Unification in the Empty and Flat Theories with Sequence Variables and Flexible Arity Symbols*

Temur Kutsia

Research Institute for Symbolic Computation
Johannes Kepler University Linz
A-4040, Linz, Austria
kutsia@risc.uni-linz.ac.at

Abstract. We define equational theory with sequence variables and flexible arity symbols and consider a general unification problem in two such theories: in an empty theory and in a flat theory (a theory with a flat flexible arity symbol). We prove that unifiability in both theories is decidable and describe unification procedures, which enumerate the minimal and complete set of solutions of the problem and terminate, if the set is finite.

1 Introduction

The problem of studying equational theories with sequence variables and flexible arity symbols was posed by Bruno Buchberger and was inspired by a mathematical software system Mathematica [20], the programming language of which is a functional (or rewrite) language [3] with an additional feature - sequence variables.

Sequence variables are used in the Theorema system [5] - an integrated environment for proving, solving and computing, built on top of Mathematica. The proving context, where they occur in Theorema, makes it important to study unification problem with sequence variables.

We use the term "flexible arity symbols" for symbols that can have an arbitrary arity. Such symbols often occur in programming languages, term rewriting systems, etc. and sometimes are called "variable arity symbols" (see e.g. [10]). We build terms over two types of variables - individual and sequence variables, object constants and two types of function symbols - of fixed and flexible arity. For sequence variables one can substitute any (possibly empty) sequence of terms. We do not introduce an additional constructor for sequences in the theory. Presence of fixed arity symbols and sequence variables makes it necessary to solve a system of linear Diophantine equations over naturals in order to decide whether an expression of the theory is a term (or an equation) of the theory.

In the usual variants of predicate logic sequence variables are not included. However, they are a convinient feature that often is used implicitely in everyday mathematics. In fact, sequence variables together with a natural induction

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principle for sequence variables (see [4]), are quite strong because they expand the power of logic by an implicit formalization of natural numbers. Sequence variables in the context of term rewriting were studied in [11]. In [6] sequence variables are used in inductive theorem proving over lists.

We address the problem of solving equations (unification) in a theory with sequence variables and flexible arity symbols. This problem can be seen as a generalization of the problems like word equations [14, 13, 16], general A-unification [8], equations over lists of atoms with concatenation [9] or Mathematica pattern matching.

We consider a general unification problem in empty and flat theories with sequence variables and flexible arity symbols (flat theory is a theory with a flat flexible arity symbol) and answer two questions: a) is it decidable whether a unification problem is solvable and b) how to obtain a minimal unification procedure.

We show decidability of unification in empty and flat theories in two steps. First, with solvability preserving transformations we reduce the problem to a "simpler" unification problem. Second, we apply Baader-Schulz combination method [7] to the reduced problem and show its decidability.

Unification procedure both in the empty and flat theories is designed as a tree generation process. We design three kinds of single steps of the processes, called projection, arity completion and transformation. This technique generalizes the "pig-pug" procedure - the basic algorithm for word unification [12, 14, 18, 17]. We prove that the procedures both in the empty and flat theories enumerate the minimal complete set of unifiers of the unification problem and terminate if the set is finite.

We have implemented the unification procedure for the empty theory in Mathematica 4.0.

2 Preliminaries

Definition 1 (Individual variables). *IV is a set of individual variables iff IV is a denumerable set of words consisting of an English letter and subsequent letters or digits, starting with x, y or z.*

Definition 2 (Sequence variables). *SV is a set of sequence variables iff SV is a denumerable set of words with overbar, consisting of an English letter and subsequent letters or digits, starting with x, y or z.*

Definition 3 (Constants). *$(CONST, FFIX, FFLEX, =, AR)$ is domain of constants iff*

- *$CONST$ is a set of symbols ("the set of object constants"),*
- *$FFIX$ is a set of symbols ("the set of function constants of fixed arity"),*
- *$FFLEX$ is a set of symbols ("the set of function constants of flexible arity"),*
- *$=$ is the "equality relation constant",*
- *$AR : (FFIX \cup \{=\}) \to \mathbb{N}$ ("the arity function"),*

- $AR(=) = 2$,
- $CONST, FFIX, FFLEX, \{=\}$ *are pairwise disjoint and disjoint from* IV *and* SV.

Let now IV be a set of individual variables, SV - a set of sequence variables, $V = (IV, SV)$ - variables, $C = (CONST, FFIX, FFLEX, =, AR)$ - a domain of constants and P - a set $\{(,),,\}$ ("parentheses and comma").

Definition 4 (Quasi-Term). *The set of quasi-terms (over* (V, C, P)*) is the smallest set of strings over* (V, C, P) *that satisfies the following conditions:*

- *If* $v \in IV \cup SV$ *then* v *is a quasi-term.*
- *If* $c \in CONST$ *then* c *is a quasi-term.*
- *If* $f \in FFIX \cup FFLEX$ *and* $t_1, \ldots, t_n$ *are quasi-terms, then* $f(t_1, \ldots, t_n)$ *is a quasi-term* $(n \geq 0)$.

Definition 5 (System of Linear Diophantine Equations Associated with a Quasi-Term). *For a quasi-term* qt*, the associated system of linear Diophantine equations* $DS(qt)$ *is defined in the following way:*

- *If* $qt \in IV \cup SV \cup CONST$ *then* $DS(qt)$ *is empty.*
- *If* qt *is* $f(t_1, \ldots, t_l)$*, where* $f \in FFIX$ *and* $AR(f) = n$*, then* $DS(qt)$ *is*

  $$k_1 x_1 + \ldots + k_s x_s = n - (l - (k_1 + \ldots + k_s)) \& DS(t_1) \& \ldots \& DS(t_l)$$

  where $k_1, \ldots, k_s$ $(s \geq 0)$ *are numbers of occurrences of all distinct sequence variables among* $t_1, \ldots, t_l$*, the variables* $x_1, \ldots, x_s$ *are metavariables corresponding to the sequence variables and* & *denotes conjunction on the meta level.*
- *If* qt *is* $f(t_1, \ldots, t_l)$*, where* $f \in FFLEX$*, then* $DS(qt)$ *is*

  $$DS(t_1) \& \ldots \& DS(t_l).$$

Example 1. Let f be ternary and g be binary. Then the linear Diophantine system associated with the quasi-term $f(\overline{x}, \overline{y}, g(\overline{x}, \overline{x}, a))$ is $x_1 + x_2 = 2 \& 2x_1 = 1$.

Definition 6 (Term). *A string* t *over* (V, C, P) *is called a term (over* (V, C, P)*) iff*

- t *is a quasi-term and*
- $DS(t)$ *has a solution in natural numbers.*

Definition 7 (Quasi-Equation). *The set of quasi-equations (over* (V, C, P)*) is the smallest set of strings over* (V, C, P) *that satisfies the following condition:*

- *If* $t_1, \ldots, t_n$ $(n \geq 0)$ *are terms over* (V, C, P)*, then* $= (t_1, \ldots, t_n)$ *is a quasi-equation.*

Definition 8 (System of Linear Diophantine Equations Associated with a Quasi-Equation). *For a quasi-equation* qe*, the associated system of linear Diophantine equations* $DS(qe)$ *is defined in the following way:*

– *If qe is $=(t_1,\ldots,t_l)$, then $DS(qe)$ is*

$$k_1x_1+\ldots+k_sx_s=2-(l-(k_1+\ldots+k_s))\&DS(t_1)\&DS(t_2)$$

where $k_1,\ldots,k_s$ ($s\geq 0$) are numbers of occurrences of all distinct sequence variables among $t_1,\ldots,t_l$, the variables $x_1,\ldots,x_s$ are metavariables corresponding to the sequence variables and & denotes conjunction on the meta level.

Definition 9 (Equation). *A string eq over (V,C,P) is called an equation (over (V,C,P)) iff*

- *eq is a quasi-equation and*
- *$DS(eq)$ has a solution in natural numbers.*

For quasi-equations of the form $=(t_1,t_2)$ we will write $t_1=t_2$. If not otherwise stated, we will use x, y and z as metavariables ranging over individual variables, $\overline{x}$, $\overline{y}$ and $\overline{z}$ - metavariables over sequence variables, v and u - over (individual or sequence) variables, c - over object constants, $ffix$, $gfix$ and $hfix$ - over function symbols of fixed arity, $fflex$, $gflex$ and $hflex$ - over function symbols of flexible arity, f, g and h - over (fixed or flexible arity) function symbols, s and t - over terms, seq - over sequences of terms, eq - over equations.

Definition 10 (Variable Replacement). *$s_{x\leftarrow t}$ ("the quasi-term resulting from s by replacing x by t for $t\notin SV$), $eq_{x\leftarrow t}$ ("the quasi-equation resulting from eq by replacing x by t for $t\notin SV$), $s_{\overline{x}\leftarrow t_1,\ldots,t_n}$ ("the quasi-term or sequence of terms resulting from s by replacing $\overline{x}$ by the finite sequence $t_1,\ldots,t_n$") and $eq_{\overline{x}\leftarrow t_1,\ldots,t_n}$ ("the quasi-equation resulting from eq by replacing $\overline{x}$ by the finite sequence $t_1,\ldots,t_n$") are inductively defined as follows: Let $t\notin SV$, then*

- $x_{x\leftarrow t}=t$.
- *If u is distinct from x then $u_{x\leftarrow t}=u$.*
- $(f(s_1,\ldots,s_n))_{x\leftarrow t}=f((s_1)_{x\leftarrow t},\ldots,(s_n)_{x\leftarrow t})$.
- $(=(s_1,\ldots,s_n))_{x\leftarrow t}==((s_1)_{x\leftarrow t},\ldots,(s_n)_{x\leftarrow t})$.
- $\overline{x}_{\overline{x}\leftarrow t_1,\ldots,t_n}=t_1,\ldots,t_n$.
- *If u is distinct from $\overline{x}$ then $u_{\overline{x}\leftarrow t_1,\ldots,t_n}=u$.*
- $(f(s_1,\ldots,s_n))_{\overline{x}\leftarrow t_1,\ldots,t_n}=f((s_1)_{\overline{x}\leftarrow t_1,\ldots,t_n},\ldots,(s_n)_{\overline{x}\leftarrow t_1,\ldots,t_n})$.
- $(=(s_1,\ldots,s_n))_{\overline{x}\leftarrow t_1,\ldots,t_n}==((s_1)_{\overline{x}\leftarrow t_1,\ldots,t_n},\ldots,(s_n)_{\overline{x}\leftarrow t_1,\ldots,t_n})$.

This notation can be naturally extended to several variables ("parallel replacements", where all variable replacements are carried out simultaneously), like

$$t_{x_1\leftarrow s_1,\ldots,x_k\leftarrow s_k,\overline{x}_1\leftarrow t^1_1,\ldots,t^1_n,\ldots,\overline{x}_m\leftarrow t^m_1,\ldots,t^m_l}$$

and

$$eq_{x_1\leftarrow s_1,\ldots,x_k\leftarrow s_k,\overline{x}_1\leftarrow t^1_1,\ldots,t^1_n,\ldots,\overline{x}_m\leftarrow t^m_1,\ldots,t^m_l}$$

where $x_1,\ldots,x_k,\overline{x}_1,\ldots,\overline{x}_m$ are restricted to represent distinct variables.

