# NEW STABILITY ESTIMATES FOR THE INVERSE ACOUSTIC INHOMOGENEOUS MEDIUM PROBLEM AND APPLICATIONS

PETER HÄHNER* AND THORSTEN HOHAGE†

**Abstract.** This paper is concerned with the scattering of time-harmonic acoustic waves by inhomogeneous media. We study the problem to recover the refractive index from far field measurements and from near field mesurements. We establish logarithmic stability estimates for these problems using a-priori information with respect to Sobolev norms and a-priori information about the support of the inhomogeneity. Our results improve previous estimates due to Stefanov by giving an explicit exponent in the logarithmic estimate, by using the $L^2$-norm for far field patterns, and by dropping the assumption that the refractive indices are close together. These improvements make it possible to prove convergence rates for iterative regularization methods.

**Key words.** inverse medium scattering, logarithmic stability, regularization methods

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**1. Introduction.** The scattering of time-harmonic acoustic waves in an inhomogeneous medium with refractive index $n$ is described by the differential equation

$$\Delta u + k^2 n u = 0. \tag{1.1}$$

Here the real part of the complex valued function $u$ describes the space-dependent part of a velocity potential, and $k > 0$ is the wave number. We assume that the refractive index $n$ is a complex valued function in $\mathbb{R}^3$ satisfying $\mathrm{Im}(n(x)) \geq 0$ for all $x \in \mathbb{R}^3$ and supp $(1-n) \subset B_1$. (For $R > 0$ we use the notation $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$.) We will also assume throughout that $1-n$ belongs to the Sobolev space $H^s(\mathbb{R}^3)$ for some fixed $s > 3/2$. Due to Sobolev's imbedding theorem this implies that $n$ is uniformly Hölder continuous in $\mathbb{R}^3$ with Hölder exponent $s - 3/2$. Since by rescaling we can always obtain that $\mathrm{supp}(1-n) \subset B_1$, this is not a severe restriction.

The following problem will be referred to as *direct scattering problem*: Given an incident wave $u^{\mathrm{i}} \in C^2(B_1)$ satisfying the Helmholtz equation $\Delta u^{\mathrm{i}} + k^2 u^{\mathrm{i}} = 0$ in $B_1$ and $k$ and $n$ as above, find a scattered wave $u^{\mathrm{s}} \in C^2(\mathbb{R}^3)$ satisfying the differential equation

$$\Delta u^{\mathrm{s}} + k^2 n u^{\mathrm{s}} = k^2 (1-n) u^{\mathrm{i}} \quad \text{in } \mathbb{R}^3, \tag{1.2a}$$

and the Sommerfeld radiation condition

$$\lim_{|x| \to \infty} |x| \left( \frac{\partial u^{\mathrm{s}}}{\partial r}(x) - iku^{\mathrm{s}}(x) \right) = 0 \tag{1.2b}$$

uniformly for all directions $\hat{x} = (1/r)x$, where $r := |x|$. The right hand side of equation (1.2a) is defined to be zero outside of $B_1$. It can be shown that this problem has a unique solution (cf. [3, Chapter 8]). Note that the total field $u := u^{\mathrm{i}} + u^{\mathrm{s}}$ satisfies (1.1).

* Inst. f. Num. u. Angew. Math., Universität Göttingen, Lotzestraße 16–18, D-37083 Göttingen, Germany. Parts of this article are taken from this author's habilitation thesis [7].

† Konrad-Zuse Zentrum, Takustr. 7, D-14195 Berlin, Germany. Part of this author's work was carried out at the University of Linz, Austria, with support of the Austrian Science Fund (FWF) under the project SFB F013 "Numerical and Symbolic Scientific Computing".

To formulate our first inverse problem, let us introduce point sources $u^{\mathrm{i}}(x) = \Phi(x,y)$, located at $y \in \mathbb{R}^3$, $|y| > 1$, as incident fields. Here

$$\Phi(x,y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x, y \in \mathbb{R}^3, x \neq y,$$

denotes the fundamental solution to the Helmholtz equation. The corresponding scattered fields $u^{\mathrm{s}}$ satisfying (1.2a) and (1.2b) for $u^{\mathrm{i}}(x) = \Phi(x,y)$ are denoted by $w_n^{\mathrm{s}}(x,y)$. The total field $w_n(x,y) := \Phi(x,y) + w_n^{\mathrm{s}}(x,y)$ is the Green function for our scattering problem. It is actually defined for all $x$, $y \in \mathbb{R}^3$, $x \neq y$, but we shall need it only for points $|y| > 1$. We consider the inverse problem to determine the refractive index $n$ from a knowledge of its corresponding Green function $w_n$ on $\partial B_R \times \partial B_R$ with some $R > 1$. This means we are probing the medium using point sources located on $\partial B_R$ and measure the scattered waves on $\partial B_R$ in order to recover the inhomogeneity. We will show the following logarithmic stability estimate:

THEOREM 1.1. *Let $C_n > 0$ and $R > 1$ be given constants. Then, there exists a positive constant $C$ (depending only on $s$, $k$, $R$, and $C_n$) such that for all refractive indices $n$, $\tilde{n}$ satisfying $\|1-n\|_{H^s}, \|1-\tilde{n}\|_{H^s} \le C_n$, supp$(1-n)$, supp$(1-\tilde{n}) \subset B_1$, the estimates*

$$\|n - \tilde{n}\|_{L^2(B_1)} \le C \left[ -\ln^- \left( \|w_n^{\mathrm{s}} - w_{\tilde{n}}^{\mathrm{s}}\|_{L^2(\partial B_R \times \partial B_R)} \right) \right]^{-\frac{s}{s+3}} \tag{1.3}$$

*and*

$$\|n - \tilde{n}\|_{\infty} \le C \left[ -\ln^- \left( \|w_n^{\mathrm{s}} - w_{\tilde{n}}^{\mathrm{s}}\|_{L^2(\partial B_R \times \partial B_R)} \right) \right]^{-\frac{2s-3}{2s+3}}, \tag{1.4}$$

*hold true. Here $\ln^-(t) := \ln(t)$ for $t \le \exp(-1)$ and $\ln^-(t) := -1$ else.*

This theorem implies that if one uses point sources located on $\partial B_R$ to probe the medium and if one measures the scattered waves on $\partial B_R$, then the refractive index depends continuously on these data under the a-priori assumptions $\|1-n\|_{H^s} \le C_n$ and $\mathrm{supp}(1-n) \subset B_1$. The logarithmic estimates (1.3) and (1.4) reflect the exponentially ill-posed nature of the problem. Let us point out that Theorem 1.1 also implies a global uniqueness result for the inverse scattering problem to determine the refractive index from measurements of the scattered fields on $\partial B_R$ using point sources located on $\partial B_R$. Namely, two refractive indices from $H^s$ having the same Green function on $\partial B_R \times \partial B_R$ must be identical.

Another interesting inverse problem results from using far field data instead of near field data. A solution $v$ to the Helmholtz equation in an exterior domain satisfying (1.2b) is known to obey

$$v(x) = \frac{e^{ik|x|}}{|x|} \left( v^\infty \left( \frac{x}{|x|} \right) + \mathrm{O} \left( \frac{1}{|x|} \right) \right)$$

(cf. [3, Section 2.2]). The function $v^\infty : S^2 \to \mathbb{C}$, $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ is called *the far field pattern* of $v$. In this context we will consider plane incident waves $u^{\mathrm{i}}(x) = u^{\mathrm{i}}(x,d) = \exp(ikx \cdot d)$, $d \in S^2$. The corresponding scattered fields are denoted by $u_n^{\mathrm{s}}(\cdot, d)$, and their far field patterns by $u_n^\infty(\cdot, d)$. Probing the medium with plane incident waves from all directions and measuring the corresponding far field patterns leads to the problem to recover $n$ from a knowledge of $u_n^\infty$ on $S^2 \times S^2$. We will prove the following result:

