MacMahon's Partition Analysis VI: A New Reduction Algorithm

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Dedicated to the memory of Gian-Carlo Rota

Abstract

In his famous book "Combinatory Analysis" MacMahon introduced Partition Analysis as a computational method for solving combinatorial problems in connection with systems of linear diophantine inequalities and equations. By developing the `Omega` package we have shown that Partition Analysis is ideally suited for being supplemented by computer algebra methods. The object of this paper is to present a significant algorithmic improvement of this package. It overcomes a problem related to the computational treatment of roots of unity. Moreover, this new reduction strategy turns out to be superior to "The Method of Elliott" which is described in MacMahon's book. In order to make this article as self-contained as possible we give a brief introduction to Partition Analysis together with a variety of illustrative examples. For instance, the generating function of magic pentagrams is computed.

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1 Introduction

The key ingredient of MacMahon's Partition Analysis is the Omega operator $\Omega_{\geqq}$.

Definition 1. The operator $\Omega_{\geqq}$ is given by

$$\underset{\geqq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r},$$

where the domain of the $A_{s_1,\ldots,s_r}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to a neighborhood of the circle $|\lambda_i| = 1$. In addition, the $A_{s_1,\ldots,s_r}$ are required to be such that any of the $2^r - 1$ sums

$$\sum_{s_{i_1}=-\infty}^{\infty} \cdots \sum_{s_{i_j}=-\infty}^{\infty} A_{s_1,\ldots,s_r}$$

is absolute convergent within the domain of the definition of $A_{s_1,\ldots,s_r}$.

We emphasize that it is essential to treat everything analytically rather than formally because the method relies on unique Laurent series representations of rational functions.

Another fundamental aspect of Partition Analysis is the use of elimination rules which describe the action of the Omega operator on certain base cases. MacMahon devoted more than eighty pages of his book to Partition Analysis. He starts out by presenting a catalog [6, Vol. II, pp. 102–103] of twelve fundamental evaluations. Subsequently he extends this table by new rules whenever he is enforced to do so. Once found, most of these fundamental rules are easy to prove. This is illustrated by the following example which is taken from MacMahon's list.

Fact 1. *For any integer $s \geq 0$,*

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda x)(1-\frac{y}{\lambda^s})} = \frac{1}{(1-x)(1-x^s y)}.$$

Remark. We want to note that in view of Definition 1 it is sufficient to choose the parameters x and y from a small neighborhood of 0; in all other examples we make similar choices without mentioning it explicitly.

Proof. By geometric series expansion the left hand side equals

$$\underset{\geqq}{\Omega} \sum_{i,j\geq 0} \lambda^{i-sj} x^i y^j = \underset{\geqq}{\Omega} \sum_{j,k\geq 0} \lambda^k x^{sj+k} y^j,$$

where the summation parameter i has been replaced by $sj+k$. But now $\Omega_{\geqq}$ sets λ to 1 which completes the proof. □

We illustrate various aspects of MacMahon's method by elementary examples chosen in the spirit of MacMahon's exposition [6, Vol. II, pp. 102 ff.].

Problem. Find all nonnegative integer solutions a, b to the inequality $2a \geq 3b$.

First of all, using geometric series summation we translate the problem into a form which MacMahon calls the *crude generating function*, namely

$$f(x,y) := \sum_{\substack{a,b\geq 0\\ 2a\geq 3b}} x^a y^b = \underset{\geqq}{\Omega} \sum_{a,b\geq 0} \lambda^{2a-3b} x^a y^b = \underset{\geqq}{\Omega} \frac{1}{(1-x\lambda^2)\left(1-\frac{y}{\lambda^3}\right)}.$$

The next step is the elimination of λ. To this end we could apply the general rule (4) which will be proved in the next section. However here, for illustrative reasons, we prefer to use partial fraction decomposition instead (assuming x and y to be positive real numbers),

$$f(x,y) = \frac{1}{2} \underset{\geqq}{\Omega} \frac{1}{(1-\sqrt{x}\,\lambda)\,(1-\frac{y}{\lambda^3})} + \frac{1}{2} \underset{\geqq}{\Omega} \frac{1}{(1+\sqrt{x}\,\lambda)\,(1-\frac{y}{\lambda^3})}. \tag{1}$$

This decomposition allows to apply Fact 1 to both parts and we obtain that

$$f(x,y) = \frac{1}{2}\,\frac{1}{(1-\sqrt{x})\,(1-\sqrt{x^3}\,y)} + \frac{1}{2}\,\frac{1}{(1+\sqrt{x})\,(1-\sqrt{x^3}\,y)} = \frac{1+x^2\,y}{(1-x)(1-x^3\,y^2)}.$$

By geometric series expansion this gives a parametrized representation of the solution set to our problem; namely, from $f(x,y) = \sum_{\alpha,\beta\geq 0} x^{\alpha+3\beta} y^{2\beta}(1+x^2y)$ we can deduce that

$$\{(a,b) \in \mathbb{N}^2 : 2a \geq 3b\} = \{(m+n+\lceil n/2 \rceil, n) : (m,n) \in \mathbb{N}^2\}.$$

Already this elementary example suggests to use Partition Analysis not only for problems related to a *single* linear diophantine inequality but also with respect to *systems* of linear diophantine inequalities. A corresponding example will be given in Section 3.2.

In addition, MacMahon's method is not restricted to inequalities, it can be also adapted to the case of linear *diophantine equations*. To this end, following MacMahon, one needs to introduce a different Omega operator as follows.

Definition 2. The operator $\Omega_=$ is given by

$$\underset{=}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r}\, \lambda_1^{s_1} \cdots \lambda_r^{s_r} := A_{0,\ldots,0}.$$

This means, all non-trivial power-products in the λ's are killed by the $\Omega_=$-operator.

As already pointed out by MacMahon [6, Vol. 2, Section VIII, p. 104], this operator is related to $\Omega_{\geqq}$ in various ways, for instance,

$$\underset{=}{\Omega}\, F(\lambda) = \underset{\geqq}{\Omega}\, F(\lambda) + \underset{\geqq}{\Omega}\, F(1/\lambda) - F(1).$$

However, for actual computations MacMahon used specific $\Omega_=$ elimination rules; we state one of those explicitly.

Fact 2. *[6, Vol. 2, Section VIII, p. 105]*

$$\underset{=}{\Omega} \frac{1}{(1-\lambda^2\,x)\big(1-\frac{y}{\lambda}\big)\big(1-\frac{z}{\lambda}\big)} = \frac{1+x\,y\,z}{(1-x\,y^2)(1-x\,z^2)}.$$

Again we illustrate the method by an elementary example.

Problem. Find all nonnegative integer solutions a, b, c to the diophantine equation $2a = 3b + c$.

Evidently this problem is equivalent to the previous one. Nevertheless, we find it instructive to solve it via $\Omega_=$ elimination.

As the first step, we again translate the problem into the corresponding crude generating function,

$$\begin{aligned} g(x,y,z) &:= \sum_{\substack{a,b,c\ge 0\\ 2a=3b+c}} x^a\, y^b\, z^c = \underset{=}{\Omega} \sum_{a,b,c\ge 0} \lambda^{2a-3b-c}\, x^a\, y^b\, z^c \\ &= \underset{=}{\Omega} \frac{1}{(1-x\,\lambda^2)\big(1-\frac{y}{\lambda^3}\big)\big(1-\frac{z}{\lambda}\big)}. \end{aligned}$$

Remark. Note that $f(x,y) = g(x,y,1)$.

