# MacMahon's Partition Analysis VII: Constrained Compositions

George E. Andrews, Peter Paule, and Axel Riese

Abstract. Our object here is to examine compositions under constraints. From this view, classical partitions arise from one set of constraints. We shall show that other quite different conditions give rise to constrained compositions with elegant generating functions that merit further investigation.

## 1. Introduction

In classical combinatorics as considered by MacMahon [**6**, Vol. 1, p. 3], the generating function for the homogeneous symmetric functions $h_n(x_1, x_2, \ldots, x_r)$ is given by

$$\sum_{n=0}^{\infty} h_n(x_1, x_2, \ldots, x_r)\, t^n = \frac{1}{(1 - x_1 t)(1 - x_2 t) \cdots (1 - x_r t)} \,.$$

Or we may say that

$$\sum x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r} = \frac{1}{(1 - x_1)(1 - x_2) \cdots (1 - x_r)}$$

is the fully parameterized generating function for compositions. In other words, every composition (i.e. ordered sum of non-negative integers) has its own unique term in the expansion.

To move from compositions (unordered sums) to partitions (sums with the sequence of parts non-increasing) MacMahon introduced his technique of Partition Analysis. We have provided previous applications of this method [**1, 2, 3, 4**] with an account of our implementation of it in [**2, 3**]. So here we content ourselves with the definition of MacMahon's operator $\Omega_{\geqq}$ [**6**, Vol. 2, p. 92]:

$$\underset{\geqq}{\Omega} \sum_{n_1, \ldots, n_r = -\infty}^{\infty} A_{n_1 n_2 \ldots n_r} \lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_r^{n_r} = \sum_{n_1, \ldots, n_r \geq 0} A_{n_1 n_2 \ldots n_r} \,.$$

With this tool, we have provided fully parameterized solutions of a number of problems. For example, suppose we want to generate partitions instead of compositions. Then as MacMahon [**6**, Vol. 2, p. 97] has shown us, we may proceed

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directly:

$$\begin{aligned}
&\sum_{a_1\geq a_2\geq\cdots\geq a_r\geq 0} x_1^{a_1}x_2^{a_2}\cdots x_r^{a_r}\\
&=\underset{\geqq}{\Omega}\sum_{a_1,a_2,\ldots,a_r\geq 0} x_1^{a_1}x_2^{a_2}\cdots x_r^{a_r}\lambda_1^{a_1-a_2}\lambda_2^{a_2-a_3}\cdots\lambda_{r-1}^{a_{r-1}-a_r}\\
&=\underset{\geqq}{\Omega}\frac{1}{(1-x_1\lambda_1)\big(1-x_2\frac{\lambda_2}{\lambda_1}\big)\big(1-x_3\frac{\lambda_3}{\lambda_2}\big)\cdots\big(1-x_{r-1}\frac{\lambda_{r-1}}{\lambda_{r-2}}\big)\big(1-\frac{x_r}{\lambda_{r-1}}\big)}\\
&=\frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)\cdots(1-x_1x_2\cdots x_r)}\\
&=\sum_{n_1,n_2,\ldots,n_r\geq 0} x_1^{n_1+n_2+\cdots+n_r}x_2^{n_2+n_3+\cdots+n_r}\cdots x_{r-1}^{n_{r-1}+n_r}x_r^{n_r}\ .
\end{aligned}$$

Or (as we have done in [**1**]) we may choose to generate triples of integers $a_1, a_2, a_3$ so that they form the sides of a non-degenerate triangle (listed in non-increasing order). Then

$$\begin{aligned}
&\sum_{\substack{a_1\geq a_2\geq a_3>0\\ a_2+a_3>a_1}} x_1^{a_1}x_2^{a_2}x_3^{a_3}\\
&=\underset{\geqq}{\Omega}\sum_{a_1,a_2,a_3\geq 0} x_1^{a_1}x_2^{a_2}x_3^{a_3}\lambda_1^{a_1-a_2}\lambda_2^{a_2-a_3}\lambda_3^{a_2+a_3-a_1-1}\\
&=\underset{\geqq}{\Omega}\frac{\lambda_3^{-1}}{\big(1-x_1\frac{\lambda_1}{\lambda_3}\big)\big(1-x_2\frac{\lambda_2\lambda_3}{\lambda_1}\big)\big(1-x_3\frac{\lambda_3}{\lambda_2}\big)}\\
&=\frac{x_1x_2x_3}{(1-x_1x_2)(1-x_1x_2x_3)(1-x_1^2x_2x_3)}\\
&\qquad\qquad\qquad\qquad\text{(see [1] for details)}\\
&=\sum_{n_1,n_2,n_3\geq 0} x_1^{n_1+n_2+2n_3+1}x_2^{n_1+n_2+n_3+1}x_3^{n_2+n_3+1}\ .
\end{aligned}$$

In many instances we have studied previously such lovely parameterizations; see [**1, 2, 3, 4**]. Indeed this situation in its full generality is discussed by MacMahon [**6**, Vol. 2, pp. 107–114]. To understand fully his view of the general case, we quote extensively [**6**, Vol. 2, pp. 107–108].

> "The simple theory of unipartite partitions has been made to depend upon $s$ Diophantine inequalities
>
> $$\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \cdots \geqslant \alpha_s \geqslant 0\,,$$
>
> $s$ being an arbitrary integer.
>
> We enlarge the theory by making the integers $\alpha_1, \alpha_2, \alpha_3, \ldots$ depend upon a number of inequalities
>
> $$\begin{aligned}
> &A_1^{(1)}\alpha_1 + A_2^{(1)}\alpha_2 + \cdots + A_s^{(1)}\alpha_s \geqslant 0,\\
> &A_1^{(2)}\alpha_1 + A_2^{(2)}\alpha_2 + \cdots + A_s^{(2)}\alpha_s \geqslant 0,\\
> &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
> &A_1^{(r)}\alpha_1 + A_2^{(r)}\alpha_2 + \cdots + A_s^{(r)}\alpha_s \geqslant 0,
> \end{aligned}$$

which involve at most $rs$ numerical magnitudes $A$, each of which may be positive, zero, or negative; but in each inequality it is clear that one at least must be positive.

For all sets of numbers $\alpha_1, \alpha_2, \dots, \alpha_s$ which satisfy the inequalities we seek the sum

$$\sum X_1^{\alpha_1} X_2^{\alpha_2} \dots X_s^{\alpha_s}.$$

By a theorem of Hilbert it appears that there is in every case a finite number of ground or fundamental solutions of the inequalities, viz.:

$$\begin{array}{ccccc} \alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \dots & \alpha_s^{(1)}, \\ \alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & \dots & \alpha_s^{(2)}, \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_1^{(m)} & \alpha_2^{(m)} & \alpha_3^{(m)} & \dots & \alpha_s^{(m)}, \end{array}$$

such that every solution

$$\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_s$$

is of the form

$$\begin{aligned} \alpha_1 &= c_1\alpha_1^{(1)} + c_2\alpha_1^{(2)} + \dots + c_m\alpha_1^{(m)}, \\ \alpha_2 &= c_1\alpha_2^{(1)} + c_2\alpha_2^{(2)} + \dots + c_m\alpha_2^{(m)}, \\ &\dots\dots\dots\dots\dots\dots \\ \alpha_s &= c_1\alpha_s^{(1)} + c_2\alpha_s^{(2)} + \dots + c_m\alpha_s^{(m)}, \end{aligned}$$

$c_1, c_2, c_3, \dots, c_m$ being positive integers.