THEOREM 1.2. *Let $C_n > 0$ and $0 < \epsilon < \frac{s}{s+3}$ be given constants. Then, there exists a positive constant $C$ (depending only on $s$, $\epsilon$, $k$, and $C_n$) such that for all refractive* *indices $n$, $\tilde{n}$ satisfying $\|1-n\|_{H^s}, \|1-\tilde{n}\|_{H^s} \le C_n$, and supp$(1-n)$, supp$(1-\tilde{n}) \subset B_1$, the estimate*

$$\|n-\tilde{n}\|_{L^2(B_1)} \le C\left[-\ln^-\left(\|u_n^\infty - u_{\tilde{n}}^\infty\|_{L^2(S^2\times S^2)}\right)\right]^{-\frac{s}{s+3}+\epsilon} \tag{1.5}$$

*holds true.*

*Moreover, for $0<\epsilon<\frac{2s-3}{2s+3}$ the maximum norm of $n-\tilde{n}$ can be estimated by*

$$\|n-\tilde{n}\|_\infty \le C\left[-\ln^-\left(\|u_n^\infty - u_{\tilde{n}}^\infty\|_{L^2(S^2\times S^2)}\right)\right]^{-\frac{2s-3}{2s+3}+\epsilon}. \tag{1.6}$$

The main idea for the proof of Theorem 1.1 goes back to Alessandrini [1] who examined the continuous dependence of the conductivity on its corresponding Dirichlet-to-Neumann map. In [17] Stefanov applied this idea to potential scattering problems and proved that the potential depends continuously on the corresponding Dirichlet-to-Neumann map. Both articles employ a lemma about holomorphic functions in several variables which prevents the exponent in the inequalities to be known explicitly. In order to obtain the explicit exponents $-s/(s+3)$ and $-(2s-3)/(2s+3)$ we present a modified and simpler proof. Moreover, contrary to Stefanov, who establishes his result only for refractive indices $n$ and $\tilde{n}$ sufficiently close to each other, we shall only need the assumption that the $H^s$ norms of $1-n$ and $1-\tilde{n}$ are bounded by a constant $C_n$. Stefanov's paper [17] also contains a stability estimate involving far field data. However, instead of the $L^2$ norm a very strong norm involving exponentially increasing weights for the Fourier coefficients of the far field patterns is used. Since no estimate on the measurement error with respect to this norm will be available in practice, Stefanov's result is primarily of theoretical interest.

Let us point out that the function $[-\ln t]^{-1}$ is converging very slowly to 0 as $t\to 0$. The power $0<p<1$ in the function $[-\ln t]^{-p}$ is even deteriorating the convergence. However, since it is possible to prove convergence rates for iterative regularization methods which are based on stability estimates and since our numerical example indicates that the convergence of an iterative regularization method is not much faster than predicted by our estimate, we have strong evidence that our stability estimate cannot be significantly improved.

We also want to emphasize that assuming high regularity for $n$, i.e., using the a-priori information $1-n\in H_0^s(B_1)$ with a very large $s$, will improve the estimate somewhat but not very much.

In the next section we briefly collect the necessary results for the Green function. The following two sections are dedicated to the proofs of the two main theorems. In the final section of this paper we show how Theorem 1.2 can be used to obtain convergence rates for iterative regularization methods and present numerical computations comparing the actual speed of convergence to our estimates.

**2. The Green function.** In this section we want to briefly review the properties of the Green function that we shall use in the sequel for our stability estimates.

Let us first introduce the *Lippmann-Schwinger equation*, which is an equivalent formulation of (1.2a), (1.2b) as an integral equation of the second kind:

$$u(x)+k^2\int_{B_1}\Phi(x,y)\,(1-n(y))\,u(y)\,\mathrm{d}y = u^{\mathrm{i}}(x), \quad x\in B_1. \tag{2.1}$$

If $u \in C(B_1)$ is a solution to (2.1), then

$$u^{\mathrm{s}}(x) := -k^2 \int_{B_1} \Phi(x,y)\,(1-n(y))\,u(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^3, \tag{2.2}$$

is the solution to (1.2a), (1.2b) and vice versa (cf. [3, Section 8.2]). Using the incident waves $u^{\mathrm{i}}(x) = \Phi(x,y)$, it follows that $w_n^s$ is infinitely smooth for $|x|, |y| > 1$, $x \neq y$. From (2.2) and the asymptotic behavior of $\Phi(x,y)$ we obtain

$$u^{\infty}(\hat{x}) = -\frac{k^2}{4\pi} \int_{B_1} e^{-ik\hat{x}\cdot y}\,(1-n(y))\,u(y)\,\mathrm{d}y, \quad \hat{x} \in S^2.$$

Hence, the far field $u^\infty$ is $C^\infty$ on $S^2 \times S^2$.

For a fixed radius $R > 1$ let us introduce the single layer potential having the Green function $w_n(x,y) = \Phi(x,y) + w_n^s(x,y)$ as kernel, that is the function

$$u(x) := \int_{\partial B_R} w_n(x,y)\,\varphi(y)\ \mathrm{d}s(y), \quad x \in \mathbb{R}^3. \tag{2.3}$$

Here, we assume that the density $\varphi$ is a continuous function on $\partial B_R$. For convenience we also introduce the operator $S_n$ which maps $\varphi$ to $u|_{\partial B_R}$, i.e.,

$$(S_n\varphi)(x) := \int_{\partial B_R} w_n(x,y)\,\varphi(y)\ \mathrm{d}s(y), \quad x \in \partial B_R. \tag{2.4}$$

LEMMA 2.1. *Suppose $1 < R$. Then*

*(a) The Green function satisfies $w_n(x,y) = w_n(y,x)$. Especially, the operator $S_n$ is symmetric: $\int_{\partial B_R} g\,(S_n f)\ \mathrm{d}s = \int_{\partial B_R} (S_n g) f\ \mathrm{d}s$ for all $f$, $g \in C(\partial B_R)$.*

*(b) The mapping $1 - n \mapsto w_n|_{\partial B_R \times \partial B_R}$ regarded as a mapping from $H_0^s(B_1)$ to $L^2(\partial B_R \times \partial B_R)$ is completely continuous. Especially, bounded sets are mapped into bounded sets.*

*Proof.* The symmetry of $w_n^{\mathrm{s}}$ and then of the Green function $w_n$ can be shown with the help of Green's theorem (cf. [16, Section 8, Chapter 1.2.2 and Section 3, Chapter 1.2.3]) This immediately implies the symmetry of $S_n$ by interchanging the order of integration.

For part (b) we observe that the embedding $H_0^s(B_1) \to H_0^{s'}(B_1)$ with $s > s' > 3/2$ is a compact linear mapping. Since the mapping $H_0^{s'}(B_1) \to L^2(\partial B_R \times \partial B_R)$, $1 - n \mapsto w_n|_{\partial B_R \times \partial B_R}$ is continuous, the composition is completely continuous and maps bounded sets into bounded sets. ☐

The next lemma states that the single layer potential having as kernel the Green function $w_n$ is the unique solution to a certain transmission problem.

LEMMA 2.2. *For $f \in C(\partial B_R)$ the function*

$$u(x) := \int_{\partial B_R} w_n(x,y)\,f(y)\ \mathrm{d}s(y), \quad x \in \mathbb{R}^3, \tag{2.5}$$

*is the unique solution to the boundary value problem*

$$\Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial B_R, \tag{2.6a}$$

$$\frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu} = f \quad \text{on } \partial B_R, \tag{2.6b}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{2.6c}$$

*in* $C(\mathbb{R}^3) \cap C^2(\mathbb{R}^3 \backslash \partial B_R)$*. Here,* $\nu$ *denotes the unit normal vector on* $\partial B_R$ *directed into the exterior of* $B_R$*.*

*Proof.* In order to prove that the boundary value problem (2.6) has at most one solution we assume that $u$ is a solution of (2.6) with $f = 0$. Then we can follow the first part of the reasoning for the uniqueness proof of (1.2a), (1.2b) ([3, Theorem 8.7]) and we obtain $u = 0$ in the exterior of $B_R$, whence $u = \frac{\partial u_-}{\partial \nu} = 0$ on $\partial B_R$. Now, Green's representation formula (cf. [3, Theorem 2.1]),

$$\begin{aligned} u(x) = &\int_{\partial B_R} \left\{ \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y) - u(y)\, \frac{\partial \Phi(x,y)}{\partial \nu(y)}(x,y) \right\} \, \mathrm{d}s(y) \\ &- \int_{B_R} (\Delta u(y) + k^2 u(y))\, \Phi(x,y) \, \mathrm{d}y, \quad x \in B_R, \end{aligned}$$

together with the differential equation (2.6a) implies that $u$ is a solution of the homogeneous Lippmann-Schwinger equation. Thus, we also have $u = 0$ in $B_R$.