In order to eliminate λ we apply a reduction strategy which has been described by MacMahon as *The Method of Elliott*; see [6, Vol. 2, Section VIII, pp. 111–114]. More precisely, we make use of the following fact.

Fact 3 (Elliott Reduction). *For positive integers j and k,*

$$\frac{1}{(1-x\,\lambda^j)(1-y\,\lambda^{-k})} = \frac{1}{1-x\,y\,\lambda^{j-k}}\Big(\frac{1}{1-x\,\lambda^j} + \frac{1}{1-y\,\lambda^{-k}} - 1\Big).$$

Applying Fact 3 to the first two denominator factors $1-x\,\lambda^2$ and $1-y\,\lambda^{-3}$ of the crude generating function yields

$$g(x,y,z) = \underset{=}{\Omega} \frac{1}{1-\frac{x\,y}{\lambda}} \left(\frac{1}{1-x\,\lambda^2} + \frac{1}{1-\frac{y}{\lambda^3}} - 1\right) \frac{1}{1-\frac{z}{\lambda}}.$$

This simplifies to

$$g(x,y,z) = \underset{=}{\Omega} \frac{1}{(1-x\,\lambda^2)\big(1-\frac{x\,y}{\lambda}\big)\big(1-\frac{z}{\lambda}\big)},$$

since both of the remaining $\Omega_=$ expressions involve only negative powers of λ in the denominator factors which $\Omega_=$ trivially reduces to 1.

The final step is done by using Fact 2 which results in

$$g(x,y,z) = \frac{1+x^2\,y\,z}{(1-x^3\,y^2)(1-x\,z^2)}.$$

In Section 3.1 we describe how the solutions to $2a \ge 3b$ and $2a = 3b+c$ are obtained in automatic fashion by using the `Omega` package.

More generally, MacMahon discusses the problem of evaluating the crude generating function of the form

$$\underset{=}{\Omega} \frac{1}{(1-x_1\,\lambda^{j_1})\cdots(1-x_n\,\lambda^{j_n})(1-y_1\,\lambda^{-k_1})\cdots(1-y_m\,\lambda^{-k_m})},$$

where n and m are nonnegative and the j_i and k_i are positive integers. He describes Elliott's algorithm, i.e., the termination of this reduction when applied iteratively, as follows [6, Vol. 2, Section VIII, p. 112]:

> "To reduce this expression [Elliott] makes use of the equation of [Fact 3], choosing j and k to be the greatest of the quantities j_i, k_i respectively. The generating function is thus expressed as the sum of three fractions, each with either ± 1 for numerator. Each of the three fractions is on the whole simpler

than the original. The process is continued with each fraction but it cannot be indefinitely applied. Eventually the original fraction will be replaced by a sum of fractions each with either ± 1 for numerator, in none of which is there in the denominator *both* positive and negative powers of λ. The factors of a denominator will either involve both factors without λ and with positive powers of λ *or* both factors without λ and with negative powers of λ. No single denominator will involve *both* positive and negative powers of λ. Putting all factors, which involve positive or negative powers of λ, equal to unity in these fractions gives the completion of the operation $\Omega_=$ [...]"

MacMahon never mentioned explicitly that this reduction algorithm can be also applied to the $\Omega_{\geqq}$ case. As an illustrative example we present an alternative evaluation of the crude generating function associated to $f(x,y)$ from the example above.

Example. Applying Fact 3 to

$$f(x,y) = \underset{\geqq}{\Omega} \frac{1}{(1 - x\lambda^2)\left(1 - \frac{y}{\lambda^3}\right)}$$

results in

$$f(x,y) = \underset{\geqq}{\Omega} \frac{1}{1 - \frac{x\,y}{\lambda}} \left(\frac{1}{1 - x\lambda^2} + \frac{1}{1 - \frac{y}{\lambda^3}} - 1 \right).$$

This simplifies to

$$f(x,y) = \underset{\geqq}{\Omega} \frac{1}{(1 - x\lambda^2)\left(1 - \frac{x\,y}{\lambda}\right)},$$

since both of the remaining $\Omega_{\geqq}$ expressions involve only negative powers of λ in the denominator factors which $\Omega_{\geqq}$ trivially reduces to 1. For further reduction we apply Fact 3 again which gives

$$f(x,y) = \underset{\geqq}{\Omega} \frac{1}{1 - x^2\,y\,\lambda} \left(\frac{1}{1 - x\lambda^2} + \frac{1}{1 - \frac{x\,y}{\lambda}} - 1 \right).$$

This simplifies to

$$f(x,y) = \frac{1}{(1 - x^2\,y)(1 - x)} + \underset{\geqq}{\Omega} \frac{1}{(1 - x^2\,y\,\lambda)\left(1 - \frac{x\,y}{\lambda}\right)} - \frac{1}{1 - x^2\,y}.$$

Now we could apply again Fact 3 in order to arrive at the base cases discussed by MacMahon. Alternatively, one can use Fact 1 with $s = 1$ to derive

$$f(x,y) = \frac{1}{(1 - x^2\,y)(1 - x)} + \frac{1}{(1 - x^2\,y)(1 - x^3\,y^2)} - \frac{1}{1 - x^2\,y} = \frac{1 + x^2\,y}{(1 - x)(1 - x^3\,y^2)}$$

as computed above. □

We have examined MacMahon's use of Partition Analysis with the object of providing a proper algorithmic setting for the problems that he considers. As a result we developed an algorithm [2] which avoids table look-up and improves significantly upon Elliott reduction. The latter is achieved by using a variant of partial fraction decomposition which splits the elimination problem into *two* parts instead of *three* as in Elliott reduction. Nevertheless, this approach carried along two major shortcomings which we succeeded to overcome by a new reduction strategy. The details of this new algorithm are explained in Section 2. In Section 3 we illustrate the algorithm and the usage of the corresponding computer algebra package by applications involving systems of linear diophantine inequalities and equations. It is important to note that this package replaces the package described in [2]. In Section 4 we comment on future developments and possible extensions of MacMahon's method.

2 The New Reduction Algorithm

As already discussed in [2], the problem of finding nonnegative integer solutions to linear systems of diophantine inequalities and equations is equivalent to applying $\Omega_{\geqq}$ or $\Omega_{=}$ to expressions of the form

$$\frac{P(x_1,\ldots,x_n;\lambda_1,\ldots,\lambda_r)}{\prod_{i=1}^n (1-x_i\,\lambda_1^{v_1(i)}\cdots\lambda_r^{v_r(i)})},$$

where P is a Laurent polynomial in the $n+r$ variables and the $v_h(i)$ are integers not necessarily positive. Note that as long as the x_i (which may be power products in other variables) are restricted to a small neighborhood of 0, we are guaranteed that we have avoided any singularities inside the annuli that provide the domain for the λ_i. This problem amounts to the successive elimination of the variables λ_i.

Theoretically the order in which the variables λ_i are eliminated is irrelevant; however, in practice it turns out that certain orders are preferable with respect to performance. But in order to limit technicalities we restrict ourselves to the description of the new reduction step. To this end we assume that the λ_i to be eliminated is denoted in short by λ.

