\mu(A) = 0$ for each measurable subset A of (τ,∞) is not optimal for each $(x_0,K_0) \in (0,\bar{x}) \times (0,\infty)$.*

Proof: Let (u,μ) be a policy with the property that there exists $\tau > 0$ with $\mu(A) = 0$ for all $A \subset [\tau,\infty)$. Let x, K be the associate state variables. Then $\lim_{t\to\infty} x(t) = \bar{x}$, $\lim_{t\to\infty} K(t) = 0$; see Remark 3. It is sufficient due to the Bellmann optimality principle (see appendix) that a policy of this kind is not optimal for initial values (x_0,K_0) in the neighbourhood of $(\bar{x},0)$. Consider first the case when we have the initial value $(\bar{x},0)$. Let

$$I(h) := \int_0^\infty e^{-\delta t}(c - px^h(t))he^{-\gamma t}u(t)dt + rh\,,\ h > 0\,,$$

where x^h solves

$$x' = F(x) - u(t)he^{-\gamma t}x\,,\ x(0) = \bar{x}\,.$$

This is the objective function for the pair of controls $(u,h\delta_0)$, where δ_0 is the Dirac measure in $t = 0$. It is easy to see that

$$I'(0) := \lim_{h\to 0} h^{-1}(I(h) - I(0)) = r + (c - p\bar{x})\frac{1}{\gamma+\delta} = \frac{1}{\gamma+\delta}(c^* - p\bar{x}) < 0\,.$$

From the assumption (V3) it follows that the assertion of the lemma is true for the initial value $(\bar{x},0)$.

Since $c^* - px \leq -\alpha$ for some $\alpha > 0$ in the neighborhood of $\bar{x}$, the assertion may be proved in the same way for initial values (x_0,K_0) in the neighborhood of $(\bar{x},0)$. ■

Corollary 5 *Let (x, K, u, μ) be an optimal process with adjoint variables η, z, λ. If $\eta = 0$, then there exists for any $\tau > 0$ some $t > \tau$ with $\lambda(t) = 0$.*

Proof: If there exists a $\tau > 0$ such that $\lambda(t) < 0$, for all $t > \tau$, then $\mu(A) = 0$ for each measurable subset A of (τ, ∞). This contradicts the optimality of the policy (u, μ) (see Lemma 4), proving the corollary. ■

Theorem 6 *Let (x, K, u, μ) be an optimal process with adjoint variables η, z, λ. Then $\eta \neq 0$.*

Proof: If $\eta = 0$, we have the necessary conditions $z_0^2 + \lambda_0^2 \neq 0$,

$$\begin{aligned} z' &= zg(x)\,,\ z(0) = z_0\,, \\ \lambda' &= \kappa\lambda - zu(t)\,,\ \lambda(0) = \lambda_0 \\ \lambda(t) &\leq 0 \text{ for all } t \in [0, \infty)\,, \\ \lambda(t) &= 0\ \mu\text{–a.e. in } [0, \infty)\,. \end{aligned}$$

We distinguish six cases according to the initial conditions (λ_0, z_0):
i) $\lambda_0 = 0$, $z_0 = 0$: Here we have $z(t) = 0$, $\lambda(t) = 0$, for all $t > 0$. This is a contradiction, since not all μ, λ, z may vanish.
ii) $\lambda_0 = 0$, $z_0 < 0$: Clearly, $\lambda'(0) = 0$ and $z'(0) < 0$. This implies $z(t) < 0$, $u(t) = 0$, $\lambda(t) = 0$ for all $t > 0$. Therefore the policy (u, μ) is not better than $(u, \hat{\mu})$ with $\hat{\mu} \equiv 0$. From Corollary 5 we know that already this policy is not optimal.
iii) $\lambda_0 = 0$, $z_0 > 0$: Note that $z'(0) > 0$, $\lambda'(0) < 0$ and we have $z(t) > 0$, $u(t) = 1$, $\lambda(t) < 0$ for all $t > 0$. From Corollary 5 we obtain that this policy is not optimal.
iv) $\lambda_0 < 0$, $z_0 = 0$: We have $z'(0) = 0$, $\lambda'(0) < 0$. This implies $z(t) = 0$, $\lambda(t) < 0$ for all $t > 0$. From Corollary 5 follows that this policy is not optimal.
v) $\lambda_0 < 0$, $z_0 < 0$: We have $z'(0) < 0$, $\lambda'(0) < 0$. This implies $z(t) < 0$, $\lambda(t) < 0$ for all $t > 0$ and again due to Corollary 5 this policy is not optimal.
vi) $\lambda_0 < 0$, $z_0 > 0$: In this case $z'(0) > 0$, $\lambda'(0) < 0$ and we have $z(t) > 0$, $\lambda(t) < 0$ for all $t > 0$. Applying Corollary 5, we conclude that this policy is not optimal.
Therefore in all cases we have a contradiction and the theorem is proved. ■

3 Exploitation of the necessary conditions

An immediate consequence of Theorem 6 is the fact that the Lagrange multiplier η can be chosen equal to one. In this case we have to analyze the following necessary conditions:

$$\begin{aligned} x' &= F(x) - u(t)Kx\ ,\ x(0) = x_0\,, \\ dK &= -\gamma K dt + \mu(dt)\,,\ K(0) = K_0\,, \\ z' &= zg(x) - \psi(x)\,,\ z(0) = z_0\,, \\ \lambda' &= (\lambda - r)\kappa - zu(t) + r'\,,\ \lambda(0) = \lambda_0\,, \\ \lambda(t) &\leq r\,,\ \text{for all } t \in [0, \infty)\,, \\ \lambda(t) &= r\,,\ \mu\text{–a.e. in } [0, \infty)\,, \\ z(t)K(t)u(t) &= \max_{w \in [0,1]} z(t)K(t)w \ \text{ a.e. in } [0, \infty)\,. \end{aligned}$$

The central problem in the analysis of the necessary conditions consists in finding out the initial conditions $\lambda_0 = \lambda(0)$, $z_0 = z(0)$ which are in agreement with the condition

$$(R) \qquad \lambda(t) \le r, \text{ for all } t \in [0,\infty).$$

One can easily check that if $\lambda(\tau) = r$ for some $\tau > 0$, then $\lambda'(\tau) = 0$ and $\lambda''(\tau) \le 0$. It is also clear that $K_{0,+} \ne K_0$ at most if we have $\lambda(0) = r$. Knowing that it is possible to chose $\eta = 1$, we formulate Corollary 5 again:

Corollary 7 *Let (x, K, u, μ) be an optimal process with adjoint variables z, λ. Then for each $\tau > 0$ there exists $t > \tau$ with $\lambda(t) = r$.*

Proof: See the proof for Corollary 5. ■

The rest of this section is devoted to the analysis of the relationship between the initial states (x_0, K_0) and the initial conditions (z_0, λ_0) of the adjoint variables. We prove a series of auxiliary lemmas, that will allow us in the next section to detect the optimal trajectories and correspondent policies.

Lemma 8 *Let (x, K, u, μ) be optimal with adjoint variables z, λ. Then there exists no $\tau > 0$ with $x(\tau) > x^*$ and $F(x(\tau)) \ge K(\tau)x(\tau)$.*

Proof: Assume the contrary. We consider two cases separately:
i) If $\lambda(\tau) = r$, then it follows from (R) that $\lambda'(\tau) = 0$. Consequently, $z(\tau)u(\tau) = r'$ and $z(\tau) = r'$. Now, it follows from $(V4)$ and the assumption $x(\tau) > x^*$ that $\psi_*(x(\tau)) > 0$ and substituting in the dynamic of z we have

$$z'(\tau) = (z(\tau) - r')g(x(\tau)) - \psi_*(x(\tau)) < 0\,.$$

Because of $\lambda''(\tau) = -z'(\tau) > 0$ we have a contradiction to (R).
ii) If $\lambda(\tau) < r$, then we obtain from Corollary 7 a $\tau_1 > \tau$ with $\lambda(\tau_1) = r$, $\lambda(t) < r$, for $t \in [\tau, \tau_1)$. Observing the dynamic of the pair $(x(t), K(t))$, we conclude that $0 \le F(x(t)) - K(t)x(t) \le x'(t)$, $t \ge \tau$. Then for $t = \tau_1$ we have $F(x(\tau_1)) \ge K(\tau_1)x(\tau_1)$, $x(\tau_1) \ge x(\tau) > x^*$. To obtain a contradiction, we argument as in *i)*, setting $\tau := \tau_1$. ■

Lemma 9 *Let (x, K, u, μ) be optimal with adjoint variables z, λ. Then for $x_0 = x^*$, $K_0 > K^*$ the initial conditions $\lambda_0 = r$, $z_0 \ge r'$ are not possible.*

Proof: Assume that $\lambda_0 = r$, $z_0 = r'$. From the differential equations for x, λ and z follows $x'(0) < 0$, $\lambda'(0) = 0$, $z'(0) = 0$. Then we have

$$\lambda''(0) = \lambda'(0)\kappa - z'(0) = 0\,, \quad \lambda'''(0+) = \psi_*'(x^*)[F(x^*) - K_{0,+}x^*] < 0\,.$$

Therefore $\lambda(t) \not\equiv r$ does not occur. This holds also if $z_0 > r'$ since in this case we have $\lambda'(0) < 0$. Due to Corollary 7, $\lambda(t) < r$ for all $t > 0$ is not possible. Let

$$\tau := \sup\{\sigma \ge 0 \,|\, \lambda(t) < r,\ 0 < t < \sigma\}\,.$$

We know that $\tau < \infty$, $\lambda(\tau) = r$, $\lambda'(\tau) = 0$, $z(\tau) = r'$. Since $x_0 = x^*$ and $x'(0) < 0$, we have three possible cases:
i) If $x(t) < x^*$ for all $t \in (0, \tau]$, then $z'(t) > 0$, for $t \in [0, \tau)$ and $z(0) = z(\tau) = r'$ which is not possible.
ii) If $x(t) < x^*$, for all $t \in (0, \tau)$ and $x(\tau) = x^*$, then we may argument as in *i)*.
iii) If $x(\tau) > x^*$, then we can choose $\sigma \in (0, \tau)$ with $x(t) < x^*$, for $t \in [0, \sigma)$, and $x(\sigma) = x^*$. Note that $z(t) > r'$ (and consequently $u(t) = 1$) for $t \in (0, \sigma]$. From the definition of σ we have