This arises from the circumstance that every term

$$X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} \dots X_s^{\alpha_s}$$

of the summation is found to be expressible as a product

$$\begin{aligned} &\{X_1^{\alpha_1^{(1)}} X_2^{\alpha_2^{(1)}} X_3^{\alpha_3^{(1)}} \dots X_s^{\alpha_s^{(1)}}\}^{c_1} \\ \times &\{X_1^{\alpha_1^{(2)}} X_2^{\alpha_2^{(2)}} X_3^{\alpha_3^{(2)}} \dots X_s^{\alpha_s^{(2)}}\}^{c_2} \\ \times &\dots\dots\dots\dots\dots\dots \\ \times &\{X_1^{\alpha_1^{(m)}} X_2^{\alpha_2^{(m)}} X_3^{\alpha_3^{(m)}} \dots X_s^{\alpha_s^{(m)}}\}^{c_m}. \end{aligned}$$

Denoting this product by

$$P_1^{c_1} P_2^{c_2} P_3^{c_3} \dots P_m^{c_m},$$

the sum (or generating function of solutions) takes the form

$$\begin{aligned} &\big(1 - \{Q_1^{(1)} + Q_1^{(2)} + Q_1^{(3)} + \dots\} + \{Q_2^{(1)} + Q_2^{(2)} + Q_2^{(3)} + \dots\} - \\ &\quad \{Q_3^{(1)} + Q_3^{(2)} + Q_3^{(3)} + \dots\} + \dots\big) \,/ \\ &\big((1-P_1)(1-P_2)(1-P_3)\dots(1-P_m)\big), \end{aligned}$$

and we have what is termed a *syzygetic theory*."

MacMahon calls the $P_1, P_2, \dots, P_m$ the $m$ ground or fundamental solutions.

Now the examples we have considered so far are effectively syzygy-less. In this paper, we shall restrict our considerations to the instance of MacMahon's diophantine inequalities where $r = s$, i.e. the number of diophantine inequalities equals the number of integer variables. We remark that this restriction applies to our examples of compositions, partitions and integer sided triangles.

In Section 2 we examine the two variable system of diophantine inequalities that exist tacitly in the 2000 Putnam Examination problem B3 [**5**, p. 728]. We both solve and generalize the problem using Partition Analysis. The resulting solutions naturally suggest a more combinatorial consideration, and we present this exploration in Section 3. The combinatorial analysis leads us to natural $T$-dimensional generalizations of the original two-dimensional problem in Section 4. Section 5 contains our ruminations on the implications of our Section 4 results for further development of the Omega package, our Mathematica implementation of MacMahon's Partition Analysis.

## 2. The Putnam Problem

Problem B3 on the 2000 Putnam Examination [**5**, p. 728] reads as follows:

Problem. Let $A = \{(x, y) : 0 \leq x, y < 1\}$. For $(x, y) \in A$, let

$$S(x,y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$

where the sum ranges over all pairs $(m, n)$ of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{\{(x,y)\to(1,1),(x,y)\in A\}} (1-x^2y)(1-xy^2)S(x,y)\,.$$

This problem is easily solved using MacMahon's Partition Analysis; for the corresponding automatic evaluation of the limit see Section 3. Indeed the only formula required is the fourth rule from Section 348 of [**6**, Vol. 2, p. 102],

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda^2 x)\left(1-\frac{y}{\lambda}\right)} = \frac{1+xy}{(1-x)(1-xy^2)}\,.$$

Hence

$$\begin{aligned}
1 + S(x,y) &= \sum_{\substack{m,n\geq 0\\ 2m\geq n,\, 2n\geq m}} x^m y^n \\
&= \underset{\geq}{\Omega} \sum_{m,n\geq 0} x^m y^n \lambda_1^{2m-n} \lambda_2^{2n-m} \\
&= \underset{\geq}{\Omega} \frac{1}{\left(1-x\frac{\lambda_1^2}{\lambda_2}\right)\left(1-y\frac{\lambda_2^2}{\lambda_1}\right)} \\
&= \underset{\geq}{\Omega} \frac{1+xy\lambda_2}{\left(1-\frac{x}{\lambda_2}\right)(1-xy^2\lambda_2^3)} \\
&= \underset{\geq}{\Omega} \left( \frac{1}{\left(1-\frac{x}{\lambda_2}\right)(1-y\lambda_2^2)} - \frac{y\lambda_2^2}{(1-y\lambda_2^2)(1-xy^2\lambda_2^3)} \right) \\
&= \frac{1+xy}{(1-y)(1-x^2y)} - \frac{y}{(1-y)(1-xy^2)} \\
&= \frac{1+xy+x^2y^2}{(1-xy^2)(1-x^2y)}\,.
\end{aligned}$$

Immediately we see that

$$\lim_{\{(x,y)\to(1,1),(x,y)\in A\}}(1-xy^2)(1-x^2y)S(x,y)=3\,.$$

This suggests immediately that we should consider integers $K, L \geq 2$ and

$$\begin{aligned}S_{K,L}(x,y) &= \sum_{\substack{m,n\geq 0\\ Km\geq n, Ln\geq m}} x^m y^n \\ &= \underset{\geq}{\Omega} \sum_{m,n\geq 0} x^m y^n\, \lambda_1^{Km-n}\,\lambda_2^{Ln-m} \\ &= \underset{\geq}{\Omega} \frac{1}{\left(1-x\frac{\lambda_1^K}{\lambda_2}\right)\left(1-y\frac{\lambda_2^L}{\lambda_1}\right)}\,.\end{aligned}$$

And now we require the seventh formula in Section 348 of [**6**, Vol. 2, p. 102],

$$\begin{aligned}&\underset{\geq}{\Omega} \frac{1}{(1-\lambda^s x)\left(1-\frac{y}{\lambda}\right)} = \frac{1+xy\frac{1-y^{s-1}}{1-y}}{(1-x)(1-xy^s)} \\ &= \frac{1}{(1-x)(1-y)} - \frac{y}{(1-y)(1-xy^s)}, \qquad (s\geq 1)\,.\end{aligned}$$

Therefore

$$\begin{aligned}S_{K,L}(x,y) &= \underset{\geq}{\Omega}\left(\frac{1}{\left(1-\frac{x}{\lambda_2}\right)(1-y\lambda_2^L)} - \frac{y\lambda_2^L}{(1-y\lambda_2^L)(1-xy^K\lambda_2^{KL-1})}\right) \\ &= \frac{1+xy\frac{1-x^{L-1}}{1-x}}{(1-y)(1-x^Ly)} - \frac{y}{(1-y)(1-xy^K)} \\ &= \frac{1+xy\frac{(1-x^L)(1-y^K)}{(1-x)(1-y)} - xy^K - x^Ly}{(1-xy^K)(1-x^Ly)} \\ &= \left\{1+\sum_{\substack{1\leq i\leq L,\ 1\leq j\leq K\\ (L,1)\neq(i,j)\neq(1,K)}} x^i y^j\right\}\Big/\left((1-xy^K)(1-x^Ly)\right)\,.\end{aligned}$$

## 3. Experiments with the Omega Package

A fundamental feature of MacMahon's Partition Analysis consists in the fact that it is ideally suited for being supplemented by computer algebra methods.

In [**2**] we presented a corresponding new algorithmic approach to MacMahon's method together with a Mathematica implementation. In [**3**] this approach and the Omega package have been improved significantly. It is this version that will be used below in order to demonstrate how the Omega package can be used for exploration and generalization of the Putnam problem.

Before starting computations one has to load the package by

```
In[1]:= <<Omega2.m
Out[1]= Axel Riese's Omega implementation version 2.30 loaded
```

We note that the package is freely available from the Web via *http://www.risc.uni-linz.ac.at/research/combinat/risc/software/Omega.*

As the first application we show how the original Putnam problem discussed in detail in Section 2 can be solved automatically; i.e. without any thought only by applying commands of the package.