Problem. We need an algorithm that evaluates the application of $\Omega_{\geqq}$ and $\Omega_{=}$ to the term

$$\frac{\lambda^a}{(1-x_1\,\lambda^{j_1})\cdots(1-x_n\,\lambda^{j_n})(1-y_1\,\lambda^{-k_1})\cdots(1-y_m\,\lambda^{-k_m})}, \tag{2}$$

where n and m are nonnegative integers, the j_i and k_i are positive integers, and a is any integer.

In the previous paper [2] we suggested the following way of solving the problem. As a preprocessing step, factor the terms $(1-x_i\,\lambda^{j_i})$ into *linear* factors with respect to λ. Doing so, one ends up with an expression of form (2) where all j_i are equal to 1. With this condition satisfied apply partial fraction decomposition, as e.g. in (1) above, to break the whole term additively into two parts of the same type again, but with n decreased by 1. Repeat this process (recursively) until the case $n=1$ is reached and evaluate the base case via an explicit formula.

This method comes along with two major shortcomings. The preprocessing might not only produce many more denominator factors than the ones in the original term (2), it might also introduce roots of unity, which slows down the computation enormously. With our new approach we avoid both of these difficulties by utilizing a more general partial fraction decomposition that no longer requires the λ to appear linearly in the denominator of (2). Furthermore, in each step it can be applied to either the factors with positive or negative exponents of λ, thus decreasing either n or m by 1 in all steps. In other words we can choose our reduction strategy according to the minimum of n and m.

Let us first first look at the degenerate cases of (2) when m or n is 0. For this we define variants of the homogeneous symmetric functions, denoted by $h_i(z_1,\ldots,z_n;\zeta_1,\ldots,\zeta_n)$ and given by

$$\sum_{i=0}^{\infty} h_i(z_1,\ldots,z_n;\zeta_1,\ldots,\zeta_n)\,t^i=\frac{1}{(1-z_1\,t^{\zeta_1})\cdots(1-z_n\,t^{\zeta_n})}.$$

Case $m = 0$. We immediately obtain

$$\underset{\geqq}{\Omega} \frac{\lambda^a}{(1 - x_1 \lambda^{j_1}) \cdots (1 - x_n \lambda^{j_n})} = \underset{\geqq}{\Omega} \sum_{i=0}^{\infty} h_i(x_1, \dots, x_n; j_1, \dots, j_n) \lambda^{a+i}$$

$$= \begin{cases} \dfrac{1}{(1 - x_1) \cdots (1 - x_n)}, & \text{if } a \geq 0, \\ \dfrac{1}{(1 - x_1) \cdots (1 - x_n)} - \displaystyle\sum_{i=0}^{-a-1} h_i(x_1, \dots, x_n; j_1, \dots, j_n), & \text{if } a < 0, \end{cases}$$

and

$$\underset{=}{\Omega} \frac{\lambda^a}{(1 - x_1 \lambda^{j_1}) \cdots (1 - x_n \lambda^{j_n})} = \underset{=}{\Omega} \sum_{i=0}^{\infty} h_i(x_1, \dots, x_n; j_1, \dots, j_n) \lambda^{a+i}$$

$$= \begin{cases} 0, & \text{if } a > 0, \\ h_{-a}(x_1, \dots, x_n; j_1, \dots, j_n), & \text{if } a \leq 0. \end{cases}$$

Case $n = 0$. Similarly as above we get

$$\underset{\geqq}{\Omega} \frac{\lambda^a}{(1 - y_1 \lambda^{-k_1}) \cdots (1 - y_m \lambda^{-k_m})} = \underset{\geqq}{\Omega} \sum_{i=0}^{\infty} h_i(y_1, \dots, y_m; k_1, \dots, k_m) \lambda^{a-i}$$

$$= \begin{cases} 0, & \text{if } a < 0, \\ \displaystyle\sum_{i=0}^{a} h_i(y_1, \dots, y_m; k_1, \dots, k_m), & \text{if } a \geq 0, \end{cases}$$

and

$$\underset{=}{\Omega} \frac{\lambda^a}{(1 - y_1 \lambda^{-k_1}) \cdots (1 - y_m \lambda^{-k_m})} = \underset{=}{\Omega} \sum_{i=0}^{\infty} h_i(y_1, \dots, y_m; k_1, \dots, k_m) \lambda^{a-i}$$

$$= \begin{cases} 0, & \text{if } a < 0, \\ h_a(y_1, \dots, y_m; k_1, \dots, k_m), & \text{if } a \geq 0. \end{cases}$$

Next we investigate the base cases of our recurrence, i.e., the cases when m or n is 1.

Case $m = 1$. For sake of simplicity, let $y := y_1$ and $k := k_1$. If $a \leq -k$, then we may write $a = -p\,k - s$ with $p \geq 1$ and $0 \leq s < k$ to obtain

$$\frac{\lambda^a}{(1 - x_1 \lambda^{j_1}) \cdots (1 - x_n \lambda^{j_n})(1 - y \lambda^{-k})} = -\frac{y^{-p} \lambda^{-s} (1 - y^p \lambda^{-p\,k}) - y^{-p} \lambda^{-s}}{(1 - x_1 \lambda^{j_1}) \cdots (1 - x_n \lambda^{j_n})(1 - y \lambda^{-k})}$$

$$= -\frac{y^{-p} \lambda^{-s} \sum_{h=0}^{p-1} y^h \lambda^{-h\,k}}{(1 - x_1 \lambda^{j_1}) \cdots (1 - x_n \lambda^{j_n})} + \frac{y^{-p} \lambda^{-s}}{(1 - x_1 \lambda^{j_1}) \cdots (1 - x_n \lambda^{j_n})(1 - y \lambda^{-k})}.$$

While the result of applying the Ω-operator to the first term can be computed by means of the $m = 0$ case, for the second term we now have $a > -k$. We claim that in this case

$$\underset{\geqq}{\Omega} \frac{\lambda^a}{(1 - x_1 \lambda^{j_1}) \cdots (1 - x_n \lambda^{j_n})(1 - y \lambda^{-k})}$$

$$= \frac{1}{(1 - x_1) \cdots (1 - x_n)(1 - y)} - \frac{\sum_{\tau_1, \dots, \tau_n = 0}^{k-1} \prod x_i^{\tau_i} \, y^{\left\lfloor \frac{\sum j_i \tau_i + a}{k} \right\rfloor + 1}}{(1 - x_1^k y^{j_1}) \cdots (1 - x_n^k y^{j_n})(1 - y)},$$

where in all sums and products without explicit bounds i runs from 1 to n. We want to remark that after bringing the right-hand side over a common denominator it is easily seen that the factor $(1-y)$ divides the numerator.

