Identification of nonlinear $B - H$ curves based on magnetic field computations and multigrid methods for ill-posed problems

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Our task is the identification of the reluctivity $\nu = \nu(B)$ in $\vec{H} = \nu(B)\vec{B}$, $(B = |\vec{B}|)$ from measurements of the magnetic flux for different excitation currents in a driving coil, in the context of a nonuniform magnetic field distribution. This is a nonlinear inverse problem and ill-posed in the sense of unstable data dependence, whose solution is done numerically by a Newton type iterative scheme, regularized by an appropriate stopping criterion. The computational complexity of this method is determined by the number of necessary forward evaluations, i.e., the number of numerical solutions to the 3d magnetic field problem. We keep the effort minimal by applying a special discretization strategy to the inverse problem, that is based on multigrid methods for ill-posed problems. Numerical results demonstrate the efficiency of the proposed method.

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1 Introduction

In many practical applications of electromagnetism, the relation between the magnetic field strength $\vec{H}$ and the magnetic flux density $\vec{B}$ is nonlinear, i.e., the magnetic permeability μ or its reciprocal, the reluctivity ν in

$$\begin{aligned} \vec{B} &= \mu(H)\vec{H} \\ \vec{H} &= \nu(B)\vec{B} \end{aligned}$$

where $H = |\vec{H}|$, $B = |\vec{B}|$ are not constant. This happens especially in the case of large excitations, as they occur e.g. in magneto-mechanical actuators. A typical $B - H$ curve is approximately linear only in the range of relatively low magnetic flux density but with increasing H changes to a nonlinear behavior to finally go into saturation as H gets large; see Figure 1 for an example.

In order to be able to do reliable numerical magnetic field simulations, it is crucial to precisely know the shape of this curve. So far, this is being obtained by building up a special experimental setup — the so-called Epstein apparatus — using very thin samples of the material under investigation, arranged in an appropriate pattern, in order

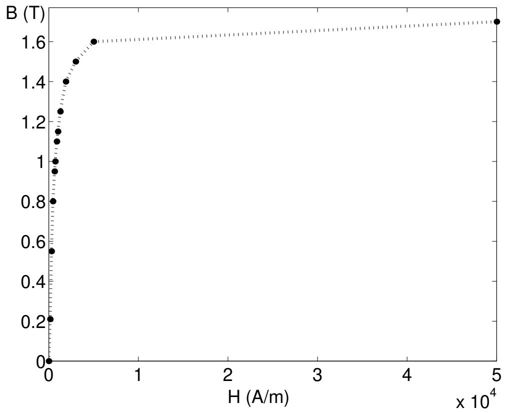

FIGURE 1. Typical $B-H$-curve shape

to achieve a uniform distribution of the magnetic flux density. Namely, if B is constant over the whole test sample, then the relations

$$B = \frac{n_w}{A_t}\Phi \ , \quad H = \frac{n_w}{l_t} I \ , \tag{1.1}$$

hold, where n_w is the number of coil windings, A_t a typical cross sectional area of the test sample, and l_t a typical length of the magnetic flux lines in the experimental setup. These allow an explicit computation of the ratio ν between H and B from measured data of the magnetic flux $\Phi = \Phi(I)$ corresponding to different values of the impressed current I, that yield a collection of points on the searched curve $\nu = \nu(B)$.

This measurement setup is, however, often hard or impossible to realize (cf. e.g. [9]), or yields too inexact approximations of the curve. We therefore propose to replace the combination of "complicated measurement setup with simple mathematics", by a combination of "simple measurement setup with more sophisticated mathematical methods" of extracting the parameter information from the given data. More precisely, we use an inversion scheme that is based on full 3d simulations of the magnetic field problem and are therefore able to encounter for variations of the magnetic flux density over the region covered by the probe and its surroundings.

The concrete mathematical model we deal with is given by Maxwell's equations (cf., e.g., [8]) in the steady state case for isotropic media which results in

$$\begin{array}{rcll} \bigtriangledown \times (\nu(B) \bigtriangledown \times \vec{A}) & = & \vec{J} & \text{in } D \\ \vec{n} \times \vec{A} & = & 0 & \text{on } \Gamma_B \\ \vec{n} \cdot \vec{A} & = & 0 & \text{on } \Gamma_H = \partial D \setminus \Gamma_B \end{array} \tag{1.2}$$

where

$$\begin{array}{l} \vec{J} = \left\{ \begin{array}{ll} \frac{I}{|\Omega_c|} \vec{e}_J & \text{in } D_c \\ 0 & \text{else} \end{array} \right. \quad \ldots \text{ impressed current density} \\ \Phi = \oint_c \vec{A} \cdot \vec{ds} = \int_{\Omega_c} \vec{B} \cdot \vec{n} d\sigma \ \ldots \ \text{magnetic flux} \\ \vec{B} = \bigtriangledown \times \vec{A} \ , \quad B = |\vec{B}| \ , \end{array}$$

$\vec{A}$ is the magnetic vector potential, $\vec{e}_J$ the unit vector in current direction, D_c the region covered by the excitation coil, Ω_c its cross sectional area, and the equality of the integral $\oint_c$ of $\vec{A}$ along the coil winding with the integral over Ω_c of $\vec{B}$ holds due to Stokes' theorem.

We have to identify the nonlinear real function $\nu = \nu(B)$ from measurements of the magnetic flux Φ for different currents I in an excitation coil, i.e., the data also consists of (in reality discrete measurements of) a scalar function $\Phi = \Phi(I)$ of one variable. This is a nonlinear inverse problem and ill-posed as will be pointed out below. Consequently small perturbations in the data (as they naturally turn up due to measurement noise), can lead to large deviations in the solution, which can obviously cause serious problems in a naive numerical approach. We therefore have to implement some appropriate numerical approximation scheme, i.e., a so-called regularization method (see e.g. [3], [5], [16], [17], [18], [19], [20]) for its stable solution. This is here done by a Newton type method with iterative solution of the linearized equations, where both the outer and the inner iteration is stopped at an a posteriori chosen appropriate index in order to prevent the increasingly propagated data noise from destroying the approximation.