First of all, note that $1 + S(x,y) = S_{2,2}(x,y)$. Already the preprocessing step, i.e. the conversion of the $S_{2,2}(x,y)$ sum into an Omega expression is carried out automatically:

```
In[2]:= OSum[x^m y^n, {2m ≥ n, 2n ≥ m}, λ]

        Assuming m ≥ 0
        Assuming n ≥ 0
```

Out[2]=

$$\underset{\lambda_1,\lambda_2}{\Omega_{\geq}} \frac{1}{\left(1 - \frac{x\,\lambda_1^2}{\lambda_2}\right)\left(1 - \frac{y\,\lambda_2^2}{\lambda_1}\right)}$$

Now the task of eliminating $\lambda_1$ and $\lambda_2$ is done in one stroke by the command

```
In[3]:= OR[%]

        Eliminating λ2...
        Eliminating λ1...
```

Out[3]=

$$\frac{1 + x\,y + x^2\,y^2}{(1 - x^2\,y)\,(1 - x\,y^2)}$$

We have seen in Section 2 that this solves Problem B3 from the 2000 Putnam Exam.

Already from this elementary example it is evident that the $(K, L)$ generalization from Section 2 can be discovered without any effort. We restrict to showing only two cases, $(K, L) = (4, 2)$ and $(K, L) = (3, 4)$:

```
In[4]:= OR[ OSum[x^m y^n, {4m ≥ n, 2n ≥ m}, λ] ]
```

Out[4]=

$$\frac{1 + x\,y + x\,y^2 + x^2\,y^2 + x\,y^3 + x^2\,y^3 + x^2\,y^4}{(1 - x^2\,y)\,(1 - x\,y^4)}$$

```
In[5]:= OR[ OSum[x^m y^n, {3m ≥ n, 4n ≥ m}, λ] ]
```

Out[5]=

$$\frac{1 + x\,y + x^2\,y + x^3\,y + x\,y^2 + x^2\,y^2 + x^3\,y^2 + x^4\,y^2 + x^2\,y^3 + x^3\,y^3 + x^4\,y^3}{(1 - x^4\,y)\,(1 - x\,y^3)}$$

We remark that computations of this type are done by the package within a few seconds on an SGI Octane using Mathematica 4.0.1. However, it turns out that the computations related to Theorem 1 discussed in Section 4 are more involved as we will show later in this section.

At the beginning of our efforts to view the Putnam problem in a more general setting, we tried a number of possible systems of linear diophantine inequalities. For example, one natural way to extend the Putnam problem would be to consider $T$ variables $n_1, n_2, \ldots, n_T$ with $n_i \leq 2n_j$ for each $i$ and $j$. We can ask Omega to consider the next case, $T = 3$:

```
In[6]:= OR[ OSum[x1^n1 x2^n2 x3^n3, {n1 ≤ 2n2, n2 ≤ 2n1, n1 ≤ 2n3, n3 ≤ 2n1, n2 ≤ 2n3,
                n3 ≤ 2n2}, λ] ]
```

Out[6]=

$$
\begin{aligned}
&(1 + x_1 x_2 x_3 + x_1^2 x_2^2 x_3^2 - x_1^3 x_2^3 x_3^2 - x_1^3 x_2^2 x_3^3 - x_1^2 x_2^3 x_3^3 - 2\, x_1^3 x_2^3 x_3^3 - x_1^4 x_2^3 x_3^3 - \\
&\quad x_1^3 x_2^4 x_3^3 - x_1^4 x_2^4 x_3^3 - x_1^3 x_2^3 x_3^4 - x_1^4 x_2^3 x_3^4 - x_1^3 x_2^4 x_3^4 + x_1^5 x_2^4 x_3^4 + x_1^4 x_2^5 x_3^4 + \\
&\quad x_1^5 x_2^5 x_3^4 + x_1^4 x_2^4 x_3^5 + x_1^5 x_2^4 x_3^5 + x_1^4 x_2^5 x_3^5 + 2\, x_1^5 x_2^5 x_3^5 + x_1^6 x_2^5 x_3^5 + x_1^5 x_2^6 x_3^5 + \\
&\quad x_1^5 x_2^5 x_3^6 - x_1^6 x_2^6 x_3^6 - x_1^7 x_2^7 x_3^7 - x_1^8 x_2^8 x_3^8) \,/ \\
&((1 - x_1^2 x_2 x_3)(1 - x_1 x_2^2 x_3)(1 - x_1^2 x_2^2 x_3)(1 - x_1 x_2 x_3^2)(1 - x_1^2 x_2 x_3^2)(1 - x_1 x_2^2 x_3^2))
\end{aligned}
$$

Now this does not have the simplicity associated with the original Putnam problem. However, it does suggest that somehow we are confronted with a rather messy amalgam of two nice problems. One having as denominator

$$(1 - x_1^2 x_2 x_3)(1 - x_1 x_2^2 x_3)(1 - x_1 x_2 x_3^2)$$

and the other

$$(1 - x_1^2 x_2^2 x_3)(1 - x_1^2 x_2 x_3^2)(1 - x_1 x_2^2 x_3^2)\,.$$

If we now look back to our solution of the Putnam problem, we see that a fully parameterized solution follows from the Partition Analysis solution

$$
\begin{aligned}
&\frac{1 + xy + x^2 y^2}{(1 - x^2 y)(1 - x y^2)} \\
&= \sum_{r,s \geq 0} x^{2s+r} y^{2r+s} + \sum_{r,s \geq 0} x^{2s+r+1} y^{2r+s+1} + \sum_{r,s \geq 0} x^{2s+r+2} y^{2r+s+2} \\
&= \sum_{\substack{m,n \geq 0 \\ 2m \geq n,\ 2n \geq m \\ m+n \equiv 0 \pmod 3}} x^m y^n + \sum_{\substack{m,n \geq 0 \\ 2m \geq n,\ 2n \geq m \\ m+n \equiv 1 \pmod 3}} x^m y^n + \sum_{\substack{m,n \geq 0 \\ 2m \geq n,\ 2n \geq m \\ m+n \equiv 2 \pmod 3}} x^m y^n \\
&= \sum_{\substack{m,n \geq 0 \\ 2m \geq n,\ 2n \geq m}} x^m y^n\,.
\end{aligned}
$$

Indeed, from

$$m = 2s + r, \quad n = s + 2r$$

we see that equivalently

$$s = \frac{2m - n}{3}, \quad r = \frac{2n - m}{3}\,,$$

and this suggests that the right inequalities equivalent to $r \geq 0$ and $s \geq 0$ are $2m \geq n$ and $2n \geq m$.

So an expansion like

$$
\begin{aligned}
&\frac{1}{(1 - x_1^2 x_2 x_3)(1 - x_1 x_2^2 x_3)(1 - x_1 x_2 x_3^2)} \\
&= \sum_{N_1, N_2, N_3 \geq 0} x_1^{2N_1 + N_2 + N_3} x_2^{N_1 + 2N_2 + N_3} x_3^{N_1 + N_2 + 2N_3}
\end{aligned}
$$

suggests we consider

$$n_1 = 2N_1 + N_2 + N_3, \quad n_2 = N_1 + 2N_2 + N_3, \quad n_3 = N_1 + N_2 + 2N_3$$

which is equivalent to

$$N_1 = \frac{3n_1 - n_2 - n_3}{4}, \quad N_2 = \frac{3n_2 - n_1 - n_3}{4}, \quad N_3 = \frac{3n_3 - n_1 - n_2}{4}\,,$$

and this suggests that the right inequalities equivalent to $N_1, N_2, N_3 \geq 0$ should be

$$3n_1 \geq n_2 + n_3, \quad 3n_2 \geq n_1 + n_3, \quad 3n_3 \geq n_1 + n_2 .$$

When we ask Omega to provide the generating function related to these inequalities we find what is shown in (1) below.