Proof. First observe that

$$\underset{\geqq}{\Omega} \frac{\lambda^a}{(1-x_1\,\lambda^{j_1})\cdots(1-x_n\,\lambda^{j_n})(1-y\,\lambda^{-k})} = \underset{\geqq}{\Omega} \sum_{s_1,\ldots,s_n\geq 0} \sum_{r\geq 0} \prod x_i^{s_i}\, y^r \lambda^{a+\sum j_i s_i - k\,r}$$
$$= \sum_{s_1,\ldots,s_n\geq 0} \prod x_i^{s_i} \sum_{0\leq r\leq \left\lfloor \frac{\sum j_i s_i + a}{k}\right\rfloor} y^r, \tag{3}$$

where the last equality follows from the fact that

$$\frac{\sum j_i\, s_i + a}{k} \geq \frac{a}{k} > -1.$$

Next the right-hand side of (3) may be rewritten as

$$\sum_{s_1,\ldots,s_n\geq 0} \prod x_i^{s_i} \left(\sum_{r\geq 0} y^r - \sum_{r> \left\lfloor \frac{\sum j_i s_i + a}{k}\right\rfloor} y^r \right)$$
$$= \frac{1}{(1-x_1)\cdots(1-x_n)(1-y)} - \frac{1}{(1-y)} \sum_{s_1,\ldots,s_n\geq 0} \prod x_i^{s_i}\, y^{\left\lfloor \frac{\sum j_i s_i + a}{k}\right\rfloor + 1}$$

and after substituting $k\,\sigma_i + \tau_i$ $(0 \leq \tau_i < k)$ for s_i this expression turns into

$$\frac{1}{(1-x_1)\cdots(1-x_n)(1-y)} - \frac{1}{(1-y)} \sum_{\sigma_1,\ldots,\sigma_n\geq 0} \sum_{\tau_1,\ldots,\tau_n=0}^{k-1} \prod x_i^{k\sigma_i+\tau_i}\, y^{\left\lfloor \frac{\sum j_i (k\sigma_i+\tau_i) + a}{k}\right\rfloor + 1}$$
$$= \frac{1}{(1-x_1)\cdots(1-x_n)(1-y)} - \frac{\sum_{\tau_1,\ldots,\tau_n=0}^{k-1} \prod x_i^{\tau_i}\, y^{\left\lfloor \frac{\sum j_i \tau_i + a}{k}\right\rfloor + 1}}{(1-x_1^k\, y^{j_1})\cdots(1-x_n^k\, y^{j_n})(1-y)}.$$

□

By similar reasoning we find that for $a > -k$

$$\underset{=}{\Omega} \frac{\lambda^a}{(1-x_1\,\lambda^{j_1})\cdots(1-x_n\,\lambda^{j_n})(1-y\,\lambda^{-k})} = \frac{{}^*\sum_{\tau_1,\ldots,\tau_n=0}^{k-1} \prod x_i^{\tau_i}\, y^{\frac{\sum j_i \tau_i + a}{k}}}{(1-x_1^k\, y^{j_1})\cdots(1-x_n^k\, y^{j_n})},$$

where ${}^*\sum$ only sums those terms for which the exponent of y is an integer.

Case $n=1$. Again, for sake of simplicity, let $x := x_1$ and $j := j_1$. If $a \geq j$, then we may write $a = p\,j + r$ with $p \geq 1$ and $0 \leq r < j$ to obtain

$$\frac{\lambda^a}{(1-x\,\lambda^j)(1-y_1\,\lambda^{-k_1})\cdots(1-y_m\,\lambda^{-k_m})} = -\frac{x^{-p}\,\lambda^r\,(1-x^p\,\lambda^{p\,j}) - x^{-p}\,\lambda^r}{(1-x\,\lambda^j)(1-y_1\,\lambda^{-k_1})\cdots(1-y_m\,\lambda^{-k_m})}$$
$$= -\frac{x^{-p}\,\lambda^r\,\sum_{h=0}^{p-1} x^h\,\lambda^{h\,j}}{(1-y_1\,\lambda^{-k_1})\cdots(1-y_m\,\lambda^{-k_m})} + \frac{x^{-p}\,\lambda^r}{(1-x\,\lambda^j)(1-y_1\,\lambda^{-k_1})\cdots(1-y_m\,\lambda^{-k_m})}.$$

Thus we may assume that $a < j$. We claim that in this case

$$\underset{\geqq}{\Omega} \frac{\lambda^a}{(1-x\,\lambda^j)(1-y_1\,\lambda^{-k_1})\cdots(1-y_m\,\lambda^{-k_m})} = \frac{\sum_{\tau_1,\ldots,\tau_m=0}^{j-1} \prod y_i^{\tau_i}\, x^{\left\lceil \frac{\sum k_i \tau_i - a}{j}\right\rceil}}{(1-x)(1-x^{k_1}\, y_1^j)\cdots(1-x^{k_m}\, y_m^j)}, \tag{4}$$

where in all sums and products without explicit bounds i runs from 1 to m.

Proof. First observe that

$$\underset{\geq}{\Omega} \frac{\lambda^a}{(1-x\,\lambda^j)(1-y_1\,\lambda^{-k_1})\cdots(1-y_m\,\lambda^{-k_m})} = \underset{\geq}{\Omega} \sum_{r\geq 0} \sum_{s_1,\ldots,s_m\geq 0} x^r \prod y_i^{s_i}\,\lambda^{a+j\,r-\sum k_i s_i}$$
$$= \sum_{s_1,\ldots,s_m\geq 0} \prod y_i^{s_i} \sum_{r\geq\left\lceil \frac{\sum k_i s_i - a}{j}\right\rceil} x^r, \tag{5}$$

where the last equality follows from the fact that

$$\frac{\sum k_i\,s_i - a}{j} \geq \frac{-a}{j} > -1.$$

Next the right-hand side of (5) may be rewritten as

$$\frac{1}{(1-x)} \sum_{s_1,\ldots,s_m\geq 0} \prod y_i^{s_i}\,x^{\left\lceil \frac{\sum k_i s_i - a}{j}\right\rceil}$$

and after substituting $j\,\sigma_i + \tau_i$ $(0 \leq \tau_i < j)$ for s_i this expression turns into

$$\frac{1}{(1-x)} \sum_{\sigma_1,\ldots,\sigma_m\geq 0} \sum_{\tau_1,\ldots,\tau_m=0}^{j-1} \prod y_i^{j\sigma_i+\tau_i}\,x^{\left\lceil \frac{\sum k_i (j\sigma_i+\tau_i)-a}{j}\right\rceil}$$
$$= \frac{\sum_{\tau_1,\ldots,\tau_m=0}^{j-1} \prod y_i^{\tau_i}\,x^{\left\lceil \frac{\sum k_i \tau_i - a}{j}\right\rceil}}{(1-x)(1-x^{k_1}\,y_1^j)\cdots(1-x^{k_m}\,y_m^j)}.$$

□

By similar reasoning we find that for $a < j$

$$\underset{=}{\Omega} \frac{\lambda^a}{(1-x\,\lambda^j)(1-y_1\,\lambda^{-k_1})\cdots(1-y_m\,\lambda^{-k_m})} = \frac{{}^*\sum_{\tau_1,\ldots,\tau_m=0}^{j-1} \prod y_i^{\tau_i}\,x^{\frac{\sum k_i \tau_i - a}{j}}}{(1-x^{k_1}\,y_1^j)\cdots(1-x^{k_m}\,y_m^j)},$$

where ${}^*\sum$ only sums those terms for which the exponent of x is an integer.

Finally, the essential reduction step of our new algorithm is the following partial fraction decomposition.