Definition 11 (Substitution). *A substitution is a finite set*

$$\{x_1 \leftarrow s_1, \ldots, x_n \leftarrow s_n, \overline{x}_1 \leftarrow t_1^1, \ldots, t_{k_1}^1, \ldots, \overline{x}_m \leftarrow t_1^m, \ldots, t_{k_m}^m\}$$

where

- $n \geq 0$, $m \geq 0$ *and for all* $i \in \{1, \ldots, m\}$, $k_i \geq 0$,
- $x_1, \ldots, x_n$ *are distinct individual variables,*
- $\overline{x}_1, \ldots, \overline{x}_m$ *are distinct sequence variables,*
- *for all* $i \in \{1, \ldots, n\}$ s_i *is a term,* $s_i \notin SV$ *and* s_i *is not* x_i,
- *for all* $j \in \{1, \ldots, m\}$ $t_1^j, \ldots, t_{k_j}^j$ *is a sequence of terms and if* k_j*=1 then* $t_{k_j}^j$ *is different from* $\overline{x}_j$.

The substitution is called ground iff all $s_1, \ldots, s_n, t_1^1, \ldots, t_{k_1}^1, t_1^m, \ldots, t_{k_m}^m$ are variable-free terms. The substitution is called empty iff $n = 0$ and $m = 0$.

Greek letters are used to denote substitutions. The letter ε denotes the empty substitution.

Definition 12 (Applicable Substitutions). *Let*

$$\theta = \{x_1 \leftarrow s_1, \ldots, x_n \leftarrow s_n, \overline{x}_1 \leftarrow t_1^1, \ldots, t_{k_1}^1, \ldots, \overline{x}_m \leftarrow t_1^m, \ldots, t_{k_m}^m\}$$

be a substitution.

- θ *is applicable on* s *iff* $s_{x_1 \leftarrow s_1, \ldots, x_n \leftarrow s_n, \overline{x}_1 \leftarrow t_1^1, \ldots, t_{k_1}^1, \ldots, \overline{x}_m \leftarrow t_1^m, \ldots, t_{k_m}^m}$ *is a term or a sequence of terms.*
- θ *is applicable on* eq *iff* $eq_{x_1 \leftarrow s_1, \ldots, x_n \leftarrow s_n, \overline{x}_1 \leftarrow t_1^1, \ldots, t_{k_1}^1, \ldots, \overline{x}_m \leftarrow t_1^m, \ldots, t_{k_m}^m}$ *is an equation.*

It is easy to see that a substitution of the form $\{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$ is applicable to any term or equation.

We say that θ is applicable on a sequence of terms $t_1, \ldots, t_n$ iff θ is applicable on a term $fflex(t_1, \ldots, t_n)$, where $fflex$ is a new flexible arity function symbol.

Definition 13 (Instance). *Let*

$$\theta = \{x_1 \leftarrow s_1, \ldots, x_n \leftarrow s_n, \overline{x}_1 \leftarrow t_1^1, \ldots, t_{k_1}^1, \ldots, \overline{x}_m \leftarrow t_1^m, \ldots, t_{k_m}^m\}$$

be a substitution, and A *be either a term or an equation such that* θ *is applicable to* A*. Then by an instance of* A *w.r.t.* θ*, denoted by* $A\theta$*, we mean* $A_{x_1 \leftarrow s_1, \ldots, x_n \leftarrow s_n, \overline{x}_1 \leftarrow t_1^1, \ldots, t_{k_1}^1, \ldots, \overline{x}_m \leftarrow t_1^m, \ldots, t_{k_m}^m}$.

If seq is a sequence of terms $t_1, \ldots, t_n$ and θ is applicable on seq, by an instance of seq w.r.t. θ, denoted be $seq\theta$ we mean the sequence $t_1\theta, \ldots, t_n\theta$ of instances of $t_1, \ldots, t_n$.

3 Equational Theory with Sequence Variables and Flexible Arity Symbols

A set of equations E (called representation) defines an equational theory, i.e. the equality of terms induced by E. We use the term E-theory for the equational theory defined by E. We will write $s =_E t$ for $s = t$ modulo E. Below we will concentrate on solving equations in the following E-theories:

1. empty theory: $E = \emptyset$;
2. flat theory (i.e. a theory with flexible arity flat symbol(s)):

$$E = \{fflex(\dots fflex(t_1, \dots, t_n) \dots) = fflex(\dots t_1, \dots, t_n \dots)\}.$$

We will refer the empty theory as $\emptyset$-theory and the flat theory as F-theory.

Solving equations in an E-theory is called E-unification. We will write equations as pairs of terms and will use the term "unification problem" for equation.

We say that two sequences of terms $t_1, \dots, t_n$ and $s_1, \dots, s_m$ are equal modulo E (written $t_1, \dots, t_n =_E s_1, \dots, s_m$) iff $m = n$ and for all $i \in \{1, \dots, n\}$, we have $t_i =_E s_i$.

Definition 14 (Arity completing substitution). *A substitution θ is an arity completing substitution for a term t (resp. for an equation eq) iff for each $\overline{x}$ occurring as an argument of a subterm of t (of eq) with a fixed arity head, $\overline{x}\theta = t_1, \dots, t_n$, $n \geq 0$, where $t_1, \dots, t_n$ are terms with no sequence variables among them and θ is applicable on t (on eq).*

Definition 15 (Unifier). *A substitution θ is called an E-unifier of terms s and t iff θ is an arity completing substitution for $s =_E t$ and $s\theta =_E t\theta$. The terms s and t are called E-unifiable iff there exists an E-unifier of s and t.*

Example 2. $\theta = \{\overline{x} \leftarrow z_1, z_2, \overline{y} \leftarrow z_1, z_2\}$, where z_1 and z_2 are individual variables, is an $\emptyset$-unifier for the terms $f(\overline{x})$ and $f(\overline{y})$, with f a binary function symbol.

Note that by Definition 15, the substitution $\sigma = \{\overline{x} \leftarrow \overline{y}\}$ is not an $\emptyset$-unifier of $f(\overline{x})$ and $f(\overline{y})$, because σ is not an arity completing substitution for $f(\overline{x}) =_\emptyset f(\overline{y})$.

Now, by solving an E-unification problem $\langle s, t \rangle$ (or, equivalently, by solving an equation $s =_E t$ in an E-theory) we mean finding an E-unifier of s and t.

Definition 16 (Composition). *Let*

$$\theta = \{x_1 \leftarrow s_1, \dots, x_n \leftarrow s_n, \overline{x}_1 \leftarrow seq_1^1, \dots, \overline{x}_m \leftarrow seq_m^1\}$$

and

$$\lambda = \{y_1 \leftarrow d_1, \dots, y_n \leftarrow d_l, \overline{y}_1 \leftarrow seq_1^2, \dots, \overline{y}_r \leftarrow seq_r^2\}$$

be two substitutions such that λ is applicable on terms $s_1, \ldots, s_n$ and on the sequences $seq_1^1, \ldots, seq_m^1, seq_1^2, \ldots, seq_r^2$. Then the composition of θ and λ is the substitution, denoted by $\theta \circ \lambda$, that is obtained from the set

$$\{x_1 \leftarrow s_1\lambda, \ldots, x_n \leftarrow s_n\lambda, \overline{x}_1 \leftarrow seq_1^1\lambda, \ldots, \overline{x}_m \leftarrow seq_m^1\lambda,$$
$$y_1 \leftarrow d_1, \ldots, y_l \leftarrow d_l, \overline{y}_1 \leftarrow seq_1^2, \ldots, \overline{y}_r \leftarrow seq_r^2\}$$

by deleting

- *all the elements $x_i \leftarrow s_i\lambda$ ($i \in \{1, \ldots, n\}$) for which x_i is identical to $s_i\lambda$,*
- *all the elements $\overline{x}_i \leftarrow seq_i^1\lambda$ ($i \in \{1, \ldots, m\}$) for which $seq_i^1\lambda$ consists of one term only and and $\overline{x}_i$ is identical to $seq_i^1\lambda$,*
- *all the elements $y_i \leftarrow d_i$ ($i \in \{1, \ldots, l\}$) such that $y_i \in \{x_1, \ldots, x_n\}$,*
- *all the elements $\overline{y}_i \leftarrow seq_i^2$ ($i \in \{1, \ldots, r\}$) such that $\overline{y}_i \in \{\overline{x}_1, \ldots, \overline{x}_m\}$.*

Definition 17 (More General Substitution). *θ is more general than λ on a finite set of variables Var modulo a theory E (denoted $\theta \ll_E^{Var} \lambda$) iff there exists σ such that for all $\overline{x} \in Var$, $\overline{x}\sigma$ is not an empty sequence and for all $v \in Var$ we have $v\lambda =_E v\theta \circ \sigma$.*

Note that this definition differs from the standard definition of more general substitution by requiring from σ not to substitute sequence variables from Var with the empty sequence. For the substitutions without sequence variables this definition coincides with the standard one.

Example 3. $\theta = \{\overline{x} \leftarrow \overline{y}\}$ is more general than $\lambda = \{\overline{x} \leftarrow a, b,\ \overline{y} \leftarrow a, b\}$ on $Var = \{\overline{x}, \overline{y}\}$ modulo $\emptyset$-theory, but is not more general than $\eta = \{\overline{x} \leftarrow, \overline{y} \leftarrow\}$.

Below we do not distinguish between substitutions σ and λ if σ is more general than λ and λ is more general than σ.

Definition 18 (Minimal Set of Substitutions). *A set of substitutions Σ is called E-minimal with respect to a set of variables Var iff for all $\theta, \sigma \in \Sigma$, $\theta \ll_E^{Var} \sigma$ implies $\theta = \sigma$.*

Definition 19 (The Minimal Complete Set of Unifiers). *The minimal complete set of E-unifiers of s and t, denoted $MCU_E(s,t)$ is an E-minimal set of substitutions w.r.t. the set Var of all variables occurring in s or t, satisfying the following conditions:*

E-Correctness - for all $\theta \in MCU_E(s,t)$, θ is an E-unifier of s and t.
E-Completeness - for any E-unifier σ of s and t there exists $\theta \in MCU_E(s,t)$ such that $\theta \ll_E^{VAR} \sigma$.

Example 4. Compute the minimal complete set of unifiers in the empty and flat theories ($\overline{x}, \overline{y}, \overline{u}, \overline{v}$ are sequence variables, y and $newvar$ are an individual variables, a, b, c are constants, f and g are flexible arity function symbols, h is a flat flexible arity function symbol, s is a binary function symbol):

1. $MCU_{\emptyset}(f(\overline{x},y),f(a,b,c))=\{\overline{x}\leftarrow a,b,\ y\leftarrow c\}$.
2. $MCU_{\emptyset}(f(\overline{x},\overline{y}),f(a,b)) = \{\{\overline{x}\leftarrow,\ \overline{y}\leftarrow a,b\},\ \{\overline{x}\leftarrow a,\overline{y}\leftarrow b\},\ \{\overline{x}\leftarrow a,b,\overline{y}\leftarrow\}\}$.
3. $MCU_{\emptyset}(f(\overline{x},a),f(a,\overline{x}))=\{\{\overline{x}\leftarrow\},\ \{\overline{x}\leftarrow a\},\ \{\overline{x}\leftarrow a,a\},\ldots\}$.
4. $MCU_{\emptyset}(f(g(a,\overline{x}),\ g(\overline{y},c)),\ f(\overline{u},\ g(b,\overline{v}))) = \{\{\overline{u}\leftarrow g(a,\overline{x}),\overline{y}\leftarrow b,\overline{v}\leftarrow c\},\ \{\overline{x}\leftarrow,\overline{u}\leftarrow g(a),\overline{y}\leftarrow b,\overline{v}\leftarrow c\},\ \{\overline{u}\leftarrow g(a,\overline{x}),\overline{y}\leftarrow b,\overline{y},\ \overline{v}\leftarrow \overline{y},c\},\ \{\overline{x}\leftarrow,\overline{u}\leftarrow g(a),\overline{y}\leftarrow b,\overline{y},\ \overline{v}\leftarrow\overline{y},c\}\}$.
5. $MCU_{\emptyset}(f(s(a,\overline{x}),s(\overline{y},\ c)),f(\overline{u},s(b,\overline{v})))=\{\{\overline{x}\leftarrow newvar,\ \overline{u}\leftarrow s(a,newvar),\ \overline{y}\leftarrow b,\ \overline{v}\leftarrow c\}\}$.
6. $MCU_{F}(h(\overline{x},y),\ h(a,\ b,\ c)) = \{\{\overline{x}\leftarrow,\ y\leftarrow h(a,\ b,\ c)\},\ \{\overline{x}\leftarrow a,\ y\leftarrow h(b,\ c)\},\ \{\overline{x}\leftarrow h(a),\ y\leftarrow h(b,\ c)\},\ \{\overline{x}\leftarrow a,b,\ y\leftarrow c\},\ \{\overline{x}\leftarrow h(a),\ b,\ \ y\leftarrow c\},\ \{\overline{x}\leftarrow a,h(b),\ y\leftarrow c\},\ \{\overline{x}\leftarrow h(a),h(b),\ y\leftarrow c\},\ \{\overline{x}\leftarrow h(a,b),\ y\leftarrow c\},\ \{\overline{x}\leftarrow a,b,\ y\leftarrow h(c)\},\ \{\overline{x}\leftarrow h(a),b,\ y\leftarrow h(c)\},\ \{\overline{x}\leftarrow a,h(b),\ y\leftarrow h(c)\},\ \{\overline{x}\leftarrow h(a),h(b),\ y\leftarrow h(c)\},\ \{\overline{x}\leftarrow h(a,b),\ y\leftarrow h(c)\}\}$.