In addition, an expansion like

$$\frac{1}{(1-x_1^2x_2^2x_3)(1-x_1^2x_2x_3^2)(1-x_1x_2^2x_3^2)} = \sum_{N_1,N_2,N_3\geq 0} x_1^{2N_1+2N_2+N_3} x_2^{2N_1+N_2+2N_3} x_3^{N_1+2N_2+2N_3}$$

leads from

$$n_1 = 2N_1 + 2N_2 + N_3, \quad n_2 = 2N_1 + N_2 + 2N_3, \quad n_3 = N_1 + 2N_2 + 2N_3$$

to

$$N_1 = \frac{2n_1 + 2n_2 - 3n_3}{5}, \quad N_2 = \frac{2n_1 + 2n_3 - 3n_2}{5}, \quad N_3 = \frac{2n_2 + 2n_3 - 3n_1}{5},$$

and this suggests that the right inequalities equivalent to $N_1, N_2, N_3 \geq 0$ should be

$$2n_1 + 2n_2 \geq 3n_3, \quad 2n_1 + 2n_3 \geq 3n_2, \quad 2n_2 + 2n_3 \geq 3n_1 .$$

When we ask Omega to provide the generating function related to these inequalities we find what is shown in (2) below.

In both cases Omega tells us that succinct and appealing generating functions correspond to each of these extensions of the Putnam problem to 3 dimensions.

Once these observations have been provided by Omega, one can try a few more cases which lead directly to conjecturing the results in the next section. This process — and also the run-time behavior mentioned above — is illustrated by the following examples. In all these applications we specify the dimension $T$ and the parameters $k$ and $c$ according to Theorem 1 and the corresponding inequalities (3).

- The case $T = 3$, $k = 1$, $c = 2$ is solved quickly:

In[7]:= `OR[ OSum[`$x_1^{n_1}\, x_2^{n_2}\, x_3^{n_3}$`, {`$n_1+n_2 \leq 3n_3,\ n_1+n_3 \leq 3n_2,\ n_2+n_3 \leq 3n_1$`}, λ] ] // Timing`

Out[7]=

$$\left\{5.24\ \text{Second}, \frac{1 + x_1\,x_2\,x_3 + x_1^2\,x_2^2\,x_3^2 + x_1^3\,x_2^3\,x_3^3}{(1-x_1^2\,x_2\,x_3)\,(1-x_1\,x_2^2\,x_3)\,(1-x_1\,x_2\,x_3^2)}\right\} \tag{1}$$

- For $T = 3$, $k = 1$, $c = 3$ the computation takes considerably longer:

In[8]:= `OR[ OSum[`$x_1^{n_1}\, x_2^{n_2}\, x_3^{n_3}$`, {`$n_1+n_2 \leq 4n_3,\ n_1+n_3 \leq 4n_2,\ n_2+n_3 \leq 4n_1$`}, λ] ] // Timing`

Out[8]=

$$\{100.55\ \text{Second},$$
$$(1 + x_1\,x_2\,x_3 + x_1^2\,x_2\,x_3 + x_1\,x_2^2\,x_3 + x_1^2\,x_2^2\,x_3 + x_1\,x_2\,x_3^2 + x_1^2\,x_2\,x_3^2 + x_1\,x_2^2\,x_3^2 +$$
$$x_1^2\,x_2^2\,x_3^2 + x_1^3\,x_2^2\,x_3^2 + x_1^2\,x_2^3\,x_3^2 + x_1^3\,x_2^3\,x_3^2 + x_1^2\,x_2^2\,x_3^3 + x_1^3\,x_2^2\,x_3^3 + x_1^2\,x_2^3\,x_3^3 +$$
$$x_1^3\,x_2^3\,x_3^3 + x_1^4\,x_2^4\,x_3^3 + x_1^4\,x_2^3\,x_3^4 + x_1^3\,x_2^4\,x_3^4 + x_1^4\,x_2^4\,x_3^4)\ /$$
$$((1-x_1^3\,x_2\,x_3)\,(1-x_1\,x_2^3\,x_3)\,(1-x_1\,x_2\,x_3^3))\}$$

- For $T = 3$, $k = 1$, $c = 4$ we finally run out of memory.
- Next we turn to the case $T = 3$, $k = 2$, $c = 1$:

```
In[9]:= OR[ OSum[x1^n1 x2^n2 x3^n3, {2n1+2n2 ≥ 3n3, 2n1+2n3 ≥ 3n2, 2n2+2n3 ≥ 3n1}, λ] ] //
        Timing
```

Out[9]=

$$\left\{74.59\ \text{Second}, \frac{1+x_1x_2x_3+x_1^2x_2^2x_3^2+x_1^3x_2^3x_3^3+x_1^4x_2^4x_3^4}{(1-x_1^2x_2^2x_3)(1-x_1^2x_2x_3^2)(1-x_1x_2^2x_3^2)}\right\} \tag{2}$$

- Concerning run-time, things are getting worse for $T=3$, $k=2$, $c=3$:

```
In[10]:= OR[ OSum[x1^n1 x2^n2 x3^n3, {2n1+2n2 ≤ 5n3, 2n1+2n3 ≤ 5n2, 2n2+2n3 ≤ 5n1}, λ] ] //
         Timing
```

Out[10]=

$$\left\{785.57\ \text{Second}, \frac{1+x_1x_2x_3+x_1^2x_2^2x_3^2+x_1^3x_2^3x_3^3+x_1^4x_2^4x_3^4+x_1^5x_2^5x_3^5+x_1^6x_2^6x_3^6}{(1-x_1^3x_2^2x_3^2)(1-x_1^2x_2^3x_3^2)(1-x_1^2x_2^2x_3^3)}\right\}$$

- For $T=3$, $k=3$, $c=1$ we need more than 30 minutes:

```
In[11]:= OR[ OSum[x1^n1 x2^n2 x3^n3, {3n1+3n2 ≥ 4n3, 3n1+3n3 ≥ 4n2, 3n2+3n3 ≥ 4n1}, λ] ] //
         Timing
```

Out[11]=

$$\begin{aligned}\{&1959.27\ \text{Second},\\ &(1+x_1x_2x_3+x_1^2x_2^2x_3+x_1^2x_2x_3^2+x_1x_2^2x_3^2+x_1^2x_2^2x_3^2+x_1^3x_2^2x_3^2+x_1^2x_2^3x_3^2+\\ &\quad x_1^3x_2^3x_3^2+x_1^2x_2^2x_3^3+x_1^3x_2^2x_3^3+x_1^2x_2^3x_3^3+x_1^3x_2^3x_3^3+x_1^4x_2^3x_3^3+x_1^3x_2^4x_3^3+\\ &\quad x_1^4x_2^4x_3^3+x_1^3x_2^3x_3^4+x_1^4x_2^3x_3^4+x_1^3x_2^4x_3^4+x_1^4x_2^4x_3^4+x_1^5x_2^4x_3^4+x_1^4x_2^5x_3^4+\\ &\quad x_1^4x_2^4x_3^5+x_1^5x_2^5x_3^5+x_1^6x_2^5x_3^5+x_1^5x_2^6x_3^5+x_1^5x_2^5x_3^6+x_1^6x_2^6x_3^6)\ /\\ &((1-x_1^3x_2^3x_3)(1-x_1^3x_2x_3^3)(1-x_1x_2^3x_3^3))\}\end{aligned}$$

The main reason for this run-time explosion is that in our implementation we eliminate one $\lambda_i$ after the other. It emerges that even for applications with rather simple solutions, the size of the intermediate results after eliminating some of the $\lambda_i$'s usually grows enormously. Therefore we are currently working on a new elimination strategy which avoids this problem. For this we split the term which the $\Omega$ operator acts on additively into subterms whenever possible. In each of these subterms we then eliminate the $\lambda_i$ that is optimal in a certain sense and proceed recursively. Clearly elimination works much faster with this approach. However, in the end we are faced with a sum of many multivariate rational functions, each of quite simple form. It turns out that trying to bring the sum over a common denominator is a real bottle-neck that often cannot be accomplished by Mathematica.