Theorem 1 (Generalized PFD). *If $j_1 \geq j_2 \geq 1$ and $\gcd(j_1, j_2) = 1$, then*

$$\frac{1}{(1-x_1\,\lambda^{j_1})(1-x_2\,\lambda^{j_2})}$$
$$= \frac{1}{x_2^{j_1} - x_1^{j_2}} \left(\frac{\alpha_0 + \alpha_1\,\lambda + \cdots + \alpha_{j_1-1}\,\lambda^{j_1-1}}{1-x_1\,\lambda^{j_1}} + \frac{\beta_0 + \beta_1\,\lambda + \cdots + \beta_{j_2-1}\,\lambda^{j_2-1}}{1-x_2\,\lambda^{j_2}} \right), \tag{6}$$

where

$$\alpha_i = \begin{cases} -x_1^{j_2}\,x_2^{i/j_2}, & \text{if } j_2 \mid i \text{ or } i = 0, \\ -x_2^{\operatorname{rmd}\left((j_2^{-1} \bmod j_1)i, j_1\right)}\,x_1^{\operatorname{rmd}\left((j_1^{-1} \bmod j_2)i, j_2\right)}, & \text{otherwise}, \end{cases} \tag{7}$$

$$\beta_i = \begin{cases} x_2^{j_1}, & \text{if } i = 0, \\ x_2^{\operatorname{rmd}\left((j_2^{-1} \bmod j_1)i, j_1\right)}\,x_1^{\operatorname{rmd}\left((j_1^{-1} \bmod j_2)i, j_2\right)}, & \text{otherwise}, \end{cases} \tag{8}$$

and $\operatorname{rmd}(k,l)$ *denotes the least nonnegative residue of* k *and* l *given by*

$$\operatorname{rmd}(k,l) := k - l\left\lfloor \frac{k}{l} \right\rfloor.$$

Before we prove the theorem a few remarks are appropriate. First of all, by replacing λ by λ^{-1} we immediately get the corresponding relation for

$$\frac{1}{(1-y_1\,\lambda^{-k_1})(1-y_2\,\lambda^{-k_2})}.$$

Furthermore, the assumption that j_1 and j_2 are relatively prime is not at all a restriction. To see this, suppose that $d := \gcd(j_1, j_2) > 1$. Then we can use Theorem 1 to find the decomposition of

$$\frac{1}{(1-x_1\,\lambda^{j_1/d})(1-x_2\,\lambda^{j_2/d})}.$$

Substituting λ^d for λ in the result gives the desired relation.

It might happen that $(x_2^{j_1} - x_1^{j_2})$ vanishes. In this case we simply replace x_1 or x_2 by a new variable and substitute back after the prefactor has been canceled from the result.

Finally, for computing $(j_1^{-1} \bmod j_2)$ and $(j_2^{-1} \bmod j_1)$ we may use the fact that $a^{-1} \equiv a^{\phi(b)-1} \pmod b$, if $\gcd(a,b) = 1$. Here ϕ denotes Euler's totient function.

Proof of Theorem 1. First we note that (6) is completely equivalent to

$$\begin{aligned} x_2^{j_1} - x_1^{j_2} &= (1-x_2\,\lambda^{j_2})(\alpha_0 + \alpha_1\,\lambda + \cdots + \alpha_{j_1-1}\,\lambda^{j_1-1}) \\ &\quad + (1-x_1\,\lambda^{j_1})(\beta_0 + \beta_1\,\lambda + \cdots + \beta_{j_2-1}\,\lambda^{j_2-1}). \end{aligned} \tag{9}$$

All that is required is showing that both sides of (9) match up for each coefficient of each power of λ. Hence we must show that the following $j_1 + j_2$ equations hold:

$$\alpha_0 + \beta_0 = x_2^{j_1} - x_1^{j_2},, \tag{10}$$
$$\alpha_h + \beta_h = 0, \qquad 0 < h < j_2, \tag{11}$$
$$\alpha_h - x_2\,\alpha_{h-j_2} = 0, \qquad j_2 \le h < j_1, \tag{12}$$
$$-x_2\,\alpha_{h-j_2} - x_1\,\beta_{h-j_1} = 0, \qquad j_1 \le h < j_1 + j_2. \tag{13}$$

The top line of (7) and the top line of (8) for $i = 0$ together establish (10). Next we note that (11) follows directly from the second lines of (7) and (8) with $0 < i = h < j_2$.

From now on, for sake of brevity, we shall write j_1^{-1} for $(j_1^{-1} \bmod j_2)$ and j_2^{-1} for $(j_2^{-1} \bmod j_1)$. As for (12), we note that if $j2 \nmid h$, then

$$\begin{aligned} \alpha_{h-j_2} &= -x_2^{\operatorname{rmd}(j_2^{-1}(h-j_2),j_1)}\,x_1^{\operatorname{rmd}(j_1^{-1}(h-j_2),j_2)} \\ &= -x_2^{\operatorname{rmd}(j_2^{-1}h-1,j_1)}\,x_1^{\operatorname{rmd}(j_1^{-1}h,j_2)} \\ &= -x_2^{-1}\,x_2^{\operatorname{rmd}(j_2^{-1}h,j_1)}\,x_1^{\operatorname{rmd}(j_1^{-1}h,j_2)} \\ &= x_2^{-1}\,\alpha_h, \end{aligned}$$

which is equivalent to (12) if $j_2 \nmid h$. If $j_2 \mid h$, then

$$\alpha_{h-j_2} = -x_1^{j_2}\,x_2^{(h-j_2)/j_2} = x_2^{-1}\,\alpha_h,$$

so (12) is proved in full generality.

There are three cases for (13).

Case 1. $h = j_1$.

$$
\begin{aligned}
-x_2\,\alpha_{j_1-j_2} - x_1\,\beta_0 &= x_2\,x_2^{\mathrm{rmd}(j_2^{-1}(j_1-j_2),j_1)}\,x_1^{\mathrm{rmd}(j_1^{-1}(j_1-j_2),j_2)} - x_1\,x_2^{j_1}\\
&= x_2\,x_2^{j_1-1}\,x_1 - x_1\,x_2^{j_1}\\
&= 0.
\end{aligned}
$$

Case 2. $j_1 < h < j_1 + j_2$ and $j_2 \mid h$.

$$
\begin{aligned}
-x_2\,\alpha_{h-j_2} - x_1\,\beta_{h-j_1} &= -x_2\,(-x_1^{j_2} x_2^{h/j_2-1}) - x_1\,x_2^{\mathrm{rmd}(j_2^{-1}(h-j_1),j_1)}\,x_1^{\mathrm{rmd}(j_1^{-1}(h-j_1),j_2)}\\
&= x_1^{j_2}\,x_2^{h/j_2} - x_1\,x_2^{h/j_2}\,x_1^{j_2-1} \qquad \text{(note } \mathrm{rmd}(-1,j_2) = j_2 - 1\text{)}\\
&= 0.
\end{aligned}
$$

Case 3. $j_1 < h < j_1 + j_2$ and $j_2 \nmid h$.

$$
\begin{aligned}
-x_2\,\alpha_{h-j_2} - x_1\,\beta_{h-j_1} &= x_2\,x_2^{\mathrm{rmd}(j_2^{-1}(h-j_2),j_1)}\,x_1^{\mathrm{rmd}(j_1^{-1}(h-j_2),j_2)}\\
&\quad - x_1\,x_2^{\mathrm{rmd}(j_2^{-1}(h-j_1),j_1)}\,x_1^{\mathrm{rmd}(j_1^{-1}(h-j_1),j_2)}\\
&= x_2\,x_2^{\mathrm{rmd}(j_2^{-1}h-1,j_1)}\,x_1^{\mathrm{rmd}(j_1^{-1}h,j_2)}\\
&\quad - x_1\,x_2^{\mathrm{rmd}(j_2^{-1}h,j_1)}\,x_1^{\mathrm{rmd}(j_1^{-1}h-1),j_2)}\\
&= x_2^{\mathrm{rmd}(j_2^{-1}h,j_1)}\,x_1^{\mathrm{rmd}(j_1^{-1}h,j_2)}\\
&\quad - x_2^{\mathrm{rmd}(j_2^{-1}h,j_1)}\,x_1^{\mathrm{rmd}(j_1^{-1}h,j_2)}\\
&= 0.
\end{aligned}
$$

Note that we used the fact that if $a \nmid b$, then $\mathrm{rmd}(b-1,a) = \mathrm{rmd}(b,a) - 1$.