$x'(\sigma) \geq 0$, i.e. $K(\sigma) \leq F(x^*)/x^*$. Now, argumenting as in case *ii)* in the proof of Lemma 8, we obtain the inequality $K(\sigma) > F(x^*)/x^*$, again a contradiction.[1] ■

Lemma 10 *Let (x, K, u, μ) be optimal with adjoint variables z, λ. If $x_0 = x^*$, $K_0 = K^*$, then $\mu = \gamma K^* dt$ and $x(t) = x^*$, $K(t) = K^*$, $\lambda(t) = r$, $z(t) = r'$, $u(t) = 1$, for $t \geq 0$.*

Proof: Assume $\lambda_0 < r$. From Corollary 7 we obtain $\tau > 0$ with $\lambda(\tau) = r$, $\lambda(t) < r$, for $t \in (0, \tau)$. Condition (R) implies $\lambda'(\tau) = 0$ and therefore $z(\tau) = r'$; especially we have $u(t) \equiv 1$ in a neighborhood of $t = \tau$. From $x' = F(x) - Kx$ we obtain $x(\tau) > x^*$ and $\lambda''(\tau) = -z'(\tau) = \psi_*(x(\tau)) > 0$, contradicting (R). Therefore we must have $\lambda_0 = r$. Next verify that $z_0 = r'$:
i) If $z_0 < r'$, then $\lambda'(0) = r' - z_0 > 0$, contradicting (R).
ii) If $z_0 > r'$, then $\lambda'(0) < 0$. Argumenting with Corollary 7 (see the beginning of this proof), we obtain a $\tau > 0$ with $\lambda(\tau) = r$, $\lambda'(\tau) = 0$ but $\lambda''(\tau) > 0$, again contradicting (R).
Therefore we must have $z_0 = r'$. Now define

$$\tau := \sup\{\sigma \geq 0 \,|\, \lambda(t) = r,\ 0 \leq t \leq \sigma\}.$$

If $\tau = 0$, then $\lambda'(0) < 0$ and argumenting as in *ii)* above we obtain a contradiction. Therefore we must have $\tau > 0$. Note that if $\sigma \in (0, \tau)$, then $\lambda(t) = r$, $\lambda'(t) = 0$, $z(t) = r'$, $z'(t) = 0$, for $t \in [0, \sigma]$. It follows $\psi_*(x(t)) = 0$, for $t \in [0, \sigma]$, and with condition (V4) we have $x(t) = x^*$, for $t \in [0, \sigma]$. From the differential equation $x' = F(x) - Kx$ follows $K(t)x^* = F(x^*)$, for $t \in [0, \sigma]$, and with $dK = -\gamma K dt + \mu(dt)$ we finally obtain $\mu_{|_{[0,\sigma]}} = -\gamma K^* dt$.
Clearly, it is enough to prove $\tau = \infty$. If this were not the case, we would have either $K(\tau+) > K^*$ or $K(\tau+) = K^*$. Due to Lemma 9, $K(\tau+) > K^*$ is not possible. The other case, $K(\tau+) = K^*$, cannot be true, since otherwise we could repeat the complete argumentation for the initial time $t = \tau$, contradicting the maximality of τ. ■

Lemma 11 *Let (x, K, u, μ) be optimal with adjoint variables z, λ. Then for $x_0 = x^*$, $K_0 \leq K^*$ we must have the initial values $\lambda_0 = r$, $z_0 = r'$, $\lambda'(0) = 0$, $z'(0) = 0$. Further we have $K_{0,+} = K^*$.*

Proof: The equality $\lambda'(0) = 0$, $z'(0) = 0$ follow from $x_0 = x^*$, $\lambda_0 = r$, $z_0 = r'$ in an obvious way. Consequently, we only have to prove $\lambda_0 = r$, $z_0 = r'$.
If $\lambda_0 < r$, then there exists $\tau > 0$ such that $\lambda(t) < r$, for $t \in [0, \tau)$. Then $K(t) < K_0 < K^*$ for $t \in [0, \tau]$, and consequently $x(\tau) > x^*$, $x'(\tau) = F(x(\tau)) - K(\tau)x(\tau) > 0$. But this cannot occur due to Lemma 8. Therefore $\lambda_0 = r$.
If $z_0 < r'$, then $\lambda'(0) < 0$. Argumenting as before (assumption $\lambda_0 < r$) we obtain a contradiction. Therefore $z_0 \geq r'$. Next we exclude the case $z_0 > r'$.
If $K_0 = K^*$, then Lemma 10 implies $z_0 = r'$ proving the theorem. If $K_0 < K^*$, we consider the following cases:
i) $K_{0,+} > K^*$ is not possible due to Lemma 9.
ii) If $K_{0,+} = K^*$, then $z_0 = r'$ follows from Lemma 10.
iii) $K_{0,+} \in [K_0, K^*)$. We obtain from the dynamic of the pair (x, K) some $\tau > 0$ with $x(\tau) > x^*$, $F(x(\tau)) - K(\tau)x(\tau) > 0$. But this is not possible due to Lemma 8.
Therefore only $K_{0,+} = K^*$ is possible and the theorem is proved. ■

Lemma 12 *Let (x, K, u, μ) be optimal with adjoint variables z, λ. If $x_0 > x^*$ and $K_0 < F(x_0)/x_0$, then we must have: $\lambda_0 = r$, $z_0 > r'$, $K_{0,+} > K_0$.*

Proof: Since $K_{0,+} = K_0$ is not allowed (see Lemma 8), we have $K_{0,+} > K_0$ and $\lambda_0 = r$. Then

[1] One should note that the development in case *ii)* in the proof of Lemma 8 still holds if $x(\tau) = x^*$.

$\lambda'(0) \le 0$, and we obtain $z_0 \ge r'$. If $z_0 = r'$, we would have $z'(0) = -\psi_*(x_0) < 0$, $\lambda'(0) = 0$ and $\lambda''(0) = -z'(0) > 0$. This however contradicts the condition (R). ■

Let (x, K) be a solution of the system

$$\begin{cases} x' = -F(x) + Kx, \ x(0) = x^* \\ K' = \gamma K, \ K(0) = K^* \end{cases} \tag{4}$$

with interval of existence $[0, \tau)$. Since $K(t) = K^* e^{\gamma t}$, $x'(0) = 0$ and $x''(0) = K^* e^{\gamma t} \gamma x^* > 0$, we have $(x(t), K(t)) \in (x^*, \bar{x}) \times (K^*, \infty)$ for $t > 0$ small. Therefore it is easy to see that there exists some $\bar{t} \in (0, \tau)$ with $x(\bar{t}) = \bar{x}$ and $x^* \le x(t) \le \bar{x}$, $t \in [0, \bar{t}]$. The curve defined by

$$[0, \bar{t}] \ni t \longmapsto (x(t), K(t)) \in [x^*, \bar{x}] \times [K^*, \infty)$$

is denoted by Γ_1. Note that at $t = 0$ we have $(x'(0), K'(0)) = (0, K^*\gamma)$, where $K^*\gamma > 0$. In Figure 1 we illustrate the construction of the curve Γ_1 for the case of the logistic function.

Lemma 13 *There exists a function $h_1 : [x^*, \bar{x}] \to [K^*, \infty)$ such that*

(a) $\Gamma_1 = \{(x, h_1(x)) \mid x \in [x^*, \bar{x}]\}$;

(b) h_1 is twice continuous differentiable in $(x^*, \bar{x})$ and monotone increasing.

Proof: Note that the system (4) can be transformed into the scalar equation:

$$\frac{dK}{dx} = \frac{\gamma K}{-F(x) + Kx}, \ K(x^*) = K^*. \tag{5}$$

It becomes clear that Γ_1 has a parametrization $[x^*, \bar{x}] \ni x \mapsto (x, h_1(x)) \in [x^*, \bar{x}] \times [K^*, \infty)$ and the assertions follow. ■

Lemma 14 *Let (x, K, u, μ) be optimal with adjoint variables z, λ. For $x_0 > x^*$ and $K_0 < h_1(x_0)$, the initial conditions have to satisfy $\lambda_0 = r$, $z_0 > r'$, $K_{0,+} \ge h_1(x_0)$.*

Proof: Assume $\lambda_0 < r$. We define $\tau := \sup\{\sigma \ge 0 \mid \lambda(t) < r, \ t \in [0, \sigma)\}$. Due to Corollary 7, τ is positive and finite. Then we have $\lambda(\tau) = r$, $\lambda'(\tau) = 0$, $z(\tau) = r'$. Considering the definition of Γ_1, we obtain $x(t) > x^*$, $t \in (0, \tau]$ due to the values of λ. Now, from assumption (V4), the differential equation for z and $x(\tau) > x^*$, follow $z'(\tau) < 0$. Again from the differential equation for z we obtain $z(t) > r'$, $t \in (0, \tau)$. This implies $\lambda'(t) < 0$, $t \in (0, \tau)$, which is a contradiction to the definition of τ. Therefore $\lambda_0 = r$ must occur. This implies $0 \ge \lambda'(0) = r' - z_0$ and $z_0 \ge r'$ follows.
If $z_0 = r'$, we would have $z'(0) = -\psi_*(x_0) < 0$, $\lambda'(0) = 0$ and $\lambda''(0) = -z'(0) > 0$. This however contradicts the condition (R). Therefore we must have $z_0 > r'$.
Assume $K_{0,+} = K_0$. We know already that $z_0 > r'$. Then $\lambda'(0) < 0$ and by the same arguments as above (see $\lambda_0 < r$) we obtain a contradiction. Therefore $K_{0,+} > K_0$ must hold. If $K_{0,+} \in (x_0, h_1(x_0))$ we repeat the argumentation above with $K_0 := K_{0,+} < h_1(x_0)$, obtaining again a contradiction. Thus, we must have $K_{0,+} \ge h_1(x_0)$. ■

We denote the curve

$$[0, K^*] \ni K \longmapsto (x^*, K) \in [0, \infty) \times [0, K^*]$$

by Σ^*. Next we verify that when an optimal trajectory meets the curve Σ^*, some properties have to be satisfied.