Since $u$ defined as in (2.5) is a superposition of the solutions $w_n(\cdot, y)$ to $\Delta_x w_n + k^2 n w_n = 0$, $u$ itself also satisfies this differential equation in $\mathbb{R}^3 \setminus \partial B_R$. Moreover, $u$ is a radiating solution. Finally, we can conclude from the regularity properties of the single-layer potential with kernel $\Phi$ (cf. [3, Section 3.1]) that $u$ as defined in (2.5) satisfies the boundary conditions. Hence it is the solution of (2.6). ☐

Our final lemma of this section states a series representation of $w_n^s$ and of $u_n^\infty$ in terms of spherical harmonics. For convenience we want to use the following notation:

$$\mathcal{M} := \{(l_1, m_1, l_2, m_2) \in \mathbb{N}_0 \times \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{Z} : |m_1| \le l_1 \text{ and } |m_2| \le l_2\}$$

and $Y_l^m$, $l = 0, 1, 2, \ldots$, $-l \le m \le l$, are a basis of the spherical harmonics. $h_l^{(1)}$ denotes the spherical Hankel of the first kind and of order $l$ (cf., e.g., [3, Section 2.4]).

LEMMA 2.3.

*(a) For* $|x|$, $|y| > 1$ *the scattered part of the Green function can be represented as*

$$\begin{aligned} w_n^s(x,y) = -\frac{k^2}{4\pi} \sum_{(l_1,m_1,l_2,m_2) \in \mathcal{M}} & i^{l_1 - l_2}\, \alpha_{l_1,m_1,l_2,m_2}\, h_{l_1}^{(1)}(k|x|)\, h_{l_2}^{(1)}(k|y|) \times \\ & \times Y_{l_1}^{m_1}\left(\frac{x}{|x|}\right) Y_{l_2}^{m_2}\left(\frac{y}{|y|}\right) \end{aligned}$$

*with suitable constants* $\alpha_{l_1,m_1,l_2,m_2}$*. The series converges absolutely and uniformly on compact subsets of* $\mathbb{R}^3 \setminus \overline{B_1}$ *and can be differentiated termwise.*

*(b) The far field* $u_n^\infty(\hat{x}, d)$ *has the expansion*

$$u_n^\infty(\hat{x}, d) = \sum_{(l_1,m_1,l_2,m_2) \in \mathcal{M}} \alpha_{l_1,m_1,l_2,m_2}\, Y_{l_1}^{m_1}(\hat{x})\, Y_{l_2}^{m_2}(d),$$

*where the coefficients* $\alpha_{l_1,m_1,l_2,m_2}$ *are the same as in part (a).*

For a proof we refer to Stefanov [17, Proposition 2.2].

**3. Estimating the refractive index with the help of near field data (Proof of Theorem 1.1).** The main idea for the stability estimate is to estimate the Fourier coefficients in the Fourier expansion of $n - \tilde{n}$ with the help of special solutions to eq. (1.1) which depend in a particular way on a complex parameter vector $\zeta \in \mathbb{C}^3$. We call these solutions *geometrical optics solutions.*

LEMMA 3.1. *Suppose $k > 0$, $1 < R'$, a positive constant $C_n$, and a refractive index $n$ with $\|1-n\|_{H^s} \le C_n$ are given. Then there are constants $M_1, M_2 > 0$, depending only on $s$, $k$, $R'$, and $C_n$ with the following property: For all $\zeta \in \mathbb{C}^3$ satisfying*

$$|\operatorname{Im}(\zeta)| \ge M_1 \quad \textit{and} \quad \zeta \cdot \zeta = k^2$$

*there exists a function $v(\cdot\,,\zeta) \in C^2(B_{R'})$ such that*

$$U(x,\zeta) := e^{i\zeta\cdot x}(1 + v(x,\zeta)), \quad x \in B_{R'}, \tag{3.1}$$

*is a solution to $\Delta u + k^2 n u = 0$ in $B_{R'}$, and the estimate*

$$\|v(\cdot\,,\zeta)\|_{L^2(B_{R'})} \le \frac{M_2}{|\operatorname{Im}(\zeta)|} \tag{3.2}$$

*holds.*

These solutions have a long history and were employed by different authors dealing with inverse scattering problems. We refer the reader to [15] for a brief overview. A proof of the above lemma for the equation $\Delta u + qu = 0$ can be found in [18]. For our case we refer the reader to Lemma 5 in [6].

We have

$$n(x) - \tilde{n}(x) = \sum_{\gamma\in\mathbb{Z}^3} \widehat{(n-\tilde{n})}(\gamma)\, \frac{1}{(2\pi)^{3/2}}\, \exp(i\gamma\cdot x), \quad x \in B_1, \tag{3.3}$$

where $\hat{f}(\gamma)$ denotes the Fourier coefficients of a function $f \in L^2((-\pi,\pi)^3)$ with respect to the orthonormal bases $(2\pi)^{-3/2}\exp(i\gamma\cdot x)$, $x \in (-\pi,\pi)^3$, $\gamma \in \mathbb{Z}^3$. We first bound the sum (3.3) over those multiindices $\gamma$ whose norm is larger than a given constant $\rho \ge 2$. It is immediately seen that

$$\begin{aligned}\sum_{|\gamma|>\rho} |\widehat{(n-\tilde{n})}(\gamma)|^2 &\le \frac{1}{(1+\rho^2)^s} \sum_{|\gamma|>\rho} (1+\gamma\cdot\gamma)^s |\widehat{(n-\tilde{n})}(\gamma)|^2 \\ &\le \frac{c^2}{\rho^{2s}},\end{aligned} \tag{3.4}$$

because the second factor can be bounded with the help of $\|1-n\|_{H^s} \le C_n$, $\|1-\tilde{n}\|_{H^s} \le C_n$. For the estimate of $\|n-\tilde{n}\|_\infty$ we use the Cauchy-Schwarz inequality and obtain

$$\begin{aligned}&\sum_{|\gamma|>\rho} |\widehat{(n-\tilde{n})}(\gamma)| \\ &\le \Big(\sum_{|\gamma|>\rho} (1+\gamma\cdot\gamma)^s |\widehat{(n-\tilde{n})}(\gamma)|^2\Big)^{1/2} \Big(\sum_{|\gamma|>\rho} \frac{1}{(1+\gamma\cdot\gamma)^s}\Big)^{1/2} \\ &\le \frac{c}{\rho^{s-3/2}}.\end{aligned} \tag{3.5}$$

Here, for the second factor we have employed the inequality

$$\sum_{|\gamma|>\rho} \frac{1}{(1+\gamma\cdot\gamma)^s} \le \frac{c}{\rho^{2s-3}}.$$

Note, that $c$ may denote different constants in the sequel.

Next, we study the remaining sum of (3.3) over those multiindices $\gamma$ with $|\gamma| \le \rho$. To this end we need the following lemma which together with the geometrical optics solutions is the main tool to establish a relation between the Fourier coefficients and the operator $S_n - S_{\tilde n}$.

