MacMahon's Partition Analysis VIII: Plane Partition Diamonds

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DEDICATED TO DOMINIQUE FOATA AT THE OCCASION OF HIS 65TH BIRTHDAY

In his famous book "Combinatory Analysis" MacMahon introduced Partition Analysis as a computational method for solving combinatorial problems in connection with systems of linear diophantine inequalities and equations. However, MacMahon failed in his attempt to use his method for a satisfactory treatment of plane partitions. It is the object of this article to show that nevertheless Partition Analysis is of significant value when treating non-standard types of plane partitions. To this end "plane partition diamonds" are introduced. Applying Partition Analysis a simple closed form for the full generating function is derived. In the discovering process the `Omega` package developed by the authors has played a fundamental rôle.

* Partially supported by NSF Grant DMS-9206993
† Supported by SFB Grant F1305 of the Austrian FWF.

1. INTRODUCTION

In his famous book "Combinatory Analysis" [6, Vol. II, Section VIII, pp. 91–170] MacMahon introduced Partition Analysis as a computational method for solving combinatorial problems in connection with systems of linear diophantine inequalities and equations. In Chapter II of Section IX he starts out to consider plane partitions as a natural application domain for his method. MacMahon begins by discussing the "most simple case" [6, Vol. II, p. 183], namely where non-negative integers a_i are placed at the corner of a square such that the following order relations are satisfied:

$$a_1 \geq a_2,\ a_1 \geq a_3,\ a_2 \geq a_4 \text{ and } a_3 \geq a_4. \tag{1}$$

By using Partition Analysis he derives that

$$\begin{aligned} D_1 &:= \sum x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} \\ &= \frac{1 - x_1^2 x_2 x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}, \end{aligned} \tag{2}$$

where the sum is taken over all non-negative integers a_i satisfying (1). Furthermore, he observes that if $x_1 = x_2 = x_3 = x_4 = q$, the resulting generating function is

$$\frac{1}{(1-q)(1-q^2)^2(1-q^3)}.$$

In order to see how Partition Analysis works on (2) we need to recall the key ingredient of MacMahon's method, the Omega operator $\Omega_{\geqq}$.

Definition 1.1. The operator $\Omega_{\geqq}$ is given by

$$\underset{\geqq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\dots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\dots,s_r},$$

where the domain of the $A_{s_1,\dots,s_r}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to a neighborhood of the circle $|\lambda_i| = 1$. In addition, the $A_{s_1,\dots,s_r}$ are required to be such that any of the series involved is absolute convergent within the domain of the definition of $A_{s_1,\dots,s_r}$.

We emphasize that it is essential to treat everything analytically rather than formally because the method relies on unique Laurent series representations of rational functions.

Another fundamental aspect of Partition Analysis is the use of elimination rules which describe the action of the Omega operator on certain base cases. MacMahon begins the discussion of his method by presenting a catalog [6, Vol. II, pp. 102–103] of twelve fundamental evaluations. Subsequently he extends this table by new rules whenever he is forced to do so. Once found, most of these fundamental rules are easy to prove. This is illustrated by the following examples which are taken from MacMahon's list.

Proposition 1.1. *For integer $s \geq 1$,*

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda A)\left(1-\frac{B}{\lambda^s}\right)} = \frac{1}{(1-A)(1-A^sB)}; \tag{3}$$

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda A)(1-\lambda B)\left(1-\frac{C}{\lambda}\right)} = \frac{1-ABC}{(1-A)(1-B)(1-AC)(1-BC)}. \tag{4}$$

We prove (3); the proof of (4) is analogous and is left to the reader.

Proof (of (3)). By geometric series expansion the left hand side equals

$$\underset{\geqq}{\Omega} \sum_{i,j\geq 0} \lambda^{i-sj} A^i B^j = \underset{\geqq}{\Omega} \sum_{j,k\geq 0} \lambda^k A^{sj+k} B^j,$$

where the summation parameter i has been replaced by $sj+k$. But now $\Omega_{\geqq}$ sets λ to 1 which completes the proof. ▮

Now we are ready for deriving the closed form expression for D_1 with Partition Analysis.