Lemma 15 *Let (x, K, u, μ) be optimal with adjoint variables z, λ. Let $x(\sigma) = x^*$, $K(\sigma) \in (0, K^*)$ for some $\sigma > 0$. Then*

$$\lambda(\sigma) = r\,,\ \ \lambda'(\sigma) = 0\ ,\ \ z(\sigma) = r'\,,\ \ z'(\sigma) = 0\,.$$

Proof: Assume $\lambda(\sigma) < r$. From the differential equation for x and K we obtain $\tau > \sigma$ with

$$x(\tau) > x^*,\ \ F(x(\tau)) - K(\tau)x(\tau) > 0,$$

which is in contradiction to Lemma 8. Thus we must have $\lambda(\sigma) = r$. Therefore $\lambda'(\sigma) = 0$, $z(\sigma) = r'$. Finally, $z'(\sigma) = 0$ follows from $\psi_*(x^*) = 0$. ■

In the neighbourhood of Γ_1 and Σ^* we have obtained a lot of information concerning the behaviour of an extremal trajectory. Lemma 14, ensures that if $x_0 > x^*$ and $K_0 < h_1(x_0)$, there must be a jump at $t = 0$. Furthermore Lemma 8 says that an optimal trajectory $(x(t), K(t))$ does not enter the dashed region in Figure 1.

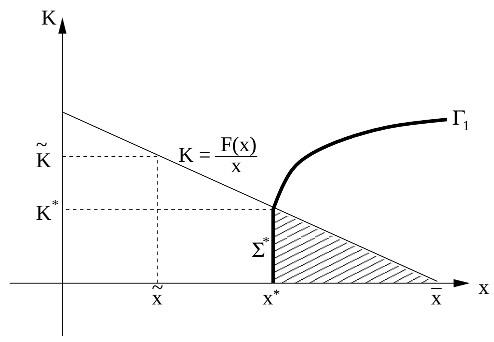

Figure 1: Curves Γ_1 and Σ^*.

Our next step is to analyze the behaviour of the optimal trajectories that meet the curve Σ^*, i.e. $(x(\tau), K(\tau)) \in \Sigma^*$ for some $\tau > 0$. Since the curve Σ^* is reached by an optimal trajectory with $z = r'$, $\lambda = r$, $x = x^*$ and $K \in [0, K^*]$, we use this information to solve the differential equations for x, K, z and λ backwards in time:

$$\begin{cases} x' &= -F(x) + Kx \qquad\qquad\ ,\ x(0) = x^*, \\ K' &= \gamma K \qquad\qquad\qquad\qquad ,\ K(0) = K_1, \\ z' &= -(z - r')g(x) + \psi_*(x),\ z(0) = r', \\ \lambda' &= -(\lambda - r)\kappa + z - r' \quad\ ,\ \lambda(0) = r, \end{cases}$$

where $K_1 \in [0, K^*]$. Such a trajectory eventually comes close to $(x, K) = (0, 0)$ where we expect an optimal control $u \equiv 0$. Therefore, the zeros of the switching variable z are of interest in this region. Since for $z(t) < r'$ a value $\lambda(t) = r$ is not allowed, the behavior of the adjoint variable λ is therefore not so important in this region.

Lemma 16 *For each $K_1 \in [0, K^*]$, let (x, z) be the solution of*

$$\begin{cases} x' = -F(x) + K_1 e^{\gamma t} x \qquad\ ,\ x(0) = x^*, \\ z' = -(z - r')g(x) + \psi_*(x),\ z(0) = r'. \end{cases} \tag{6}$$

Then there exists for each $K_1 \in (0, K^*)$ $\tau := \tau_{K_1} > 0$ with $x(\tau) = x^*$. Moreover there exists $\widetilde{K}_1 \in (0, K^*)$ such that the following assertions hold:

(a) If $K_1 \in (0, \widetilde{K}_1)$, then z has a uniquely determined zero $\sigma \in (0, \tau)$, where $z'(\sigma) < 0$ and $x(\sigma) \in (0, \tilde{x})$;

(b) If $K_1 \in (\widetilde{K}_1, K^*)$, then $z(t) > 0$ for all $t \in [0, \tau]$;

(c) If $K_1 = \widetilde{K}_1$, then there is a uniquely determined σ in $(0, \tau)$ with $z(\sigma) = z'(\sigma) = 0$; moreover $x(\sigma) = \tilde{x}$ and $\widetilde{\widetilde{K}} := K_1 e^{\gamma\sigma} > F(\tilde{x})/\tilde{x} = \widetilde{K}$.

Proof: Consider the solution (x, z) of (6) with $K_1 = 0$. Since $x_0 = 0$ is an attracting equilibrium point of $x' = -F(x)$ (see Remark 3), the solution x exists for all times $t \geq 0$ and $\lim_{t\to\infty} x(t) = 0$. Due to this fact z is also defined for all $t \geq 0$. Assume $z(t) > 0$ for all $t > 0$. As we know, the differential equation may be formulated as $z' = -zg(x) + \psi(x)$. Since ψ is negative and continuous in $[0, \tilde{x})$ (see (V4)), there exists some $a > 0$ such that

$$\psi(\xi) \leq -a, \text{ for } \xi \in [0, \tfrac{\tilde{x}}{2}].$$

Since $\lim\limits_{t\to\infty} x(t) = 0$, there is some $t_0 > 0$ with $x(t) \in [0, \frac{\tilde{x}}{2}]$, $t \geq t_0$. This implies for $t \geq t_0$

$$z(t) - z(t_0) \;=\; \int_{t_0}^{t} \big[-z(s)g(x(s)) + \psi(x(s))\big]ds \;\leq\; \int_{t_0}^{t} \psi(x(s))ds \;\leq\; -a(t - t_0).$$

But this contradicts the hypothesis on z. Thus, there must be a $\sigma > 0$ with $z(\sigma) = 0$ and $z(t) > 0$, $t \in [0, \sigma)$. Then $z'(\sigma) \leq 0$, $\psi(x(\sigma)) \leq 0$, and therefore $x(\sigma) \leq \tilde{x}$. Due to

$$0 \leq z''(\sigma) = -z'(\sigma)g(x(\sigma)) + \psi'(x(\sigma))x'(\sigma) = -z'(\sigma)g(x(\sigma)) - \psi'(x(\sigma))F(x(\sigma))$$

and assumption (V5), $z'(\sigma) = 0$ cannot occur. Therefore $z'(\sigma) < 0$, $x(\sigma) < \tilde{x}$ and $z(t) < 0$ for $t > \sigma$ due to the differential equation for z. By continuity arguments, we may choose a maximal $\widetilde{K}_1 > 0$ such that for the solution (x, z) of (6) with $K_1 \in [0, \widetilde{K}_1)$ there exists $\tau > 0$ and $\sigma \in (0, \tau]$ with $x(\tau) = x^*$,

$$z(t) > 0,\; t \in [0, \sigma), \quad z(\sigma) = 0, \quad z(t) < 0,\; t \in (\sigma, \tau];$$

proving *(a)* and *(b)*.
Now we prove *(a)*. From the construction of $\widetilde{K}_1$ and due to the differential equation for x we obtain $z(\sigma) = 0$ and $x(\sigma) > 0$. Since $\widetilde{K}_1$ is maximal, we have $z(\sigma) = z'(\sigma) = 0$. We cannot have $x(\sigma) > \tilde{x}$, since $z'(\sigma) = -\psi(x(\sigma)) < 0$. The case $x(\sigma) < \tilde{x}$ cannot be occur, since $z'(\sigma) = -\psi(x(\sigma)) > 0$. Therefore we must have $x(\sigma) = \tilde{x}$.
We cannot have $K(\sigma)x(\sigma) < F(x(\sigma))$ since, due to $z''(\sigma) = \psi'(x(\sigma))x'(\sigma) < 0$, z would be negative in a neighbourhood of σ. Otherwise, if we had $K(\sigma)x(\sigma) = F(x(\sigma))$, i.e. $K(\sigma)\tilde{x} = F(\tilde{x})$, then z would be positive in a neighbourhood of σ, due to $z''(\sigma) = 0$ and $z'''(\sigma) = \psi'(x(\sigma))x''(\sigma) = \psi'(x(\sigma))\gamma F(\tilde{x}) > 0$. Thus, we must have $K(\sigma) > F(\tilde{x})/\tilde{x}$ and *(c)* is proved. ■

Corollary 17 *Let $\widetilde{K}_1 \in (0, K^*)$ be chosen as in Lemma 16. Let $K_1 \in [0, \widetilde{K}_1]$ and (x, z) the corresponding solution of* (6)*. Then there exists a continuous mapping $l : [0, \widetilde{K}_1] \to (0, \infty)$ and $\breve{x} \in (0, \tilde{x})$, such that*

(a) $l(0) = \breve{x}$, $l(\widetilde{K}_1) = \tilde{x}$;

(b) $z(l(K_1)) = 0$, $K_1 \in [0, \widetilde{K}_1]$, $\quad z'(l(K_1)) < 0$, $K_1 \in [0, \widetilde{K}_1)$, $\quad z'(l(\widetilde{K}_1)) = 0$;

(c) *l is continuous differentiable in* $(0, \widetilde{K}_1)$ *and* $l'(K_1) > 0$, $K_1 \in (0, \widetilde{K}_1)$.