LEMMA 3.2. *Assume $1 < R < R'$. Moreover, $n$, $\tilde n$ are refractive indices with supp$(1-n)$, supp$(1-\tilde n) \subset B_1$. Then, there exists a positive constant $M_3$ (depending only on $k$, $R$, and $R'$) such that for all solutions $u \in C^2(B_{R'}) \cap L^2(B_{R'})$ to $\Delta u + k^2 n u = 0$ and all solutions $\tilde u \in C^2(B_{R'}) \cap L^2(B_{R'})$ to $\Delta \tilde u + k^2 \tilde n \tilde u = 0$ in $B_{R'}$ the estimate*

$$\Big| \int_{B_1} (n - \tilde n)\, u\, \tilde u \,\mathrm{d}x \Big| \le M_3 \, \|S_n - S_{\tilde n}\|_{L^2(\partial B_R)} \, \|u\|_{L^2(B_{R'})} \, \|\tilde u\|_{L^2(B_{R'})} \tag{3.6}$$

*holds.*

*Proof.* Given $u$ as in the lemma we first define a function $v$ by extending $u|_{\overline{B_R}}$ continuously as a radiating solution to the Helmholtz equation in the exterior of $B_R$, i.e., we set $v|_{\overline{B_R}} = u|_{\overline{B_R}}$ in $B_R$ and $v|_{\mathbb{R}^3 \setminus B_R}$ to be the radiating solution to the Helmholtz equation $\Delta v + k^2 v = 0$ in $\mathbb{R}^3 \setminus \overline{B_R}$ with Dirichlet data $v|_{\partial B_R} = u|_{\partial B_R}$. From Lemma 2.2 we obtain

$$v = S_n\Big(\frac{\partial v_-}{\partial \nu} - \frac{\partial v_+}{\partial \nu}\Big) \quad \text{on } \partial B_R. \tag{3.7}$$

Proceeding analogously with $\tilde u$ to define a function $\tilde v$ and establishing the analog of relation (3.7) for $\tilde v$, we use Green's second integral theorem to compute

$$\begin{aligned}
&\int_{\partial B_R} \Big(\frac{\partial v_-}{\partial \nu} - \frac{\partial v_+}{\partial \nu}\Big)(S_n - S_{\tilde n})\Big(\frac{\partial \tilde v_-}{\partial \nu} - \frac{\partial \tilde v_+}{\partial \nu}\Big) \,\mathrm{d}s \\
&= \int_{\partial B_R} v \Big(\frac{\partial \tilde v_-}{\partial \nu} - \frac{\partial \tilde v_+}{\partial \nu}\Big) \,\mathrm{d}s - \int_{\partial B_R} \tilde v \Big(\frac{\partial v_-}{\partial \nu} - \frac{\partial v_+}{\partial \nu}\Big) \,\mathrm{d}s \\
&= k^2 \int_{B_1} (n - \tilde n)\, u\, \tilde u \,\mathrm{d}x,
\end{aligned}$$

and therefore

$$\begin{aligned}
&\Big| \int_{B_1} (n - \tilde n)\, u\, \tilde u \,\mathrm{d}x \Big| \\
&\le \frac{1}{k^2} \, \|S_n - S_{\tilde n}\|_{L^2(\partial B_R)} \Big\| \frac{\partial v_-}{\partial \nu} - \frac{\partial v_+}{\partial \nu} \Big\|_{L^2(\partial B_R)} \Big\| \frac{\partial \tilde v_-}{\partial \nu} - \frac{\partial \tilde v_+}{\partial \nu} \Big\|_{L^2(\partial B_R)}.
\end{aligned} \tag{3.8}$$

In order to bound the norms of the normal derivatives in (3.8) we apply standard regularity results (Weyl's lemma) for the Helmholtz equation in the shells

$$\Gamma_1 := \Big\{ x \in \mathbb{R}^3 : \frac{1+R}{2} < |x| < \frac{R'+R}{2} \Big\} \subset \Gamma_2 := \{ x \in \mathbb{R}^3 : 1 < |x| < R' \}$$

to obtain $\|u\|_{C^2(\Gamma_1)} \le c\|u\|_{L^2(\Gamma_2)} \le c\|u\|_{L^2(B_{R'})}$. Hence, $\|\partial v_- / \partial \nu\|_{L^2(\partial B_R)}$ can be bounded by $\|u\|_{L^2(B_{R'})}$. Regularity properties for the solution to the exterior boundary value problem for the Helmholtz equation with smooth boundary data $u|_{\partial B_R}$ imply that $\|\partial v_+ / \partial \nu\|_{L^2(\partial B_R)}$ can also be bounded by $\|u\|_{L^2(B_{R'})}$. The same reasoning applies to $\tilde v$. Now, the lemma follows from inequality (3.8). ◻

We are now in a position to estimate the Fourier coefficients for the multiindices $|\gamma| \le \rho$.

LEMMA 3.3. *Assume $\rho \ge 2$ and $C_n$ are positive constants, and $n$, $\tilde{n}$ are refractive indices satisfying $\|1-n\|_{H^s}, \|1-\tilde{n}\|_{H^s} \le C_n$ and supp$(1-n)$, supp$(1-\tilde{n}) \subset B_1$. Moreover, suppose $1 < R$ and define $t_0 := \sqrt{M_1^2+k^2}$ ($M_1$ being the constant from Lemma 3.1 for $R' = 2R$). Then, there exists a positive constant $M_4$ (depending only on $s$, $k$, $R$, and $C_n$) such that for all $\gamma \in \mathbb{Z}^3$ with $|\gamma| \le \rho$ and all $t > t_0$ the estimate*

$$\left|\widehat{(n-\tilde{n})}(\gamma)\right| \le M_4 \left( e^{4R(t+\rho)} \|S_n - S_{\tilde{n}}\|_{L^2(\partial B_R)} + \frac{1}{t} \right) \tag{3.9}$$

*holds true.*

*Proof.* For $t > t_0 > k$ and a fixed multiindex $\gamma \in \mathbb{Z}^3$ we choose unit vectors $d_1$, $d_2 \in \mathbb{R}^3$ such that $d_1 \cdot d_2 = d_1 \cdot \gamma = d_2 \cdot \gamma = 0$ and define the complex vectors

$$\zeta_t := -\frac{1}{2}\,\gamma + i\sqrt{t^2 - k^2 + \frac{|\gamma|^2}{4}}\; d_1 + t\, d_2 \in \mathbb{C}^3, \tag{3.10}$$

$$\tilde{\zeta}_t := -\frac{1}{2}\,\gamma - i\sqrt{t^2 - k^2 + \frac{|\gamma|^2}{4}}\; d_1 - t\, d_2 \;\in \mathbb{C}^3. \tag{3.11}$$

Then, the relations $\zeta_t + \tilde{\zeta}_t = -\gamma$, $\zeta_t \cdot \zeta_t = \tilde{\zeta}_t \cdot \tilde{\zeta}_t = k^2$ are satisfied. Furthermore $t > t_0 = \sqrt{M_1^2+k^2}$ implies $|\operatorname{Im}(\zeta_t)|, |\operatorname{Im}(\tilde{\zeta}_t)| \ge M_1$. Therefore, according to Lemma 3.1 there exist the geometrical optics solutions

$$U(x,\zeta_t) = e^{i\zeta_t \cdot x}(1 + v(x,\zeta_t)), \quad x \in B_{2R},$$

to $\Delta u + k^2 n u = 0$ and

$$\tilde{U}(x,\tilde{\zeta}_t) = e^{i\tilde{\zeta}_t \cdot x}(1 + \tilde{v}(x,\tilde{\zeta}_t)), \quad x \in B_{2R},$$

to $\Delta\tilde{u} + k^2\tilde{n}\tilde{u} = 0$. Using Lemma 3.1 once more we infer

$$U(x,\zeta_t)\,\tilde{U}(x,\tilde{\zeta}_t) = e^{-i\gamma\cdot x}(1 + p(x,t)),$$

where the $L^1$ norm of the remainder

$$p(x,t) = v(x,\zeta_t) + \tilde{v}(x,\tilde{\zeta}_t) + v(x,\tilde{\zeta}_t)\,\tilde{v}(x,\tilde{\zeta}_t)$$