Proof (of (2)). First, in order to get rid of the diophantine constraints, one rewrites the sum expression in (2) into what MacMahon called the "crude form" of the generating function,

$$\begin{aligned} D_1 &= \underset{\geqq}{\Omega} \sum_{a_1,a_2,a_3,a_4\geq 0} \lambda_1^{a_1-a_2}\lambda_2^{a_1-a_3}\lambda_3^{a_2-a_4}\lambda_4^{a_3-a_4} x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4} \\ &= \underset{\geqq}{\Omega} \frac{1}{\left(1-\lambda_1\lambda_2 x_1\right)\left(1-\frac{\lambda_3}{\lambda_1}x_2\right)\left(1-\frac{\lambda_4}{\lambda_2}x_3\right)\left(1-\frac{x_4}{\lambda_3\lambda_4}\right)}. \end{aligned} \tag{5}$$

Next by rule (3) we eliminate successively λ_1, λ_3, and λ_4,

$$\begin{aligned} D_1 &= \underset{\geqq}{\Omega} \frac{1}{\left(1-\lambda_2 x_1\right)\left(1-\lambda_2\lambda_3 x_1 x_2\right)\left(1-\frac{\lambda_4}{\lambda_2}x_3\right)\left(1-\frac{x_4}{\lambda_3\lambda_4}\right)} \\ &= \underset{\geqq}{\Omega} \frac{1}{\left(1-\lambda_2 x_1\right)\left(1-\lambda_2 x_1 x_2\right)\left(1-\frac{\lambda_4}{\lambda_2}x_3\right)\left(1-\frac{\lambda_2 x_1 x_2 x_4}{\lambda_4}\right)} \end{aligned}$$

$$= \underset{\geqq}{\Omega} \frac{1}{\left(1-\lambda_2 x_1\right)\left(1-\lambda_2 x_1 x_2\right)\left(1-\frac{x_3}{\lambda_2}\right)\left(1-x_1 x_2 x_3 x_4\right)}.$$

Finally, applying rule (4) eliminates λ_2 and completes the proof of (2). ▮

After considering some further questions about plane partitions on a square, MacMahon writes down the "crude form" for the general case, i.e. the $\Omega_{\geqq}$ expression for the full generating function for plane partitions of m rows, l columns and each part not exceeding n; see [6, Vol. II, p. 186] and also [1]. But a few lines later MacMahon writes: "Our knowledge of the Ω operation is not sufficient to enable us to establish the final form of result. This will be accomplished by the aid of new ideas which will be brought forward in the following chapters."

Despite MacMahon's negative statement, in this article our object is to show that Partition Analysis nevertheless is an extremely valuable tool in studying plane partitions of non-standard type. In Section 2 we will consider *plane partition diamonds* for which Partition Analysis enables to derive an elegant expression for the full generating function. In the concluding Section 3 we present two further types of possible plane partition generalizations.

2. PLANE PARTITION DIAMONDS

It will be convenient to introduce alternative descriptions for "$\geq$" relations. For instance, an alternative description of the inequalities (1) is Figure 1 below. It is understood that an arrow pointing from a_i to a_j is

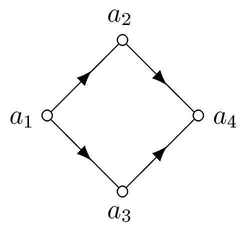

FIG. 1. The inequalities (1)

interpreted as $a_i \geq a_j$.

In the spirit of MacMahon's Partition Analysis we introduce another equivalent description of the inequalities (1), namely

$$\lambda_1^{a_1-a_2}\lambda_2^{a_1-a_3}\lambda_3^{a_2-a_4}\lambda_4^{a_3-a_4}. \tag{6}$$

Each λ variable stands for an arrow, i.e. if its exponent is $a_i - a_j$, it is interpreted as $a_i \geq a_j$.