Proof: Let $K_1 = 0$ and let $\sigma > 0$ with $z(\sigma) = 0$ (see proof of Lemma 16). Now define $\breve{x} := x(\sigma)$. Given $K_1 \in (0, \widetilde{K}_1)$, we denote by $x := x(\cdot; K_1)$, $z := z(\cdot; K_1)$ the corresponding solution of (6) and define $l(K_1) := \sigma$, where $\sigma > 0$ is the uniquely determined zero of $z(\cdot; K_1)$ (see Lemma 16); then *(b)* holds. The mapping $K_1 \mapsto l(K_1)$ is continuous since z depends continuously on K_1 (notice that $z'(\sigma) < 0$). Now we may extend $l : (0, \widetilde{K}_1) \to (0, \infty)$ to a continuous map on $[0, \widetilde{K}_1]$, proving *(a)*. Notice that the mapping $\Psi : [0, \tau] \times [0, \widetilde{K}_1] \to \mathbb{R}$ defined by $\Psi(s, K_1) := z(s; K_1)$ is differentiable and

$$\frac{\partial \Psi}{\partial s}(s, K_1) = z'(s; K_1)\,, \quad \frac{\partial \Psi}{\partial K_1}(s, K_1) = v(s; K_1),$$

where $v := v(\cdot; K_1) := \frac{\partial z}{\partial K_1}(\cdot; K_1)$ is the solution of the initial value problem

$$v' = -vg(x(t;K_1)) + \underbrace{\big[-(z(t;K_1) - r')g'(x(t;K_1)) + \psi_*'(x(t;K_1))\big]}_{(*)} w,\ v(0) = 0,$$

and $w := \frac{\partial x}{\partial K_1}$ solves

$$w' = \big[-F'(x(t;K_1)) + K_1 e^{\gamma t} x(t;K_1)\big] w + e^{-\gamma t} x(t;K_1),\ w(0) = 0.$$

By the implicit function theorem applied on $\Psi(l(K_1), K_1) = 0$ for $K_1 \in (0, \widetilde{K}_1)$ we have

$$z'(l(K_1); K_1)\, l'(K_1) = -v(l(K_1); K_1), \quad z'(l(K_1); K_1) < 0\,,\ K_1 \in (0, \widetilde{K}_1), \tag{7}$$

It is obvious that $w(t) > 0$ for all $t > 0$ since $(*)$ is positive for $t \in (0, l(K_1))$. From (7) we obtain $l'(K_1) > 0$, $K_1 \in (0, \widetilde{K}_1)$, and *(c)* is proved. ■

Corollary 17 allow us to define a curve Σ_0, parameterized by

$$\begin{array}{rcl} \xi : [0, \widetilde{K}_1] & \longrightarrow & [0, \tilde{x}] \times [0, \widetilde{K}] \\ K_1 & \longmapsto & (x(l(K_1); K_1), K_1 e^{\gamma l(K_1)}) \end{array}$$

(see the proof of Corollary 17 for the notation). From Lemma 16 and Corollary 17 we conclude that for the solutions (x, K, z) of

$$\begin{cases} x' = -F(x) + Kx & , x(0) = x^*, \\ z' = -(z - r')g(x) + \psi_*(x), & z(0) = r', \\ K' = -\gamma K & , K(0) = K_1 \in [0, K^*] \end{cases}$$

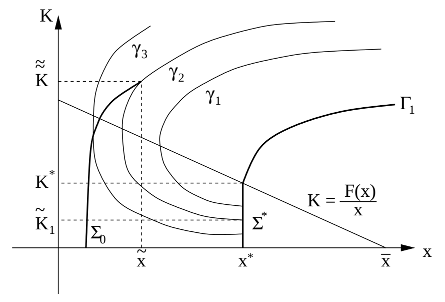

Figure 2: Curve Σ_0.

one of the following alternatives holds:
(i) if $K_1 \in (\widetilde{K}_1, K^*]$, then $z(t) > 0$, for all $t \geq 0$ (see curve γ_1 in Figure 2);
(ii) if $K_1 = \widetilde{K}_1$, then $z(t) > 0$ except at a single time point, where $(x, K) = (\tilde{x}, \tilde{\tilde{K}})$ holds; (see curve γ_2 in Figure 2);
(iii) if $K_1 \in [0, \widetilde{K}_1)$, then $z(t) > 0$, before the trajectory intercept Σ_0 and $z(t) < 0$ after that (see curve γ_3 in Figure 2).

Corollary 18 *Let $\widetilde{K}_1$, $\tilde{\tilde{K}}$, $\breve{x}$ and l be defined as in Lemma 16 and in the proof of Corollary 17. Then there exists $\hat{x} \in (\breve{x}, \tilde{x})$, $\widehat{K} > 0$ and a continuous mapping $h_0 : [\hat{x}, \tilde{x}] \to [\widehat{K}, \tilde{\tilde{K}}]$ with*

$$\Sigma_0 \cap \{(x_0, K_0) \,|\, F(x_0) \leq K_0 x_0\} = \{(x_0, h_0(x_0)) \,|\, x_0 \in [\hat{x}, \tilde{x}]\}.$$

Moreover:

(a) $h_0(\hat{x}) = \widehat{K}$, $h_0(\tilde{x}) = \tilde{\tilde{K}}$, $F(\hat{x}) = \widehat{K}\hat{x}$;

(b) h_0 *is continuous differentiable in* $(\hat{x}, \tilde{x})$ *and* $h_0'(x) > 0$, $x \in [\hat{x}, \tilde{x})$.

Proof: Consider the mapping

$$\xi : [0, \widetilde{K}_1] \ni K_1 \longmapsto (x(l(K_1); K_1), z(l(K_1); K_1)) \in [0, \tilde{x}] \times I\!R$$

where $x(\cdot;\cdot)$ and $z(\cdot;\cdot)$ are defined as in the proof of Corollary 17. Then

$$\xi'(K_1) = \Big(x'(l(K_1); K_1)l'(K_1) + \frac{\partial x(\cdot;\cdot)}{\partial K_1}(l(K_1); K_1), z'(l(K_1); K_1)l'(K_1) + \frac{\partial z(\cdot;\cdot)}{\partial K_1}(l(K_1), K_1)\Big)$$

and we see by the implicit function theorem that the region $\Sigma_0 \cap \{(x_0, K_0) \,|\, F(x_0) \leq K_0 x_0\}$ may be reparametrized by a function h_0. Notice that the implicit function theorem may be used due to Corollary 17. The condition $h_0'(x) > 0$ follows from the same corollary. ■

Let (x, K) be the solution of the initial value problem

$$\begin{cases} x' = -F(x) + Kx, & x(0) = \tilde{x}, \\ K' = \gamma K & , \; K(0) = \tilde{\tilde{K}} \end{cases} \tag{8}$$

in the interval $[0, \tau)$. Similar to the definition of Γ_1 we obtain $\bar{t} \in (0, \tau)$ with $x(\bar{t}) = \bar{x}$. We denote the curve

$$[0, \bar{t}] \ni t \longmapsto (x(t), K(t)) \in [\tilde{x}, \bar{x}] \times [\tilde{\tilde{K}}, \infty)$$

by Γ_3. (In Figure 2 one can recognize Γ_3 as the part of the curve γ_2 with $x > \tilde{x}$ and $K > \tilde{\tilde{K}}$.) Now let (x, K) be the solution of the initial value problem

$$\begin{cases} x' = F(x) - Kx, & x(0) = \tilde{x}, \\ K' = -\gamma K & , \; K(0) = \tilde{\tilde{K}}. \end{cases} \tag{9}$$

This solution meets the curve Σ^* in $(x^*, \widetilde{K}_1)$ for some $\tau > 0$ (see Lemma 16). The curve defined by this trajectory is called Γ_2. (In Figure 2, the curve Γ_2 corresponds to the part of γ_2 with $K < \tilde{\tilde{K}}$.) The last curve we define is Γ_4, which is parametrized by the solution (x, K) of

$$\begin{cases} x' = -F(x), & x(0) = \tilde{x}, \\ K' = \gamma K & , \; K(0) = \tilde{\tilde{K}}. \end{cases} \tag{10}$$

Note that the solution exists in $[0, \infty)$ and $x'(0) < 0$, $K'(0) > 0$. (See Figure 3)

Now we come back to the region $x > x^*$, more specifically, above the curve Γ_1. We already know that if the initial condition (x_0, K_0) lays below Γ_1, then $K_{0,+} \geq h_1(x_0)$ must hold. We want to determine a curve Σ_s, above Γ_1, upon which the trajectories must jump for this initial conditions, i.e. $(x_0, K_{0,+}) \in \Sigma_s$. In order to be able to construct such a curve Σ_s, we need the fact that there exists $(x_0, K_0) \in (x^*, \bar{x}) \times (K^*, \infty)$ such that the solution (x, K, z, λ) of

$$\begin{cases} x' = F(x) - Kx & , \; x(0) = x_0 \,, \\ K' = -\gamma K & , \; K(0) = K_{0,+} \,, \\ z' = (z - r')g(x) - \psi_*(x), & z(0) = z_0 \,, \\ \lambda' = (\lambda - r)\kappa - z + r' & , \; \lambda(0) = r \,, \end{cases}$$

meets the curve Σ^*.

Lemma 19 *There exists* $\widetilde{K}_2 \in (0, \widetilde{K}_1)$, $\widetilde{K}_3 \in (\widetilde{K}_1, K^*)$ *and* $a > 0$ *such that the solution* $(x(\cdot; K_1), z(\cdot; K_1), \lambda(\cdot; K_1))$ *of the system*

$$\begin{cases} x' = -F(x) + K_1 e^{\gamma t} x \quad , \; x(0) = x^*, \\ z' = -(z - r')g(x) + \psi_*(x), \; z(0) = r', \\ \lambda' = -(\lambda - r)\kappa + z - r' \quad , \; \lambda(0) = r\,, \end{cases} \tag{11}$$

exists in $[0, a]$ *for each* $K_1 \in (\widetilde{K}_2, K^*)$. *Moreover, for each* $K_1 \in (\widetilde{K}_3, K^*)$ *there exist numbers* $\rho(K_1)$, $\sigma(K_1)$, $\tau(K_1)$ *with*

- *(a)* $0 < \rho(K_1) < \sigma(K_1) < \tau(K_1) < a$;
- *(b)* $x(t; K_1) \le x^*$, $t \in (0, \rho(K_1))$, $x(\rho(K_1); K_1) = x^*$, $x(t; K_1) > x^*$, $t \in (\rho(K_1), a]$ *and* $x(a; K_1) \ge \bar{x}$;
- *(c)* $0 < z(t; K_1) < r'$, $t \in (0, \sigma(K_1))$, $z(\sigma(K_1); K_1) = r'$, $z(t; K_1) > r'$, $t \in (\sigma(K_1), a]$;
- *(d)* $\lambda(t; K_1) < r$, $t \in (0, \tau(K_1))$, $\lambda(\tau(K_1); K_1) = r$, $\lambda'(\tau(K_1); K_1) > 0$;
- *(e)* $\lim_{K_1 \uparrow K^*} \tau(K_1) = 0$.