is bounded by

$$\int_{B_1} |p(x,t)|\ \mathrm{d}x \le \frac{c}{t} \tag{3.12}$$

with a suitable constant $c$. Then, we compute

$$\begin{aligned}
&\left|\widehat{(n-\tilde{n})}(\gamma)\right| \\
&= \frac{1}{(2\pi)^{3/2}} \left| \int_{B_1} (n-\tilde{n})(x)\, e^{-i\gamma\cdot x}\ \mathrm{d}x \right| \\
&= \frac{1}{(2\pi)^{3/2}} \left| \int_{B_1} (n-\tilde{n})(x)\, U(x,\zeta_t)\, \tilde{U}(x,\tilde{\zeta}_t)\ \mathrm{d}x \right. \\
&\qquad\qquad \left. - \int_{B_1} (n-\tilde{n})(x)\, e^{-i\gamma\cdot x}\, p(x,t)\ \mathrm{d}x \right| \\
&\le c'\, \|S_n - S_{\tilde{n}}\|_{L^2(\partial B_R)}\, \|U(\cdot,\zeta_t)\|_{L^2(B_{2R})}\, \|\tilde{U}(\cdot,\tilde{\zeta}_t)\|_{L^2(B_{2R})} + \frac{c''}{t} \\
&\le M_4 \left( e^{4R(t+\rho)} \|S_n - S_{\tilde{n}}\|_{L^2(\partial B_R)} + \frac{1}{t} \right),
\end{aligned} \tag{3.13}$$

where we have used the estimates (3.6) and (3.12) for the first inequality. For the last line in (3.13) we note that

$$\begin{aligned}\|U(\cdot,\zeta_t)\|_{L^2(B_{2R})} &= \|e^{i\zeta_t\cdot x}\,(1+v(\cdot,\zeta_t))\|_{L^2(B_{2R})}\\ &\le \|e^{i\zeta_t\cdot x}\|_{\infty,B_{2R}}\|1+v(\cdot,\zeta_t)\|_{L^2(B_{2R})}\\ &\le c''' e^{2R(t+\rho)}\end{aligned}$$

for all $t \ge t_0$ and all $\gamma \in \mathbb{Z}^3$ with $|\gamma| \le \rho$, because of $|\operatorname{Im}(\zeta_t)| \le t + |\gamma| \le t + \rho$. Together with the analogous estimate for $\|\tilde{U}(\cdot,\tilde{\zeta}_t)\|_{L^2(B_{2R})}$ we obtain (3.13), i.e., we have proved inequality (3.9). □

Finally, we can prove Theorem 1.1 by choosing the parameters $\rho$ and $t$ in (3.4), (3.5), and (3.9) appropriately:

*Proof. (Theorem 1.1)* In view of the Fourier expansion (3.3) for $n-\tilde{n}$, Parseval's equation, the inequalities (3.4) and (3.9), and the fact that there are less than $c\rho^3$ multiindices $\gamma \in \mathbb{Z}^3$ with $|\gamma| \le \rho$ we have

$$\begin{aligned}\|n-\tilde{n}\|^2_{L^2(B_1)} &\le \frac{c^2}{\rho^{2s}} + \left(\sum_{|\gamma|\le\rho} |\widehat{(n-\tilde{n})}(\gamma)|\right)^2\\ &\le \frac{c^2}{\rho^{2s}} + \left(c\rho^3\left[e^{4R(t+\rho)}\|S_n - S_{\tilde{n}}\|_{L^2(\partial B_R)} + \frac{1}{t}\right]\right)^2\\ &\le c\left(e^{(4R+1)(t+\rho)}\|S_n - S_{\tilde{n}}\|_{L^2(\partial B_R)} + \frac{\rho^3}{t} + \frac{1}{\rho^s}\right)^2, \qquad (3.14)\end{aligned}$$

where we have also used $\rho^3 e^{4R(t+\rho)} \le e^{(4R+1)(t+\rho)}$ (due to $\rho^3 \le 6e^\rho$) in the last line. Note that the value of the constants $c$ will change during the proof. For the maximum norm, in view of (3.3) and (3.5) we compute

$$\begin{aligned}\|n-\tilde{n}\|_\infty &\le c\sum_{|\gamma|\le\rho} |\widehat{(n-\tilde{n})}(\gamma)| + \frac{c}{\rho^{s-3/2}}\\ &\le c\rho^3\left(e^{4R(t+\rho)}\|S_n - S_{\tilde{n}}\|_{L^2(\partial B_R)} + \frac{1}{t}\right) + \frac{c}{\rho^{s-3/2}}\\ &\le c\left(e^{(4R+1)(t+\rho)}\|S_n - S_{\tilde{n}}\|_{L^2(\partial B_R)} + \frac{\rho^3}{t} + \frac{1}{\rho^{s-3/2}}\right). \qquad (3.15)\end{aligned}$$

For $t > t_0 + 2^{s+3}$ ($t_0$ from Lemma 3.3) and $\rho := t^{1/(s+3)}$ the condition $\rho \ge 2$ is satisfied and inequality (3.14) becomes

$$\|n-\tilde{n}\|_{L^2(B_1)} \le c\left(e^{(8R+2)t}\|w_n^{\mathrm{s}} - w_{\tilde{n}}^{\mathrm{s}}\|_{L^2(\partial B_R\times\partial B_R)} + \frac{2}{t^{s/(s+3)}}\right) \qquad (3.16)$$

because of $\rho = t^{1/(s+3)} \le t$ and $\|S_n - S_{\tilde{n}}\|_{L^2(\partial B_R)} \le \|w_n^{\mathrm{s}} - w_{\tilde{n}}^{\mathrm{s}}\|_{L^2(\partial B_R\times\partial B_R)}$.

We may restrict our attention to the case that $\|w_n^{\mathrm{s}} - w_{\tilde{n}}^{\mathrm{s}}\|_{L^2}$ is sufficiently small such that

$$t := -\frac{1}{s+3}\,\frac{3}{8R+2}\,\ln\left(\|w_n^{\mathrm{s}} - w_{\tilde{n}}^{\mathrm{s}}\|_{L^2}\right) > t_0 + 2^{s+3}.$$

Otherwise, the bound on $\|n-\tilde{n}\|_\infty$ from the a-priori information about $n$ and $\tilde{n}$ and the monotonicity of $[-\ln^-(t)]^{-p}$, $0<p<1$, imply

$$\|n-\tilde{n}\|_{L^2(B_1)} \le c\,\frac{[-\ln^-(\|w_n^{\mathrm{s}} - w_{\tilde{n}}^{\mathrm{s}}\|_{L^2})]^{-p}}{[(t_0+2^{s+3})\,\frac{s+3}{3}\,(8R+2)]^{-p}}.$$

Inserting $t$ into (3.16) yields the inequality

$$\begin{aligned}\|n-\tilde{n}\|_{L^2(B_1)} &\leq c\left\{\|w_n^{\mathrm{s}}-w_{\tilde{n}}^{\mathrm{s}}\|_{L^2}^{s/(s+3)}+(-\ln\|w_n^{\mathrm{s}}-w_{\tilde{n}}^{\mathrm{s}}\|_{L^2})^{-s/(s+3)}\right\}\\ &\leq c\,[-\ln(\|w_n^{\mathrm{s}}-w_{\tilde{n}}^{\mathrm{s}}\|_{L^2})]^{-s/(s+3)}\end{aligned}$$

because $x \leq (-\ln(x))^{-1}$ for $0 < x < 1$.

The analogous analysis with $\rho := t^{2/(2s+3)}$ and

$$t := -\frac{1}{2s+3}\,\frac{3}{4R+1}\,\ln(\|w_n^{\mathrm{s}}-w_{\tilde{n}}^{\mathrm{s}}\|_{L^2}) > t_0 + 2^{s+3}.$$

reveals

$$\begin{aligned}\|n-\tilde{n}\|_\infty &\leq c\left\{\|w_n^{\mathrm{s}}-w_{\tilde{n}}^{\mathrm{s}}\|_{L^2}^{(2s-3)/(2s+3)}+(-\ln\|w_n^{\mathrm{s}}-w_{\tilde{n}}^{\mathrm{s}}\|_{L^2})^{-(2s-3)/(2s+3)}\right\}\\ &\leq c\,[-\ln(\|w_n^{\mathrm{s}}-w_{\tilde{n}}^{\mathrm{s}}\|_{L^2})]^{-(2s-3)/(2s+3)}\,.\end{aligned}$$

This finishes the proof. ☐

**4. Estimating the refractive index with the help of far field data (Proof of Theorem 1.2).** Our proof of Theorem 1.2 is based on the following abstract linear stability result. It has been proved in [11, 14] for bounded operators. For the convenience of the reader we include the short proof for compact operators which is sufficient for our purposes.