Now we are in the position to describe "plane partition diamonds". Instead of gluing such squares together as in the case of standard plane partitions, we consider the configurations shown in Figure 2. Note that

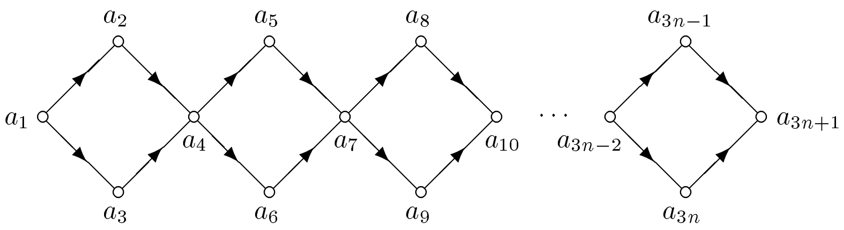

FIG. 2. Diamond of length n

the arrows can be also interpreted as a "flow", so a_1 is considered as the "source" and a_{3n+1} as the "sink". For $n \geq 1$ we call such a configuration a *diamond of length n*. A more formal description can be given as follows.

Definition 2.1. For $n \geq 1$,

$$\Lambda_n := \lambda_{4n-3}^{a_{3n-2}-a_{3n-1}} \lambda_{4n-2}^{a_{3n-2}-a_{3n}} \lambda_{4n-1}^{a_{3n-1}-a_{3n+1}} \lambda_{4n}^{a_{3n}-a_{3n+1}}.$$

Obviously, Λ_1 is the product in (6) and represents one-to-one the four inequalities (1). In view of Figure 2 this leads to the following definition.

Definition 2.2. For $n \geq 1$ the product

$$\mathcal{D}_n := \Lambda_1 \Lambda_2 \cdots \Lambda_n$$

is called *diamond of length n*.

Obviously, $\Lambda_1 \Lambda_2 \cdots \Lambda_n$ gives a precise description of the $4n$ inequalities described by Figure 2.

Definition 2.3. For $n \geq 1$ we define

$$D_n := D_n(x_1, \ldots, x_{3n+1}) := \sum x_1^{a_1} \cdots x_{3n+1}^{a_{3n+1}},$$

where the sum ranges over all non-negative integers $a_1, \ldots, a_{3n+1}$ which satisfy the inequalities encoded by $\mathcal{D}_n$.

Thus D_n is the full generating function for all diamonds of fixed length n. We are also interested in diamonds with sink $\geq \rho$, $\rho \in \mathbb{N}$.

Definition 2.4. For $n \geq 1$, $\rho \geq 0$ we define

$$D_n^{(\rho)} := D_n^{(\rho)}(x_1, \ldots, x_{3n+1}) := \sum x_1^{a_1} \cdots x_{3n+1}^{a_{3n+1}},$$

where the sum ranges over all non-negative integers $a_1, \ldots, a_{3n+1}$ which satisfy the inequalities encoded by $\mathcal{D}_n$ and where $a_{3n+1} \geq \rho$.

2.1. The Crude Generating Function

In this subsection we will define the "crude forms" of D_n and $D_n^{(\rho)}$, i.e. the $\Omega_{\geqq}$ expressions for D_n and $D_n^{(\rho)}$. To this end we need a few definitions.

Definition 2.5. For $k, n \geq 1$,

$$f_k := \frac{1}{\left(1 - \frac{\lambda_{4k-1}}{\lambda_{4k-3}} x_{3k-1}\right)\left(1 - \frac{\lambda_{4k}}{\lambda_{4k-2}} x_{3k}\right)\left(1 - \frac{\lambda_{4k+1}\lambda_{4k+2}}{\lambda_{4k-1}\lambda_{4k}} x_{3k+1}\right)},$$

$$g_n := \frac{1 - \frac{\lambda_{4n+1}\lambda_{4n+2}}{\lambda_{4n-1}\lambda_{4n}} x_{3n+1}}{1 - \frac{x_{3n+1}}{\lambda_{4n-1}\lambda_{4n}}},$$

and

$$h := \frac{1}{1 - \lambda_1 \lambda_2 x_1}.$$

Note that if x_{3n+1} is replaced by $\lambda_{4n+1}\,\lambda_{4n+2}\,x_{3n+1}$ then