In the next sections we will answer two questions for $\emptyset$- and F-theories:

Decidability: Is it decidable whether a unification problem is solvable?
Unification procedure: How can we obtain a (preferably minimal) unification procedure?

4 Decidability of Unification Problem in the Empty Theory with Sequence Variables and Flexible Arity Symbols

Let $UP = \langle term_1, term_2\rangle$ be a unification problem in the empty theory with sequence variables and flexible arity symbols. We assume without loss of generality that no sequence variable occurs as an argument of a fixed arity symbol. The problem is to prove that UP is decidable. To do so, first we try to reduce UP to a "simpler" unification problem RUP (reduced unification problem) such that UP is solvable if and only if RUP is solvable, and then try to prove that solvability of RUP is decidable.

4.1 Reduction

We reduce the problem of solvability of UP to the problem of solvability of a reduced unification problem RUP.

Let $fflex_1,\ldots,fflex_s$ $(s\geq 0)$ be all flexible arity function symbols occurring in UP. First we reduce UP to an intermediate unification problem IUP by performing the following steps:

1. Introduce
   - a new flat flexible arity symbol Seq,
   - a new unary function symbol fun_i for each $fflex_i$ $(i\in\{1,\ldots,s\})$.
2. Replace each term $fflex_i()$ in UP by $fun_i(Seq())$.
3. Replace each term $fflex_i(t_1,\ldots,t_n)$ in UP by $fun_i(Seq(t_1,\ldots,t_n))$ $(n\geq 1)$.

Note that in IUP sequence variables occur only as arguments of terms with the head Seq, or can be $term_1$ or $term_2$. We impose *individual variable restrictions on terms* on IUP demanding that, for a solution θ of IUP and for any individual variable x, $x\theta$ must not have Seq as a head.

Theorem 1. *UP is solvable iff IUP with individual variable restrictions on terms is solvable.*

Proof. ($\Rightarrow$) Let θ be a unifier of UP. Then from θ we can get a unifier of IUP with individual variable restrictions on terms by repeating the steps 2 and 3 of the reduction procedure on θ.

($\Leftarrow$) Let IUP with individual variable restrictions on terms have a solution ϕ. Show that UP is solvable.

Let θ be a substitution obtained from ϕ by

- replacing each unary function symbol fun introduced at the first step of the reduction procedure above by the corresponding flexible arity symbol $fflex$,
- replacing each term of the form $Seq()$ by the empty sequence,
- replacing each term of the form $Seq(t_1, \ldots, t_n)$ $(n \geq 1)$ by the sequence of terms $t_1, \ldots, t_n$.

Obviously θ is a unifier of UP. $\square$

Remark. Note that solvability of IUP (without individual variable restrictions on terms) does not imply solvability of UP: Let UP be the unification problem $\langle fflex(x), fflex(a, b)\rangle$, then IUP is $\langle fun(Seq(x)), fun(Seq(a, b))\rangle$. Clearly, UP does not have a solution, while $x \leftarrow Seq(a, b)$ is a solution of IUP because the flatness of Seq implies $Seq(Seq(a, b)) = Seq(a, b)$.

Next, our goal is to reduce IUP with individual variable restrictions on terms to a finite set $SRUP$ of reduced unification problems with the property that IUP with individual variable restrictions on terms is solvable iff there exists a $RUP \in SRUP$ which is solvable.

First, we define two new sets:

1. Let $CONST_{IUP}$ be the set of constants of IUP and $const$ be a new constant. Then
$$C`_{IUP} = CONST_{IUP} \cup \{const\}.$$
2. Let $FFIX_{IUP} = \{f_1, \ldots, f_m\}$, $m \geq 1$ be the set of all fixed arity function symbols in IUP and $y_1^1, \ldots, y_{AR(f_1)}^1, \ldots, y_1^m, \ldots, y_{AR(f_m)}^m$ be new distinct individual variables. Then
$$T`_{IUP} = \{f_1(y_1^1, \ldots, y_{AR(f_1)}^1), \ldots, f_m(y_1^m, \ldots, y_{AR(f_m)}^m)\},$$

Let $IV_{IUP} = \{x_1, \ldots, x_n\}$ be the set of all individual variables of IUP. Then $SRUP$ is defined as:

$$SRUP = \{IUP \cup \{\langle x_1, t_1\rangle, \ldots, \langle x_n, t_n\rangle\} \mid \forall i \in \{1, \ldots, n\}\, t_i \in C`_{IUP} \vee t_i \in T`_{IUP}\}.$$

Theorem 2. *IUP with individual variable restrictions on terms is solvable iff there exists a $RUP \in SRUP$ which is solvable.*

Proof. ($\Rightarrow$) Let θ be a unifier of IUP with individual variable restrictions on terms. Then for each $x \in IV_{IUP}$, $x\theta$ does not have Seq as a head, i.e. $x\theta$ is either a individual variable y, a constant c or has a form $f(s_1, \ldots, s_k)$ for some function symbol f and terms $s_1, \ldots, s_k$. We can assume without loss of generality that $y \in IV_{IUP}$, $c \in CONST_{IUP}$ and $f \in FFIX_{IUP}$.

Let σ be the substitution $\{x \leftarrow c \,|\, x \in IV_{IUP}\}$. Then $\theta \circ \sigma$ is a solution of IUP such that for each $x \in IV_{IUP}$, $x\theta \circ \sigma$ is either a constant from $C`_{IUP}$ or has a form $f(s_1\sigma, \ldots, s_{AR(f)}\sigma)$ for some $f \in FFIX_{IUP}$ and terms $s_1, \ldots, s_{AR(f)}$.

We introduce the following denotations:

- X_C - for the subset of IV_{IUP} such that for each $x \in X_C$, $x\theta \circ \sigma \in C`_{IUP}$;
- X_F - for the subset of IV_{IUP} such that for each $x \in X_F$, $x\theta \circ \sigma$ has a form $f(s_1\sigma, \ldots, s_{AR(f)}\sigma)$ for some $f \in FFIX_{IUP}$ and terms $s_1, \ldots, s_{AR(f)}$;
- c_x for each $x \in X_C$ - for a constant in $C`_{IUP}$ such that $x\theta \circ \sigma = c_x$;
- t_x for each $x \in X_F$ - for a term in $T`_{IUP}$ such that t_x and $x\theta \circ \sigma$ have the same fixed arity function symbol as the head.
- λ_x for each $x \in X_F$ - for a substitution $\{y_1 \leftarrow s_1\sigma, \ldots, y_k \leftarrow s_k\sigma\}$, such that for some $f \in FFIX_{IUP}$, individual variables $y_1, \ldots, y_k$ and terms $s_1, \ldots, s_k$, we have $k = AR(f)$, $t_x = f(y_1, \ldots, y_k)$ and $x\theta \circ \sigma = f(s_1\sigma, \ldots, s_k\sigma)$.

We choose $RUP \in SRUP$ such that

$$RUP = IUP \cup \bigcup_{x \in X_C} \{\langle x, c_x \rangle\} \cup \bigcup_{x \in X_F} \{\langle x, t_x \rangle\}.$$

Then the substitution

$$\theta \circ \sigma \circ \bigcup_{x \in X_F} \lambda_x$$

is a solution of RUP. Thus, solvability of IUP with individual variable restrictions on terms implies solvability of some RUP from $SRUP$.

($\Leftarrow$) Let ϕ be a solution of some $RUP \in SRUP$. Then, obviously, ϕ is a solution of IUP. Moreover, ϕ binds every individual variable of IUP either with a constant or a term with fixed arity head. Hence, ϕ is a solution of IUP with individual variable restrictions on terms. □

Thus, the problem of decidability of the unification problem UP is reduced to the problem of decidability of a reduced unification problem RUP.

4.2 Decidability of the Reduced Problem

We prove decidability of the reduced unification problem RUP using Baader-Schulz combination method [7]:

Theorem 3 (Baader and Schulz).
Let $E_1, \dots, E_n$ be equational theories over disjoint signatures such that solvability of E_i-unification problems with linear constant restriction is decidable for $i = 1, \dots, n$. Then unifiability is decidable for the combined theory $E_1 \cup \ldots \cup E_n$.

Linear constant restrictions are induced by a linear order $<$ on the set of variables and constants, demanding that, for a unifier θ, a constant c and a variable x, c must not occur in $x\theta$ if $c > x$.

Let Ω_1 be the set $\{Seq\}$ and Ω_2 be the set of fixed arity function symbols and object constants which occur in RUP. Let E_1 be an equational theory over the signature Ω_1 where terms are built from individual variables, sequence variables and the symbol Seq from Ω_1. Since Seq is a flat symbol every non-variable term of E_1 has a form $Seq(t_1, \dots, t_n)$ $(n \geq 0)$ where each t_i is either a sequence variable or an individual variable. Let E_2 be a free theory over Ω_2, where terms are built from individual variables and symbols from Ω_2. Then we can consider RUP as a unification problem in the combined theory $E_1 \cup E_2$. Since $\Omega_1 \cap \Omega_2 = \emptyset$, by Theorem 3, in order to show that solvability of RUP in $E_1 \cup E_2$ is decidable we need to show that solvability of E_1- and E_2-unification problems with linear constant restrictions is decidable.

Decidability of E_1-unification problem with linear constant restrictions. The E_1-unification problem can be seen as a word equation with an additional restriction which demands that for a unifier θ and a variable x in the word equation which corresponds to an individual variable in the unification problem, the length of $x\theta$ should be 1. We call the additional restrictions *individual variable restrictions on length.* Thus, the E_1-unification problem with linear constant restrictions can be considered as a word equation problem with linear constant restrictions and individual variable restrictions on length.

Note that we can consider ground unifiability case only. Let V be a set of (sequence or individual) variables, C be a set of constants and C' be the set $C \cup \{c\}$, where c is a constant not occurring in C. Suppose that a word equation WE is given with variables in V, constants in C, with linear constant restrictions induced by a linear ordering $<$ on $V \cup C$ and with individual variable restrictions on length. WE has a unifier θ over $V \cup C$ which satisfies the linear constant restrictions induced by $<$ and the individual variable restrictions on length iff WE has a ground unifier θ' over C' which satisfies the linear constant restrictions induced by $<$ and the individual variable restrictions on length.