The main challenge in this approach lies in the fact that the corresponding forward problem, that repeatedly appears in the course of the inversion procedure, demands high computational effort: For each value of I, the large scale nonlinear 3d magnetic field equation (1.2), has to be solved. Even if, as done here, (1.2) is numerically solved by methods that are known to be most efficient, and the accuracy for the nonlinear solution is chosen adaptively according to the state of convergence in the inversion iteration, there is still a need for taking care of efficiency by means of an improved inversion procedure. At this point the usefulness of multilevel methods for this inverse problem turns out: By an adaptive choice of the discretization in data space, using appropriate multigrid operators for ill-posed problems, we can minimize the number of necessary boundary value problem (1.2) solutions and therewith significantly decrease the overall computational effort for the parameter identification.

The paper will be organized as follows. The inversion issue in Section 2 forms the main emphasis of this paper and is parted into three subsections: After formulating the inverse problem in Subsection 2.1 we motivate the use of multigrid methods in Subsection 2.2, by proposing either to combine a Newton type method with a full multigrid method as a linear solver (i.e., iterative regularization by coarse discretization), or to use directly a nonlinear full multigrid method. Subsection 2.3 contains the necessary results on some multigrid methods for ill-posed problems. In Section 3 we describe some important realization aspects, namely measurement setup, numerical solution of the forward problem, and curve approximation. Section 4 shows numerical tests verifying the performance of multigrid methods for the ill-posed equations relevant for the identification problem under consideration and provides numerical results on our $B - H$ curve reconstruction technique. The final Section 5 gives an outlook, containing especially the applicability of our approach to other parameter identification problems.

In the following we will use the notation c or C, for positive constants that are typically "small" or "large", respectively. They may have different values whenever they appear in the text.

2 The Inverse Problem of $B-H$ Curve Identification

2.1 Formulation as a Nonlinear Operator Equation

We denote by F the forward operator, i.e., the application of the measurement operator (mapping a magnetic vector potential to the corresponding magnetic flux through the coil) to the parameter-to solution map for (1.2), i.e.,

$$\begin{aligned} F:\mathcal{D}(F)\subseteq X &\to Y \\ \nu &\mapsto (\oint_c \vec{A}^I\cdot \vec{ds})_{I\in[\underline{I},\overline{I}]} \end{aligned}$$

where for all $I\in[\underline{I},\overline{I}]$, $\vec{A}^I$ solves

$$\int_D \nabla\times\vec{A}'\cdot\nu(|\nabla\times\vec{A}^I|)\ \nabla\times\vec{A}^I\ dV \quad = I\frac{1}{|\Omega_c|}\int_{D_c}\vec{A}'\cdot\vec{e}_J\ dV \quad \forall\vec{A}'\in H_0(curl)\ , \quad (2.1)$$

i.e., the weak formulation of (1.2), with

$$H_0(curl):=\{\vec{A}\in L^2(D)\ :\ \nabla\times\vec{A}\in L^2(D)\ \wedge\ \vec{n}\times\vec{A}=0 \text{ on } \Gamma_B\}\ ,$$

where X and Y are appropriate function spaces to be defined below, and $\mathcal{D}(F)\subseteq X$ is such that $F:\mathcal{D}(F)\to Y$ is well-defined.

Well-posedness of the forward problem is essential not only in the context of our identification problem: Our final aim is of course to employ the reconstructed reluctivity ν in simulations of general application problems beyond our experimental setup, so problems of the form (1.2) with general $\vec{J}$ and D should be well-posed.

In order to achieve this, we have to guarantee unique solvability of (2.1) via the calculus of variations and the fact that a solution $\vec{A}^I$ to (2.1) minimizes the functional

$$\vec{A}\mapsto\frac{1}{2}\int_D\Big(\int_0^B\vec{H}\cdot d\vec{B}-\vec{A}\cdot\vec{J}\Big)dV\ .$$

For this purpose we need strict monotonicity of the $H-B$ curve — a physically natural requirement, which in our numerical computations has to be explicitely maintained by imposing uniform bounds on the first derivative of the function $B\mapsto H(B)=\nu(B)B$, more precisely by bounding ν'. Moreover, there will be a need for evaluating ν' whenever we want to solve general nonlinear boundary value problems (BVPs) of the form (1.2) by a Newton type method. Therefore, our domain $\mathcal{D}(F)$ in the parameter space X has to be a subset of a space containing W^1_∞ (the space of a.e. uniformly bounded functions with a.e. uniformly bounded weak first derivative). In order to work with an inner product space we choose a Sobolev space of order larger than $3/2$:

$$\mathcal{D}(F):=\{\tilde{\nu}\in H^2([\underline{B},\overline{B}])\mid\|\tilde{\nu}-\nu_0\|_{W^1_\infty}\le\rho\}\ ,\quad X=H^2([\underline{B},\overline{B}]),$$

where ν_0 is an initial guess that is supposed to be sufficiently close to the searched for exact ν, and $\rho>0$ is a small radius such that on one hand $\mathcal{D}(F)$ contains ν, and on the other hand $B\mapsto\tilde{\nu}(B)B$ is strictly monotone for all $\tilde{\nu}\in\mathcal{D}(F)$. The interval $[\underline{B},\overline{B}]$ on which we expect to be able to identify ν depends on the measurement interval $[\underline{I},\overline{I}]$. Motivated by the formulas (1.1) (which are valid for the special case of uniformly distributed magnetic field) we set

$$[\underline{B},\overline{B}]:=[\frac{n_w}{A_t}\Phi(\underline{I}),\frac{n_w}{A_t}\Phi(\overline{I})]\ ,$$

which will (also in the non-uniform situation) contain all B-values of interest, as long as we use a sufficiently wide measurement interval $[\underline{I}, \overline{I}]$ covering the region of saturation, where the reluctivity does not change any more.

We mention that in order to evaluate the measurement operator

$$\vec{A} \mapsto \oint_c \vec{A} \cdot \vec{ds} = \int_{\Omega_c} \vec{B} \cdot \vec{n} d\sigma \ ,$$

we require a bit more smoothness of the solution $\vec{A}$, than only $\vec{A} \in H_0(curl)$, see Remark 1 in the Appendix.