LEMMA 4.1. *Let $X, Y$ be Hilbert spaces and $T : X \to Y$ a compact linear operator. Moreover, assume that $f \in C([0, \|T^*T\|])$ is monotonically increasing with $f(0) = 0$ and that the function*

$$\phi_f : [0, f(\|T^*T\|)^2] \to [0, \|T^*T\| f(\|T^*T\|)^2]$$

*defined by $\phi_f(\xi) := \xi \cdot (f \cdot f)^{-1}(\xi)$ is convex. Then, the source condition*

$$w = f(T^*T)v, \qquad \|v\| \leq \rho \tag{4.1}$$

*implies the stability estimate*

$$\|w\|^2 \leq \rho^2 \phi_f^{-1}\left(\frac{\|Tw\|^2}{\rho^2}\right). \tag{4.2}$$

*Proof.* We can assume that $x_\mu$, $\mu \in \mathcal{M}$, is a Hilbert basis of $X$ with $T^*Tx_\mu = \lambda_\mu x_\mu$, and $\lambda_\mu \geq 0$ for all $\mu \in \mathcal{M}$. Replacing $w$ by $(1/\rho)w$ we see that we can also assume $\rho = 1$. If $v = \sum_{\mu\in\mathcal{M}} \alpha_\mu x_\mu$, then $w = \sum_{\mu\in\mathcal{M}} f(\lambda_\mu)\alpha_\mu x_\mu$. We first assume that $\|v\|^2 = \sum_{\mu\in\mathcal{M}} |\alpha_\mu|^2 = 1$. Then Jensen's inequality yields

$$\begin{aligned}\phi_f(\|w\|^2) = \phi_f\left(\sum_{\mu\in\mathcal{M}}(f(\lambda_\mu))^2|\alpha_\mu|^2\right) &\leq \sum_{\mu\in\mathcal{M}}\phi_f((f\cdot f)(\lambda_\mu))\,|\alpha_\mu|^2\\ &= \sum_{\mu\in\mathcal{M}}(f\cdot f)(\lambda_\mu)\,\lambda_\mu\,|\alpha_\mu|^2 = \|Tw\|^2.\end{aligned}$$

Since $\phi_f$ is convex and $\phi_f(0) = 0$, this inequality is also valid for $\|v\| < 1$. Applying $\phi_f^{-1}$ to both sides of the inequality yields the assertion since $\phi_f^{-1}$ is increasing under the given assumptions. ☐

We want to apply this result to the operator $T$ which maps the difference $w_n - w_{\tilde{n}} = w_n^{\mathrm{s}} - w_{\tilde{n}}^{\mathrm{s}}$ of two Green functions on $\partial B_{2R} \times \partial B_{2R}$ to the corresponding difference of far field patterns $u_n^\infty - u_{\tilde{n}}^\infty$. For the function $f$ we choose $f = g_\theta$, $0 < \theta < 1$, defined by

$$g_\theta(\lambda) := \begin{cases} \exp(-\frac{1}{2}(-\ln\lambda)^\theta), & 0 < \lambda \le \exp(-1), \\ 0, & \lambda = 0. \end{cases} \tag{4.3}$$

The corresponding functions $\phi_\theta := \phi_{g_\theta}$ and their second derivatives are given by

$$\begin{aligned} \phi_\theta(\xi) &= \xi \exp(-(-\ln\xi)^{\frac{1}{\theta}}), \qquad 0 < \xi \le \exp(-1), \\ \phi_\theta''(\xi) &= \exp\left(-(-\ln\xi)^{\frac{1}{\theta}}\right)\left(\frac{(-\ln\xi)^{\frac{1}{\theta}-2}}{\theta\xi}\right)\left(1 - \ln\xi + \frac{(-\ln\xi)^{\frac{1}{\theta}} - 1}{\theta}\right). \end{aligned}$$

It is obvious that the functions $g_\theta$ are continuous and monotonically increasing and that $\phi_\theta''(\xi) > 0$ for $0 < \xi < \exp(-1)$ and $0 < \theta < 1$.

With the help of the orthonormal basis

$$Y_{r,\mu}(x,y) = \frac{1}{r^2}\, Y_{l_1}^{m_1}\left(\frac{x}{|x|}\right) Y_{l_2}^{m_2}\left(\frac{y}{|y|}\right), \quad \mu = (l_1, m_1, l_2, m_2) \in \mathcal{M},$$

of $L^2(\partial B_r \times \partial B_r)$ and the basis $Y_\mu := Y_{1,\mu}$ in $L^2(S^2 \times S^2)$ we infer from the series expansions in Lemma 2.3 that $T$ defined by

$$Y_{2R,\mu} \mapsto -\frac{\pi}{k^2R^2}\, i^{l_2 - l_1}\, \frac{1}{h_{l_1}^{(1)}(2kR) h_{l_2}^{(1)}(2kR)}\, Y_\mu \tag{4.4}$$

maps $(w_n - w_{\tilde{n}})|_{\partial B_{2R}\times\partial B_{2R}}$ to $u_n^\infty - u_{\tilde{n}}^\infty$. Because $g_\theta$ is only defined on $[0, \exp(-1)]$ we introduce the scaling factor $\omega := \|T\| \exp(1/2)$ and use the operator $\omega^{-1}T$. In order to apply Lemma 4.1 we must check that a function $w_n - w_{\tilde{n}}$ from which it is a-priori known that it originates from refractive indices $n, \tilde{n}$ such that $\|1-n\|_{H^s}, \|1-\tilde{n}\|_{H^s} \le C_n$, $\operatorname{supp}(1-n)$, $\operatorname{supp}(1-\tilde{n}) \subset B_1$, satisfies the source condition, i.e., there is a constant $\rho > 0$ such that $(w_n - w_{\tilde{n}})|_{\partial B_{2R}\times\partial B_{2R}} = g_\theta(\omega^{-2}T^*T)v$ for an appropriate $v \in L^2(\partial B_{2R} \times \partial B_{2R})$ with $\|v\| \le \rho$. We do this in the following lemma.

LEMMA 4.2. *Let $1 < R$, $C_n > 0$ and $0 < \theta < 1$ be given. Furthermore define $\omega := \|T\| \exp(1/2)$ where $T$ denotes the operator from (4.4). Then there exists a constant $\rho > 0$ such that all $(w_n - w_{\tilde{n}})|_{\partial B_{2R}\times\partial B_{2R}}$ originating from refractive indices $n$, $\tilde{n}$ such that $\|1-n\|_{H^s}, \|1-\tilde{n}\|_{H^s} \le C_n$, supp$(1-n)$, supp$(1-\tilde{n}) \subset B_1$, satisfy the source condition $(w_n - w_{\tilde{n}})|_{\partial B_{2R}\times\partial B_{2R}} = g_\theta(\omega^{-2}T^*T)v$ for an appropriate $v \in L^2(\partial B_{2R} \times \partial B_{2R})$ with $\|v\| \le \rho$.*

*Proof.* We know from Lemma 2.3 that

$$(w_n - w_{\tilde{n}})(r\hat{x}, r\hat{y}) = \sum_{\mu\in\mathcal{M}} \alpha_\mu\, r^2\, h_{l_1}^{(1)}(kr)\, h_{l_2}^{(1)}(kr)\, Y_{r,\mu}(r\hat{x}, r\hat{y})$$

for all $r \ge R$ with suitable coefficients $\alpha_\mu$. For convenience we set

$$\gamma_\mu := -\pi(kR\, i^{l_1} h_{l_1}^{(1)}(2kR))^{-1}(kR(-i)^{l_2}\, h_{l_2}^{(1)}(2kR))^{-1}$$

and

$$\delta_\mu = 4\, \frac{h_{l_1}^{(1)}(2kR)}{h_{l_1}^{(1)}(kR)}\, \frac{h_{l_2}^{(1)}(2kR)}{h_{l_2}^{(1)}(kR)}.$$

Note that $|\gamma_\mu|$ are the singular values of $T$. If we can verify that the function

$$v := \sum_{\mu\in\mathcal{M}} \alpha_\mu \, R^2 \, h_{l_1}^{(1)}(kR)\, h_{l_2}^{(1)}(kR)\, \frac{\delta_\mu}{g_\theta(|\gamma_\mu|^2/\omega^2)}\, Y_{2R,\mu}$$

belongs to $L^2(\partial B_{2R}\times\partial B_{2R})$ and $\|v\| \le \rho$, a straightforward computation shows $g_\theta(\omega^{-2}T^*T)v = w_n - w_{\tilde n}$ on $\partial B_{2R}\times\partial B_{2R}$ and we have proved the source condition.