The following general result of [15] plays the crucial role in proving decidability of E_1-unification problem with linear constant restrictions.

Theorem 4 (Schulz). *If WE is a word equation with variables $v_1, \dots, v_n$ and constants in the alphabet C', and if $L_1, \dots, L_n$ are regular languages over C', then it is decidable whether WE has a solution θ such that $v_i\theta \in L_i$ for $i = 1, \dots, n$.*

Using this result we can prove the theorem which solves the decidability problem for E_1-unification with linear constant restrictions:

Theorem 5. *Solvability of a word equation with linear constant restrictions and individual variable restrictions on length is decidable.*

Proof. Let WE be a word equation with variables $x_1, \dots, x_m, \overline{y}_1, \dots, \overline{y}_k$ and constants in the alphabet C', where the variables $x_1, \dots, x_m$ are subjects of individual variable restrictions on length. Let $L_1, \dots, L_{m+k}$ be regular languages over C' such that

- for all $i \in \{1, \dots, m\}$, $L_i = \{a \in C \mid a < x_i\} \cup \{c\}$,
- for all $i \in \{1, \dots, k\}$, $L_{m+i} = (\{a \in C \mid a < \overline{y}_i\} \cup \{c\})^+$.

A solution θ of WE satisfies the linear constant restrictions and the individual variable restrictions on length iff $\theta(x_i) \in L_i$ for all $i \in \{1, \dots, m\}$ and $\theta(\overline{y}_i) \in L_{m+i}$ for all $i \in \{1, \dots, k\}$. Thus, Theorem 4 implies that solvability of a word equation with linear constant restrictions and individual variable restrictions on length is decidable. $\square$

Decidability of E_2-unification problem with linear constant restrictions. Obviously, the E_2-unification problem is a Robinson unification problem. Solvability of Robinson unification problems with linear constant restrictions is decidable (see [8]).

Thus, solvability of unification problem in the $\emptyset$-theory with sequence variables and flexible arity symbols is decidable.

5 Decidability of Unification Problem in the Flat Theory with Sequence Variables and Flexible Arity Symbols

Let $UP = \langle term_1, term_2 \rangle$ be a unification problem in the flat theory with sequence variables and flexible arity symbols. Let $flat$ be a flexible arity flat symbol in UP. We assume without loss of generality that no sequence variable occurs as an argument of a fixed arity symbol. The problem is to prove that UP is decidable. Again, as in the case with the empty theory, first we try to reduce UP to a "simpler" unification problem RUP (reduced unification problem) such that UP is solvable if and only if RUP is solvable, and then try to prove that solvability of RUP is decidable.

5.1 Reduction

Let $fflex_1, \dots, fflex_s$ $s \geq 0$ be all non-flat flexible arity function symbols occurring in UP. First we reduce UP to an intermediate unification problem IUP by performing the following steps:

1. Introduce
   - a new flat flexible arity symbol Seq
   - a new unary function symbol fun_i for each $fflex_i$ ($i \in \{1, \dots, s\}$).
2. Replace each term $fflex_i()$ in UP by $fun_i(Seq())$.

3. Replace each term $fflex_i(t_1, \ldots, t_n)$ in UP by $fun_i(Seq(t_1, \ldots, t_n))$ $(n \geq 1)$.

Note that sequence variables occur only as arguments of terms with the head $flat$ or Seq, or can be $term_1$ or $term_2$. As in the case of the empty theory, we impose *individual variable restrictions on terms* on IUP demanding that, for a solution θ of IUP and for any individual variable x, $x\theta$ must not have Seq as the head.

Theorem 6. *UP is solvable iff IUP with individual variable restrictions on terms is solvable.*

Proof. Analogous to the proof of Theorem 1. □

Next, our goal is to reduce IUP with individual variable restrictions on terms to a finite set $SRUP$ of reduced unification problems (RUP-s) with the property that IUP with individual variable restrictions on terms is solvable iff at least one RUP is solvable. We define $SRUP$ in the same way as in the case of the empty theory.

Theorem 7. *IUP with individual variable restrictions on terms is solvable iff there exists a $RUP \in SRUP$ which is solvable.*

Proof. Analogous to the proof of Theorem 2. □

5.2 Decidability of the Reduced Problem

We prove decidability of the reduced unification problem RUP using Baader-Schulz combination method.

Let Ω_1 be the set $\{flat\}$, Ω_2 be the set $\{Seq\}$, and Ω_3 be the set of fixed arity function symbols and object constants which occur in RUP. Let E_1 be an equational theory over the signature Ω_1, where terms are built from individual variables, sequence variables and the symbol $flat$ from Ω_1. Every non-variable term of E_1 has a form $flat(t_1, \ldots, t_n)$ $(n \geq 0)$ where each t_i is either a sequence variable or an individual variable. Let E_2 be an equational theory over the signature Ω_2, where terms are built from individual variables, sequence variables and the symbol Seq from Ω_2. Every non-variable term of E_2 has a form $Seq(t_1, \ldots, t_n)$ $(n \geq 0)$ where each t_i is either a sequence variable or an individual variable. Let E_3be a free theory over Ω_3, where terms are built from individual variables and symbols from Ω_3. Then we can consider RUP as a unification problem in the combined theory $E_1 \cup E_2 \cup E_3$. Since Ω_1, $\Omega_2$and Ω_3 are pairwise disjoint, by Theorem 3, in order to show that solvability of RUP in $E_1 \cup E_2 \cup E_3$ is decidable we need to show for each $i \in \{1, 2, 3\}$ that solvability of E_i-unification problem with linear constant restrictions is decidable.

The E_1-unification problem with linear constant restrictions can be seen as a word equation problem with linear constant restrictions, which is decidable (see [8]).

The E_2-unification problem with linear constant restrictions can be seen as a word equation problem with linear constant restrictions and individual variable restrictions on length. Then, by Theorem 5 we have that solvability of the E_2-unification problem with linear constant restrictions is decidable.

The E_3−unification problem is a Robinson unification problem solvability of which with linear constant restrictions is decidable (see [8]).

Thus, solvability of RUP is decidable, which, by Theorem 6 and Theorem 7 implies that solvability of unification problem in the F-theory with sequence variables and flexible arity symbols is decidable.

6 Unification Procedure in the Empty Theory with Sequence Variables and Flexible Arity Symbols

We design unification procedure as a tree generation process based on three basic steps: projection, arity completion and transformation. Below we describe these steps in details.

For a unification problem UP, we denote by $SUC(UP)$ the successors of UP under projection, arity completion or transformation. $SUC(UP)$ can be $SUCCESS$, $FAILURE$ or a tuple of unification problems. $SUB(UP)$ denotes the tuple of substitutions which were applied on UP to get the successors of UP.

Let us fix the unification problem $UP = \langle term_1, term_2 \rangle$.

6.1 Projection

The idea of projection is to eliminate some sequence variables from the given unification problem ([1]). Let $S_1, \dots, S_n$ be all subsets of the set of all sequence variables of UP and Θ be the set of substitutions $\{\theta_1, \dots, \theta_n\}$ such that for all $i \in \{1, \dots, n\}$ $\theta_i = \{\overline{x} \leftarrow \ \mid \overline{x} \in S_i\}$. Then under projection $SUB(UP) = \langle \sigma_1, \dots, \sigma_k \rangle$ and $SUC(UP) = \langle UP\sigma_1, \dots, UP\sigma_k \rangle$, where $\sigma_1, \dots, \sigma_k$ are all the substitutions from Θ applicable on UP.

6.2 Arity Completion

The idea of arity completion is to get rid of sequence variables which have occurrences as arguments of terms with fixed arity heads.

We denote by $ACS_E(UP)$ the E-minimal complete set of arity completing substitutions for UP with respect to the set of all sequence variables occurring as arguments of subterms of UP with fixed arity heads. Note that each substitution from $ACS_E(UP)$ replaces a sequence variable $\overline{x}$, occurring as an argument of a subterm of UP with a fixed arity head, with a (possibly empty) sequence of new distinct individual variables. $ACS_E(UP)$ is finite and unique up to renaming individual variables in its elements.

For UP we define $SUB(UP)$ and $SUC(UP)$ under arity completion in the empty theory as follows:

- if $ACS_\emptyset(UP) = \emptyset$, then $SUB(UP) = \langle\rangle$ and $SUC(UP) = FAILURE$;
- if $ACS_\emptyset(UP) = \{\sigma_1, \dots, \sigma_k\} \neq \emptyset$, then $SUB(UP) = \langle \sigma_1, \dots, \sigma_k \rangle$ and $SUC(UP) = \langle UP\sigma_1, \dots, UP\sigma_k \rangle$.

6.3 Transformation

Given a unification problem $UP = \langle term_1, term_2 \rangle$, to find $SUC(UP)$ and $SUB(UP)$ under transformation we distinguish the following three cases:

1. $term_1$ and $term_2$ are identical. Then $SUB(UP) = \langle \varepsilon \rangle$ and $SUC(UP) = SUCCESS$.
2. $term_1$ and $term_2$ are not neither identical terms nor non-variable terms with the same head. Then $SUB(UP)$ and $SUC(UP)$ are defined according to Table 1.
3. $term_1$ and $term_2$ are non-identical non-variable terms with the same head. Let f be the head. Then we have the following two cases:
   (a) Only one from $term_1$ and $term_2$ has the form $f()$. Then $SUB(UP) = \langle\rangle$ and $SUC(UP) = FAILURE$.
   (b) None of $term_1$ and $term_2$ is a term of the form $f()$. Let $term_1$ be $f(t_1, \overline{t})$ and $term_2$ be $f(s_1, \overline{s})$, where $\overline{t}$ and $\overline{s}$ are (possibly empty) sequences of terms. Then we have the following cases:
      i. t_1 and s_1 are identical. Then $SUB(UP) = \langle \varepsilon \rangle$ and $SUC(UP) = \langle\langle g(\overline{t}), g(\overline{s})\rangle\rangle$, where g is a new $n-1$-ary function symbol if f is an n-ary function symbol, and g is f if f is a flexible arity function symbol.
      ii. t_1 and s_1 are not identical. Then we define $SUB(UP)$ and $SUC(UP)$ according to Table 2.

6.4 Unification Procedure - Tree Generation

Projection, arity completion and transformation can be seen as single steps in a tree generation process. Each node of the tree is labeled either with a unification problem, $SUCCESS$ or $FAILURE$. The edges of the tree are labeled by substitutions. The nodes labeled with $SUCCESS$ or $FAILURE$ are terminal nodes. The nodes labeled with unification problems are non-terminal nodes. The children of a non-terminal node are constructed in the following way:

Given a nonterminal node, let UP be a unification problem attached to it. First, we decide whether UP is unifiable. If the answer is negative, we replace UP with the new label $FAILURE$. If UP is unifiable, we apply projection, arity completion or transformation on UP and get $SUB(UP)$ and $SUC(UP)$. If $SUC(UP)$ is $SUCCESS$, the node has a single child with the label $SUCCESS$ and the edge to that node is labeled with $SUB(UP)$. If $SUC(UP) = \langle p_1, \ldots, p_n \rangle$ and $SUB(UP) = \langle \sigma_1, \ldots, \sigma_k \rangle$, the node UP has n children, labeled respectively with $p_1, \ldots, p_n$ and the edge to the p_i node is labeled with σ_i $(1 \le i \le n)$.