Due to the fact that only values but not higher order derivatives of Φ can be measured, and since we want to work in Hilbert spaces, the data space is naturally

$$Y = L^2([\underline{I}, \overline{I}]).$$

By means of the forward operator F we can formulate our inverse problem as a nonlinear operator equation

$$F(\nu) = \Phi \ , \tag{2.2}$$

which is ill-posed as can be seen from the special case of a uniformly distributed magnetic field: In this case relation (1.1) is valid and we see that a reconstruction in $X = H^2([\underline{B}, \overline{B}])$ of the function

$$B \mapsto \nu(B) = \frac{H(B)}{B} = \frac{n_w}{l_t} \frac{\Phi^{-1}(\frac{A_t}{n_w} B)}{B} \tag{2.3}$$

from L^2 measurements of Φ is about as ill-posed as twice differentiation — for a more precise quantification of the ill-posedness of the parameter identification problem, see Lemma 1 below. Note that by (2.3) we would expect well-posedness if we could choose the same order of smoothness for the parameter and for the measurement space, which, however, would not make sense, as the argumentation for our choice of the spaces X and Y shows.

Due to this ill-posedness and since the measurements will typically be contaminated with noise of some noise level $\delta > 0$, i.e., we have Φ^δ with

$$\|\Phi - \Phi^\delta\|_Y \leq \delta \ , \tag{2.4}$$

we must regularize (2.2) (see [3, 5, 16, 17, 18, 19, 20]). Some typical regularizing principles for general ill-posed operator equations

$$F(\nu) = \Phi \tag{2.5}$$

with noisy data Φ^δ satisfying (2.4) are:

(1) Tikhonov method: $\|F(\nu_\alpha) - \Phi^\delta\|_Y^2 + \alpha\|\nu_\alpha\|_X^2$, $\alpha > 0$; regularization by *spectrum shift.*

(2) iterative solution: $\nu_n = \psi(\nu_{n-1}, \Phi^\delta)$; regularization by *early stopping.*

(3) projection onto finite-dimensional space $Y_l \subset Y$: $Q^l F(\nu^l) = Q^l \Phi^\delta$; regularization by *coarse discretization.*

Here, the first of these three items is quoted due to the fact that it is still certainly the most well-known one. The second and third ones, since they are closely related to the regularization we are going to apply to our parameter identification problem (and its linearization). A crucial point is the correct choice of the regularization parameter α, or, in our situation, of the stopping index n or discretization level l, in order to achieve an optimal trade-off between approximation and stability. A well known and effective optimal a posteriori strategy for this purpose is Morozov's discrepancy principle (cf. e.g. [3], [17]), where out of a family (parameterized by α or n or l) of regularized approximations the most stable one is chosen, under the restriction that the residual is of the order of magnitude of the noise level (see (2.8) below).

2.2 Motivation of Multigrid Methods for the Inverse Problem

The iterative solution of (2.2) by an (inexact) Newton type method

$$\nu_{k+1} = \nu_k + s_k \text{ where } F'(\nu_k)[s_k] \approx \Phi - F(\nu_k)$$

involves the (approximate) solution of the linear operator equation

$$T_k s = b_k \tag{2.6}$$

with

$$\begin{array}{rcl} T_k : X & \to & Y \\ s & \mapsto & (\oint_c \vec{S}^I \cdot \vec{ds})_{I\in[\underline{I},\overline{I}]} \ , \end{array}$$

where for all $I \in [\underline{I},\overline{I}]$, $\vec{S}^I$ solves

$$\int_D \nabla\times\vec{A}'\cdot\Big(\nu_k(B_k^I) + \nu_k'(B_k^I)B_k^I\, proj_{\vec{e}_{B_k^I}}\Big)(\nabla\times\vec{S}^I)\, dV = -\int_D \nabla\times\vec{A}'\cdot s(B_k^I)B_k^I\, dV \ , \tag{2.7}$$

$$b_k(I) := \Phi(I) - \oint_c \vec{A}_k^I \cdot \vec{ds} \ , \quad I \in [\underline{I},\overline{I}] \ ,$$

where

$$\vec{B}_k^I = \nabla\times\vec{A}_k^I \ , \quad B_k^I = |\vec{B}_k^I| \ , \quad \vec{e}_{B_k^I} = \frac{1}{B_k^I}\vec{B}_k^I \ , \quad proj_{\vec{e}_{B_k^I}} := \vec{e}_{B_k^I}\vec{e}_{B_k^I}^T \ ,$$

and $\vec{A}_k^I$ solves (2.1) with $\nu := \nu_k$.

To account for the ill-posedness of the nonlinear problem (2.2), we regularize the Newton iteration by stopping it according to a discrepancy principle as described above, i.e., at

$$k = \text{min! s.t. } ||F(\nu_k) - \Phi^\delta||_Y \le \tau\delta \tag{2.8}$$

with some a priori chosen constant $\tau > 1$.

It has to be noticed that also the linear problems (2.6) in each Newton step are ill-posed: Roughly speaking, the linear operator T_k acts like the inverse of a differential operator, which implies that, like in the underlying nonlinear equation, a solution to (2.6) in general does not depend continuously on the data Φ. A quantification of this ill-posedness in a form in which we will need it for constructing appropriate multigrid operators, can be given in terms of the smoothing property of the adjoint T_k^*, that, as can be easily checked, is for any $z\in Y=L^2([\underline{I},\overline{I}])$, given by

$$(T_k^*z)(\lambda)=\int_{[\underline{I},\overline{I}]} z(I)\int_D \eta(\lambda-B_k^I)\vec{B}_k^I\cdot(\nabla\times\vec{Z}^I)\;dV\;dI \tag{2.9}$$

where

$$(\mathcal{F}\eta)(\xi)=\frac{1}{(1+\xi^2)^2}(\mathcal{F}\delta_0)(\xi)$$

with $\mathcal{F}$ denoting the Fourier transform and δ_0 the Dirac delta distribution (not to be confused with the noise level δ) i.e.,

$$\eta(\lambda)=\frac{1}{2}\sqrt{\frac{\pi}{2}}(1+|\lambda|)\exp(-|\lambda|)$$

and $\vec{Z}^I$ solves

$$\int_D \nabla\times\vec{A}'\cdot\Big(\nu_k(B_k^I)\;id\;+\nu_k'(B_k^I)B_k^I\;proj_{\vec{e}_{B_k^I}}\Big)(\nabla\times\vec{Z}^I)\;dV=\oint_c\vec{A}'\cdot\vec{ds}\;.$$

Proposition 1 *Under the conditions* $I\to\vec{B}_k^I\in W^1_\infty(\underline{I},\overline{I};L^\infty(D))\cap H^2(\underline{I},\overline{I};L^2(D))$, $I\to\vec{Z}^I\in H^2(\underline{I},\overline{I};H_0(curl))$, *the operator* T_k^* *can be extended to a continuous linear operator from* $(H^\sigma([\underline{I},\overline{I}]))^*$ *to* $H^2([\underline{B},\overline{B}])$ *for any* $\sigma\in(0,\frac{3}{2})$.