$$f_n \cdot g_n \text{ turns into } f_n. \tag{7}$$

Proposition 2.1. *For $n \geq 1$,*

$$D_n = \underset{\geqq}{\Omega}\ h \cdot f_1 \cdots f_n \cdot g_n. \tag{8}$$

Proof. We proceed by induction on n. The case $n = 1$ corresponds to the "most simple" plane partition case (5). Suppose (8) is true for n, then

$$D_{n+1} = \underset{\geqq}{\Omega} \sum_{a_i \geq 0} \Lambda_1 \cdots \Lambda_{n+1}\, x_1^{a_1} \cdots x_{3n+4}^{a_{3n+4}}$$

$$
\begin{aligned}
&= \underset{\geqq}{\Omega} \sum_{a_i \geq 0} \Lambda_1 \cdots \Lambda_n\, x_1^{a_1} \cdots x_{3n}^{a_{3n}} (\lambda_{4n+1}\lambda_{4n+2}x_{3n+1})^{a_{3n+1}} \\
&\qquad\qquad \cdot \left(\frac{\lambda_{4n+3}}{\lambda_{4n+1}}\, x_{3n+2}\right)^{a_{3n+2}} \left(\frac{\lambda_{4n+4}}{\lambda_{4n+2}}\, x_{3n+3}\right)^{a_{3n+3}} \\
&\qquad\qquad \cdot \left(\frac{x_{3n+4}}{\lambda_{4n+3}\lambda_{4n+4}}\right)^{a_{3n+4}} \\
&= \underset{\geqq}{\Omega}\ h \cdot f_1 \cdots f_n \cdot f_{n+1} \cdot g_{n+1},
\end{aligned}
$$

where the last line is by the induction hypothesis and by (7). ▮

Proposition 2.2. *For $n \geq 1$ and $\rho \geq 0$,*

$$
D_n^{(\rho)} = \underset{\geqq}{\Omega}\ h \cdot f_1 \cdots f_n \cdot g_n \cdot \left(\frac{x_{3n+1}}{\lambda_{4n-1}\lambda_{4n}}\right)^{\rho}. \tag{9}
$$

Proof. The induction proof with respect to n is entirely analogous to the proof of Proposition 2.1 and is left to the reader. ▮

2.2. The Diamond Generating Function

In this subsection we will prove our main theorem.

Theorem 2.1. *For $n \geq 1$,*

$$
\begin{aligned}
D_n(x_1, \ldots, x_{3n+1}) = {} & \frac{1}{(1-X_1)(1-X_2)\cdots(1-X_{3n+1})} \\
& \cdot \frac{1-X_1X_3}{1-\frac{X_3}{x_2}} \cdot \frac{1-X_4X_6}{1-\frac{X_6}{x_5}} \cdots \frac{1-X_{3n-2}X_{3n}}{1-\frac{X_{3n}}{x_{3n-1}}},
\end{aligned}
$$

where $X_k = x_1 x_2 \cdots x_k$, $k \geq 1$.

Before we turn to the proof of Theorem 2.1, we introduce two corollaries.

Corollary 2.1. *For $n \geq 1$,*

$$
D_n(q, \ldots, q) = \frac{(1+q^2)(1+q^5)(1+q^8)\cdots(1+q^{3n-1})}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^{3n+1})}.
$$

Proof. The proof is immediate from Theorem 2.1. ▮

Corollary 2.2. *For $n \geq 1$ and $\rho \geq 0$,*

$$
D_n^{(\rho)}(x_1, \ldots, x_{3n+1}) = X_{3n+1}^{\rho} D_n(x_1, \ldots, x_{3n+1}), \tag{10}
$$

where $X_k = x_1 x_2 \cdots x_k$ *as in Theorem* 2.1.