We design the unification procedure as a breadth first (level by level) tree generation process. Let UP be a unification problem. We label the root of the tree with UP (zero level). First level nodes (the children of the root) of the tree are obtained from the original problem by projection. Second level nodes are obtained by arity completion, completing arity for each unification problem attached to the first level nodes. Starting from the second level, we apply only

Table 1. Transformation table for the unification problem $UP = \langle term_1, term_2 \rangle$ where $term_1$ and $term_2$ are not non-variable terms with the same head.

$term_1$	$term_2$	$SUB(UP)$	$SUC(UP)$
Ind. variable	Ind. variable	$\langle\{term_1 \leftarrow term_2\}\rangle$	$SUCCESS$
Ind. variable	Seq. variable	$\langle\{term_2 \leftarrow term_1\}\rangle$	$SUCCESS$
Ind. variable	Non-variable term	$\langle\rangle$, if $term_1$ occurs in $term_2$ $\langle\{term_1 \leftarrow term_2\}\rangle$, otherwise	$FAILURE$ if $term_1$ occurs in $term_2$ $SUCCESS$ otherwise
Seq. variable	Ind. variable	$\langle\{term_1 \leftarrow term_2\}\rangle$	$SUCCESS$
Seq. variable	Seq. variable	$\langle\{term_1 \leftarrow term_2\}\rangle$	$SUCCESS$
Seq. variable	Non-variable term	$\langle\rangle$, if $term_1$ occurs in $term_2$ $\langle\{term_1 \leftarrow term_2\}\rangle$, otherwise	$FAILURE$ if $term_1$ occurs in $term_2$ $SUCCESS$ otherwise
Non-variable term	Ind.variable	$\langle\rangle$, if $term_2$ occurs in $term_1$ $\langle\{term_2 \leftarrow term_1\}\rangle$, otherwise	$FAILURE$ if $term_2$ occurs in $term_1$ $SUCCESS$ otherwise
Non-variable term	Seq.variable	$\langle\rangle$, if $term_2$ occurs in $term_1$ $\langle\{term_2 \leftarrow term_1\}\rangle$, otherwise	$FAILURE$ if $term_2$ occurs in $term_1$ $SUCCESS$ otherwise
Non-variable term	Non-variable term	$\langle\rangle$	$FAILURE$

Table 2. Transformation table for the unification problem $UP = \langle term_1, term_2 \rangle$ where $term_1 = f(t_1, \overline{t})$ and $term_2 = f(s_1, \overline{s})$ are non-variable terms with the same head. The function symbol g in $SUC(UP)$ is a new $n-1$-ary function symbol if f is an n-ary function symbol, and g is f if f is a flexible arity function symbol.

t_1	s_1	$SUB(UP)$	$SUC(UP)$
Ind. variable	Ind. variable	$\sigma = SUB(\langle t_1, s_1 \rangle)$	$\langle\langle g(\overline{t}\sigma), g(\overline{s}\sigma)\rangle\rangle$
Ind. variable	Seq. variable	$\langle \sigma_1, \sigma_2 \rangle$, where $\langle \sigma_1 \rangle = SUB(\langle t_1, s_1 \rangle)$, $\sigma_2 = \{s_1 \leftarrow t_1, s_1\}$,	$\langle\langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1)\rangle$, $\langle f(\overline{t}\sigma_2), f(s_1, \overline{s}\sigma_2)\rangle\rangle$
Ind. variable	Non-variable term	$\langle\rangle$, if $SUB(\langle t_1, s_1 \rangle) = \langle\rangle$ $\langle \sigma \rangle = SUB(\langle t_1, s_1 \rangle)$, otherwise	$FAILURE$ if $SUC(\langle t_1, s_1 \rangle) = FAILURE$ $\langle\langle g(\overline{t}\sigma), g(\overline{s}\sigma)\rangle\rangle$ otherwise
Seq. variable	Ind. variable	$\langle \sigma_1, \sigma_2 \rangle$, where $\langle \sigma_1 \rangle = SUB(\langle t_1, s_1 \rangle)$, $\sigma_2 = \{t_1 \leftarrow s_1, t_1\}$,	$\langle\langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1)\rangle$, $\langle f(t_1, \overline{t}\sigma_2), f(\overline{s}\sigma_2)\rangle\rangle$
Seq. variable	Seq. variable	$\langle \sigma_1, \sigma_2, \sigma_3 \rangle$, where $\langle \sigma_1 \rangle = SUB(\langle t_1, s_1 \rangle)$, $\sigma_2 = \{t_1 \leftarrow s_1, t_1\}$, $\sigma_3 = \{s_1 \leftarrow t_1, s_1\}$,	$\langle\langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1)\rangle$, $\langle f(t_1, \overline{t}\sigma_2), f(\overline{s}\sigma_2)\rangle\rangle$ $\langle f(\overline{t}\sigma_3), f(s_1, \overline{s}\sigma_2)\rangle\rangle$
Seq. variable	Non-variable term	$\langle\rangle$, if $SUB(\langle t_1, s_1 \rangle) = \langle\rangle$ $\langle \sigma_1, \sigma_2 \rangle$, where $\langle \sigma_1 \rangle = SUB(\langle t_1, s_1 \rangle)$, $\sigma_2 = \{t_1 \leftarrow s_1, t_1\}$, otherwise	$FAILURE$ if $SUC(\langle t_1, s_1 \rangle) = FAILURE$ $\langle\langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1)\rangle$, $\langle f(t_1, \overline{t}\sigma_2), f(\overline{s}\sigma_2)\rangle\rangle$ otherwise
Non-variable term	Ind.variable	$\langle\rangle$, if $SUB(\langle t_1, s_1 \rangle) = \langle\rangle$ $\langle \sigma \rangle = SUB(\langle t_1, s_1 \rangle)$, otherwise	$FAILURE$ if $SUC(\langle t_1, s_1 \rangle) = FAILURE$ $\langle\langle g(\overline{t}\sigma), g(\overline{s}\sigma)\rangle\rangle$ otherwise
Non-variable term	Seq. variable	$\langle\rangle$, if $SUB(\langle t_1, s_1 \rangle) = \langle\rangle$ $\langle \sigma_1, \sigma_2 \rangle$, where $\langle \sigma_1 \rangle = SUB(\langle t_1, s_1 \rangle)$, $\sigma_2 = \{s_1 \leftarrow t_1, s_1\}$, otherwise	$FAILURE$ if $SUC(\langle t_1, s_1 \rangle) = FAILURE$ $\langle\langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1)\rangle$, $\langle f(t_1, \overline{t}\sigma_2), f(\overline{s}\sigma_2)\rangle\rangle$ otherwise
Non-variable term	Non-variable term	$SUB(\langle t_1, s_1 \rangle)$	$FAILURE$ if $SUC(\langle t_1, s_1 \rangle) = FAILURE$ $\langle\langle g(\overline{t}\sigma), g(\overline{s}\sigma)\rangle\rangle$, if $SUC(\langle t_1, s_1 \rangle) = SUCCESS$ and $SUB(\langle t_1, s_1 \rangle) = \sigma$ $\langle\langle f(suc_{11}, \overline{t}\sigma_1), f(suc_{12}, \overline{s}\sigma_1)\rangle, \dots,$ $\langle f(suc_{k1}, \overline{t}\sigma_k), f(suc_{k2}, \overline{s}\sigma_k)\rangle\rangle$, if $SUC(\langle t_1, s_1 \rangle) =$ $\langle\langle suc_{11}, suc_{12}\rangle, \dots, \langle suc_{k1}, suc_{k2}\rangle\rangle$ and $SUB(\langle t_1, s_1 \rangle) = \langle \sigma_1, \dots, \sigma_k \rangle$

a transformation step to a unification problem of each node, thus getting new successor nodes. The branch which ends with a node labeled by $SUCCESS$ is called a successful branch. The branch which ends with a node labeled by $FAILURE$ is a failed branch. For each node in the tree, we compose substitutions (top-down) displayed on the edges of the branch which leads to this node, restrict the composed substitution on the variables from UP and attach the obtained substitution to the node together with the unification problem the node was labeled with. The empty substitution is attached to the root. For a node N, the substitution attached to N in such a way is called the associated substitution of N. Let $\Sigma(UP)$ denote the set of all substitutions which are associated with $SUCCESS$ nodes. We call the tree a $\emptyset$-unification tree for UP and denote it $UTree_{\emptyset}(UP)$.

Our goal is to prove that $\Sigma(UP)$ is a minimal complete set of $\emptyset$-unifiers for UP. In fact, we will prove a stronger statement: $\Sigma(UP)$ is a disjoint complete set of $\emptyset$-unifiers for UP, where disjointness is defined as follows:

Definition 20 (Disjoint Set of Substitutions). *A set of substitutions Σ is called disjoint modulo E with respect to a set of variables Var iff for all θ, $\sigma \in \Sigma$, if there exist substitutions λ_1, λ_2 such that for all sequence variables $\overline{x} \in Var$, $\overline{x}\lambda_1$ and $\overline{x}\lambda_2$ are not the empty sequences and for all variables $v \in Var$, $v\theta \circ \lambda_1 =_E v\sigma \circ \lambda_2$, then $\theta = \sigma$.*

First we need to establish some preliminary results.

Lemma 1. *If a set of substitutions Σ is E-disjoint with respect to a set of variables Var, then Σ is E-minimal w.r.t. Var.*

Proof. Let θ and σ be two substitutions from Σ such that $\theta \ll_E^{Var} \sigma$. We will show that $\theta = \sigma$.

$\theta \ll_E^{Var} \sigma$ implies that there exists a substitution φ such that for all $\overline{x} \in Var$, $\overline{x}\varphi$ is not the empty sequence and for all $v \in Var$, $v\sigma =_E v\theta \circ \varphi$. On the other hand, for the empty substitution ε, $v\sigma =_E v\sigma \circ \varepsilon$ and for all $\overline{x} \in Var$, $\overline{x}\varepsilon$ is not the empty sequence. Thus, we found substitutions ε and φ such that for all sequence variables $\overline{x} \in Var$, $\overline{x}\varepsilon$ and $\overline{x}\varphi$ are not the empty sequences and for all variables $v \in Var$, $v\sigma \circ \varepsilon =_E v\theta \circ \varphi$. Then, since Σ is disjoint, by Definition 20 we get $\theta = \sigma$. □

Lemma 2. *For every $\emptyset$-unifier ϕ of UP there exists a branch β in $UTree_{\emptyset}(UP)$ with the following property: if P is a unification problem occurring in β with the associated substitution θ, then $\theta \ll_{\emptyset}^{Var} \phi$ where Var is the set of variables of UP.*

Proof. Let ϕ be an arbitrary $\emptyset$-unifier of UP and Var be the set of variables of UP. The first node in β is the root, labeled with UP with the associated substitution ε. Obviously, $\varepsilon \ll_{\emptyset}^{Var} \phi$.

Suppose now for the induction hypothesis that P is a unification problem in β with the associated substitution σ such that $\sigma \ll_{\emptyset}^{Var} \phi$. Thus, there exists a substitution λ such that for all $\overline{x} \in Var$, $\overline{x}\lambda$ is not an empty sequence and for all $v \in Var$, $v\phi =_{\emptyset} v\sigma \circ \lambda$.

To made the induction step, we have to find the successor of P with respect to ϕ and show that the substitution associated with the successor is more general than ϕ.

We can have projection, arity compeltion or transformation to make the step.

It can be easily seen that if we derive the successor of P using projection or arity completion with respect to ϕ then the substitution θ associated to the successor is more general then ϕ.

As for deriving the successor of P with respect to ϕ by transformation, there are the following 7 cases according to the form of the P.

1. P is a pair of identical terms.
2. P is a pair of individual variables.
3. P is a pair of an individual variable and a sequence variable.
4. P is a pair of an individual variable and non-variable term.
5. P is a pair of sequence variables.
6. P is a pair of an sequence variable and non-variable term.
7. P is a pair of non-variable terms.