(Here $W_p^s(\underline{I},\overline{I};X)$ denotes the usual Sobolev space of functions on the interval $[\underline{I},\overline{I}]$ with values in the Banach space X.)

Proof see the Appendix. □

Proposition 1 states that T_k^* is smoothing "of order"

$$p=\frac{3}{2}\;,$$

and by

$$\begin{aligned}\|T_ks\|_{H^\sigma([\underline{I},\overline{I}])} &= \sup_{z\in Y}\frac{\langle T_ks,z\rangle_{L^2([\underline{I},\overline{I}])}}{\|z\|_{(H^\sigma([\underline{I},\overline{I}]))^*}}=\sup_{z\in Y}\frac{\langle s,T_k^*z\rangle_X}{\|z\|_{(H^\sigma([\underline{I},\overline{I}]))^*}}\\ &\le \|T_k^*\|_{(H^\sigma([\underline{I},\overline{I}]))^*\to H^2([\underline{B},\overline{B}])}\|s\|_X\end{aligned}$$

for any $s\in X$, $\sigma\in(0,\frac{3}{2})$, implies that the range T_kX of T_k is non-closed in Y and hence, (2.6) is ill-posed, so that regularization is necessary also for the linearized problems.

On the other hand, note that in order to formulate (2.6), for each value of I a linear magnetic field problem (2.7) for setting up the system matrix, and a nonlinear one (2.1) for computing the right hand side, has to be solved. Therefore it is desirable to keep the dimension of the discretized measurement space (i.e., the number of different $I-\Phi$ pairs entering into the computations) as low as possible — subject to approximation requirements as usual in an inexact Newton method (namely such that the linear residual is by some factor $\eta_k\in(0,1)$ smaller than the nonlinear residual in the actual Newton step, see (2.12) below). This just corresponds to regularization by coarse discretization, with the refinement level chosen by a discrepancy principle (cf. [10]).

More precisely, consider the projection of (2.6) onto a space Y_l out of a sequence

$(Y_l)_{l=1,2,3,\ldots}$ of nested finite dimensional subspaces of Y, whose union is dense in Y:

$$Y_1 \subseteq Y_2 \subseteq Y_3 \ldots \subseteq Y \ , \quad \overline{\bigcup_l Y_l} = Y.$$

In our concrete parameter identification problem, the spaces Y_l consist of piecewise linear splines on a not necessarily equidistant partition of the interval $[\underline{B}, \overline{B}]$, where the successive refinement is done by bisection of the partition intervals. Choosing out of the (infinitely many) solutions to the projected equation the one with minimal norm, (cf. Section 3.3 in [3]) leads to a finite dimensional version

$$s_k^l = (T_k^l)^* u_k^l \tag{2.10}$$

with

$$A_k^l u_k^l = b_k^l \tag{2.11}$$

of (2.6), where

$$A_k^l := T_k^l (T_k^l)^* \ , \quad T_k^l = Q^l T_k \ , \quad b_k^l := Q^l b_k$$

and Q^l is the orthogonal projection onto Y_l

$$Q^l := Proj_{Y_l} \ .$$

In order to find the optimal discretization level according to the discrepancy principle

$$l = \min! \text{ s.t. } ||F(\nu_k) + T_k s_k^l - \Phi^\delta|| \leq \eta_k ||F(\nu_k) - \Phi^\delta|| \ , \tag{2.12}$$

some iterative procedure has to be used, starting at a very coarse level of discretization and successively refining until the linear residual is sufficiently small. Here it would be clumsy to solve the full discrete equation at each refinement level from scratch. Instead, we use the solution obtained on the coarser level as a starting guess for an iterative process finding the solution on the next finer level. The direct solution of system (2.11) then only has to be done on the coarsest level. In principle any iterative regularization method such as conjugate gradient (CG) or Landweber could be used for approximately solving the linear system on each finer discretization level; due to the small size of the systems, such a single iteration step does not count in the overall effort as compared to a forward solution. However, the ill-posedness can cause a very slow convergence of this iteration, i.e., a large number of iterations may be necessary to get a reasonable approximation of the Newton step, and thus lead to a considerable contribution to the overall computational effort. Therefore it is advisable to precondition the iteration, which can be done in a very effective way by means of multigrid operators (see the numerical experiments in Subsection 4.1). In this way we arrive at a nested iteration technique (or full multigrid method FMGM, cf., e.g., [6], and, for ill-posed problems, [11], [15]) which is summarized in the following Algorithm 1, where we leave away the Newton step subscripts k for simplicity of notation.

Algorithm 1

$$
\begin{array}{l}
Set\ u^0 := (A^0)^{-1}b^0 =: u^0_{(0)}, \quad q(0) := 0 \\
For\ m = 1 \ldots l \\
\qquad Set\ u^m_{(0)} := u^{m-1}_{(q(m-1))} \\
\qquad For\ i = 1 \ldots q(m) \\
\qquad\qquad Set\ u^m_{(i)} = u^m_{(i-1)} + M^m(b^m - A^m u^m_{(i-1)}) \\
Set\ s^l := (T^l)^* u^l_{(q(l))}
\end{array}
$$

where $q(m) \geq 1$, and $b^m = Q^m b$. In the simple but most important situation of $q(m) \equiv 1$ and V-cycle multigrid preconditioners M^m, this can be schematically represented as in Figure 2.

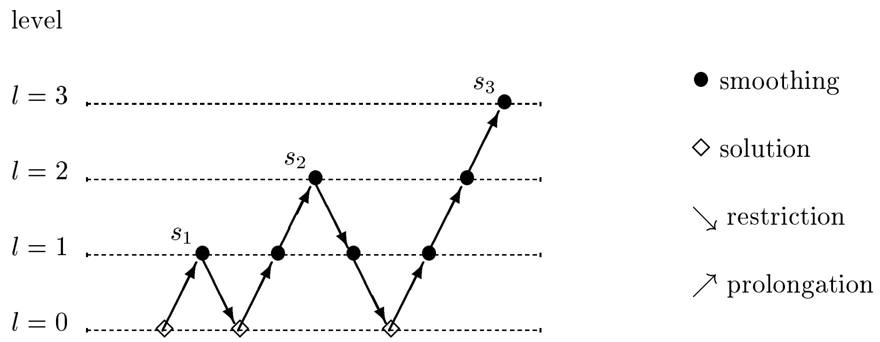

FIGURE 2. The full multigrid method with $q(m) \equiv 1$ and V-cycle multigrid preconditioning.