Since we know from Lemma 2.1 (b) that

$$\sum_{\mu\in\mathcal{M}} \left|\alpha_\mu \, R^2 \, h_{l_1}^{(1)}(kR)\, h_{l_2}^{(1)}(kR)\right|^2 = \|(w_n - w_{\tilde n})|_{\partial B_R\times\partial B_R}\|^2 \le c^2$$

for all refractive indices satisfying the assumptions, it suffices to prove

$$\sup_{\mu\in\mathcal{M}} \frac{|\delta_\mu|}{g_\theta(|\gamma_\mu/\omega|^2)} \le c, \tag{4.5}$$

in other words that the supremum in (4.5) is finite. ($c$ denotes various constants during the proof).

To see this, we use the asymptotic formulae

$$\begin{aligned} \frac{1}{c}\left(\frac{l}{kRe}\right)^l &\le |h_l^{(1)}(2kR)| \le c\left(\frac{l}{kRe}\right)^l, \qquad l\to\infty, \\ \frac{h_l^{(1)}(2kR)}{h_l^{(1)}(kR)} &= \frac{1}{2^{l+1}}\left(1+\mathrm{O}\left(\frac{1}{l}\right)\right), \qquad l\to\infty, \end{aligned} \tag{4.6}$$

which follow from the series representation of the spherical Hankel functions (cf. [3]). Therefore, we have

$$|\delta_\mu| \le c\, 2^{-l_1-l_2}. \tag{4.7}$$

Using the monotonicity of $g_\theta$, eq. (4.6), and the inequality $(a+b)^\theta \le a^\theta + b^\theta$, which holds for $a, b \ge 0$ and $0\le\theta\le 1$ (for a proof consider the case $a+b=1$), we obtain the estimate

$$\begin{aligned} g_\theta(|\gamma_\mu/\omega|^2)^{-2} &\le g_\theta\left(c^2\left(\frac{ekR}{l_1}\right)^{2l_1}\left(\frac{ekR}{l_2}\right)^{2l_2}\right)^{-2} \\ &= \exp\left(\left\{\ln\left(c^2\left(\frac{l_1}{ekR}\right)^{2l_1}\left(\frac{l_2}{ekR}\right)^{2l_2}\right)\right\}^\theta\right) \\ &\le \exp\left(\left\{\ln c + 2l_1 \ln\frac{l_1}{ekR}\right\}^\theta + \left\{\ln c + 2l_2\ln\frac{l_2}{ekR}\right\}^\theta\right). \end{aligned}$$

Since $\lim_{l\to\infty}\left(\{\ln c + 2l\ln(l/(ekR))\}^\theta - 2l\ln 2\right) = -\infty$, it follows from (4.7) that

$$\frac{|\delta_\mu|^2}{g_\theta(|\gamma_\mu/\omega|^2)^2} \le c$$

for all $\mu\in\mathcal{M}$. This shows (4.5) and finishes the proof. □

*Proof.* *(Theorem 1.2)* Lemma 4.2 yields

$$(w_n - w_{\tilde{n}})|_{\partial B_{2R}\times\partial B_{2R}} = g_\theta(\omega^{-2}T^*T)v, \quad \|v\| \le \rho.$$

Hence, using $T((w_n - w_{\tilde{n}})|_{\partial B_{2R}\times\partial B_{2R}}) = u_n^\infty - u_{\tilde{n}}^\infty$, according to Lemma 4.1 we have

$$\begin{aligned}\|(w_n - w_{\tilde{n}})|_{\partial B_{2R}\times\partial B_{2R}}\|^2 &\le \rho^2\phi_\theta^{-1}\left(\frac{\|T((w_n - w_{\tilde{n}})|_{\partial B_{2R}\times\partial B_{2R}})\|^2}{\omega^2\rho^2}\right)\\ &= \rho^2\phi_\theta^{-1}\left(\frac{\|u_n^\infty - u_{\tilde{n}}^\infty\|^2}{\omega^2\rho^2}\right).\end{aligned}$$

To estimate $\phi_\theta^{-1}$, we write $\phi_\theta(\xi) = \exp(-\psi(-\ln\xi))$ with $\psi(x) := x + x^{\frac{1}{\theta}}$. Obviously $\psi(x) \le 2x^{\frac{1}{\theta}}$ for $x \ge 1$, so $\psi^{-1}(y) \ge \left(\frac{y}{2}\right)^\theta$ for $y \ge 2$. It follows that

$$\phi_\theta^{-1}(\lambda) = \exp(-\psi^{-1}(-\ln\lambda)) \le \exp\left(-\left(\frac{-\ln\lambda}{2}\right)^\theta\right)$$

for $0 < \lambda \le \exp(-2)$ and we get

$$\|w_n^{\mathrm{s}} - w_{\tilde{n}}^{\mathrm{s}}\|^2_{L^2(\partial B_{2R}\times\partial B_{2R})} \le \rho^2\exp\left(-\left(-\ln\frac{\|u_n^\infty - u_{\tilde{n}}^\infty\|_{L^2(S^2\times S^2)}}{\omega\rho}\right)^\theta\right)$$

for sufficiently small $\|u_n^\infty - u_{\tilde{n}}^\infty\|_{L^2(S^2\times S^2)}$. Inserting this into (1.3) and choosing $\theta$ such that $\theta\frac{s}{s+3} = \frac{s}{s+3} - \epsilon$ yields the assertion. By adjusting the constant similarly to the proof of Theorem 1.1 this inequality holds for all refractive indices satisfying the assumptions. The estimate for $\|n - \tilde{n}\|_\infty$ is derived analogously. □

**5. Applications.** In this final section we discuss the numerical solution of the inverse problem to recover the refractive index $n(x) = 1 - a(x)$ from measurements of the far field pattern $u^\infty(\hat{x}, d)$ by iterative regularization methods. Introducing the Hilbert spaces $X = H_0^s(B_1)$, $Y = L^2(S^2\times S^2)$ and the operator $F : X \to Y$, $F(a) := u_{1-a}^\infty$, this problem can be formulated as a nonlinear operator equation

$$F(a) = u^\infty. \tag{5.1}$$

Usually the measured data will be contaminated by noise. We use the notation $u^{\infty,\delta}$ with $\|u^{\infty,\delta} - u^\infty\| \le \delta$ for noisy data with noise level $\delta$.

A number of methods have been developed to solve general problems of the form (5.1). Among the most popular are *Landweber iteration*

$$a_{j+1}^\delta := a_j^\delta + \mu F'[a_j^\delta]^*(u^{\infty,\delta} - F(a_j^\delta)) \tag{5.2}$$

($\mu \le \|F'[a]\|^{-1}$ for $a$ in a neighborhood of the exact solution $a^\dagger$), the *Levenberg-Marquardt algorithm*

$$a_{j+1}^\delta := a_j^\delta + (\alpha_j I + F'[a_j^\delta]^*F'[a_j^\delta])^{-1}F'[a_j^\delta]^*(u^{\infty,\delta} - F(a_j^\delta)), \tag{5.3}$$

and the *iteratively regularized Gauß-Newton method* (IRGNM)

$$a_{j+1}^\delta := a_0 + (\alpha_j I + F'[a_j^\delta]^*F'[a_j^\delta])^{-1}F'[a_j^\delta]^*\left(u^{\infty,\delta} - F(a_j^\delta) + F'[a_j^\delta](a_j^\delta - a_0)\right). \tag{5.4}$$

For the last two methods there are different strategies to choose the regularization parameters $\alpha_j$, e.g. $\alpha_j = 2^{-j}\alpha_0$.