In the following we demonstrate the speed-up achieved with an unofficial prototype which utilizes this alternative method.

```
In[1]:= <<Omega3.m
Out[1]= Axel Riese's Omega prototype version 3.3 loaded
```

- First we take another look at the $T=3$, $k=2$, $c=3$ case:

```
In[2]:= OR[ OSum[x1^n1 x2^n2 x3^n3, {2n1+2n2 ≤ 5n3, 2n1+2n3 ≤ 5n2, 2n2+2n3 ≤ 5n1}, λ] ] //
        Timing
```

Out[2]=

$$\left\{58.95\ \text{Second}, \frac{-1-x_1x_2x_3-x_1^2x_2^2x_3^2-x_1^3x_2^3x_3^3-x_1^4x_2^4x_3^4-x_1^5x_2^5x_3^5-x_1^6x_2^6x_3^6}{(-1+x_1^3x_2^2x_3^2)(-1+x_1^2x_2^3x_3^2)(-1+x_1^2x_2^2x_3^3)}\right\}$$

- The case $T=3$, $k=3$, $c=1$ now works surprisingly fast:

In[3]:= `OR[ OSum[`$x_1^{n_1}\, x_2^{n_2}\, x_3^{n_3}$`, {`$3n_1+3n_2 \geq 4n_3$`, `$3n_1+3n_3 \geq 4n_2$`, `$3n_2+3n_3 \geq 4n_1$`}, λ] ] // Timing`

Out[3]=

$$\{7.14\ \text{Second},$$
$$(-1 - x_1 x_2 x_3 - x_1^2 x_2^2 x_3 - x_1^2 x_2 x_3^2 - x_1 x_2^2 x_3^2 - x_1^2 x_2^2 x_3^2 - x_1^3 x_2^2 x_3^2 - x_1^2 x_2^3 x_3^2 -$$
$$x_1^3 x_2^3 x_3^2 - x_1^2 x_2^2 x_3^3 - x_1^3 x_2^2 x_3^3 - x_1^2 x_2^3 x_3^3 - x_1^3 x_2^3 x_3^3 - x_1^4 x_2^3 x_3^3 - x_1^3 x_2^4 x_3^3 -$$
$$x_1^4 x_2^4 x_3^3 - x_1^3 x_2^3 x_3^4 - x_1^4 x_2^3 x_3^4 - x_1^3 x_2^4 x_3^4 - x_1^4 x_2^4 x_3^4 - x_1^5 x_2^4 x_3^4 - x_1^4 x_2^5 x_3^4 -$$
$$x_1^4 x_2^4 x_3^5 - x_1^5 x_2^5 x_3^5 - x_1^6 x_2^5 x_3^5 - x_1^5 x_2^6 x_3^5 - x_1^5 x_2^5 x_3^6 - x_1^6 x_2^6 x_3^6) \,/$$
$$((-1 + x_1^3 x_2^3 x_3)(-1 + x_1^3 x_2 x_3^3)(-1 + x_1 x_2^3 x_3^3))\}$$

- And also the case $T = 3$, $k = 1$, $c = 4$ can be handled now:

In[4]:= `OR[ OSum[`$x_1^{n_1}\, x_2^{n_2}\, x_3^{n_3}$`, {`$n_1+n_2 \leq 5n_3$`, `$n_1+n_3 \leq 5n_2$`, `$n_2+n_3 \leq 5n_1$`}, λ] ] // Timing`

Out[4]=

$$\{643.86\ \text{Second},$$
$$(-1 - x_1 x_2 x_3 - x_1^2 x_2 x_3 - x_1^3 x_2 x_3 - x_1 x_2^2 x_3 - x_1^2 x_2^2 x_3 - x_1^3 x_2^2 x_3 - x_1 x_2^3 x_3 -$$
$$x_1^2 x_2^3 x_3 - x_1 x_2 x_3^2 - x_1^2 x_2 x_3^2 - x_1^3 x_2 x_3^2 - x_1 x_2^2 x_3^2 - x_1^2 x_2^2 x_3^2 - x_1^3 x_2^2 x_3^2 -$$
$$x_1^4 x_2^2 x_3^2 - x_1 x_2^3 x_3^2 - x_1^2 x_2^3 x_3^2 - x_1^3 x_2^3 x_3^2 - x_1^4 x_2^3 x_3^2 - x_1^2 x_2^4 x_3^2 - x_1^3 x_2^4 x_3^2 -$$
$$x_1^4 x_2^4 x_3^2 - x_1 x_2 x_3^3 - x_1^2 x_2 x_3^3 - x_1 x_2^2 x_3^3 - x_1^2 x_2^2 x_3^3 - x_1^3 x_2^2 x_3^3 - x_1^4 x_2^2 x_3^3 -$$
$$x_1^2 x_2^3 x_3^3 - x_1^3 x_2^3 x_3^3 - x_1^4 x_2^3 x_3^3 - x_1^2 x_2^4 x_3^3 - x_1^3 x_2^4 x_3^3 - x_1^4 x_2^4 x_3^3 - x_1^5 x_2^5 x_3^3 -$$
$$x_1^2 x_2^2 x_3^4 - x_1^3 x_2^2 x_3^4 - x_1^4 x_2^2 x_3^4 - x_1^2 x_2^3 x_3^4 - x_1^3 x_2^3 x_3^4 - x_1^4 x_2^3 x_3^4 - x_1^2 x_2^4 x_3^4 -$$
$$x_1^3 x_2^4 x_3^4 - x_1^4 x_2^4 x_3^4 - x_1^5 x_2^4 x_3^4 - x_1^4 x_2^5 x_3^4 - x_1^5 x_2^5 x_3^4 - x_1^5 x_2^3 x_3^5 - x_1^4 x_2^4 x_3^5 -$$
$$x_1^5 x_2^4 x_3^5 - x_1^3 x_2^5 x_3^5 - x_1^4 x_2^5 x_3^5 - x_1^5 x_2^5 x_3^5) \,/$$
$$((-1 + x_1^4 x_2 x_3)(-1 + x_1 x_2^4 x_3)(-1 + x_1 x_2 x_3^4))\}$$

## 4. The General Theorem

Our object here is to consider the non-negative integer solutions of

$$(k-c)(k(n_1+n_2+\cdots+n_T) - ((T-1)k+c)n_j) \geq 0, \qquad 1 \leq j \leq T, \tag{3}$$

where $k$ and $c$ are fixed unequal positive integers. Note that the only effect of multiplication by $(k-c)$ is to change the direction of the "$\geq$" depending on the sign of $(k-c)$. Furthermore we may assume that $\gcd(k,c) = 1$ because division of (3) by $\gcd(k,c)^2$ leaves the solutions unaltered.

Definition 1. We denote by $\mathcal{P}_T$ the set of all $T$-tuples of non-negative integers $(n_1, \ldots, n_T)$ that satisfy (3).