Proof: Since the trajectory for $K_1 := K^*$ corresponds to Γ_1 we can choose $a > 0$ and $\widetilde{K}_2 \in (0, \widetilde{K}_1)$ with $x(a; K_1) \ge \bar{x}, K_1 \in (\widetilde{K}_2, K^*)$. It follows from the differential equations for z and λ that $z(t; K^*) > r'$, $\lambda(t; K^*) > r$, $t \in (0, a]$.
Let $\varepsilon \in (0, a)$ be given. Choose $\delta > 0$ and $\widetilde{K}_3 \in (\widetilde{K}_1, K^*)$ such that for each $K_1 \in (\widetilde{K}_3, K^*)$

$$z(t; K_1) \ge r' + \delta, \lambda(t; K_1) \ge r + \delta, \; t \in [\varepsilon, a].$$

This can be done due to the fact that the solution of (11) depends continuously on the parameter K_1.
Let $K_1 \in (\widetilde{K}_3, K^*)$ and set $(x, z, \lambda) := (x(\cdot; K_1), z(\cdot; K_1), \lambda(\cdot; K_1))$. Then we see that there exists $\rho(K_1) > 0$ with the property in *(b)*.
Since $z''(0) = \psi_*(x^*)x'(0) < 0$ and since $z(a) \ge r' + \delta$ we obtain $\sigma(K_1) > 0$ with $z(\sigma(K_1)) = r'$and $0 < z(t) < r'$, $t \in (0, \sigma(K_1))$. $\sigma(K_1) < \rho(K_1)$ cannot hold since $z'(\sigma(K_1)) = \psi_*(x(\sigma(K_1))) < 0$. $\sigma(K_1) = \rho(K_1)$ cannot hold since $z''(\sigma(K_1)) = z''(\rho(K_1)) = \psi_*(x^*)x'(\rho(K_1)) > 0$. From the differential equation for z and the fact that $\rho(K_1) < \sigma(K_1)$ we conclude $z(t) > r'$, $t > \sigma(K_1)$. Repeating the argumentation above we obtain $\tau(K_1)$ with $\lambda(\tau(K_1)) = r$, $\lambda(t) < r$, $0 < t < \tau(K_1)$. $\tau(K_1) < \sigma(K_1)$ cannot hold since $\lambda(\tau(K_1)) = z(\tau(K_1)) - r' < 0$. Assume $\tau(K_1) = \sigma(K_1)$. Then $\lambda'(\tau(K_1)) = z'(\sigma(K_1)) = \psi_*(x(\sigma(K_1))) > 0$. This contradicts the fact that $\lambda(t) < r$, $t \in (0, \tau(K_1))$. From the differential equation for λ and the fact that $z(t) > r'$, $t \in \tau(K_1)$, we obtain $\lambda(t) > r$, $t > \tau(K_1)$. Clearly, from the continouous dependency we obtain $\lim_{K_1 \uparrow K^*} \tau(K_1) = 0$. ■

The value $\tau(K_1)$ according to Lemma 19 is locally uniquely determined and the same is true for $K(\tau(K_1))$. We want to identify $K(\tau(K_1))$ as a value $K_{0,+}$, when an extremal trajectory starts in the initial value $(x_0, K_0) \in (x^*, \bar{x}) \times [0, \infty)$, with $K_0 < h_1(K_0)$. To do this we need more information concerning the mapping $K_1 \mapsto \tau(K_1)$.

Consider the system (11) for $K_1 \in (\widetilde{K}_3, K^*)$ and let (x, λ, z) be the corresponding solution. To make clear the dependence of λ on K_1, we denote it by $G(\cdot; K_1)$. The equation

$$G(\tau; K_1) = r \tag{12}$$

describes the fact that the solution $G(\cdot;K_1)$ has the value r at time τ. In order to find the jump curve Σ_s we try to resolve (12) with respect to τ. Note that from Lemma 19, we know that there are solutions of (12).

Lemma 20 *Using the notations of Lemma 19 the following assertions hold:*

(a) There exists $\widehat{K_1} \in [\widetilde{K_1}, K^*)$ *and a continuous differentiable mapping* $g_\tau : (\widehat{K_1}, K^*) \to (0,\infty)$ *with*

$$G(g_\tau(K_1);K_1) = r,\ \ K_1 \in (\widehat{K_1}, K^*); \tag{13}$$

(b) $g_\tau'(K_1) < 0$ *for all* $K_1 \in (\widehat{K_1}, K^*)$*;*

(c) Either $x(g_\tau(\widehat{K_1});\widehat{K_1}) < \bar{x}$ *(in which case we can assume* $\big(x(g_\tau(\widehat{K_1});\widehat{K_1}), \widehat{K_1}e^{\gamma g_\tau(\widehat{K_1})}\big) \in \Gamma_3$*), or* $x(g_\tau(\widehat{K_1});\widehat{K_1}) = \bar{x}$*.*

Proof: Let $K_1 \in (\widetilde{K_3}, K^*)$. Since $\lambda = G(\cdot;K_1)$, the function G is obviously differentiable and we have

$$\frac{\partial G}{\partial \tau}(\tau(K_1);K_1) = \lambda'(\tau(K_1)) > 0$$

(see Lemma 19). With the implicit function theorem we obtain a neighbourhood U of K_1 and a function $g_\tau : U \to (0,\infty)$ such that $G(g_\tau(K_1);K_1) = r$, for $K_1 \in U$, holds. The implicit function theorem implies more: g_τ can be extended in a maximal way to $(\widehat{K_1}, K^*)$ with $\widehat{K_1} \in [\widetilde{K_1}, K^*)$, proving (13). The properties in *(c)* are a consequence of the maximality of the extension. Moreover:

$$\lambda'(g_\tau(K_1))g_\tau'(K_1) = -v(g_\tau(K_1)),$$

where $v := \dfrac{\partial \lambda}{\partial K_1}(\cdot)$ solves

$$v' = -\kappa v + w,\ \ v(0) = 0,$$

$w := \dfrac{\partial z}{\partial K_1}(\cdot)$ solves

$$w' = -wg(x(t)) + \big[-(z(t) - r')g'(x(t)) + \psi_*'(x(t))\big]y,\ \ w(0) = 0,$$

and $y := \dfrac{\partial x}{\partial K_1}(\cdot)$ solves

$$y' = \big[-F'(x(t)) + K_1e^{\gamma t}\big]y + e^{\gamma t}x(t),\ \ y(0) = 0.$$

Obviously $y(t) > 0$, $t \in (0, g_\tau(K_1)]$. Since $-(z(t) - r')g'(x(t)) + \psi_*'(x(t)) > 0$ for each $t \in (0, g_\tau(K_1)]$, we have $w(t) > 0$, $t \in (0, g_\tau(K_1)]$, and therefore $v(t) = \dfrac{\partial \lambda}{\partial K_1}(t) > 0$, $t \in (0, g_\tau(K_1)]$. It follows $g_\tau'(K_1) < 0$. ■

Now, we have to distiguish two cases:

Case I $\big(x(g_\tau(\widehat{K_1});\widehat{K_1}), \widehat{K_1}e^{\gamma g_\tau(\widehat{K_1})}\big) \in \Gamma_3$ and $x(g_\tau(\widehat{K_1});\widehat{K_1}) < \bar{x}$.

Case II $x(g_\tau(\widehat{K_1});\widehat{K_1}) = \bar{x}$.

In each case we have a curve Σ_s defined by

$$[\widehat{K_1}, K^*] \ni K_1 \ \longmapsto\ \big(x(g_\tau(K_1);K_1)\,,\ K_1e^{\gamma g_\tau(K_1)}\big) \in [x^*, \bar{x}] \times [0,\infty).$$

In case II this curve ends at the "boundary" $x = \bar{x}$. In the case I, we want to continue this curve such that the continuation ends at the "boundary" $x = \bar{x}$ too. For this continuation the trajectories starting in $(\tilde{x}, K)$, $K \geq \tilde{\tilde{K}}$, come into consideration. To analize the situation we need the curve $\widetilde{\Sigma}$ which is defined by

$$\widetilde{\Sigma} : [0,\infty) \ni K \longmapsto (\tilde{x}, K) \in [0,\bar{x}] \times [0,\infty)$$

(see Figure 3). As we will see in Theorem 29 an optimal trajectory (x, K) with adjoint variables (z, λ) meets the curve $\widetilde{\Sigma}$ with the values

$$z(\sigma) = 0,\ z'(\sigma) = 0,\ \lambda(\sigma) < r.$$

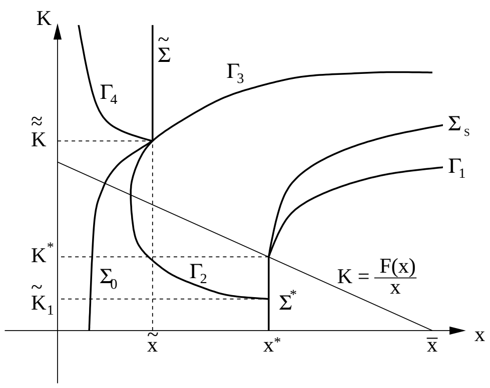

Figure 3: Main curves.