We have to show that the substitution associated with the successor is more general than ϕ.

1. There is only one possible choice: successor of P is $SUCCESS$ with the accosiated substitution $\theta = \sigma \circ \varepsilon$. Then $\theta = \sigma \ll_{\emptyset}^{Var} \phi$.

2. Let P be $\langle x, y \rangle$. To make the step we choose the substitution μ such that $\mu = \{x \leftarrow y\}$ if $x\lambda =_{\emptyset} y$ and $\mu = \{y \leftarrow x\}$, if $y\lambda =_{\emptyset} x$. There can not be any other choice because $\sigma \circ \lambda$ is a unifier of UP and $UP\sigma$ is P. Applying the substitution $\theta = \sigma \circ \mu$ associated with the successor of P to any variable $v \in Var$ gives the same result as applying $\sigma \circ \lambda$ on it: $v\theta =_{\emptyset} v\sigma \circ \mu =_{\emptyset} v\sigma \circ \lambda =_{\emptyset} v\phi$, which implies $\theta \ll_{\emptyset}^{Var} \phi$.

3. Let P be $\langle \overline{x}, y \rangle$. To make the step we can only choose the substitution $\mu = \{\overline{x} \leftarrow y\}$. Since $\sigma \circ \lambda$ is an $\emptyset$-unifier on UP, $\overline{x}\lambda =_{\emptyset} y$. Applying the substitution $\theta = \sigma \circ \mu$ associated with the successor of P to any variable $v \in Var$ gives the same result as applying $\sigma \circ \lambda$ on it: $v\theta =_{\emptyset} v\sigma \circ \mu =_{\emptyset} v\sigma \circ \lambda =_{\emptyset} v\phi$, which implies $\theta \ll_{\emptyset}^{Var} \phi$.

Cases 4-6 can be handled analogously.

7. Let P have the form $\langle f(t_1, \overline{t}), f(s_1, \overline{s}) \rangle$, where $\overline{t}$ and $\overline{s}$ are (possibly empty) sequences of terms. Again, with respect to t_1 and s_1, the pair $\langle t_1, s_1 \rangle$ can have one of the 7 forms above. Let us consider the case when it is a pair of a sequence variable and a non-variable term (case 6) (The proof in cases 1-5 for $\langle t_1, s_1 \rangle$ are similar to the proof of the case 6 for $\langle t_1, s_1 \rangle$ and the case 7 for $\langle t_1, s_1 \rangle$ can be handled as the case 7 for P):

Suppose t_1 is a sequence variable $\overline{x}$ and s_1 is a non-variable term. To find the successor of P with respect to ϕ we choose the substitution μ such that $\mu = \{\overline{x} \leftarrow s_1\}$ if $\overline{x}\lambda =_{\emptyset} s_1$ and $\mu = \{\overline{x} \leftarrow s_1, \overline{x}\}$, if $\overline{x}\lambda =_{\emptyset} s_1, l_1, \ldots, l_k$, where $l_1, \ldots, l_k$ is a (possibly empty) sequence of terms. Note that λ is an $\emptyset$-unifier of P, because $\sigma \circ \lambda$ is an $\emptyset$-unifier of UP and P can be reached from $UP\sigma$ by iterated deletion of identical first arguments of the members of the pair. Therefore, the two choices of μ are the only possible choices to made the step from P with respect to λ.

Let us define a new substitution ν as follows: for all $v \in Var$, $v \neq \overline{x}$,

- $\nu = \{v \leftarrow v\lambda\}$ if $\mu = \{\overline{x} \leftarrow s_1\}$ and
- $\nu = \{\overline{x} \leftarrow l_1, \ldots, l_k, v \leftarrow v\lambda\}$, if $\mu = \{\overline{x} \leftarrow s_1, \overline{x}\}$.

Then $\mu \ll_{\emptyset}^{Var} \lambda$, since for all $\overline{x} \in Var$, $\overline{x}\nu$ is not an empty sequence and for all $v \in Var$, $v\lambda =_{\emptyset} v\mu \circ \nu$.

The substitution associated with the successor of P is $\sigma \circ \mu$. Let $v \in Var$. We have $v\sigma \circ \mu \circ \nu =_{\emptyset} v\sigma \circ \lambda =_{\emptyset} v\phi$, which shows that $\sigma \circ \mu \ll_{\emptyset}^{Var} \phi$. □

Theorem 8. *$\Sigma(UP)$ is a complete set of $\emptyset$-unifiers for UP.*

Proof. The theorem follows from Lemma 2 by the definition of Σ. □

Next, we want to show that $\Sigma(UP)$ is a disjoint set of $\emptyset$-unifiers. For this, first we introduce the notion of D-preserving (disjointness-preserving) substitution due to Schulz [16]

Definition 21 (D-preserving substitution). *A substitution θ is D-preserving modulo E with respect to a set Var of variables iff for any two substitutions σ_1 and σ_2, which are disjoint modulo E with respect to the set of variables $\cup_{v \in Var} VS(v\theta)$ (where $VS(v\theta)$ is the set of variables of $v\theta$), the substitutions $\theta \circ \sigma_1$ and $\theta \circ \sigma_2$ are disjoint modulo E with respect to Var.*

Lemma 3. *For an $\emptyset$-unification problem P any of the projection, arity completion and transformation substitutions are D-preserving modulo $\emptyset$-theory with respect to the set of variables of P.*

Proof. It can be easily shown that the projection and arity completion substitutions preserve disjointness. Let us consider the transformation substitutions.

There are the following 7 possible forms of the unification problem P the transformation step of the unification procedure deals with:

1. P is a pair of identical terms.
2. P is a pair of individual variables.
3. P is a pair of an individual variable and a sequence variable.
4. P is a pair of an individual variable and non-variable term.
5. P is a pair of sequence variables.
6. P is a pair of an sequence variable and non-variable term.
7. P is a pair of non-variable terms.

We will prove that the cases 3 and 7 of transformation preserve disjointness. The rest of the cases can be proven similarly to the case 3.

3. Let $Varset(P)$ be the set of variables of P. Let P be $\langle \overline{x}, y \rangle$. Then the transformation substitution is $\{\overline{x} \leftarrow y\}$. Let σ_1 and σ_2 be two disjoint substitutions with respect to $\bigcup_{v \in Varset(P)} Varset(v\{\overline{x} \leftarrow y\}) = \{y\}$. We have to show that

$\{\overline{x} \leftarrow y\} \circ \sigma_1$ and $\{\overline{x} \leftarrow y\} \circ \sigma_2$ are disjoint w.r.t. $Varset(P)$. Assume by contradiction that they are not disjoint. Then there exist substitutions λ_1 and λ_2 such that for all sequence variables $\overline{x} \in Varset(P)$, $\overline{x}\lambda_1$ and $\overline{x}\lambda_2$ are not the empty sequences and for all $v \in Varset(P)$, $v\{\overline{x} \leftarrow y\} \circ \sigma_1 \circ \lambda_1 =_\emptyset v\{\overline{x} \leftarrow y\} \circ \sigma_2 \circ \lambda_2$. In particular, since $Varset(P) = \{\overline{x}, y\}$

$$\overline{x}\{\overline{x} \leftarrow y\} \circ \sigma_1 \circ \lambda_1 =_\emptyset \overline{x}\{\overline{x} \leftarrow y\} \circ \sigma_2 \circ \lambda_2,$$

$$y\{\overline{x} \leftarrow y\} \circ \sigma_1 \circ \lambda_1 =_\emptyset y\{\overline{x} \leftarrow y\} \circ \sigma_2 \circ \lambda_2.$$

Hence,

$$y \circ \sigma_1 \circ \lambda_1 =_\emptyset y \circ \sigma_2 \circ \lambda_2,$$

which contradicts the assumption that σ_1 and σ_2 are two disjoint substitutions with respect to $\{y\}$.

7. Let P have the form $\langle f(t_1, \overline{t}), f(s_1, \overline{s})\rangle$, where $\overline{t}$ and $\overline{s}$ are (possibly empty) sequences of terms. Again, with respect to t_1 and s_1, the pair $\langle t_1, s_1\rangle$ can have one of the 7 forms above. Let us consider the case when it is a pair of an individual variable and a sequence variable (case 3) (The proof of the cases 1, 2, 4-5 for $\langle t_1, s_1\rangle$ are similar to the proof of the case 3 for $\langle t_1, s_1\rangle$ and the case 7 for $\langle t_1, s_1\rangle$ can be handled as the case 7 for P):

Suppose t_1 is a sequence variable $\overline{x}$ and s_1 is an individual variable y. Then there are two possibilities to make the transformation: 1) using the substitution $\{\overline{x} \leftarrow y\}$ and 2) using the substitution $\{\overline{x} \leftarrow y, \overline{x}\}$. The case 1) is similar to the case 3 for UP we have considered above. Consider 2). Let σ_1 and σ_2 be two disjoint substitutions with respect to $\bigcup_{v \in Varset(P)} Varset(v\{\overline{x} \leftarrow y, \overline{x}\})$. We have to show that $\{\overline{x} \leftarrow y, \overline{x}\} \circ \sigma_1$ and $\{\overline{x} \leftarrow y, \overline{x}\} \circ \sigma_2$ are disjoint w.r.t. $Varset(P)$. Assume by contradiction that they are not disjoint. Then there exist substitutions λ_1 and λ_2 such that for all sequence variables $\overline{x} \in Varset(P)$, $\overline{x}\lambda_1$ and $\overline{x}\lambda_2$ are not the empty sequences and for all $v \in Varset(P)$, $v\{\overline{x} \leftarrow y, \overline{x}\} \circ \sigma_1 \circ \lambda_1 =_\emptyset v\{\overline{x} \leftarrow y, \overline{x}\} \circ \sigma_2 \circ \lambda_2$. In particular, for $\overline{x}$ and y we have

$$\overline{x}\{\overline{x} \leftarrow y, \overline{x}\} \circ \sigma_1 \circ \lambda_1 =_\emptyset \overline{x}\{\overline{x} \leftarrow y, \overline{x}\} \circ \sigma_2 \circ \lambda_2,$$

$$y\{\overline{x} \leftarrow y, \overline{x}\} \circ \sigma_1 \circ \lambda_1 =_\emptyset y\{\overline{x} \leftarrow y, \overline{x}\} \circ \sigma_2 \circ \lambda_2.$$

Therefore,

$$\overline{x} \circ \sigma_1 \circ \lambda_1 =_\emptyset \overline{x} \circ \sigma_2 \circ \lambda_2.$$

Since for all $v \neq \overline{x}$, $v\{\overline{x} \leftarrow y, \overline{x}\} =_\emptyset v$, what we got is that σ_1 and σ_2 are not disjoint substitutions with respect to $\cup_{v \in Varset(P)} Varset(v\{\overline{x} \leftarrow y, \overline{x}\})$, which is a contradiction. □

Theorem 9. *$\Sigma(UP)$ is a disjoint set of $\emptyset$-unifiers for UP with respect to the set of variables of UP.*

Proof. The projection, arity completion and transformation substitutions, applied on a unification problem P are D-preserving w.r.t. variables of P. Suppose $\sigma_1$and σ_2 are substitutions associated with distinct successful leaves of

$UTree_\emptyset(UP)$ and let π be the common part (edges) leading to the respective leaves. We prove that $\sigma_1$and σ_2 are disjoint with respect to the set of variables of UP.

The substitutions σ_1 and σ_2 can be represented respectively as $\gamma \circ \theta_1 \circ \cdots \circ \theta_k$ and $\gamma \circ \lambda_1 \circ \cdots \circ \lambda_n$, where γ is the composition of substitutions associated with the nodes in π. Let P be the last node of π.