Definition 2. We define $A(q) = (\alpha_{i,j})_{1 \leq i,j \leq T}$ as the $(T \times T)$ matrix over rational numbers such that $\alpha_{i,i} = q$ and $\alpha_{i,j} = 1$ for $i \neq j$.

For the study of the set $\mathcal{P}_T$ we need various properties of the matrices $A(q)$, $q \in \mathbb{Q}$ being positive.

First of all, only $q = 1$ gives a singular matrix for which

$$A(1)^2 = T \cdot A(1)\,.$$

PROPOSITION 1. *For $q \neq 1$ the matrix $A(q)$ is non-singular and its inverse matrix is*

$$A(q)^{-1} = \frac{1}{(1-q)(T+q-1)}\, A(2-q-T)\,. \tag{4}$$

PROOF. We rewrite (4) as $A(q)^{-1} = \gamma \cdot (\beta_{i,j})_{1\leq i,j\leq T}$ with $\gamma = 1/((1-q)(T+q-1))$, $\beta_{i,i} = 2-q-T$ and $\beta_{i,j} = 1$ for $i \neq j$. Let $A(q) = (\alpha_{i,j})_{1\leq i,j\leq T}$ as in Definition 2. It suffices to show that

$$\sum_{h=1}^{T} \gamma \cdot \beta_{i,h} \cdot \alpha_{h,j} = \delta_{i,j} \qquad (1 \leq i,j \leq T)\,,$$

where $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$, if $i \neq j$. The sum equals

$$\frac{2-q-T}{(1-q)(T+q-1)}\, \alpha_{i,j} + \sum_{\substack{h=1\\h\neq i}}^{T} \frac{1}{(1-q)(T+q-1)}\, \alpha_{h,j}$$

which for $i \neq j$ reduces to

$$\frac{2-q-T}{(1-q)(T+q-1)} + \frac{q}{(1-q)(T+q-1)} + \frac{1}{(1-q)(T+q-1)} \sum_{\substack{h=1\\h\neq i,h\neq j}}^{T} 1 = 0\,,$$

whereas for $i = j$ it reduces to

$$\frac{2-q-T}{(1-q)(T+q-1)}\, q + \frac{1}{(1-q)(T+q-1)} \sum_{\substack{h=1\\h\neq i}}^{T} 1 = 1\,.$$

□

REMARK. See also Exercise 8 of [**7**, p. 411].

The singular and non-singular cases are coupled by the formula

$$A(q) = A(1) - (1-q)I\,, \tag{5}$$

where $I = (\delta_{i,j})_{1\leq i,j\leq T}$ denotes the identity matrix. Equation (5) together with Proposition 1 implies the analogous formula for $A(q)^{-1}$; namely, for $q \neq 1$

$$A(q)^{-1} = \frac{1}{(1-q)(T+q-1)}\, (A(1) - (T+q-1)I)\,. \tag{6}$$

Now we are ready to link these matrix considerations with our original problem.

PROPOSITION 2. *For $\boldsymbol{n} = (n_1, \ldots, n_T)$ the inequalities* (3) *can be rewritten in the form*

$$A\Big(\frac{c}{k}\Big)^{-1} \boldsymbol{n} \geq \boldsymbol{0}\,, \tag{7}$$

*where $\boldsymbol{0} = (0, \ldots, 0) \in \mathbb{N}^T$. In other word, the set of all $\boldsymbol{n} \in \mathbb{N}^T$ satisfying* (7) *is $\mathcal{P}_T$.*

PROOF. Obviously (3) is equivalent to

$$(k-c)(k\,A(1) - ((T-1)k+c)I)\,\boldsymbol{n} \geq \boldsymbol{0}\,. \tag{8}$$

By (6) we see that

$$\begin{aligned} k\,A(1) - ((T-1)k+c)I &= k\Big(A(1) - \big(T+\frac{c}{k}-1\big)I\Big) \\ &= (k-c)\big(T+\frac{c}{k}-1\big)A\Big(\frac{c}{k}\Big)^{-1}. \end{aligned}$$

Combining this with (8) gives (7), which completes the proof of Proposition 2. □

PROPOSITION 3. *Let $\boldsymbol{n}, \boldsymbol{i} \in \mathbb{N}^T$ such that*

$$\boldsymbol{n} = k\,A\Big(\frac{c}{k}\Big)\boldsymbol{m} + \boldsymbol{i} \tag{9}$$

*for some $\boldsymbol{m} \in \mathbb{N}^T$. Then $\boldsymbol{i} \in \mathcal{P}_T$ implies $\boldsymbol{n} \in \mathcal{P}_T$.*

PROOF. Multiplying both sides of (9) with $A(c/k)^{-1}$ gives

$$A\Big(\frac{c}{k}\Big)^{-1}\boldsymbol{n} = k\,\boldsymbol{m} + A\Big(\frac{c}{k}\Big)^{-1}\boldsymbol{i} \geq k\,\boldsymbol{m} \geq \boldsymbol{0}.$$

Applying Proposition 2 completes the proof. □

DEFINITION 3. Referring to Proposition 3, we say that $\boldsymbol{i} \in \mathcal{P}_T$ is a *reduction* of $\boldsymbol{n} \in \mathcal{P}_T$ if there exists an $\boldsymbol{m} \in \mathbb{N}^T$ such that

$$\boldsymbol{n} = k\,A\Big(\frac{c}{k}\Big)\boldsymbol{m} + \boldsymbol{i}.$$

We shall say that $\boldsymbol{n} \in \mathcal{P}_T$ is *irreducible* if it has no reduction except the trivial case where $\boldsymbol{m} = \boldsymbol{0}$.

Note that $n_1 + \cdots + n_T > i_1 + \cdots + i_T \geq 0$ if $\boldsymbol{i} = (i_1, \ldots, i_T)$ is a reduction of $\boldsymbol{n} = (n_1, \ldots, n_T)$; moreover, the reduction relation is transitive.

DEFINITION 4. We let $\mathcal{S}_T$ denote that subset of $\mathcal{P}_T$ consisting of those $(i_1, \ldots, i_T)$ such that: (1) $\max\limits_{1\leq r,s\leq T} |i_r - i_s| < |k-c|$ and (2) $\max\limits_{1\leq s\leq T} i_s < k(T-1)+c$.

DEFINITION 5. We let $\mathcal{I}_T$ denote that subset of $\mathcal{S}_T$ consisting of all the irreducibles in $\mathcal{S}_T$.

REMARK. It is clear that $\mathcal{S}_T$ (and consequently $\mathcal{I}_T$) are finite sets. Indeed by condition (2) in Definition 4 we see that $\mathcal{S}_T$ certainly has at most $(k(T-1)+c)^T$ elements.

The next proposition is immediate from above. Nevertheless it will be convenient to state it explicitly.