This motivates the construction of the continuation of Σ_s in the following way: Compute trajectories (x, K, z) backwards in time starting with initial values

$$x(0) = \tilde{x},\ K(0) = K_1 > \tilde{\tilde{K}},\ z(0) = 0.$$

Lemma 21 *There exists* $\tilde{\tilde{K}}_1 > \tilde{\tilde{K}}$ *and* $a > 0$ *such that for each* $K_1 \geq \tilde{\tilde{K}}_1$

$$x(a; K_1) = \bar{x},\ z(t; K_1) < r',\ t \in [0, a],$$

where $(x(\cdot; K_1), z(\cdot; K_1))$ *is the solution of*

$$\left\{ \begin{array}{ll} x' = -F(x) + K_1 e^{\gamma t} x & ,\ x(0) = \tilde{x}, \\ z' = -(z - r')g(x) + \psi_*(x), & z(0) = 0. \end{array} \right. \tag{14}$$

Proof: Let $\varepsilon > 0$. It follows from the differential equation for x that the solution $x(\cdot; K_1)$ reaches $x = \bar{x}$ for a time $t_1 < \varepsilon$ if K_1 is sufficiently large. Since the differential equation for z may be considered as a linear equation (if we plug in $x(\cdot; K_1)$), the value $z = r'$ cannot be reached in the time intervall $[0, \varepsilon]$ if K_1 is sufficiently large. ■

As we know from the results above an optimal trajectory (x, K) with adjoint variables (z, λ) starts in $(\tilde{x}, \tilde{\tilde{K}})$ in the following way:

$$x(0) = \tilde{x},\ K(0) = \tilde{\tilde{K}},\ z(0) = 0,\ z'(0) = 0,\ \tilde{\lambda} := \lambda(0) < r.$$

For each $K_1 > \tilde{\tilde{K}}$ there exists a time $\xi = \xi(K_1)$ with $K_1 e^{-\gamma\xi} = \tilde{\tilde{K}}$. Set $\Lambda(K_1) := \lambda(\xi; K_1)$ where $\lambda(\cdot; K_1)$ is the solution of

$$\lambda' = -(\lambda - r)\kappa - r',\ \lambda(0) = \tilde{\lambda}\,. \tag{15}$$

Now consider for each $K_1 > \tilde{\tilde{K}}$ the solution of

$$\left\{ \begin{array}{ll} x' = -F(x) + K_1 e^{\gamma t} x & ,\ x(0) = \tilde{x}, \\ z' = -(z - r')g(x) + \psi_*(x), & z(0) = 0, \\ \lambda' = -(\lambda - r)\kappa + z - r' & ,\ \lambda(0) = \Lambda(K_1)\,. \end{array} \right. \tag{16}$$

We want to find for $K_1 > \tilde{\tilde{K}}$ some time $\tau(K_1)$ such that $\lambda(\tau(K_1); K_1) = r$ holds. Again we use the notation $G(\cdot; K_1) := \lambda(\cdot; K_1)$.

Lemma 22 *In the case I the following assertions hold:*

(a) *There exists* $\tilde{\tilde{K}}_1 \in (\tilde{\tilde{K}}, \infty)$ *and a continuous differentiable mapping* $g_\tau : (\tilde{\tilde{K}}, \tilde{\tilde{K}}_1) \to (0, \infty)$ *with* $G(g_\tau(K_1); K_1) = r$, $K_1 \in (\tilde{\tilde{K}}, \tilde{\tilde{K}}_1)$;

(b) $g'_\tau(K_1) < 0$ *for all* $K_1 \in (\tilde{\tilde{K}}, \tilde{\tilde{K}}_1)$;

(c) $x(g_\tau(\tilde{\tilde{K}}_1); \tilde{\tilde{K}}_1) = \bar{x}$.

Proof: This can be proved similar to Lemma 20. The main observation is that we have $z(\tau; K_1) > r'$ if $\lambda(\tau; K_1) = r$. The result of *(c)* is a consequence of Lemma 21. ■

We have constructed a curve Σ_s connecting (x^*, K^*) with $(\bar{x}, \bar{K})$, where $\bar{K} := \tilde{\tilde{K}}_1 e^{\gamma g_\tau(\tilde{\tilde{K}}_1)}$. Since this curve should be used as a jump curve in the region $[x^*, \bar{x}] \times [0, \infty)$ we want to reparametrize this curve in such a way that $[x^*, \bar{x}]$ is the parameter interval. But to do this we need the fact that the function

$$[\widehat{K}_1, K^*] \cup [\tilde{\tilde{K}}, \tilde{\tilde{K}}_1] \ni K_1 \mapsto x(g_\tau(K_1); K_1) \in [x^*, \bar{x}]$$

is monotone increasing. Unfortunately, we are not able to prove this. The fact we can prove is that the mapping

$$H : [\widehat{K}_1, K^*] \ni K_1 \mapsto x(g_\tau(K_1); K_1) \in [x^*, \bar{x}]$$

is monotone increasing in a neighbourhood of K^*.

Lemma 23 *There exists* $K_+ \in [\widehat{K}_1, K^*)$ *and* $m_0, m_1 > 0$ *such that*

a) $\tau(K^*) = 0$;

b) $\tau'(K^*) = -\frac{4}{\gamma K^*}$, τ *is differentiable in* $[K_+, K^*]$;

c) $-m_1(K^* - K) \le \tau(K) \le -m_0(K^* - K)$ *for all* $K \in [K_+, K^*]$.

Proof: *a)* is obvious. Clearly, τ is differentiable in $[K_+, K^*)$. From the identity

$$\lambda(\tau(K); K) = 0, K \in [\widehat{K}_1, K^*],$$

we obtain

$$\lambda'(\tau(K); K)\tau'(K) + v(\tau(K); K) = 0\,,\ K \in [\widehat{K}_1, K^*).$$

Here we have used the notation of Lemma 19. Due to the fact that $\lambda'(\tau(K); K) > 0$ for each $K \in [\hat{K}_1, K^*)$ we have

$$\tau'(K) = -\frac{v(\tau(K); K)}{\lambda'(\tau(K); K)}\,,\ K \in [\widehat{K}_1, K^*).$$

Using taylor's expansion for v and λ we obtain

$$\tau'(K) = \frac{-\frac{1}{6} v^{(''')}(\xi; K)}{\frac{1}{24}\lambda^{(iv)}(\eta; K)}\,,\ K \in [\widehat{K}_1, K^*)\,,$$

where $\xi, \eta \in (0, \tau(K))$. From this we can conclude

$$\tau'(K^*) \;=\; -4\frac{v^{(''')}(0; K^*)}{\lambda^{(iv)}(0; K^*)} \;=\; -\frac{4}{\gamma K^*}.$$

The result in *c)* is a consequence of the estimate in *b)* by using continuity arguments. ■

Lemma 24 *There exists* $K_+ \in [\hat{K}_1, K^*)$ *such that*

$$H(K^*) = x^* \,,\ H'(K^*) = 0 \,;\ H'(K) > 0 \,,\ K \in (K_+, K^*).$$

Proof: Let K_+ be chosen as in Lemma 23. We have

$$H'(K) \ = \ x'(\tau(K);K)\tau'(K) + y(\tau(K);K) \,,\ K \in [K_+, K^*] \,,$$

and by Lemma 23 we have $H'(K^*) = 0$. From the differential equation for y we obtain that there exists $M > 0$ such that

$$y(\tau(K);K) \geq M\tau(K) \,,\ K \in [K_+, K^*].$$

Since τ' is bounded in $[K_+, K^*]$ and since $x'(\tau(K);K) = 0$ we obtain the assertion, eventually by making K_+ larger. ∎

Corollary 25 *There exists* $x_s \in (x^*, \bar{x}]$ *and a continuously differentiable mapping* $h_s : [x^*, x_s]$ *such that*

$$(x, h_s(x)) \in \Sigma_s \,,\ x \in [x^*, x_s].$$

Proof: h_s can be found by reparametrizing the curve

$$K \ \longmapsto \ (x(g_\tau(K);K), K(g_\tau(K);K))$$

using the results of Lemma 24. ∎

Now the jump curve Σ_s allow us to find, given (x_0, K_0) with $x_0 \in (x^*, x_s)$ and $K_0 < h_s(x_0)$, the optimal initial value $K_{0,+} := h_s(K_0)$. Notice that in the region around Γ_1, Σ^*, and Σ_s, the behaviour of the extremals is rather clear. Notice also that $h_s(x) > h_1(x)$, for $x \in (x^*, \bar{x})$, since Σ_s is above Γ_1. From the construction of Σ_1 and Γ_3, we may have two cases: Γ_1 and Γ_3 have a point in common; Γ_1 and Γ_3 do not touch. In the next section we are able to find for each initial condition $(x_0, K_0) \in [0, \bar{x}) \times (0, \infty)$ the corresponding optimal trajectory.

4 Optimal trajectories

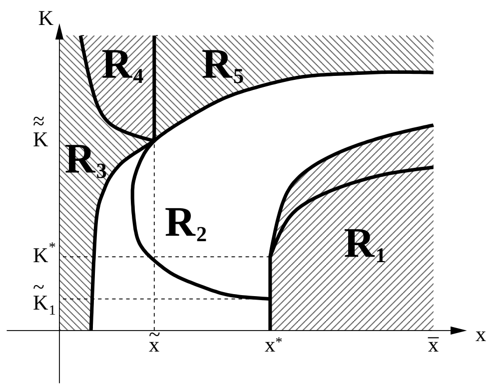

Figure 4: Main regions.

In this section we summarize the previous results, in order to design a complete picture of the optimal trajectories. We are also able to determine the correspondent optimal controls, based on the information we obtain from the switch variables. Let us consider the regions defined by

$$\Sigma^*, \widetilde{\Sigma}, \Sigma_s, \Sigma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$$

in $[0, \bar{x}] \times [0, \infty)$. In Figure 4 we present a sketch of the five main regions for the case of the logistic function. One should not forget that that the curves Σ_s and Γ_3 may have a point in common.

Domain (R1): boundaries Σ^*, Σ_s, $\{(x, K) \in [0, \bar{x}] \times [0, \infty) \mid x = \bar{x}\}$;

Domain (R2): boundaries Σ_0, Σ^*, Σ_s, Γ_3 and eventually $\{(x,k) \in [0,\bar{x}] \times [0,\infty) \mid x = \bar{x}\}$;

Domain (R3): boundaries $\{(x,K) \in [0,\bar{x}] \times [0,\infty) \mid x = 0\}$, Σ_0, Γ_4;

Domain (R4): boundaries $\{(x,K) \in [0,\bar{x}] \times [0,\infty) \mid x = 0\}$, Γ_4, $\widetilde{\Sigma}$;

Domain (R5): boundaries $\widetilde{\Sigma}$, Γ_3, $\{(x,K) \in [0,\bar{x}] \times [0,\infty) \mid x = \bar{x}\}$.