If the preconditioners M^m are effective enough to satisfy

$$\|I - (T^m)^* M^m T^m\|_X \leq c < 1 \ , \quad \forall m \leq l \tag{2.13}$$

with a constant c independent of the refinement level and $q(m) \equiv 1$, or

$$\|I - (T^m)^* M^m T^m\|_X \leq c(m) < 1 \ , \quad \forall m \leq l$$

with $\frac{1}{1-c(m)}$ growing only moderately with m so that

$$\|(I - (T^m)^* M^m T^m)^{q(m)}\|_X \leq c < 1 \ , \quad \forall m \leq l$$

holds uniformly with a slowly increasing sequence $q(m)$, we get (cf. e.g. [11]) for the difference between s^l and the exact solution $\bar{s}^l = (T^l)^\dagger b^l$ of (2.10), (2.11)

$$\|s^l - \bar{s}^l\|_X \leq c^m \tfrac{1}{1-c} \|\bar{s}^l\| + \tfrac{1}{\sqrt{1-c^2}} \|(Proj_{(T^m)^* Y_m} - I)\bar{s}^l\|_X \ ,$$

for any $m \leq l$, hence, (e.g., with $m := l/2$)

$$\|s^l - \bar{s}^l\|_X \to 0 \quad \text{ as } l \to \infty \ .$$

As it was shown in [11] the number of iterations necessary for reaching an appropriate stopping index (e.g. defined by a discrepancy principle, i.e., l as given by (2.12)) is much smaller than for conventional iterative regularization methods like Landweber or CG. The necessary finest level of discretization for obtaining the required accuracy is theoretically the same. However, it would not be clear at all, how to find it practically if standard CG or Landweber were used. It is mainly for this latter reason that we consider full multigrid as a most effective solution method in our actual context, where it is crucial to keep the data space dimension as small as possible.

Alternatively we can also directly apply a *nonlinear full multigrid method* (cf. [1], [6], for ill-posed problems [12]) for the iterative solution of (2.2). This is defined by generalizing the linear case to

Algorithm 2

Set $\nu_{start} := \nu^0$
Set $\nu^0 := SOL(\nu_{start}, Q^0\Phi^\delta)$
For $m = 1, \dots l$
  Set $\nu_{start} := \nu^{m-1}$
  Set $\tilde{f}^{m-1} := Q^{m-1}F(\nu^{m-1})$
  Set $\nu^m := NMGM(k, \nu_{start}, \tilde{f}^{m-1}, Q^m\Phi^\delta)$

Here $NMGM$ is a nonlinear multigrid operator (see the end of Subsection 2.3 below) for the projected equation on the m-th level

$$Q^m F(\nu) = Q^m \Phi^\delta \ , \tag{2.14}$$

starting from $\nu_{start} \in Y_m$, and SOL is an approximation method — usually just Newton's method — for the solution of (2.14) on the coarsest level $m = 0$. It can be shown in a general framework of nonlinear ill-posed operator equations (2.5), that under certain conditions, this iterative procedure equipped with a stopping rule of discrepancy type defines a regularization method for the underlying infinite-dimensional operator equation, i.e., $\nu_{l(\delta)} \to \nu$ as $\delta \to 0$.

2.3 Multigrid Methods for Ill-Posed Problems

Multigrid operators, as they are well-known to be extremely efficient tools for well-posed problems such as elliptic partial differential equations (PDEs) or second kind integral equations (cf., e.g. [6]), have until recently not been so well understood in the context of ill-posed problems, though. The reason for the difficulties lies in the proper construction of smoothing operators. To make this point clear we recall the definition of multigrid operators by means of the most simple two-grid case (see Figure 3). Usually, the

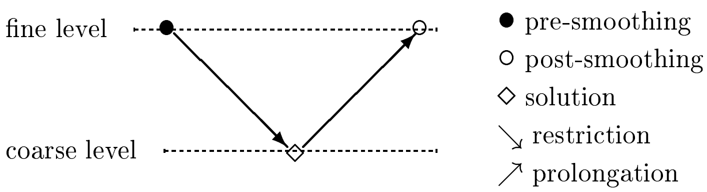

FIGURE 3. Two grid method

smoothing operators are defined by e.g. Jacobi, Gauß-Seidel, or SOR iterations that in the case of standard elliptic PDEs damp out the high frequency components of the defect on the finer grid to make it well representable on the coarser grid. In ill-posed problems, these methods lose their smoothing properties, due to the fact that small eigenvalues of compact operators correspond to high frequency eigenvalues, see, e.g. Figure 4 for the first 33 eigenvalues and three of the corresponding eigenfunctions in case of the operator T_k given by the linearization of our $B-H$ curve identification problem.

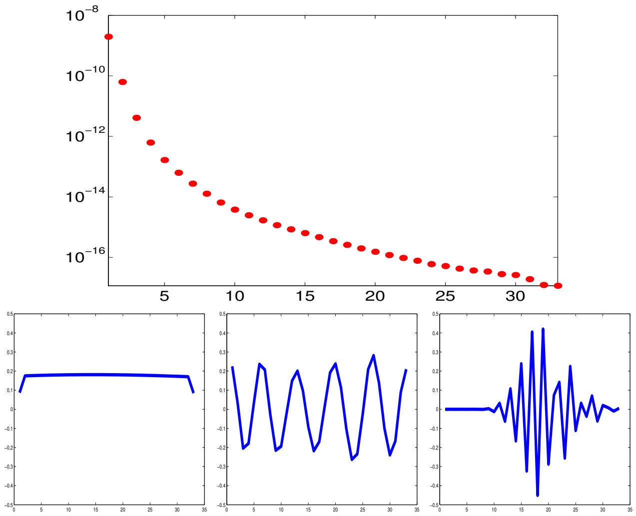

FIGURE 4. Singular values as well as 1st, 11th and 33rd eigenfunction of T_k.