PROPOSITION 4. *For any $\boldsymbol{l} = (l_1, \ldots, l_T)$, $\boldsymbol{m} = (m_1, \ldots, m_T) \in \mathbb{Q}^T$ such that*

$$\boldsymbol{l} = k\,A\Big(\frac{c}{k}\Big)\boldsymbol{m} \tag{10}$$

*we have*

$$\boldsymbol{m} = \frac{1}{(k-c)((T-1)k+c)}\,(k\,A(1) - ((T-1)k+c)I)\,\boldsymbol{l}. \tag{11}$$

*In other words, for $1 \leq j \leq T$,*

$$m_j = \frac{k\sum_{i=1}^{T} l_i - ((T-1)k+c)l_j}{(k-c)((T-1)k+c)}. \tag{12}$$

PROOF. From (10), $\boldsymbol{m} = k^{-1}A(c/k)^{-1}\boldsymbol{l}$ which by (6) equals the right hand side of (11). The reformulation of (11) to (12) is straight-forward. □

PROPOSITION 5. *Every $\boldsymbol{n} \in \mathcal{P}_T$ has a reduction to an element of $\mathcal{S}_T$.*

PROOF. Define

$$\boldsymbol{N} := \frac{1}{k}\, A\Big(\frac{c}{k}\Big)^{-1} \boldsymbol{n}\,. \tag{13}$$

By Proposition 2, $\boldsymbol{N} \in \mathbb{Q}^T$ with non-negative entries. Define

$$\boldsymbol{\mu} := \boldsymbol{N} - \lfloor \boldsymbol{N} \rfloor\,,$$

where the floor function is taken component-wise. Finally we define

$$\boldsymbol{i} := k\, A\Big(\frac{c}{k}\Big)\, \boldsymbol{\mu}\,. \tag{14}$$

Obviously $\boldsymbol{i} \in \mathbb{Q}^T$ with non-negative entries; but in addition we have $\boldsymbol{i} \in \mathbb{N}^T$ since

$$\boldsymbol{i} = k\, A\Big(\frac{c}{k}\Big)\, \boldsymbol{N} - k\, A\Big(\frac{c}{k}\Big) \lfloor \boldsymbol{N} \rfloor = \boldsymbol{n} - k\, A\Big(\frac{c}{k}\Big) \lfloor \boldsymbol{N} \rfloor \tag{15}$$

by (13). Hence (15) gives

$$\boldsymbol{n} = k\, A\Big(\frac{c}{k}\Big)\, \boldsymbol{m} + \boldsymbol{i} \quad \text{with } \boldsymbol{m} = \lfloor \boldsymbol{N} \rfloor \in \mathbb{N}^T\,.$$

Therefore $\boldsymbol{i}$ is a reduction of $\boldsymbol{n}$ if $\boldsymbol{i} \in \mathcal{P}_T$. But this is easy to check since

$$A\Big(\frac{c}{k}\Big)^{-1} \boldsymbol{i} = A\Big(\frac{c}{k}\Big)^{-1} \boldsymbol{n} - k\, \boldsymbol{m} = k\, \boldsymbol{N} - k\, \lfloor \boldsymbol{N} \rfloor \geq \boldsymbol{0}$$

and $\boldsymbol{i} \in \mathcal{P}_T$ by Proposition 2.

Finally it remains to show that $\boldsymbol{i} \in \mathcal{S}_T$. Suppose $\boldsymbol{i} = (i_1, \ldots, i_T)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_T)$, hence (14) implies

$$i_r - i_s = (c-k)\mu_r + (k-c)\mu_s\,,$$

and since $0 \leq \mu_r, \mu_s < 1$, we see immediately that

$$|i_r - i_s| < |k - c|\,.$$

Since each component $\mu_j$ of $\boldsymbol{\mu}$ lies in $[0,1)$ we see, using (14) again, that

$$0 \leq i_j < k + \cdots + k + c + k + \cdots + k = (T-1)k + c\,.$$

This completes the proof of Proposition 5. □

We have two further steps before we are ready for the generating function associated with $\mathcal{P}_T$.

PROPOSITION 6. *Every $\boldsymbol{n} \in \mathcal{P}_T$ has a reduction to an element of $\mathcal{I}_T$.*

PROOF. By Proposition 5, $\boldsymbol{n} = (n_1, \ldots, n_T)$ has a reduction $(i_1, \ldots, i_T) \in \mathcal{S}_T$. If $(i_1, \ldots, i_T)$ is irreducible we are done. If not, then there is $(m_1, \ldots, m_T) \in \mathcal{P}_T$ that is a non-trivial reduction of $(i_1, \ldots, i_T)$. By Definition 3, $\sum i_j > \sum m_j \geq 0$. If $(m_1, \ldots, m_T) \in \mathcal{S}_T$, then let $(i'_1, \ldots, i'_T) = (m_1, \ldots, m_T)$. If $(m_1, \ldots, m_T) \notin \mathcal{S}_T$, then by Proposition 5, it has a reduction $(i'_1, \ldots, i'_T) \in \mathcal{S}_T$. Hence since the reduction relation is clearly transitive, we see that $(i'_1, \ldots, i'_T) \in \mathcal{S}_T$ is a non-trivial reduction of $(i_1, \ldots, i_T)$. If $(i'_1, \ldots, i'_T)$ is irreducible then we are done because it too is a reduction of $(n_1, \ldots, n_T)$. Now we can repeat this construction producing a sequence of $T$-tuples in $\mathcal{S}_T$. For each entry the sum of the components forms a decreasing sequence of non-negative integers. Thus the sequence must terminate, and this is only possible provided the final entry we produce is irreducible. Consequently this last entry is in $\mathcal{I}_T$ and is a reduction of $(n_1, \ldots, n_T)$. □

PROPOSITION 7. *The reduction of* $\boldsymbol{n} \in \mathcal{P}_T$ *lying in* $\mathcal{I}_T$ *is unique.*

PROOF. By Proposition 6 we know there is at least one reduction of $\boldsymbol{n} = (n_1, \ldots, n_T)$ lying in $\mathcal{I}_T$. Suppose there were two distinct ones: $(i_1, \ldots, i_T)$ and $(i'_1, \ldots, i'_T)$. So there are non-negative integers $m_1, \ldots, m_T, m'_1, \ldots, m'_T$ such that for all $1 \le j \le T$

$$n_j = km_1 + \cdots + km_{j-1} + cm_j + km_{j+1} + \cdots + km_T + i_j$$

and

$$n_j = km'_1 + \cdots + km'_{j-1} + cm'_j + km'_{j+1} + \cdots + km'_T + i'_j \, .$$

There must be at least one index $r$ where $m_r > m'_r$. Otherwise

$$i_j = k(m'_1 - m_1) + \cdots + c(m'_j - m_j) + \cdots + k(m'_T - m_T) + i'_j$$

would be a non-trivial reduction of $(i_1, \ldots, i_T)$ (remember these are two distinct elements of $\mathcal{I}_T$) contradicting the irreducibility of $(i_1, \ldots, i_T)$.

By symmetry there must be at least one index $s$ where $m'_s > m_s$. Therefore

$$(m_r - m'_r) + (m'_s - m_s) \ge 2 \, , \tag{16}$$

by the integrality of the $m_h$ and $m'_h$.

Now by Proposition 4,

$$m_r = \frac{k \sum_{j=1}^{T} (n_j - i_j) - ((T-1)k + c)(n_r - i_r)}{(k-c)((T-1)k+c)} \, .$$

Therefore

$$m_r - m_s = \frac{(n_s - i_s) - (n_r - i_r)}{(k-c)}$$

and similarly

$$m'_s - m'_r = \frac{(n_r - i'_r) - (n_s - i'_s)}{(k-c)} \, .$$

So by (16)

$$\begin{aligned}
2 &\le |(m_r - m'_r) + (m'_s - m_s)| \\
&= |(m_r - m_s) + (m'_s - m'_r)| \\
&= \Big| \frac{(n_s - i_s) - (n_r - i_r)}{k-c} + \frac{(n_r - i'_r) - (n_s - i'_s)}{k-c} \Big| \\
&= \Big| \frac{i_r - i_s}{k-c} + \frac{i'_s - i'_r}{k-c} \Big| \\
&\le \frac{|i_r - i_s|}{|k-c|} + \frac{|i'_s - i'_r|}{|k-c|} \\
&< 1 + 1 = 2 \, ,
\end{aligned}$$

by Property (1) of Definition 4. This contradiction proves that there cannot be two distinct reductions of $(n_1, \ldots, n_T)$ within $\mathcal{I}_T$. $\square$

We are now prepared to obtain the $T$-variable generating function for $\mathcal{P}_T$.