Remark 26 *In the following we may assume, without loss of generality, that $K_0 = K_{0,+}$ holds since in the case of $K_0 < K_{0,+}$ we may leave the region where we started or not. In the first case we have to apply the discussion in another region, in the second case we have to repeat the analysis in the same region with $K_0 := K_{0,+}$.* □

Theorem 27 *Let (x, K, u, μ) be optimal with adjoint variables z, λ. The following assertions hold:*

(a) Let (x_0, K_0) be in (R1). Then $z_0 > r'$, $\lambda_0 = r$, $(x_0, K_{0,+}) \in \Sigma_s$ and there exists $\tau > 0$ with $(x(\tau), K(\tau)) \in \Sigma^*$ and $u \equiv 1$, $\mu \equiv 0$ in $(0,\tau)$;

(b) Let (x_0, K_0) be in (R2). Then $z_0 > 0$, $\lambda_0 < r$ and there exists $\tau > 0$ with $(x(\tau), K(\tau)) \in \Sigma^*$ and $u \equiv 1$, $\mu \equiv 0$ in $(0,\tau)$;

(c) Let (x_0, K_0) be in (R3). Then $z_0 < 0$, $\lambda_0 < r$ and there exists $\tau > 0$ with $(x(\tau), K(\tau)) \in \Sigma_0$ and $u \equiv 0$, $\mu \equiv 0$ in $(0,\tau)$;

(d) Let (x_0, K_0) be in (R4). Then $z_0 < 0$, $\lambda_0 < r$ and there exists $\tau > 0$ with $(x(\tau), K(\tau)) \in \widetilde{\Sigma}$ and $u \equiv 0$, $\mu \equiv 0$ in $(0,\tau)$;

(e) Let (x_0, K_0) be in (R5). Then $z_0 > 0$, $\lambda_0 < r$ and there exists $\tau > 0$ with $(x(\tau), K(\tau)) \in \widetilde{\Sigma}$ and $u \equiv 1$, $\mu \equiv 0$ in $(0,\tau)$.

Proof: Ad *(a)*: We actually prove only that if (x_0, K_0) is in $(R1)$, then $K_{0,+} \geq h_s(x_0)$, i.e. we must jump either to (R2) or to (R5). Later on, in the proof of items (b) and (e), we will see that the initial condition $\lambda_0 = r$ is not allowed in these regions. The last possible case: $(x_0, K_{0,+}) \in \Gamma_3$ is excluded in Theorem 28.
From Lemma 14, it is enough to consider initial conditions (x_0, K_0) with $K_0 \geq h_1(x_0)$. Assume $\lambda_0 < r$. Then there exists a $\tau > 0$ with $\lambda(\tau) = r$ and $\lambda(t) < r$, $t \in [0,\tau)$. Consequently $\lambda'(\tau) = 0$, $z(\tau) = r'$, $z'(\tau) = -\psi_*(x(\tau))$. If $z'(\tau) < 0$, then $\lambda''(\tau) > 0$, contradicting (R). If $z'(\tau) \geq 0$, then $x(\tau) \leq x^*$ and there has to be a $\sigma \in (0,\tau)$ such that $(x(\sigma), K(\sigma)) \in \Sigma_s$. Since $\lambda(\sigma) < r$, this cannot occur, due to the construction of Σ_s. Therefore we must have $\lambda_0 = r$, what implies $z_0 \geq r'$. Note that $z_0 = r'$ is not possible, since we would have $\lambda'(0) = 0$ and $\lambda''(0) = -z'(0) > 0$, contradicting (R). The case $z_0 > r'$ cannot occur, otherwise $\lambda'(0) < 0$ and repeating the hole argumentation, we have again a contradiction.

Ad *(b)*: Assume $\lambda(0) = r$. Then $\lambda'(0) \leq 0$ and $z_0 \geq r'$.
If $z_0 = r'$, then $\lambda'(0) = 0$ and we have three possible cases: *i)* $x_0 > x^*$: we have $\lambda''(0) = \psi_*(x_0) > 0$, contradicting (R); *ii)* $x_0 = x^*$: cannot occur, due to Lemma 9; *iii)* $x_0 < x^*$: then $z'(0) > 0$ and $\lambda''(0) = -z'(0) < 0$. Since $\lambda(t) < r$ for all $t > 0$ is not possible, there exists $\tau > 0$ with $\lambda(\tau) = r$, $\lambda(t) < r$, $t \in (0,\tau)$. Then $\lambda'(\tau) = 0$, $z(\tau) = r'$ and there exists $\sigma \in (0,\tau)$ with $z'(\sigma) = 0$, $z(\sigma) > r'$. From the differential equation for z we conclude with (V4) that $x(\tau) \geq x(\sigma) > x^*$. This is a contradiciton to Lemma 8, since it is easy to see that $x(\tau)K(\tau) < F(x(\tau))$.
For $z_0 > r'$ we may repeat the arguments above.
Therefore we have $\lambda_0 < r$. Since $\lambda(t) < r$ for all $t > 0$ is not allowed, there exists $\tau > 0$ with

$\lambda(\tau) = r$, $\lambda(t) < r$, $t \in (0,\tau)$, $\lambda'(\tau) = 0$, $z(\tau) = r'$. If there is a jump in τ, i.e. $K(\tau) < K(\tau+)$, with $(x(\tau), K(\tau))$ in domain (R2), then either we leave (R2), or we stay. In the first case we have to switch to (c), (d) or (e) with $\lambda_0 = r$, but there we will see that this is not possible. Thus we know that there is no jump in (R2). Therefore, by Lemma 8, we have $x(\tau) = x^*$.

Ad *(c)*: An optimal trajectory has to leave (R3) by crossing Σ_0, if there is no jump. Let $\tau > 0$ be a time with $(x(\tau), K(\tau)) \in \Sigma_0$. Then $z(\tau) = 0$ and $z'(\tau) > 0$, from Corollary 17. Therefore $z(t) < 0$ for $t < \tau$ and $z_0 < 0$. Notice that $z(\sigma) = 0$ for $\sigma \in (0,\tau)$ is not possible by looking at the differential equation for z. Since $z_0 < 0$, then $\lambda(t) = r$ cannot hold when $x(t)$ is in (R3).

Ad *(d)*: Assume $\lambda_0 = r$. Then $z_0 \geq r'$. If $z_0 = r'$, then $\lambda'(0) = 0$, $\lambda''(0) = -z'(0) < 0$. Since $\lambda(t) < r$ for all $t > 0$ is not allowed, there exists $\tau > 0$ with $\lambda(\tau) = r$, $\lambda(t) < r$, $t \in (0,\tau)$, and $\lambda'(\tau) = 0$, $z(\tau) = r'$. Moreover, there is $\sigma \in (0,\tau)$ with $z'(\sigma) = 0$, $z(\sigma) > r'$, and from the differentiable equation for z and with assumption (V4) $x(\sigma) > \tilde{x}$. Since $z(t) > 0$, $t \in [0,\sigma)$, there exists $t_0 \in (0,\sigma)$ with $x(t_0) \in \Sigma_0$. Then $z(t_0) = 0$, which is a contradiction to $z(t_0) > 0$. If $z_0 > r'$, then $\lambda'(0) < 0$ and again this leads to a contradiction. Therefore, we have $\lambda_0 < r$. Assume $z_0 \geq 0$. Then $z'(0) > 0$ and the solution meets (R3) with $z(t) > 0$. This cannot occur, since z' may change its sign only in (R2).
Now we have $\lambda_0 < r$ and $z_0 < 0$. Then there exists $\tau > 0$ such that $z(t) < 0$, $\lambda(t) < r$, $t \in [0,\tau)$, $x(\tau) = \tilde{x}$, since otherwise we could repeat the arguments above.

Ad *(e)*: Assume $\lambda(0) = r$. Then we have $z_0 \geq r'$.
If $z_0 = r'$, then $\lambda'(0) = 0$ and we consider three cases: *i)* $x_0 > x^*$: we have $\lambda''(0) = \psi_*(x_0) > 0$, contradicting (R); *ii)* $x_0 = x^*$: cannot occur, due to Lemma 9; *iii)* $x_0 < x^*$: we have $z'(0) > 0$, $\lambda''(0) < 0$. Since $\lambda(t) < r$ for all $t > 0$ is not possible, there exists $\tau > 0$ with $\lambda(\tau) = r$, $\lambda(t) < r$, $t \in (0,\tau)$, and $\lambda'(\tau) = 0$, $z(\tau) = r'$. Then there exists $\sigma \in (0,\tau)$ with $z'(\sigma) = 0$, $z(\sigma) > r' > 0$. But this is not in agreement with the differential equation for z, if $x(\tau) > \tilde{x}$.
If $z_0 > r'$, then $\lambda'(0) < 0$ and we have again a contradiction.
Therefore $\lambda_0 < r$. Assume $z_0 \leq 0$. Then $z'(0) < 0$. Since $\lambda(t) < r$ for all $t > 0$ is not allowed, there exists $\tau > 0$ with $\lambda(\tau) = r$, $\lambda(t) < r$, $t \in (0,\tau)$. Then $z'(\tau) = \lambda''(\tau) \geq 0$ and due to the differential equation for z, $x(\tau) \leq \tilde{x}$. Thus we have shown that the trajectory meets $\widetilde{\Sigma}$ as conjectured. ■

Theorem 28 *Let (x, K, u, μ) be optimal with adjoint variables z, λ. The following assertions hold:*

(a) If (x_0, K_0) is on the curve Γ_4, then we may apply Theorem 27 (d);

(b) If (x_0, K_0) is on the curve Γ_3, then we may apply Theorem 27 (e).

Proof: Follows along the argumentation above. ■

Resuming, we know that each optimal trajectory meets Σ_0 over $\widetilde{\Sigma}$ or Σ^*. We now have to discuss the behavior for points on these curves.