Hence in all three cases, we see that (13) holds. Therefore Theorem 1 is proved. □

3 Applications

The first subsection illustrates how the elementary problems from the introduction are treated with the Omega package.

The $\Omega_{\geqq}$ application in Subsection 3.2 is motivated by J. Louck's illuminating study [5] of expansion formulae for powers of determinants. In this context magic squares play a prominent rôle. In addition, Louck's work gives rise to various questions about possible new extensions of MacMahon's method; for further details we refer to Section 4.

Finally an $\Omega_=$ application is presented in Subsection 3.3. It concerns the computation of the generating function for magic pentagrams.

3.1 Introductory Examples

The package `Omega2` is written in Mathematica and freely available via *http://www.risc.uni-linz.ac.at/research/combinat/risc/software/Omega/*.

After loading the file `Omega2.m` by

```
In[1]:= <<Omega2.m

Out[1]= Axel Riese's Omega implementation version 2.30 loaded
```

the diophantine inequality $2a \geq 3b$ from Section 1 is solved as follows:

```
In[2]:= OSum[x^a y^b, {2a ≥ 3b}, λ]

        Assuming a ≥ 0
        Assuming b ≥ 0
```

Out[2]=

$$\underset{\lambda_1}{\underset{\geq}{\Omega}} \frac{1}{\left(1-\frac{y}{\lambda_1^3}\right)\left(1-x\,\lambda_1^2\right)}$$

```
In[3]:= OR[%]

        Eliminating λ1...
```

Out[3]=

$$\frac{1+x^2\,y}{(1-x)\,(1-x^3\,y^2)}$$

Similarly, the diophantine equation $2a = 3b + c$ from Section 1 is solved automatically by:

```
In[4]:= OEqSum[x^a y^b z^c, {2a == 3b+c}, λ]

        Assuming a ≥ 0
        Assuming b ≥ 0
        Assuming c ≥ 0
```

Out[4]=

$$\underset{\lambda_1}{\underset{=}{\Omega}} \frac{1}{\left(1-\frac{y}{\lambda_1^3}\right)\left(1-\frac{z}{\lambda_1}\right)\left(1-x\,\lambda_1^2\right)}$$

```
In[5]:= OEqR[%]

        Eliminating λ1...
```

Out[5]=

$$\frac{1+x^2\,y\,z}{(1-x^3\,y^2)\,(1-x\,z^2)}$$

3.2 Magic Squares

Let $Z = (z_{i,j})_{1\leq i,j\leq n}$ be an $n \times n$ matrix of commuting indeterminates, and let $A = (a_{i,j})$ be an $n \times n$ matrix of nonnegative integers. We will use the notation

$$Z^A := \prod_{i,j} z_{i,j}^{a_{i,j}}.$$

Studying powers of 3×3 determinants, Louck [5, (3.3)] expressed the corresponding expansion coefficients $C_k(A)$, described in Section 4 below, as a multiple of a hypergeometric ${}_3F_2$ series. To this end he considered 3×3 matrices $A = (a_{ij})_{1\leq i,j\leq 3}$ such that $a_{1,1} = a$, $a_{1,2} = c + e$, $a_{1,3} = b + d$, $a_{2,1} = c + d$, $a_{2,2} = b$, $a_{2,3} = a + e$, $a_{3,1} = b + e$, $a_{3,2} = a + d$, and $a_{3,3} = c$ with integers a, b, c, d, e where a, b, c are nonnegative. Additionally one imposes that all entries $a_{i,j}$ have to be nonnegative. This gives magic squares of order 3 since all rows and columns sum up to $a+b+c+d+e$. We call this set $A_3(a+b+c+d+e)$.

In order to illustrate the algorithmic usage of the $\Omega_{\geqq}$ operator we compute the generating function

$$f(Z) := \sum_{A \in A_3(a+b+c+d+e)} Z^A$$

of all the elements of $A_3(a+b+c+d+e)$. Note that by case distinction Partition Analysis is also able to handle diophantine problems where some of the solutions are supposed to be *negative* integers. In this particular example there are two such parameters, namely d and e. Accordingly we have to consider four cases:

Case 1.

$$a+d \geq 0, a+e \geq 0, b+d \geq 0, b+e \geq 0, c+d \geq 0, c+e \geq 0;$$

Case 2. $d > 0$ and

$$a-d \geq 0, a+e \geq 0, b-d \geq 0, b+e \geq 0, c-d \geq 0, c+e \geq 0;$$

Case 3. $e > 0$ and

$$a+d \geq 0, a-e \geq 0, b+d \geq 0, b-e \geq 0, c+d \geq 0, c-e \geq 0;$$

Case 4. $e, d > 0$ and

$$a-d \geq 0, a-e \geq 0, b-d \geq 0, b-e \geq 0, c-d \geq 0, c-e \geq 0.$$

With the Omega package the corresponding generating functions are computed within seconds.

Case 1.

```
In[6]:= f1 = OR[ OSum[z11^a z12^(c+e) z13^(b+d) z21^(c+d) z22^b z23^(a+e) *
                  z31^(b+e) z32^(a+d) z33^c, {a+d ≥ 0, a+e ≥ 0,
                  b+d ≥ 0, b+e ≥ 0, c+d ≥ 0, c+e ≥ 0}, λ] ]
```

Out[6]=

$$\frac{1}{(1 - z_{13} z_{22} z_{31})(1 - z_{12} z_{23} z_{31})(1 - z_{13} z_{21} z_{32})(1 - z_{11} z_{23} z_{32})(1 - z_{12} z_{21} z_{33})}$$

Case 2.

```
In[7]:= f2 = OR[ OSum[z11^a z12^(c+e) z13^(b-d) z21^(c-d) z22^b z23^(a+e) *
                  z31^(b+e) z32^(a-d) z33^c, {d > 0, a-d ≥ 0, a+e ≥ 0,
                  b-d ≥ 0, b+e ≥ 0, c-d ≥ 0, c+e ≥ 0}, λ] ]
```

Out[7]=

$$\frac{z_{11} z_{12} z_{22} z_{23} z_{31} z_{33}}{(1 - z_{13} z_{22} z_{31})(1 - z_{12} z_{23} z_{31})(1 - z_{11} z_{23} z_{32})(1 - z_{12} z_{21} z_{33})(1 - z_{11} z_{12} z_{22} z_{23} z_{31} z_{33})}$$