THEOREM 1.

$$\sum_{(n_1, \ldots, n_T) \in \mathcal{P}_T} x_1^{n_1} x_2^{n_2} \cdots x_T^{n_T} = \frac{\sum_{(i_1, \ldots, i_T) \in \mathcal{I}_T} x_1^{i_1} x_2^{i_2} \cdots x_T^{i_T}}{\prod_{j=1}^{T} (1 - (x_1 x_2 \cdots x_T)^k x_j^{c-k})} \, .$$

PROOF. By Propositions 3 and 7,

$$\begin{aligned}
&\sum_{(n_1,\dots,n_T)\in\mathcal{P}_T} x_1^{n_1}x_2^{n_2}\cdots x_T^{n_T}\\
&=\sum_{m_1,m_2,\dots,m_T\geq 0}\ \sum_{(i_1,\dots,i_T)\in\mathcal{I}_T}\ \prod_{j=1}^{T} x_j^{km_1+\cdots+cm_j+\cdots+km_T+i_j}\\
&=\frac{\sum_{(i_1,\dots,i_T)\in\mathcal{I}_T} x_1^{i_1}x_2^{i_2}\cdots x_T^{i_T}}{\prod_{j=1}^{T}(1-(x_1x_2\cdots x_T)^k x_j^{c-k})}\ .
\end{aligned}$$

□

COROLLARY 1. *If $k$ and $c$ are positive integers with $|k-c|=1$, then with $X=x_1x_2\cdots x_T$*

$$\sum_{(n_1,\dots,n_T)\in\mathcal{P}_T} x_1^{n_1}x_2^{n_2}\cdots x_T^{n_T}=\frac{1-X^{k(T-1)+c}}{(1-X)\prod_{j=1}^{T}(1-X^k x_j^{c-k})}\ .$$

PROOF. If $|k-c|=1$, then the only candidates for membership in $\mathcal{I}_T$ are the $T$-tuples $(0,0,\dots,0)$, $(1,1,\dots,1),\dots,(k(T-1)+c-1,\dots,k(T-1)+c-1)$. These each immediately are seen to satisfy (3) plus the two further conditions of Definition 4. Therefore these are all the elements of $\mathcal{S}_T$.

Now suppose that $(j,j,\dots,j)\in\mathcal{S}_T$ is *not* irreducible. Then by Proposition 7 there must exist a unique reduction in $\mathcal{I}_T\subset\mathcal{S}_T$. I.e. there must be an irreducible $(h,h,\dots,h)\in\mathcal{I}_T$ that is the reduction of $(j,j,\dots,j)$. Therefore $j>h\geq 0$. Therefore from the definition of reduction, $(j-h,j-h,\dots,j-h)$ has $(0,0,\dots,0)$ as a reduction. Therefore by (12)

$$\frac{k\sum_{i=1}^{T}(j-h)-((T-1)k+c)(j-h)}{(k-c)((T-1)k+c)}$$

must be a non-negative integer. Simplifying we see that

$$\frac{j-h}{((T-1)k+c)}$$

must be a non-negative integer, and this is impossible because $0\leq h<j<(T-1)k+c$. Hence each entry of $\mathcal{S}_T$ is irreducible.

As a result, by Theorem 1

$$\begin{aligned}
&\sum_{(n_1,\dots,n_T)\in\mathcal{P}_T} x_1^{n_1}x_2^{n_2}\cdots x_T^{n_T}\\
&=\frac{1+X+X^2+\cdots+X^{(T-1)k+c-1}}{\prod_{j=1}^{T}(1-X^k x_j^{c-k})}\\
&=\frac{1-X^{(T-1)k+c}}{(1-X)\prod_{j=1}^{T}(1-X^k x_j^{c-k})}\ .
\end{aligned}$$

□

## 5. Conclusion

There is obviously an important fact suggested by Theorem 1. Namely, it is immediate that if $k > c$, then

$$\begin{aligned}&\sum_{(n_1,\ldots,n_T)\in\mathcal{P}_T} x_1^{n_1}x_2^{n_2}\cdots x_T^{n_T}\\&= \underset{\geqq}{\Omega}\sum_{n_1\geq 0,\ldots,n_T\geq 0} x_1^{n_1}x_2^{n_2}\cdots x_T^{n_T}\lambda_1^{k\sum n_i-(k(T-1)+c)n_1}\cdots\lambda_T^{k\sum n_i-(k(T-1)+c)n_T}\\&= \underset{\geqq}{\Omega}\frac{1}{\left(1-x_1(\lambda_1\cdots\lambda_T)^k\lambda_1^{-(k(T-1)+c)}\right)\cdots\left(1-x_T(\lambda_1\cdots\lambda_T)^k\lambda_T^{-(k(T-1)+c)}\right)}.\end{aligned}$$

As we have seen in Section 3, the evaluation of this last expression as the right-hand side of the identity in Theorem 1 is quite memory consuming in Omega even for reasonably small values of $T, k$ and $c$. It would be valuable if the algorithm for Omega could be improved so that it could do many other cases of Theorem 1.

On the theoretical side, the technique used to prove Theorem 1 may well have further application. Rather than find the full fundamental basis suggested by Hilbert's Syzygy Theorem, we found only a subset which were then used to produce an equivalence relation among all solutions (i.e. two elements of $\mathcal{P}_T$ are equivalent if they have the same reduction in $\mathcal{I}_T$). While this approach may not have the same generality as Hilbert's theorem, it may well be suited for specific combinational problems when the required generating function is particularly nice.

Finally, we note that Theorem 1 generalizes the $S_{K,L}(x,y)$ of Section 2 only in the case $K = L$. Nonetheless, the final form of $S_{K,L}(x,y)$ suggests that there should be more general results in $T$ dimensions than those in Theorem 1.

## References

[1] G.E. Andrews, *MacMahon's Partition Analysis II: Fundamental Theorems*, Ann. Comb. **4** (2000).

[2] G.E. Andrews, P. Paule and A. Riese, *MacMahon's Partition Analysis III: The Omega Package*, SFB Report **99-24**, J. Kepler University, Linz, 1999, (to appear).

[3] ———, *MacMahon's Partition Analysis VI: A New Reduction Algorithm*, (to appear).

[4] G.E. Andrews, P. Paule, A. Riese and V. Strehl, *MacMahon's Partition Analysis V: Bijections, Recursions, and Magic Squares*, SFB Report **00-18**, J. Kepler University, Linz, 2000, (to appear).

[5] L.F. Klosinski, G.L. Alexanderson, and L.C. Larson, *The sixtieth William Lowell Putnam Mathematical Competition*, Amer. Math. Monthly **107** (2000), 721–732.

[6] P.A. MacMahon, *Combinatory Analysis*, Vols. 1 and 2, Cambridge University Press, Cambridge, 1915 and 1916 (Reissued: Chelsea, New York, 1960).

[7] T. Muir, *A Treatise on the Theory of Determinants*, Dover, New York, 1960.

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

*E-mail address*: andrews@math.psu.edu

Research Institute for Symbolic Computation, Johannes Kepler University Linz, A–4040 Linz, Austria

*E-mail address*: Peter.Paule@risc.uni-linz.ac.at

Research Institute for Symbolic Computation, Johannes Kepler University Linz, A–4040 Linz, Austria

*E-mail address*: Axel.Riese@risc.uni-linz.ac.at