Theorem 29 *Let (x, K, u, μ) be optimal with adjoint variables z, λ. The following assertions hold:*

(a) Let (x_0, K_0) be on Σ^*. Then $z_0 = r'$, $\lambda = r$, $(x_0, K_{0,+}) = (x^*, K^*)$ and $u \equiv 1$, $\mu = \gamma K dt$;

(b) Let (x_0, K_0) be on Σ_0. Then $z_0 = 0$, $\lambda_0 < r$ and there exists $\tau > 0$ with $(x(\tau), K(\tau)) \in \Sigma^*$ and $u \equiv 1$, $\mu \equiv 0$ in $(0,\tau)$;

(c) Let (x_0, K_0) be on $\widetilde{\Sigma}$. Then $z_0 = 0$, $\lambda_0 < r$ and there exists $\tau_1 \geq 0$, $\tau_2 > \tau_1$ with $(x(\tau_1), K(\tau_1)) = (\tilde{x}, \tilde{\tilde{K}})$, $(x(\tau_2), K(\tau_2)) \in \Sigma^*$. Moreover

$$\mu_{|(0,\tau_1)} \equiv 0, \quad u(t) = K(t)^{-1}F(\tilde{x})\tilde{x}^{-1}, \ t \in (0, \tau_1), \quad \mu_{|(\tau_1,\tau_2)} \equiv 0, \quad u_{|(\tau_1,\tau_2)} \equiv 1.$$

Proof: Note that (a) was already proved in Lemma 11. Item (b) is proved exactly in the same way as Theorem 27 (b). Now we prove (c).
Assume $\lambda(0) = r$. Then $z_0 \geq r'$. If $z_0 = r'$, then $\lambda'(0) = 0$, $z'(0) = -\psi_*(\tilde{x}) > 0$, $\lambda''(0) = -z'(0) < 0$. If $z_0 > r'$, then $\lambda'(0) < 0$. In each case there exists $\tau > 0$ with $\lambda(\tau) = r$, $\lambda(t) < r$, $t \in (0, \tau)$, and we obtain $\lambda'(\tau) = 0$, $z(\tau) = r'$ and $\sigma \in (0, \tau)$ with $z'(\sigma) = 0$. Moreover $\lambda(t) < r$, $t \in (0, \tau)$, $z(t) > r'$, $t \in (0, \tau)$. Thus we arrive in (R4) with $z(t) > 0$, $\lambda(t) < r$. But this is not in agreement with the results for domain (R4). Therefore $\lambda_0 < r$ must hold.
If $z_0 < 0$, then the trajectory meets (R5) with $z(t) < 0$, $\lambda(t) < r$, which is not possible due to the results for (R5). If $\lambda_0 < r$ and $z_0 > 0$, the trajectory reaches (R4) with $z(t) > 0$, $\lambda(t) < r$, but this is not allowed due the results for (R4).
Thus we have $\lambda_0 < r$, $z_0 = 0$ and, consequently, $z'(0) = 0$. Repeating the arguments above, we conclude that the trajectory cannot leave $\widetilde{\Sigma}$ as long as $K(t) \geq \tilde{\tilde{K}}$ holds. Therefore, there exists $\tau_1 > 0$ with $x(t) = x^*$, $u(t) = F(\tilde{x})\tilde{x}^{-1}K(t)^{-1}$, $t \in [0, \tau_1]$. In $(\tilde{x}, \tilde{\tilde{K}})$ the trajectory has to leave $\widetilde{\Sigma}$, since it cannot go into (R2) with $z(t) = 0$ (see Theorem 27 (b)). Therefore, the trajectory must follow the curve Γ_2 and meets Σ^* with time $\tau_2 > \tau_1 > 0$. ■

5 Appendix: maximum principle

The optimal control problem we want to consider is the following one:

Minimize $J(x_0, K_0; \mu, u) := \int_0^\infty e^{-\delta t} r\mu(dt) + \int_0^\infty e^{-\delta t}\{c - pqx(t)\}u(t)K(t)dt$
subject to $(u, \mu) \in U_{ad} \times C^*$ and

$$x' = F(x) - qu(t)Kx, \ x(0) = x_0, \tag{17}$$

$$dK = -\gamma K dt + \mu(dt), \ K(0) = K_0, \tag{18}$$

where

$$\begin{aligned} U_{ad} &:= \{v \in L_\infty[0, \infty) \mid 0 \leq v(t) \leq 1 \ a.e. \ in \ [0, \infty)\}, \\ C^* &:= \{\mu \mid \mu \text{ non–negative Borel measure on } [0, \infty)\}. \end{aligned}$$

This problem is denoted by $P(x_0, K_0)$. Let (x, K, u, μ) be a solution of the problem. The idea for proving a maximum principle comes from [Ha] and [We]. Halkin, however, uses a different solution concept, avoiding the use of Bellman's principle to analyse problems with infinite horizon.

Define the so–called Hamilton function $\widetilde{H}$ by

$$\widetilde{H}(t, \tilde{x}, \widetilde{K}, w, \tilde{\lambda}_1, \tilde{\lambda}_2, \eta) := \tilde{\lambda}_1(F(\tilde{x}) - w\widetilde{K}\tilde{x}) - \tilde{\lambda}_2\gamma\widetilde{K} - \eta e^{-\delta t}(c - p\tilde{x})w\widetilde{K}.$$

Let $(T_k)_{k \in I\!N}$ be a sequence in $(0, \infty)$ with $\lim_k T_k = \infty$ and consider for each $k \in I\!N$ the following problem $P_k(x_0, K_0)$:

Minimize $J(x_0, K_0; \nu, w) := \int_0^{T_k} e^{-\delta t} r\nu(dt) + \int_0^{T_k} e^{-\delta t}\{c - pqy(t)\}w(t)l(t)dt$
subject to $(w, \nu) \in U_{ad,k} \times C_k^*$ and

$$\begin{aligned} y' &= F(y) - qw(t)ly,\ y(0) = x_0,\ y(T_k) = x(T_k), && (19)\\ dl &= -\gamma l dt + \nu(dt),\ l(0) = K_0,\ l(T_k) = K(T_k), && (20) \end{aligned}$$

where

$$\begin{aligned} U_{ad,k} &:= \{v \in L_\infty[0, T_k] \mid 0 \le v(t) \le 1 \ a.e.\ in\ [0, T_k]\},\\ C_k^* &:= \{\mu \mid \mu \text{ nonnegative Borel measure on } [0, T_k]\}. \end{aligned}$$

Assumption: T_k is a point of continuity for each $k \in I\!N$. (Clearly such a choice of $(T_k)_{k \in I\!N}$ is always possible since K possesses only a countable number of jumps.)

Once one has proved the Bellman's optimality principle for control problems with infinite horizon (see [BL]), one concludes that for each $k \in I\!N$, $(x_{|[0,T_k]}, K_{|[0,T_k]}, \mu_{|[0,T_k]}, u_{|[0,T_k]})$ is a solution of $P_k(x_0, K_0)$. With the maximum principle proved in [VP], we obtain $\tilde\lambda_{1,0,k}$, $\tilde\lambda_{2,0,k}$, $\eta_k \in I\!R$ such that there exists $\tilde\lambda_{1,k}$, $\tilde\lambda_{2,k}$ with

$$\begin{aligned} \tilde\lambda_{1,0,k}^2 + \tilde\lambda_{2,0,k}^2 + \eta_k^2 &= 1,\ \eta_k \ge 0,\\ x' &= F(x) - u(t)Kx,\ x(0) = x_0,\\ dK &= -\gamma K dt + \mu(dt),\ K(0) = K_0,\\ \tilde\lambda_{1,k}' &= -\tilde\lambda_{1,k}(F'(x) - u(t)K) - \eta_k e^{-\delta t} p u(t) K,\\ \tilde\lambda_{2,k}' &= \tilde\lambda_{1,k} x u + \gamma\tilde\lambda_{2,k} + \eta_k e^{-\delta t}(c - px)u(t),\\ \tilde\lambda_{2,k}(t) - \eta_k e^{-\delta t} r &\le 0 \text{ for all } t \in [0, T_k],\\ \tilde\lambda_{1,0}(0) &= \tilde\lambda_{1,0,k},\ \tilde\lambda_{2,k}(0) = \tilde\lambda_{2,0,k},\\ \tilde\lambda_2(t) - \eta e^{-\delta t} r &= 0\ \mu\text{–a.e. in } [0, T_k],\\ \widetilde H(t, x(t), K(t), u(t), \tilde\lambda_{1,k}(t), \tilde\lambda_{2,k}(t), \eta_k) &= \max_{w \in [0,1]} \widetilde H(t, x(t), K(t), w, \tilde\lambda_{1,k}(t), \tilde\lambda_{2,k}(t), \eta_k)\\ & \qquad a.e.\ in\ [0, T_k]. \end{aligned}$$

Without loss of generality we may assume that the sequences $(\tilde\lambda_{1,0,k}, \tilde\lambda_{2,0,k})_{k \in I\!N}$ and $(\eta_k)_{k \in I\!N}$ converge. Let

$$\tilde\lambda_{1,0} := \lim_{k\to\infty} \tilde\lambda_{1,0,k},\ \tilde\lambda_{2,0} := \lim_{k\to\infty} \tilde\lambda_{2,0,k},\ \eta := \lim_{k\to\infty} \eta_k.$$

Then, due to the continuous dependence of the solution on initial data and parameter (see [Sch]) we obtain

$$\tilde\lambda_1 := \lim_{k\to\infty} \tilde\lambda_{1,k},\ \tilde\lambda_2 := \lim_{k\to\infty} \tilde\lambda_{2,k}.$$

uniformly on each intervall $[0,T], T>0$. This gives the desired maximum principle for $P(x_0,K_0)$:

$$\begin{aligned}
\tilde{\lambda}_{1,0}^2+\tilde{\lambda}_{2,0}^2+\eta^2 &\neq 0,\ \eta\geq 0,\\
x' &= F(x)-u(t)Kx,\ x(0)=x_0,\\
dK &= -\gamma K dt+\mu(dt),\ K(0)=K_0,\\
\tilde{\lambda}_1' &= -\tilde{\lambda}_1(F'(x)-u(t)K)-\eta e^{-\delta t}pu(t)K,\\
\tilde{\lambda}_2' &= \tilde{\lambda}_1 xu+\gamma\tilde{\lambda}_2+\eta e^{-\delta t}(c-px)u(t),\\
\tilde{\lambda}_1(0) &= \tilde{\lambda}_{1,0},\ \tilde{\lambda}_2(0)=\tilde{\lambda}_{2,0},\\
\tilde{\lambda}_2(t)-\eta e^{-\delta t}r &\leq 0 \text{ for all } t\in[0,\infty),\\
\tilde{\lambda}_2(t)-\eta e^{-\delta t}r &= 0\ \mu\text{–a.e. in } [0,\infty),\\
\widetilde{H}(t,x(t),K(t),u(t),\tilde{\lambda}_1(t),\tilde{\lambda}_2(t),\eta) &= \max_{w\in[0,1]}\widetilde{H}(t,x(t),K(t),w,\tilde{\lambda}_1(t),\tilde{\lambda}_2(t),\eta) \text{ a.e. in } [0,\infty).
\end{aligned}$$

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