Proof. We fix n and prove the statement by induction on ρ. For $\rho = 0$ the assertion is trivial. Suppose (10) is true for ρ. Obviously,

$$D_n^{(\rho+1)} = D_n^{(\rho)} - x_{3n+1}^{\rho} \langle x_{3n+1}^{\rho} \rangle D_n^{(\rho)},$$

where $\langle x_k^{\rho} \rangle F(x_1, \dots, x_m)$ stands for the coefficient of x_k^{ρ} in $F(x_1, \dots, x_m)$.
From (9),

$$\langle x_{3n+1}^{\rho} \rangle D_n^{(\rho)} = \langle x_{3n+1}^{\rho} \rangle X_{3n+1}^{\rho} D_n = \frac{X_{3n+1}^{\rho}}{x_{3n+1}^{\rho}} \langle x_{3n+1}^{0} \rangle D_n.$$

But by Theorem 2.1 we have

$$\begin{aligned} \langle x_{3n+1}^{0} \rangle D_n &= (1 - X_{3n+1}) D_n \langle x_{3n+1}^{0} \rangle \frac{1}{1 - X_{3n+1}} \\ &= (1 - X_{3n+1}) D_n. \end{aligned}$$

Collecting these facts and applying (9) we obtain

$$D_n^{(\rho+1)} = X_{3n+1}^{\rho} D_n - X_{3n+1}^{\rho} (1 - X_{3n+1}) D_n = X_{3n+1}^{\rho+1} D_n,$$

which completes the proof. ▮

Basically we are ready for the proof of Theorem 2.1. However, it will be convenient to introduce an elementary lemma.

Definition 2.6. Let $k \geq 1$, and $y_1, \dots, y_k, z \neq 0$ be distinct elements from a suitable field. Define

$$p(\boldsymbol{y}; z) := \prod_{i=1}^{k} \left(1 - \frac{y_i}{z}\right),$$

where $\boldsymbol{y} = (y_1, \dots, y_k)$. For $1 \leq j \leq k$ define

$$p_j(\boldsymbol{y}; z) := \prod_{\substack{i=1 \\ i \neq j}}^{k} \left(1 - \frac{y_i}{z}\right)^{-1}.$$

Lemma 2.1. *Let* $k \geq 1$ *and* $\boldsymbol{y} = (y_1, \dots, y_k)$, *then*

$$\frac{1}{p(\boldsymbol{y}; z)} = \sum_{j=1}^{k} \frac{p_j(\boldsymbol{y}; y_j)}{1 - \frac{y_j}{z}}.$$

Proof. The assertion is immediate by partial fraction decomposition. ∎

Proof (of Theorem 2.1). We proceed by induction on n. For $n=1$ the statement corresponds to the simplest case of classical plane partitions, namely (2). Suppose the theorem holds for n. By Proposition 2.1,

$$\begin{aligned} D_{n+1} &= \underset{\geqq}{\Omega}\ h \cdot f_1 \cdots f_{n+1} \cdot g_{n+1} \\ &= \underset{\geqq}{\Omega}\ h \cdot f_1 \cdots f_{n-1} \cdot \frac{1}{\left(1-\frac{\lambda_{4n-1}}{\lambda_{4n-3}}\, x_{3n-1}\right)\left(1-\frac{\lambda_{4n}}{\lambda_{4n-2}}\, x_{3n}\right)} \\ &\qquad \cdot \frac{1}{\left(1-\frac{\lambda_{4n+1}\lambda_{4n+2}}{\lambda_{4n-1}\lambda_{4n}}\, x_{3n+1}\right)\left(1-\frac{\lambda_{4n+3}}{\lambda_{4n+1}}\, x_{3n+2}\right)} \\ &\qquad \cdot \frac{1}{\left(1-\frac{\lambda_{4n+4}}{\lambda_{4n+2}}\, x_{3n+3}\right)\left(1-\frac{x_{3n+4}}{\lambda_{4n+3}\lambda_{4n+4}}\right)}. \end{aligned}$$

By applying case $n=1$ of Theorem 2.1 to the last four factors we obtain,

$$\begin{aligned} D_{n+1} &= \underset{\geqq}{\Omega}\ h \cdot f_1 \cdots f_{n-1} \cdot \frac{1}{\left(1-\frac{\lambda_{4n-1}}{\lambda_{4n-3}}\, x_{3n-1}\right)\left(1-\frac{\lambda_{4n}}{\lambda_{4n-2}}\, x_{3n}\right)} \\ &\qquad \cdot \frac{1-\frac{x_{3n+1}^2 x_{3n+2} x_{3n+3}}{\lambda_{4n-1}^2\lambda_{4n}^2}}{p(\boldsymbol{y};\lambda_{4n-1}\lambda_{4n})} \end{aligned}$$

with

$$\begin{aligned} \boldsymbol{y} &= (y_1, y_2, y_3, y_4, y_5) \\ &= (x_{3n+1}, x_{3n+1}x_{3n+2}, x_{3n+1}x_{3n+3}, x_{3n+1}x_{3n+2}x_{3n+3}, \\ &\qquad x_{3n+1}x_{3n+2}x_{3n+3}x_{3n+4}). \end{aligned} \tag{11}$$