An important issue is the choice of the stopping index. The most well-known stopping rule is *Morozov's discrepancy principle* which consists in stopping the iteration at the first index $J = J(\delta, u^{\infty,\delta})$ such that

$$\|F(a_J^\delta) - u^{\infty,\delta}\| \leq \tau\delta \tag{5.5}$$

with some fixed constant $\tau \geq 1$.

We call an iterative method $a_{j+1}^\delta := \Psi_j(a_j, \dots, a_0, u^{\infty,\delta})$ together with a stopping rule $J(\delta, u^{\infty,\delta})$ an *iterative regularization method* for $F$ if for all initial guesses $a_0$ in a neighborhood of $a^\dagger$ the iterates $a_{j+1}^\delta$, $j \leq J(\delta, u^{\infty,\delta})$ are well defined and

$$\lim_{j\to\infty} a_j^0 = a^\dagger, \tag{5.6}$$

$$\lim_{\delta\to 0} \sup_{\|u^{\infty,\delta} - u^\infty\| \leq \delta} \|a^\delta_{J(\delta,u^{\infty,\delta})} - a^\dagger\| = 0. \tag{5.7}$$

It has been shown that under certain conditions restricting the degree of nonlinearity of the operator $F$ Landweber iteration (Hanke, Neubauer and Scherzer [9]), the Levenberg-Marquardt algorithm (Hanke [8]), and the iteratively regularized Gauß-Newton method (Blaschke/Kaltenbacher, Neubauer, Scherzer [2]) together with the discrepancy principle (5.5) are regularization methods in the sense of the definition above. It must be mentioned that none of these nonlinearity conditions has been verified for the inverse acoustic medium scattering problem, yet.

Since it is well-known that the convergence in (5.7) can be *arbitrarily slow* for any ill-posed problem (cf. [5, Proposition 3.11]), one is interested in proving estimates on the convergence rate under additional a-priori assumptions. In the literature assumptions of the form

$$a^\dagger - a_0 = f(F'[a^\dagger]^* F'[a^\dagger])v, \qquad \|v\| \leq \rho \tag{5.8}$$

have been studied intensively. These conditions, called *nonlinear source conditions*, are nonlinear analogues to (4.1). It turns out that for many exponentially ill-posed problems source conditions (5.8) with $f = f_p$ $(p > 0)$

$$f_p(\lambda) := \begin{cases} (-\ln\lambda)^{-p}, & 0 < \lambda \leq \exp(-1) \\ 0, & \lambda = 0 \end{cases} \tag{5.9}$$

(so called *logarithmic source conditions*) have natural interpretations as smoothness conditions in Sobolev spaces (cf. [11, 13]). The usual Hölder-type source conditions (5.8) with $f(\lambda) = \lambda^\mu$, $\mu > 0$ are appropriate for mildly ill-posed problems, but typically far too restrictive for exponentially ill-posed problems. Under the logarithmic source condition (5.8), (5.9) convergence rates of the form

$$\|a^\delta_{J(\delta,u^{\infty,\delta})} - a^\dagger\| = \mathrm{O}\left(f_p(\delta)\right)$$

have been established for the iteratively regularized Gauß-Newton method ([10]) and for Landweber iteration ([4]).

Unfortunately, it seems that logarithmic source conditions for the inverse acoustic medium scattering problem do not have an equivalent formulation with the help of known Sobolev spaces. However, Theorem 1.2 gives us a sufficient condition for

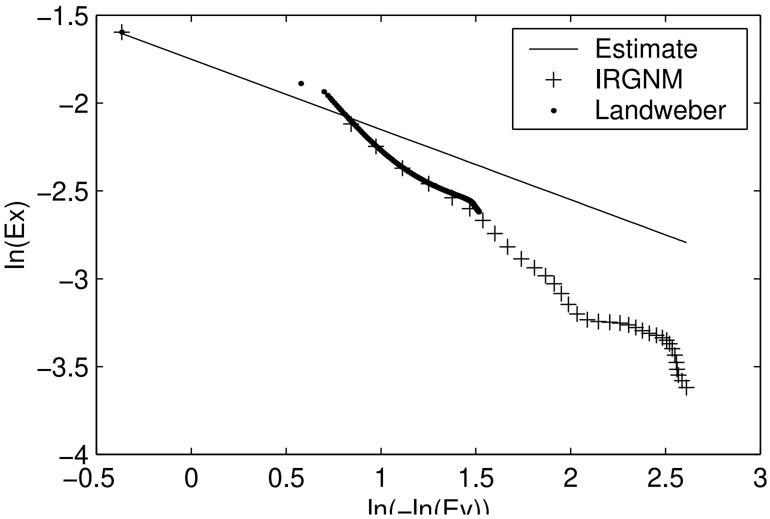


FIG. 5.1. *Plot of* $Ex := \|a_j^0 - a^\dagger\|_{L^2}$ *on a logarithmic scale over* $Ey := \|u^\infty_{1-a_j^0} - u^\infty_{1-a^\dagger}\|_{L^2}$ *on a double logarithmic scale.*

convergence rates under the simple assumption that the exact solution $n^\dagger = 1 - a^\dagger$ and the initial guess $n_0 := 1 - a_0$ belong to $H_0^s(B_1)$:

COROLLARY 5.1. *Assume that* $a^\dagger, a_0 \in H^s(B_1)$. *Let* $a_j^\delta$ *denote the iterates generated by an iterative regularization method for* $F$ *which converges for the initial guess* $a_0$, *and assume that the stopping index* $J$ *is chosen according to (5.5). Then for any* $\epsilon > 0$

$$\|a_j^0 - a^\dagger\|_{L^2(B_1)} = \mathrm{O}\left(-\ln \|u^\infty_{1-a_j^0} - u^\infty_{1-a^\dagger}\|\right)^{-\frac{s}{s+3}+\epsilon}, \qquad j \to \infty \tag{5.10}$$

$$\|a^\delta_{J(\delta,u^{\infty,\delta})} - a^\dagger\|_{L^2(B_1)} = \mathrm{O}\left((-\ln\delta)^{-\frac{s}{s+3}+\epsilon}\right), \qquad \delta \to 0. \tag{5.11}$$

*Proof.* Due to (5.6) we know that there is a constant $C_n$ such that $\|a_j^0\|_{H^s} \le C_n$ for all $j$. Hence, Theorem 1.2 with $n = 1 - a^\dagger$ and $\tilde{n} = 1 - a_j^0$ yields (5.10). Analogously, (5.11) follows from (5.7), (5.5) and Theorem 1.2. □

In Fig. 5.1 the actual values of $\|a_j^0 - a^\dagger\|_{L^2}$ and $\|u^\infty_{1-a_j^0} - u^\infty_{1-a^\dagger}\|_{L^2}$ for the IRGNM and Landweber iteration are compared to the estimate (5.10). We used the Hilbert space $X = H_0^2(B_1)$, i.e., $s = 2$. To implement the IRGNM we solved the equation system involving $\alpha_j I + F'[a_j^\delta]^* F'[a_j^\delta]$ by the conjugate gradient method using a special preconditioner (cf. [12]; our test problem is taken from Section 3 of this paper). We performed 500 Landweber iterations and 20 Newton iterations using exact synthetic data, which were created using a modified algorithm with a much finer discretization in order to avoid an inverse crime. The constant in (5.10) was chosen to fit the data. The picture shows that our estimate is of the right form. The exponent $-\frac{s}{s+3} = -\frac{2}{5}$ is fairly close to our experimental results, but it seems to be a little too pessimistic. In fact, using eq. (3.9) we cannot get an exponent greater than 1 even for a single Fourier coefficient. As long as estimate (3.9) is used we cannot expect to obtain a better stability result than

$$\|n - \tilde{n}\|_{L^2(B_1)} \leq C \left[ -\ln^- \left( \|w_n^s - w_{\tilde{n}}^s\|_{L^2(\partial B_R \times \partial B_R)} \right) \right]^{-1}.$$

To our knowledge it is still open whether there is an example which shows that no better estimate is possible or whether new ideas could improve (3.9) and then the stability estimate.

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