θ_1 and λ_1 are disjoint w.r.t the set of variables of P since they are either projection, arity completion or transformation substitutions. We will show that $\theta_1 \circ \cdots \circ \theta_k$ and $\lambda_1 \circ \cdots \circ \lambda_n$ are disjoint w.r.t. the variables of P.

Suppose $\theta_1 \circ \cdots \circ \theta_k$ and $\lambda_1 \circ \cdots \circ \lambda_n$ are not disjoint w.r.t. the set of variables of P. Then there exist substitutions μ_1 and μ_2 such that for all sequence variables $\overline{x}$ from the set of variables of P, $\overline{x}\mu_1$and $\overline{x}\mu_2$ are not the empty sequences and for all variables v from the set of variables of P, $v\theta_1 \circ \cdots \circ \theta_k \circ \mu_1 =_\emptyset v\lambda_1 \circ \cdots \circ \lambda_n \circ \mu_2$. It implies that for θ_1 and λ_1 there exist substitutions $\nu_1 = \theta_2 \circ \cdots \circ \theta_k \circ \mu_1$ and $\nu_2 = \lambda_2 \circ \cdots \circ \lambda_n \circ \mu_2$ such that for all sequence variables $\overline{x}$ from the set of variables of P, $\overline{x}\nu_1$ and $\overline{x}\nu_2$ are not the empty sequences (taking into account that $\theta_2, \ldots, \theta_k, \lambda_2, \ldots, \lambda_n$ are not projections) and for all variables v from the set of variables of P, $v\theta_1 \circ \nu_1 =_\emptyset v\lambda_1 \circ \nu_2$. It means that θ_1 and λ_1 are not disjoint w.r.t the set of variables of P, which is a contradiction.

Thus, to prove the theorem it is enough to prove that θ_1 and λ_1 are disjoint w.r.t the set of variables of UP. We prove it by induction on the length $|\pi|$ of π.

Induction base: $\pi = 0$. Then P is UP and θ_1 and λ_1 are disjoint w.r.t. the set of variables of UP.

Induction hypothesis: assume that θ_1 and λ_1 are disjoint w.r.t the set of variables of UP if $|\pi| = m > 0$.

Induction step: prove that θ_1 and λ_1 are disjoint w.r.t. the set of variables of UP when $|\pi| = m + 1$.

Let AP be an ancestor of the node P. Then θ_1 and λ_1 are disjoint w.r.t the set of variables of AP, because they are disjoint w.r.t the set of variables of P and the substitution which transformed AP to P is either a projection, arity completion or transformation substitution, which, by Lemma 3, is a D-preserving substitution w.r.t. to variables of AP. But AP belongs to π and the length of path from UP to AP is m. By induction hypothesis we conclude that θ_1 and λ_1 are disjoint w.r.t. the set of variables of UP. □

Theorem 10. $\Sigma(UP) = MCU_\emptyset(UP)$.

Proof. By Theorem 8, Theorem 9 and Lemma 1. □

7 Unification Procedure in the Flat Theory with Sequence Variables and Flexible Arity Symbols

7.1 Projection and Arity Completion

Projection and arity completion for unification in a flat theory coincide with the projection and arity completion in the empty theory.

7.2 Transformation

Given a unification problem $UP = \langle term_1, term_2 \rangle$, to find $SUC(UP)$ and $SUB(UP)$ under transformation in the flat theory we distinguish the following four cases:

1. $term_1$ and $term_2$ are identical. Then $SUB(UP) = \langle \varepsilon \rangle$ and $SUC(UP) = SUCCESS$.
2. $term_1$ and $term_2$ are not neither identical terms nor non-variable terms with the same head. Then $SUB(UP)$ and $SUC(UP)$ are defined according to Table 1.
3. $term_1$ and $term_2$ are non-identical non-variable terms with the same non-flat head. Then $SUB(UP)$ and $SUC(UP)$ are defined in the same manner as in the case 3 of the transformation procedure for the empty theory.
4. $term_1$ and $term_2$ are non-identical non-variable terms with the same flat head. Let f be the head. Then we have the following two cases:
   (a) Only one from $term_1$ and $term_2$ has the form $f()$. Then $SUB(UP)$ and $SUC(UP)$ are defined in the Table 3.
   (b) None of $term_1$ and $term_2$ is a term of the form $f()$. Let $term_1$ be $f(t_1, \overline{t})$ and $term_2$ be $f(s_1, \overline{s})$, where $\overline{t}$ and $\overline{s}$ are (possibly empty) sequences of terms. Then we have the following cases:
      i. t_1 and s_1 are identical. Then $SUB(UP) = \langle \varepsilon \rangle$ and $SUC(UP) = \langle \langle f(\overline{t}), f(\overline{s}) \rangle \rangle$.
      ii. t_1 and s_1 are not identical. Then we define $SUB(UP)$ and $SUC(UP)$ according to Table 4, Table 5 and Table 6.

Table 3. Transformation table for the unification problem $UP = \langle term_1, term_2 \rangle$ where $term_1 = f()$ and $term_2 = f(s_1, \overline{s})$ are non-variable terms with the same flat head.

s_1	$SUB(UP)$	$SUC(UP)$
Individual variable	$\langle \{s_1 \leftarrow f()\} \rangle$	$\langle \langle f(), f(\overline{s}\sigma) \rangle \rangle$
Sequence variable	$\langle \{s_1 \leftarrow f()\} \rangle$	$\langle \langle f(), f(\overline{s}\sigma) \rangle \rangle$
Non-variable term	$\langle \rangle$	$FAILURE$

7.3 Unification Procedure - Tree Generation

We design the unification procedure for the flat theory with sequence variables and flexible arity symbols in the same manner as we did for the empty theory - as a tree generation process. The single steps in this process are projection, arity completion and transformation for the flat theory. Given a unification problem UP we call the tree a F-unification tree for UP and denote it $UTree_F(UP)$. Let $\Sigma(UP)$ denote the set of all substitutions which are associated with the successful leaves of $UTree_F(UP)$. The next theorem states that $\Sigma(UP)$ is a minimal complete set of F-unifiers for UP.

Table 4. Transformation table for the unification problem $UP = \langle term_1, term_2 \rangle$ where $term_1 = f(t_1, \overline{t})$ and $term_2 = f(s_1, \overline{s})$ are non-variable terms with the same flat head and t_1 is an individual variable.

s_1	$SUB(UP)$	$SUC(UP)$
Individual variable	$\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4,$ $\sigma_5, \sigma_6, \sigma_7 \}$, where $\langle \sigma_1 \rangle = SUB(\langle t_1, s_1 \rangle)$, $\sigma_2 = \{s_1 \leftarrow f()\}$, $\sigma_3 = \{s_1 \leftarrow f(t_1)\}$, $\sigma_4 = \{t_1 \leftarrow f()\}$, $\sigma_5 = \{t_1 \leftarrow f(s_1)\}$, $\sigma_6 = \{s_1 \leftarrow f(t_1, s_1)\}$, $\sigma_7 = \{t_1 \leftarrow f(s_1, t_1)\}$	 $\langle \langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1) \rangle$, $\langle f(t_1, \overline{t}\sigma_2), f(\overline{s}\sigma_2) \rangle$, $\langle f(\overline{t}\sigma_3), f(\overline{s}\sigma_3) \rangle$, $\langle f(\overline{t}\sigma_4), f(s_1, \overline{s}\sigma_4) \rangle$, $\langle f(\overline{t}\sigma_5), f(\overline{s}\sigma_5) \rangle$, $\langle f(\overline{t}\sigma_6), f(s_1, \overline{s}\sigma_6) \rangle$, $\langle f(t_1, \overline{t}\sigma_7), f(\overline{s}\sigma_7) \rangle \rangle$
Sequence variable	$\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6,$ $\sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11} \rangle$, where $\sigma_1 = \{s_1 \leftarrow t_1\}$, $\sigma_2 = \{s_1 \leftarrow f()\}$, $\sigma_3 = \{s_1 \leftarrow f(t_1)\}$, $\sigma_4 = \{s_1 \leftarrow t_1, s_1\}$, $\sigma_5 = \{s_1 \leftarrow f(t_1), s_1\}$, $\sigma_6 = \{s_1 \leftarrow t_1, f(s_1)\}$, $\sigma_7 = \{s_1 \leftarrow f(t_1), f(s_1)\}$, $\sigma_8 = \{s_1 \leftarrow f(t_1, s_1)\}$, $\sigma_9 = \{t_1 \leftarrow f()\}$, $\sigma_{10} = \{t_1 \leftarrow f(s_1)\}$, $\sigma_{11} = \{t_1 \leftarrow f(s_1, t_1)\}$,	 $\langle \langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1) \rangle$ $\langle f(t_1, \overline{t}\sigma_2), f(\overline{s}\sigma_2) \rangle$, $\langle f(\overline{t}\sigma_3), f(\overline{s}\sigma_3) \rangle$, $\langle f(\overline{t}\sigma_4), f(s_1, \overline{s}\sigma_4) \rangle$, $\langle f(\overline{t}\sigma_5), f(s_1, \overline{s}\sigma_5) \rangle$, $\langle f(\overline{t}\sigma_6), f(s_1, \overline{s}\sigma_6) \rangle$, $\langle f(\overline{t}\sigma_7), f(s_1, \overline{s}\sigma_7) \rangle$, $\langle f(\overline{t}\sigma_8), f(s_1, \overline{s}\sigma_8) \rangle$, $\langle f(t_1, \overline{t}\sigma_9), f(\overline{s}\sigma_9) \rangle$, $\langle f(\overline{t}\sigma_{10}), f(\overline{s}\sigma_{10}) \rangle$, $\langle f(t_1, \overline{t}\sigma_{11}), f(\overline{s}\sigma_{11}) \rangle \rangle$
Non-variable term	$\langle \rangle$ if $SUB(\langle t_1, s_1 \rangle) = \langle \rangle$ $\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle$, where $\langle \sigma_1 \rangle = SUB(\langle t_1, s_1 \rangle)$, $\sigma_2 = \{t_1 \leftarrow f()\}$, $\sigma_3 = \{t_1 \leftarrow f(s_1)\}$, $\sigma_4 = \{t_1 \leftarrow f(s_1, t_1)\}$, otherwise	$FAILURE$ if $SUC(\langle t_1, s_1 \rangle) = FAILURE$ $\langle \langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1) \rangle$, $\langle f(\overline{t}\sigma_2), f(s_1, \overline{s}\sigma_2) \rangle$, $\langle f(\overline{t}\sigma_3), f(\overline{s}\sigma_3) \rangle$, $\langle f(t_1, \overline{t}\sigma_4), f(\overline{s}\sigma_4) \rangle \rangle$ otherwise

Table 5. Transformation table for the unification problem $UP = \langle term_1, term_2\rangle$ where $term_1 = f(t_1, \overline{t})$ and $term_2 = f(s_1, \overline{s})$ are non-variable terms with the same flat head and t_1 is a sequence variable.