Now applying Lemma 2.1 yields

$$\begin{aligned} D_{n+1} &= \sum_{j=1}^{5} p_j(\boldsymbol{y};y_j) D_n(x_1,\ldots,x_{3n},y_j) \\ &\qquad - x_{3n+1}^2 x_{3n+2} x_{3n+3} \cdot \sum_{j=1}^{5} \frac{p_j(\boldsymbol{y};y_j)}{y_j^2}\ D_n^{(2)}(x_1,\ldots,x_{3n},y_j). \end{aligned}$$

By Corollary 2.2 with $\rho=2$ (implied by the induction hypothesis),

$$D_{n+1} = \sum_{j=1}^{5} p_j(\boldsymbol{y};y_j) D_n(x_1,\ldots,x_{3n},y_j)$$

$$
\begin{aligned}
&- (x_1 \cdots x_{3n})^2 x_{3n+1}^2 x_{3n+2} x_{3n+3} \\
&\quad \cdot \sum_{j=1}^{5} p_j(\boldsymbol{y}; y_j) D_n(x_1, \ldots, x_{3n}, y_j) \\
&= (1 - X_{3n+1} X_{3n+3}) \cdot \sum_{j=1}^{5} p_j(\boldsymbol{y}; y_j) D_n(x_1, \ldots, x_{3n}, y_j).
\end{aligned}
$$

Observing that

$$D_n(x_1, \ldots, x_{3n}, y) = \frac{1 - X_{3n+1}}{1 - X_{3n} y} \, D_n(x_1, \ldots, x_{3n}, x_{3n+1})$$

we obtain

$$D_{n+1} = (1 - X_{3n+1} X_{3n+3})(1 - X_{3n+1}) \cdot \sum_{j=1}^{5} \frac{p_j(\boldsymbol{y}; y_j)}{1 - X_{3n} y_j} \, D_n.$$

But it is routine (computer algebra) computation that

$$
\begin{aligned}
&\sum_{j=1}^{5} \frac{p_j(\boldsymbol{y}; y_j)}{1 - X_{3n} y_j} \\
&\quad = \frac{1}{(1 - X_{3n+1})(1 - X_{3n+2})(1 - X_{3n+3})(1 - X_{3n+4})\big(1 - \frac{X_{3n+3}}{x_{3n+2}}\big)}
\end{aligned}
$$

for $\boldsymbol{y} = (y_1, \ldots, y_5)$ as in (11). Hence,

$$D_{n+1} = \frac{1}{(1 - X_{3n+2})(1 - X_{3n+3})(1 - X_{3n+4})} \, \frac{1 - X_{3n+1} X_{3n+3}}{1 - \frac{X_{3n+3}}{x_{3n+2}}} \, D_n$$

and the proof of Theorem 2.1 is completed. ▮

3. CONCLUSION

As shown in a series of articles [2, 3, 4, 5] Partition Analysis is ideally suited for being supplemented by computer algebra methods. In these papers the Mathematica package Omega which had been developed by the authors, was used as an essential tool. The package is freely available from the Web via *http://www.risc.uni-linz.ac.at/research/combinat/risc/software/Omega*.

The Omega package played a crucial rôle also in discovering Theorem 2.1 above. For instance, the simplest case (2) is treated automatically by the package as follows.