An idea for constructing smoothers for general linear ill-posed problems

$$Ts = b\ ,$$

originating from a paper by King [15] and further investigated in [11] and [13], is based on subspace decomposition: By seperating the high from the low frequency parts within some actual discretization space Y_l by simple projection, and amplifying the high frequencies with an appropriate factor ξ_l, while just removing the low frequencies, the operator

$$S^l := \xi_l[I - Q^{l-1}]Q^l \tag{2.15}$$

with

$$\xi_l \sim h_l^{-2p}\ ,$$

can indeed be shown to decrease the high frequent part in the defect to such an extent, that the rest can be well enough approximated on the coarser level. Here h_l is the mesh size on the lth level and p the degree of ill-posedness (here $p = \frac{3}{2}$, see Proposition 1). This motivates the definition of multigrid operators M^l in the usual recursive way

Algorithm 3

Set $M^0 := (A^0)^{-1}$
Assume that M^{l-1} *has been defined and define* M^l *for* $b^l \in Y_l$ *as follows:*

$$\begin{array}{l}
\textit{Set } u := 0 \\
\textit{Set } u := u + S^l(b^l - A^l u) \quad \text{(pre-smoothing)} \\
\left.\begin{array}{l}
\quad \textit{Set } s := 0 \\
\quad \textit{For } i = 1 \ldots \gamma(l) \\
\qquad\qquad \textit{Set } s := s + M^{l-1}\Big(Q^{l-1}(b^l - A^l u) - A^{l-1} s\Big) \\
\quad \textit{Set } u := u + s.
\end{array}\right\} \text{(coarse grid correction)} \\
\textit{Set } u := u + S^l(b^l - A^l u) \quad \text{(post-smoothing)} \\
\textit{Set } M^l b^l := u.
\end{array}$$

where $A^l := Q^l T (Q^l T)^*$, $b^l = Q^l b$, and $\gamma(l) \geq 1$ describes the cycle regime: $\gamma(l) \equiv 1$ corresponds to the V-cycle, $\gamma(l) \equiv 2$ to the W-cycle and $\gamma(l) = 2 - \text{mod}(l, 2)$ to the alternating cycle. For the so defined multigrid operators it can be shown (using basically the smoothing property of the forward operator T and the approximation property of the spaces Y_l), that it satisfies the preconditioning criterion (2.13) with a level independent constant c, if one uses W- or alternating MG cycles (cf. [11]). If one applies more sophisticated smoothers, one can even show V-cycle convergence with level independent contraction factor c in 2.13, see [13]. Analogous results can be obtained for the additive multigrid operators

$$M^l = \sum_{m=1}^{l} \xi_m [I - Q^{m-1}] Q^m + Q^0 \; . \tag{2.16}$$

Here it also gets clear, why we set $\xi_l \sim h_l^{-2p}$: The resulting MG operator (2.16) represents a discrete differential operator of order $2p$ and is in this sense an approximate inverse of the discrete version A^l of TT^*, so that it can be expected to serve as a good preconditioner. As a consequence, we can apply the assembled operators M^l as preconditioners in the full multigrid Algorithm 1 for the regularized solution of the Newton step equation (2.6) in our $B - H$ curve identification procedure and expect to obtain the convergence behaviour as indicated in the previous section.

In the alternative inversion approach based on a nonlinear full multigrid method (see Algorithm 2), we need nonlinear multigrid operators. These can be defined by generalizing the multigrid recursion as well as the smoothing operators according to Algorithm 4,

Algorithm 4

Set $NMGM(0, \nu_{start}, \tilde{f}^{-1}, f^0) := SOL(\nu_{start}, f^0)$.
Assume that $NMGM(l-1, \cdot, \cdot, \cdot)$ *and* ν^{l-1} *have been defined and define* $NMGM(l, \nu_{start}, \tilde{f}^{l-1}, f^l)$ *for* $\tilde{f}^{l-1} \in Y_{l-1}$ $f^l \in Y_l$ *as follows:*

Set $\nu := S^l(\nu_{start}, f^l)$
Set $d^{l-1} := Q^{l-1}(F(\nu) - f^l)$
Choose $\lambda \in \mathbb{R}^+$
Set $d^{l-1} := \tilde{f}^{l-1} - \lambda d^{l-1}$
Set $s^{l-1} := \nu^{l-1}$
For $i = 1, \ldots \gamma(l)$
    Set $s^{l-1} = NMGM(l-1, s^{l-1}, \tilde{f}^{l-1}, d^{n-1})$
Set $\nu = \nu + \frac{1}{\lambda}(s^{l-1} - \nu^{l-1})$
Set $NMGM(l, \nu_{start}, \tilde{f}^{l-1}, f^l) := \nu$

where

$$S^l(\nu_{start}, f^l) = \nu_{start} - \xi_l T_k^*[I - Q^{l-1}](Q^l F(\nu_{start}) - f^l) \ ,$$

(see [12], where also the choice of the relaxation parameter λ is discussed).

3 Some realization aspects

3.1 Measurement Setup

In order to collect the data we propose an experimental setup according to Figure 5. Here, a DC (directed current) source provides the prescribed excitation current at an

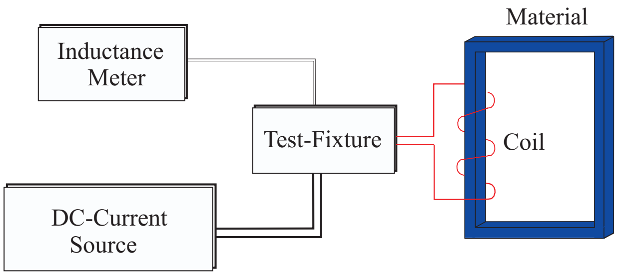

FIGURE 5. Measurement setup

actual measurement point I, which has to be in a high amplitude range in order to reach the nonlinear part of the $B - H$ curve. This is superimposed by an alternating current of very small amplitude (so that it can be neglected in the numerical simulations) in order to enable measurements of the inductance L via a Wheatstone bridge. From the relation

$$L = n_w \frac{\Phi}{I}$$

where n_w is the number of coil windings, we then get the value of the flux Φ corresponding to I. This is done repeatedly for different values of I in order to obtain a collection of points along the $I - \Phi$ curve.