s_1	$SUB(UP)$	$SUC(UP)$
Individual variable	$\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6,$	
	$\sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}\}$, where	
	$\sigma_1 = \{t_1 \leftarrow s_1\},$	$\langle\langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1)\rangle,$
	$\sigma_2 = \{t_1 \leftarrow f()\},$	$\langle f(\overline{t}\sigma_2), f(s_1, \overline{s}\sigma_2)\rangle,$
	$\sigma_3 = \{t_1 \leftarrow f(s_1)\},$	$\langle f(\overline{t}\sigma_3), f(\overline{s}\sigma_3)\rangle,$
	$\sigma_4 = \{t_1 \leftarrow s_1, t_1\},$	$\langle f(t_1, \overline{t}\sigma_4), f(\overline{s}\sigma_4)\rangle,$
	$\sigma_5 = \{t_1 \leftarrow f(s_1), t_1\},$	$\langle f(t_1, \overline{t}\sigma_5), f(\overline{s}\sigma_5)\rangle,$
	$\sigma_6 = \{t_1 \leftarrow s_1, f(t_1)\},$	$\langle f(t_1, \overline{t}\sigma_6), f(\overline{s}\sigma_6)\rangle,$
	$\sigma_7 = \{t_1 \leftarrow f(s_1), f(t_1)\}$	$\langle f(t_1, \overline{t}\sigma_7), f(\overline{s}\sigma_7)\rangle\rangle,$
	$\sigma_8 = \{t_1 \leftarrow f(s_1, t_1)\}$	$\langle f(t_1, \overline{t}\sigma_8), f(\overline{s}\sigma_8)\rangle\rangle,$
	$\sigma_9 = \{s_1 \leftarrow f()\}$	$\langle f(t_1, \overline{t}\sigma_9), f(\overline{s}\sigma_9)\rangle\rangle,$
	$\sigma_{10} = \{s_1 \leftarrow f(t_1)\}$	$\langle f(\overline{t}\sigma_{10}), f(\overline{s}\sigma_{10})\rangle\rangle,$
	$\sigma_{11} = \{s_1 \leftarrow f(t_1, s_1)\}$	$\langle f(\overline{t}\sigma_{11}), f(s_1, \overline{s}\sigma_{11})\rangle\rangle$
Sequence variable	$\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8,,$	
	$\sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}\rangle$, where	
	$\sigma_1 = \{t_1 \leftarrow s_1\},$	$\langle\langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1)\rangle$
	$\sigma_2 = \{t_1 \leftarrow f()\},$	$\langle f(\overline{t}\sigma_2), f(s_1, \overline{s}\sigma_2)\rangle,$
	$\sigma_3 = \{t_1 \leftarrow f(s_1)\},$	$\langle f(\overline{t}\sigma_3), f(\overline{s}\sigma_3)\rangle,$
	$\sigma_4 = \{t_1 \leftarrow s_1, t_1\},$	$\langle f(t_1, \overline{t}\sigma_4), f(\overline{s}\sigma_4)\rangle,$
	$\sigma_5 = \{t_1 \leftarrow f(s_1), t_1\},$	$\langle f(t_1, \overline{t}\sigma_5), f(\overline{s}\sigma_5)\rangle,$
	$\sigma_6 = \{t_1 \leftarrow s_1, f(t_1)\},$	$\langle f(t_1, \overline{t}\sigma_6), f(\overline{s}\sigma_6)\rangle,$
	$\sigma_7 = \{t_1 \leftarrow f(s_1), f(t_1)\},$	$\langle f(t_1, \overline{t}\sigma_7), f(\overline{s}\sigma_7)\rangle,$
	$\sigma_8 = \{t_1 \leftarrow f(s_1, t_1)\},$	$\langle f(t_1, \overline{t}\sigma_8), f(\overline{s}\sigma_8)\rangle,$
	$\sigma_9 = \{s_1 \leftarrow f()\},$	$\langle f(t_1, \overline{t}\sigma_9), f(\overline{s}\sigma_9)\rangle,$
	$\sigma_{10} = \{s_1 \leftarrow f(t_1)\},$	$\langle f(\overline{t}\sigma_{10}), f(\overline{s}\sigma_{10})\rangle,$
	$\sigma_{11} = \{s_1 \leftarrow f(t_1, s_1)\},$	$\langle f(\overline{t}\sigma_{11}), f(s_1, \overline{s}\sigma_{11})\rangle\rangle,$
	$\sigma_{12} = \{s_1 \leftarrow f(t_1), s_1\},$	$\langle f(\overline{t}\sigma_{12}), f(s_1, \overline{s}\sigma_{12})\rangle,$
	$\sigma_{13} = \{s_1 \leftarrow t_1, f(s_1)\},$	$\langle f(\overline{t}\sigma_{13}), f(s_1, \overline{s}\sigma_{13})\rangle,$
	$\sigma_{14} = \{s_1 \leftarrow f(t_1), f(s_1)\},$	$\langle f(\overline{t}\sigma_{14}), f(s_1, \overline{s}\sigma_{14})\rangle,$
	$\sigma_{15} = \{s_1 \leftarrow f(t_1, s_1)\},$	$\langle f(\overline{t}\sigma_{15}), f(s_1, \overline{s}\sigma_{15})\rangle\rangle$
Non-variable term	$\langle\rangle$	$FAILURE$
	if $SUB(\langle t_1, s_1\rangle) = \langle\rangle$	if $SUC(\langle t_1, s_1\rangle) = FAILURE$
	$\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\rangle$, where	
	$\langle\sigma_1\rangle = SUB(\langle t_1, s_1\rangle),$	$\langle\langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1)\rangle,$
	$\sigma_2 = \{t_1 \leftarrow f()\},$	$\langle f(\overline{t}\sigma_2), f(s_1, \overline{s}\sigma_2)\rangle,$
	$\sigma_3 = \{t_1 \leftarrow f(s_1)\},$	$\langle f(\overline{t}\sigma_3), f(\overline{s}\sigma_3)\rangle,$
	$\sigma_4 = \{t_1 \leftarrow s_1, t_1\},$	$\langle f(t_1, \overline{t}\sigma_4), f(\overline{s}\sigma_4)\rangle$
	$\sigma_5 = \{t_1 \leftarrow f(s_1), t_1\},$	$\langle f(t_1, \overline{t}\sigma_5), f(\overline{s}\sigma_5)\rangle,$
	$\sigma_6 = \{t_1 \leftarrow s_1, f(t_1)\},$	$\langle f(t_1, \overline{t}\sigma_6), f(\overline{s}\sigma_6)\rangle,$
	$\sigma_7 = \{t_1 \leftarrow f(s_1), f(t_1)\},$	$\langle f(t_1, \overline{t}\sigma_7), f(\overline{s}\sigma_7)\rangle$
	$\sigma_8 = \{t_1 \leftarrow f(s_1, t_1)\},$	$\langle f(t_1, \overline{t}\sigma_8), f(\overline{s}\sigma_8)\rangle\rangle$
	otherwise	otherwise

Table 6. Transformation table for the unification problem $UP = \langle term_1, term_2 \rangle$ where $term_1 = f(t_1, \overline{t})$ and $term_2 = f(s_1, \overline{s})$ are non-variable terms with the same flat head and t_1 is a non-variable term.

s_1	$SUB(UP)$	$SUC(UP)$
Individual variable	$\langle\rangle$ if $SUB(\langle t_1, s_1\rangle) = \langle\rangle$ $\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4\rangle$, where $\langle\sigma_1\rangle = SUB(\langle t_1, s_1\rangle)$, $\sigma_2 = \{s_1 \leftarrow f()\}$, $\sigma_3 = \{s_1 \leftarrow f(t_1)\}$, $\sigma_4 = \{s_1 \leftarrow f(t_1, s_1)\}$, otherwise	$FAILURE$ if $SUC(\langle t_1, s_1\rangle) = FAILURE$ $\langle\langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1)\rangle$, $\langle f(t_1, \overline{t}\sigma_2), f(\overline{s}\sigma_2)\rangle$, $\langle f(\overline{t}\sigma_3), f(\overline{s}\sigma_3)\rangle$, $\langle f(\overline{t}\sigma_4), f(s_1, \overline{s}\sigma_4)\rangle\rangle$ otherwise
Sequence variable	$\langle\rangle$ if $SUB(\langle t_1, s_1\rangle) = \langle\rangle$ $\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\rangle$, where $\langle\sigma_1\rangle = SUB(\langle t_1, s_1\rangle)$, $\sigma_2 = \{s_1 \leftarrow f()\}$, $\sigma_3 = \{s_1 \leftarrow f(t_1)\}$, $\sigma_4 = \{s_1 \leftarrow t_1, s_1\}$, $\sigma_5 = \{s_1 \leftarrow f(t_1), s_1\}$, $\sigma_6 = \{s_1 \leftarrow t_1, f(s_1)\}$, $\sigma_7 = \{s_1 \leftarrow f(t_1), f(s_1)\}$, $\sigma_8 = \{s_1 \leftarrow f(t_1, s_1)\}$, otherwise	$FAILURE$ if $SUC(\langle t_1, s_1\rangle) = FAILURE$ $\langle\langle f(\overline{t}\sigma_1), f(\overline{s}\sigma_1)\rangle$, $\langle f(t_1, \overline{t}\sigma_2), f(\overline{s}\sigma_2)\rangle$, $\langle f(\overline{t}\sigma_3), f(\overline{s}\sigma_3)\rangle$, $\langle f(\overline{t}\sigma_4), f(s_1, \overline{s}\sigma_4)\rangle$ $\langle f(\overline{t}\sigma_5), f(s_1, \overline{s}\sigma_5)\rangle$, $\langle f(\overline{t}\sigma_6), f(s_1, \overline{s}\sigma_6)\rangle$, $\langle f(\overline{t}\sigma_7), f(s_1, \overline{s}\sigma_7)\rangle$ $\langle f(\overline{t}\sigma_8), f(s_1, \overline{s}\sigma_8)\rangle\rangle$ otherwise
Non-variable term	$SUB(\langle t_1, s_1\rangle)$	$FAILURE$ if $SUC(\langle t_1, s_1\rangle) = FAILURE$ $\langle\langle f(\overline{t}\sigma), f(\overline{s}\sigma)\rangle\rangle$, if $SUC(\langle t_1, s_1\rangle) = SUCCESS$ and $SUB(\langle t_1, s_1\rangle) = \sigma$ $\langle\langle f(suc_{11}, \overline{t}\sigma_1), f(suc_{12}, \overline{s}\sigma_1)\rangle, \ldots,$ $\langle f(suc_{k1}, \overline{t}\sigma_k), f(suc_{k2}, \overline{s}\sigma_k)\rangle\rangle$, if $SUC(\langle t_1, s_1\rangle) =$ $\langle\langle suc_{11}, suc_{12}\rangle, \ldots, \langle suc_{k1}, suc_{k2}\rangle\rangle$ and $SUB(\langle t_1, s_1\rangle) = \langle\sigma_1, \ldots, \sigma_k\rangle$

Theorem 11. $\Sigma(UP) = MCU_F(UP)$.

Proof. We can prove the theorem analogously to the the Theorem 10, previously establishing the counterparts of Lemma 4 and Lemma 5, which is straightforward. □

8 Implementation

We have implemented the unification procedure for the empty theory in Mathematica. As a decision procedure for being a term and on the arity completion phase of the procedure we use the Omega package [2] to solve linear Diophantine equations over naturals. In the tree generation process for the efficiency reasons we did not implement the decision procedure for unifiability but put the bound on the depth of the tree which can be changed by the user - the idea which was used in the system for combining equational unification algorithms UNIMOK [19]. Also, we added a cycle detector as an option, which makes the procedure terminate in case of infinite cyclic success.

9 Future Work

We intend to study unification in orderless (theory with an orderless flexible arity symbol) and flat-orderless theories. Orderless flexible arity symbols are counterparts of commutative fixed arity symbols as well as flat flexible arity symbols correspond to associative fixed arity ones. We will study combination of flat, orderless, flat-orderless theories with associative, commutative, idempotent, associative-commutative, associative-commutative-idempotent theories with sequence variables and flexible arity symbols.

10 Conclusion

We defined equational theory with sequence variables and flexible arity symbols. We showed that in empty and flat theories general unification problem is decidable and described a unification procedure for each of these theories such that for a unification problem the procedure enumerates the minimal complete set of its unifiers and terminates if the set is finite.

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