After loading the file `Omega2.m` by

In[1]:= `<<Omega2.m`

Out[1]= `Axel Riese's Omega implementation version 2.30 loaded`

one computes the crude form of the generating function D_1 as follows:

In[2]:= $\texttt{OSum}[x_1^{a_1}\, x_2^{a_2}\, x_3^{a_3}\, x_4^{a_4},\ \{a_1 \geq a_2,\ a_1 \geq a_3,\ a_2 \geq a_4,\ a_3 \geq a_4\},\ \lambda]$

Assuming $a_1 \geq 0$
Assuming $a_2 \geq 0$
Assuming $a_3 \geq 0$
Assuming $a_4 \geq 0$

Out[2]=

$$\underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4}{\Omega_{\geq}} \frac{1}{(1 - x_1\,\lambda_1\,\lambda_2)\left(1 - \frac{x_2\,\lambda_3}{\lambda_1}\right)\left(1 - \frac{x_4}{\lambda_3\,\lambda_4}\right)\left(1 - \frac{x_3\,\lambda_4}{\lambda_2}\right)}$$

Finally, elimination can be done in one stroke and within a second:

In[3]:= `OR[%]`

Eliminating λ_4...
Eliminating λ_3...
Eliminating λ_2...
Eliminating λ_1...

Out[3]=

$$\frac{1 - x_1^2\,x_2\,x_3}{(1 - x_1)\,(1 - x_1\,x_2)\,(1 - x_1\,x_3)\,(1 - x_1\,x_2\,x_3)\,(1 - x_1\,x_2\,x_3\,x_4)}$$

Already this elementary application indicates the usefulness of the `Omega` package for a further, more detailed study of possible new plane partition generalizations. We conclude by mentioning two further examples which have been found experimentally.

The first example is a variation of the "diamond theme". Instead of arranging diamonds in a row, we arrange them in hook shape as shown in Figure 3. Note that the ordering imposed is such that we now have 2 sources, a_1 and a_{10}, and 1 sink, namely a_5. The corresponding full

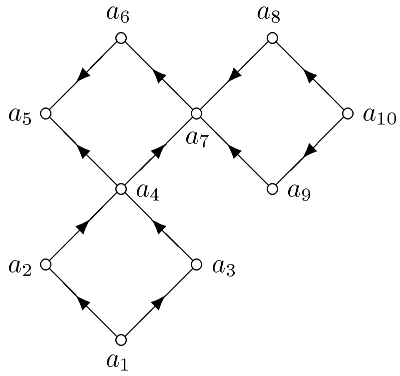

FIG. 3. Diamond hook

generating function turns out to be

$$\begin{aligned}\sum x_1^{a_1}\cdots x_{10}^{a_{10}} \\ &= \frac{(1-x_1^2x_2x_3)(1-x_8x_9x_{10}^2)}{(1-x_1)(1-x_{10})(1-x_1x_2)(1-x_1x_3)(1-x_8x_{10})} \\ &\quad\cdot\frac{1}{(1-x_9x_{10})(1-x_1x_2x_3)(1-x_8x_9x_{10})(1-x_1x_2x_3x_4)} \\ &\quad\cdot\frac{1}{(1-x_1x_2x_3x_4x_7x_8x_9x_{10})(1-x_1x_2x_3x_4x_6x_7x_8x_9x_{10})} \\ &\quad\cdot\frac{1}{(1-x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10})}.\end{aligned}$$

If we set all x_i to q the product turns into

$$\frac{1+q^2}{(1-q)^2(1-q^2)^3(1-q^3)^2(1-q^8)(1-q^9)(1-q^{10})}.$$

Other types of generalized plane partitions can be obtained by a variation of the order taken in the classical case. Consider, for instance, the reversed hook from Figure 4. Here we have again 2 sources, a_1 and a_8, and 1 sink, namely a_5. Despite the fact that the full generating function does not factor, the case $x_i = q$, $1 \le i \le 8$, again is nice:

$$\frac{1}{(1-q)^2(1-q^2)^2(1-q^3)(1-q^4)^2(1-q^5)}.$$

We conclude by the remark that without the `Omega` package such observations can hardly be made.

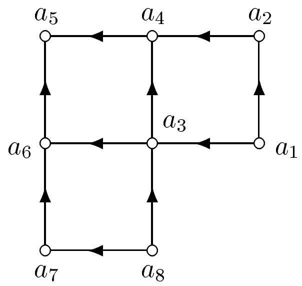

FIG. 4. Reversed standard hook

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