3.2 Numerical Solution of the Forward Problems

The forward evaluation that is necessary to determine the data misfit for a given parameter configuration as it has to be done repeatedly during the inversion iteration, consists in the numerical solution of the nonlinear vectorial 3d BVPs (1.2). By locally linearizing iterations, i.e., solution of the nonlinear field problem (1.2) by a Newton type method, we arrive at a sequence of linear boundary value problems, whose variational formulation reads as:
For all $I \in \{I_1, \ldots, I_{d(n)}\}$ find $\vec{A}^I \in H_0(curl)$ such that, for all $\vec{A}' \in H_0(curl)$:

$$\int_D \nabla \times \vec{A}' \cdot (\nu(B^I) + \nu'(B^I)B^I proj_{\vec{e}_{B^I}}) \nabla \times \vec{A}^I \; dV = I\frac{1}{|\Omega_c|}\int_{D_c} \vec{A}' \cdot \vec{e}_J \; dV, \tag{3.1}$$

where

$$\vec{B}^I = \nabla \times \vec{A}^I \; , \quad B^I = |\vec{B}^I| \; , \quad \vec{e}_{B^I} = \frac{1}{B^I}\vec{B}^I \; , \quad proj_{\vec{e}_{B^I}} := \vec{e}_{B^I}\vec{e}^{\,T}_{B^I} \; ,$$

and $\{I_1, \ldots, I_{d(n)}\}$ is the set of measurement points actually taken into account according to the discretization strategy described in the previous section.

The finite element (FE) method with edge or nodal elements

$$\vec{A}^I \approx \sum_k A_k \vec{N}_k$$

leads to a large discrete system

$$\mathbf{K}_h\{A_h\} = \{J_h\}$$

which is solved in a most efficient way by CG iterations with algebraic or geometric multigrid preconditioners — for details we refer to [14].

To save computational effort, the Newton iteration for approximating a solution $\vec{A}^I$ of (2.1) is stopped as soon as an accuracy sufficient for our inversion procedure is attained: At the beginning of the inversion iteration, i.e., "far away" from the solution, we require a lower level of precision than at a converged state. More precisely, it is sufficient to do an approximate evaluation $\tilde{F}(\nu_k)$ such that

$$\|\tilde{F}(\nu_k) - F(\nu_k)\| \le c\|\tilde{F}(\nu_k) - \Phi^\delta\| \tag{3.2}$$

for some constant $c < 1$. From the fact that the difference $\vec{D}^I_A = \tilde{\vec{A}}^I - \vec{A}^I$ between an approximate solution $\tilde{\vec{A}}^I$ and the exact one $\vec{A}^I$ of (2.1) satisfies

$$\int_D \nabla \times \vec{A}' \cdot \overline{\nu} \;\; \nabla \times \vec{D}^I_A \; dV \;\; = \;\; res^I[\vec{A}'] \quad \forall \vec{A}' \in H_0(curl) \; ,$$

with

$$res^I[\vec{A}'] := \int_D \nabla \times \vec{A}' \cdot \nu(|\nabla \times \tilde{\vec{A}}^I|) \;\; \nabla \times \tilde{\vec{A}}^I \; dV - I\frac{1}{|\Omega_c|}\int_{D_c} \vec{A}' \cdot \vec{e}_J \; dV$$

and $\overline{\nu} = \nu_k(\tilde{B}) + \int_0^1 (\nu_k'(B + \theta(\tilde{B} - B))d\theta$, (which is bounded away from zero and sufficiently smooth), we can conclude

$$\begin{aligned}\|\tilde F(\nu_k)-F(\nu_k)\|_Y^2 &\le C\int_{[\underline{I},\overline{I}]}\|\nabla\times\vec{D}_A^I\|^2_{L^2(\Omega_c)}\,dI\\ &\le C\int_{[\underline{I},\overline{I}]}\|\nabla\times\vec{D}_A^I\|^2_{H^{\frac12}(D)}\,dI\ \le C\int_{[\underline{I},\overline{I}]}\|res^I\|^2_{L^2(D)}\,dI\ ,\end{aligned}$$

where we have used the regularity result (5.1) in the second inequality. Therefore, we can guarantee (3.2) by making the $L^2([\underline{I},\overline{I}])$ norm of the residuals $\|res^I\|_{L^2(D)}$ smaller than the computable quantity $\|\tilde F(\nu_k)-\Phi^\delta\|$, multiplied with some sufficiently small but fixed factor c.

3.3 Curve Approximation

As can be seen from equations (2.7), (3.1) above, not only values but also first derivatives of ν are required for the forward simulation. Since on one hand we have to be able to evaluate ν at any point attained by B, and on the other hand can only do computations with discrete quantities, we use spline representations for both functions ν and ν'. During the inversion iteration these are obtained by appropriate updating the splines ν and ν'. A crucial point is to obtain good starting functions $B\to\nu_0(B), B\to\nu_0'(B)$. Via the formulas (1.1) we get, from our measurements of the $I-\Phi$ curve, points $(B^i,H^i)_{i=1\dots d(I)}$ defining a plausible starting curve $\nu_0=H/B$. Since we want to get an initial guess also for the derviative, the assingment of a spline to this collection of points is a nontrivial task and ill-posed (just like the $B-H$ curve identification problem itself as discussed for the case of uniform magnetic field in Section 2.1), since it involves numerical differentiation of the data: Note that we want to reconstruct first derivatives from pointwise value measurements. This is done in a stabilized way by using so-called smoothing splines (cf. [2]) or, in other words minimizing a regularized least squares functional

$$J^\alpha[B]=\alpha\|B''\|^2_{L^2}+\sum_{i=1}^{d(I)}(B(H^i)-B^i)^2$$

over the set of third order C^1 Hermite splines $B:H\to B(H)$. For determining the reguarization parameter α a posteriori, we use a discrepancy principle with bisection, i.e., set α equal to the largest member of the geometrically decreasing sequence 2^{-j} such that for the minimizer B^α of J^α the residual $\sum_{i=1}^{d(I)}(B(H^i)-B^i)^2$ is of the order of magnitude of the measurement noise; see Figure 6 for an example of such a curve approximation and its derivative with one per cent noise. Note that — unlike ν — the $B-H$ curve itself always has to be monotone for physical reasons, so in order to be able to monitor and preserve this property, we do not precompute points $(\nu^i,B^i)=(\frac{H^i}{B^i},B^i)$ and do the smoothing spline approximation for those, but prefer to apply the described smoothing spline technique directly to the collection of (B^i,H^i) points (from which we get, by $\nu_0=\frac{H_0}{B}$, $\nu_0'=\frac{H_0'B-H_0}{B^2}$ points on the curves ν_0, ν_0', whose C^1 interpolation yields the splines used in the initial step of our computations). Practically, the monotonicity of the $B-H$ curve approximation is gained by putting restrictions on the coefficients of the spline curve. These restrictions arise naturally, if we require $B'(H)>0$. Consequently, the task is to find a monotone approximation, which fulfills the discrepancy principle for a given data noise level. Note, additionally, that the noise level is given in terms of the