Case 3.

```
In[8]:= f3 = OR[ OSum[z11^a z12^(c-e) z13^(b+d) z21^(c+d) z22^b z23^(a-e) *
                  z31^(b-e) z32^(a+d) z33^c, {e > 0, a+d ≥ 0, a-e ≥ 0,
                  b+d ≥ 0, b-e ≥ 0, c+d ≥ 0, c-e ≥ 0}, λ] ]
```

Out[8]=

$$\frac{z_{11}\,z_{13}\,z_{21}\,z_{22}\,z_{32}\,z_{33}}{(1-z_{13}\,z_{22}\,z_{31})\,(1-z_{13}\,z_{21}\,z_{32})\,(1-z_{11}\,z_{23}\,z_{32})\,(1-z_{12}\,z_{21}\,z_{33})\,(1-z_{11}\,z_{13}\,z_{21}\,z_{22}\,z_{32}\,z_{33})}$$

Case 4.

```
In[9]:= f4 = OR[ OSum[z11^a z12^(c-e) z13^(b-d) z21^(c-d) z22^b z23^(a-e) *
                 z31^(b-e) z32^(a-d) z33^c, {d > 0, e > 0, a-d ≥ 0,
                 a-e ≥ 0, b-d ≥ 0, b-e ≥ 0, c-d ≥ 0, c-e ≥ 0}, λ] ]
```

Out[9]=

$$\begin{aligned}&z_{11}\,z_{22}\,z_{33}\,(1-{z_{11}}^2\,z_{12}\,z_{13}\,z_{21}\,{z_{22}}^2\,z_{23}\,z_{31}\,z_{32}\,{z_{33}}^2)/\\&\big((1-z_{13}\,z_{22}\,z_{31})\,(1-z_{11}\,z_{23}\,z_{32})\,(1-z_{12}\,z_{21}\,z_{33})\,(1-z_{11}\,z_{22}\,z_{33})\\&\;(1-z_{11}\,z_{12}\,z_{22}\,z_{23}\,z_{31}\,z_{33})\,(1-z_{11}\,z_{13}\,z_{21}\,z_{22}\,z_{32}\,z_{33})\big)\end{aligned}$$

Finally we obtain $f(Z)$ by

```
In[10]:= Factor[f1+f2+f3+f4]
```

Out[10]=

$$\begin{aligned}&-\Big((-1+z_{11}\,z_{12}\,z_{13}\,z_{21}\,z_{22}\,z_{23}\,z_{31}\,z_{32}\,z_{33})/\\&\quad\big((-1+z_{13}\,z_{22}\,z_{31})\,(-1+z_{12}\,z_{23}\,z_{31})\,(-1+z_{13}\,z_{21}\,z_{32})\,(-1+z_{11}\,z_{23}\,z_{32})\\&\quad(-1+z_{12}\,z_{21}\,z_{33})\,(-1+z_{11}\,z_{22}\,z_{33})\big)\Big)\end{aligned}$$

But this is the generating function for *all* magic squares of order 3; see, for instance, [3]. In this article, following MacMahon an alternative approach to magic squares is presented; namely, by using the $\Omega_=$ operator. For illustrative purpose a new application of this method is given in the next subsection.

3.3 Magic Pentagrams

In [4] M. Gardner discussed generalizations of magic squares to magic configurations of star and polyhedral shape. For example, for the pentagram with vertex labels a_1 to a_{10} he gave the special instance $a_1 = 12$, $a_2 = 9$, $a_3 = 6$, $a_4 = 8$, $a_5 = 10$, $a_6 = 1$, $a_7 = 4$, $a_8 = 2$, $a_9 = 5$, and $a_{10} = 3$; see Figure 1 below. Note that in this example all line segments add up to 24; e.g., $a_1 + a_6 + a_{10} + a_4 = 12 + 1 + 3 + 8 = 24$.

With the Omega package the computation of the generating function of generalized magic pentagrams is mere routine. More precisely, we compute the rational function representation of

$$\sum_{k,a_1,\dots,a_{10}\ge 0} q^{a_1+\cdots+a_{10}}\,y^k,$$

where the nonnegative integers a_i satisfy the diophantine constraints

$$\begin{aligned}&a_1+a_6+a_{10}+a_4=k,\quad a_1+a_7+a_8+a_3=k,\quad a_5+a_6+a_7+a_2=k,\\&a_5+a_{10}+a_9+a_3=k,\quad a_4+a_9+a_8+a_2=k.\end{aligned}$$

```
In[11]:= OEqR[ OEqSum[y^k Product[q^ai, {i,10}],
                {a1+a6+a10+a4 == k, a1+a7+a8+a3 == k, a5+a6+a7+a2 == k,
                 a5+a10+a9+a3 == k, a4+a9+a8+a2 == k}, λ] ]
```

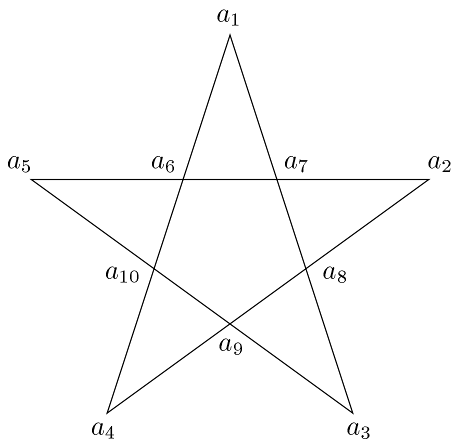

Figure 1: The Magic Pentagram

Out[11]=

$$\frac{1 + 16\,q^5\,y^2 + 41\,q^{10}\,y^4 + 16\,q^{15}\,y^6 + q^{20}\,y^8}{(1 - q^5\,y^2)^6}$$

In[12]:= `Factor[%]`

Out[12]=

$$\frac{(1 + 3\,q^5\,y^2 + q^{10}\,y^4)\,(1 + 13\,q^5\,y^2 + q^{10}\,y^4)}{(-1 + q^5\,y^2)^6}$$

In other words, from the generating function one extracts immediately that there are no magic pentagrams with odd "magic constant" k. Moreover, the number of generalized magic pentagrams with "magic constant" $k = 2j$ is the coefficient of x^j in

$$\frac{(1 + 3x + x^2)(1 + 13x + x^2)}{(1 - x)^6}.$$

4 New Aspects and Extensions of MacMahon's Method

Let $A_n(k)$ denote the set of all $n \times n$ matrices over nonnegative integers such that all row and column sums are equal to k. Let $Z = (z_{i,j})_{1 \le i,j \le n}$ be an $n \times n$ matrix of commuting indeterminates. We define

$$a_n(Z; y) := \sum_{k \ge 0, A \in A_n(k)} Z^A\,y^k$$

as the full generating function for magic squares of size n. As already mentioned above, in [3] we computed with the Omega package that

$$\begin{aligned} a_3(Z; y) = \sum_{k \ge 0, A \in A_3(k)} Z^A\,y^k &= (1 - z_{1,1}\,z_{1,2}\,z_{1,3}\,z_{2,1}\,z_{2,2}\,z_{2,3}\,z_{3,1}\,z_{3,2}\,z_{3,3}\,y^3) \\ &\times \frac{1}{(1 - z_{1,1}\,z_{2,2}\,z_{3,3}\,y)(1 - z_{1,1}\,z_{2,3}\,z_{3,2}\,y)(1 - z_{1,2}\,z_{2,1}\,z_{3,3}\,y)} \\ &\times \frac{1}{(1 - z_{1,2}\,z_{2,3}\,z_{3,1}\,y)(1 - z_{1,3}\,z_{2,1}\,z_{3,2}\,y)(1 - z_{1,3}\,z_{2,2}\,z_{3,1}\,y)}, \end{aligned}$$

as already worked out by MacMahon. We call $a_n(Z;y)$ the *full* generating function, since from the corresponding rational expression it is possible to construct all magic squares for given k. Namely, from the suitably truncated multivariate power series expansion just take the coefficient of y^k.

We also note that for $n=2$ the full generating function is very simple; in this case one has

$$a_2(Z;y) = \sum_{k\geq 0, A\in A_2(k)} Z^A\, y^k = \frac{1}{(1-z_{1,1}\, z_{2,2}\, y)(1-z_{1,2}\, z_{2,1}\, y)}.$$

Remarkably for $n=4$ the problem is still a computational challenge. Despite the fact that a variation of the reduction algorithm presented in Section 2 enables to compute $a_4(Z;y)$ as a sum of 256 multivariate rational functions, each of them of relatively simple form, the simplification of these to a single quotient is a bottle-neck. So far this task cannot be accomplished by the present computer algebra packages like Mathematica. The same observation applies when trying to compute the full generating function of magic pentagrams. In this case the simplification problem concerns to add 26 multivariate rational functions with nicely factored denominator and small numerator polynomials.

We conclude with a few comments on the use of MacMahon's method in connection with J. Louck's work [5].

Let k be a nonnegative integer, and let $Z=(z_{i,j})_{1\leq i,j\leq n}$ be as above. Louck's article is devoted to the study of properties of the expansion coefficients $C_k(A)$ for the kth power of the determinant $\det Z$. Namely, it turns out [5, (2.2)] that

$$(\det Z)^k = \sum_{A\in A_n(k)} C_k(A)\, Z^A, \tag{14}$$

where the coefficient $C_k(A)$ is a restricted sum over multinomial coefficients given by

$$C_k(A) = \sum_{\{k_\pi\geq 0\,:\,\pi\in S_n\}}^{*} (-1)^K \frac{k!}{\prod_{\pi\in S_n} k_\pi!} \quad \text{with} \quad K = \sum_{\pi\in S_n, \pi \text{ odd}} k_\pi. \tag{15}$$

The k_π are nonnegative integer summation variables indexed by permutations from the symmetric group S_n. The star on the summation quantifier indicates that the summation is a restricted one: the restriction is that the summation is over all nonnegative integers k_π such that, for a given magic square $A=(a_{i,j})\in A_n(k)$, k_π must satisfy the n^2 relations

$$\sum_{\{\pi\in S_n\,:\,\pi(i)=j\}} k_\pi = a_{i,j}, \quad 1\leq i,j\leq n.$$

MacMahon's operator method provides an elegant computational derivation of representation (15). Namely, if $\Lambda=(\lambda_{i,j})_{1\leq i,j\leq n}$ is the $n\times n$ matrix containing the λ-variables the $\Omega_=$ operator acts on, then (15) can be rewritten as

$$\begin{aligned}
C_k(A) &= \underset{=}{\Omega} \sum_{\{k_\pi\geq 0\,:\,\pi\in S_n\}} (-1)^{\sum_{\pi \text{ odd}} k_\pi} \frac{k!}{\prod_{\pi\in S_n} k_\pi!} \prod_{1\leq i,j\leq n} \lambda_{i,j}^{-a_{i,j}+\sum_{\pi\in S_n, \pi(i)=j} k_\pi} \\
&= k!\, \underset{=}{\Omega}\, \frac{1}{\Lambda^A} \prod_{\pi\in S_n} \sum_{k_\pi\geq 0} \frac{(\operatorname{sign}(\pi)\, \lambda_{1,\pi(1)}\cdots\lambda_{n,\pi(n)})^{k_\pi}}{k_\pi!} \\
&= k!\, \underset{=}{\Omega}\, \frac{1}{\Lambda^A}\, e^{\sum_{\pi\in S_n} \operatorname{sign}(\pi)\, \lambda_{1,\pi(1)}\cdots\lambda_{n,\pi(n)}} \\
&= k!\, \underset{=}{\Omega}\, \frac{e^{\det\Lambda}}{\Lambda^A}.
\end{aligned}$$

The constant term representation obtained in the last line is no surprise. However, this alternative representation of the restricted sum in (15) allows a compact proof of Louck's expansion formula. Before presenting this proof, we reformulate (14) accordingly.

Fact 4. *Let k be a nonnegative integer, and let $Z = (z_{i,j})_{1\le i,j\le n}$ be an $n \times n$ matrix of commuting indeterminates. Then*

$$(\det Z)^k = \sum_{A \in A_n(k)} C_k(A)\, Z^A \quad \textit{with} \quad C_k(A) = k!\, \underset{=}{\Omega}\, \frac{e^{\det \Lambda}}{\Lambda^A},$$

where $\Lambda = (\lambda_{i,j})_{1\le i,j\le n}$ is the $n \times n$ matrix containing the λ-variables the $\Omega_=$ operator acts on.

Proof. For each power product Z^A which arises in the expansion of $(\det Z)^k$ we have that $A \in A_n(k)$. In particular, the total degree of Z^A is $k \cdot n$. Hence we obtain for its coefficient,

$$\begin{aligned} C_k(A) &= \underset{=}{\Omega}\, \frac{1}{\Lambda^A} (\det \Lambda)^k \\ &= k!\, \underset{=}{\Omega}\, \frac{1}{\Lambda^A} \sum_{j \ge 0} \frac{(\det \Lambda)^j}{j!} = k!\, \underset{=}{\Omega}\, \frac{e^{\det \Lambda}}{\Lambda^A}. \end{aligned}$$

□

As a by-product of Fact 4 we obtain a connection between the exponential generating function for powers of a determinant with the full generating function $a_n(Z; y)$ for magic squares:

Fact 5. *For Z and Λ as in Fact 4 we have,*

$$e^{y \det Z} = \underset{=}{\Omega}\, e^{\det \Lambda}\, a_n(Z^{(\lambda)}; y),$$

where

$$Z^{(\lambda)} = \left(z_{i,j}^{(\lambda)}\right)_{1\le i,j\le n} \quad \textit{with} \quad z_{i,j}^{(\lambda)} := \frac{z_{i,j}}{\lambda_{i,j}}, \quad 1 \le i,j \le n.$$

Proof. The statement follows from Fact 4 by summing

$$y^k \frac{(\det Z)^k}{k!} = y^k\, \underset{=}{\Omega} \sum_{A \in A_n(k)} e^{\det \Lambda} \frac{Z^A}{\Lambda^A}$$

for $k \ge 0$. □

Already these elementary Omega computations indicate that MacMahon's method might be useful for further explorations of this type. For instance, Louck [5] discusses relations of the expansion coefficients $C_k(A)$ to hypergeometric series, to Clebsch-Gordan coefficients, to even powers of Vandermonde determinants, and to partitions. However, one might need to extend MacMahon's machinery by new elimination rules such as

$$\underset{=}{\Omega}\, \frac{e^{u\lambda}}{1 - v/\lambda} = e^{uv} \quad (u, v \in \mathbb{C}).$$

Similar Omega elimination rules involving non-rational expressions have been already applied successfully in [1].

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