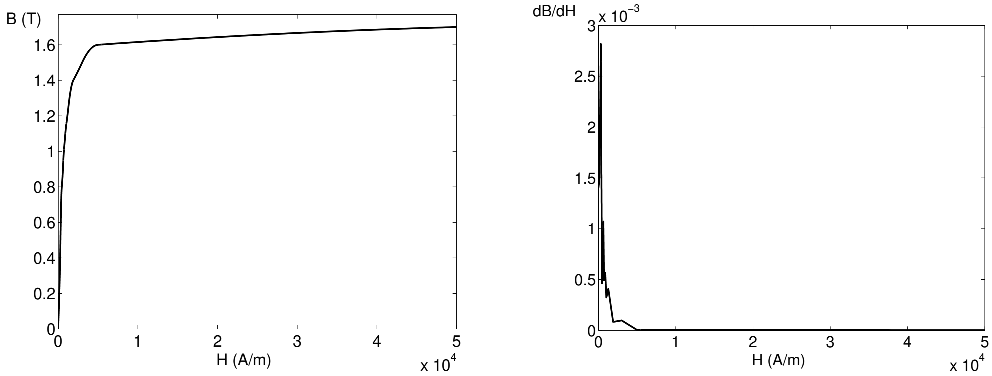

FIGURE 6. Approximation of function values and derivatives from pointwise measurements by smoothing splines.

$I - \Phi(I)$ values and therewith for the (H^i, B^i) points. If we would deduce an estimate for the data noise in terms of the (ν^i, B^i) points, this would, due to the strong variation in scale of both function and derivative values of the curve under consideration, lead to a locally too pessimistic noise estimate and hence to a bad curve approximation.

We mention that this smoothing spline technique itself can be considered as a stable solution method for the ill-posed problem of $B - H$ curve identification in the special situation of uniform magnetic field.

4 Numerical Results

4.1 Tests of multigrid operators for T_k

First of all we do some numerical experiments for testing the behavior of multigrid operators as described in Subsection 2.3 for the linearized parameter identification problems (2.6).

Figure 7 is supposed to demonstrate the main difference between multigrid methods with a conventional smoothing operator and with the one given in Subsection 2.3: While with a point Gauss-Seidel smoother the preconditioned conjugate gradient (PCG) method stalls after a few iterations (by the way a phenomenon which has also been observed for other ill-posed problems, cf. [7] for an inverse problem from computerized tomography), the smoother according to (2.15) preserves a good convergence rate.

To illustrate the preconditioning quality of the proposed multigrid operators, we directly compare the condition numbers of the operators A^l (that strongly grow due to ill-posedness) to the one of the multigrid preconditioned operators, that exhibit a significant improvement, see Figure 8.

In Figure 9, convergence of the full multigrid method is demonstrated: Both the residual and the error in pre-image space show the expected good convergence behavior. Note that we have implemented V-cycle multigrid operators with only one iteration per level ($q(m) \equiv 1$), so that already the computationally cheapest version of FMGM is satisfactory. As a right hand side we here chose the same as for our PCG tests shown in Figure 7.

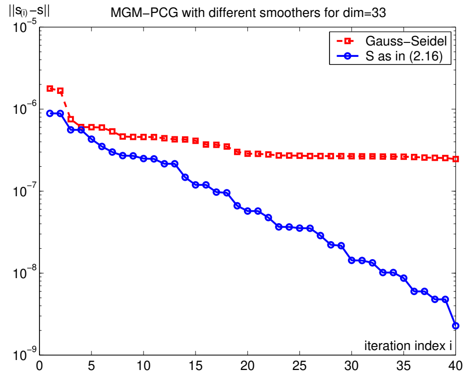

FIGURE 7. Conjugate gradient convergence with multigrid preconditioning using conventional smoother versus (2.15).

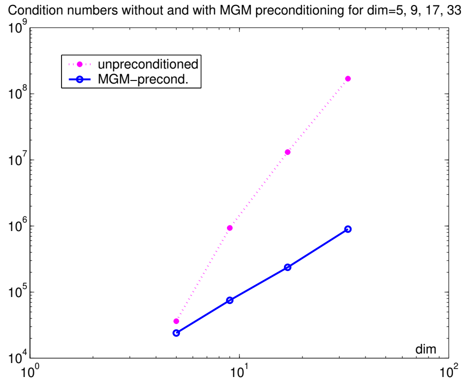

FIGURE 8. Condition numbers without and with appropriate multigrid preconditioning.

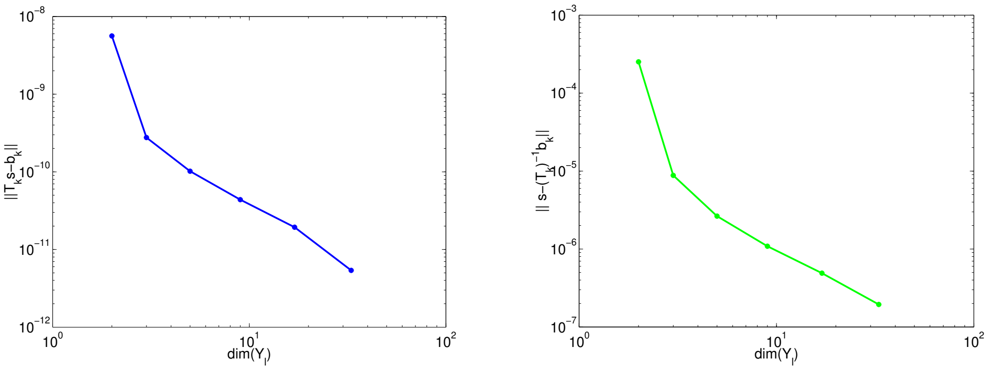

FIGURE 9. Convergence of the full multigrid method for $T_k s = b_k$.

4.2 $B-H$ curve reconstructions

First of all we do some computations with exact data. Figure 10 shows the starting curve and the first six Newton iterates (using FMGM to approximate the Newton step) compared to the exact curve. The finest discretization level corresponds to 17 measurement points.

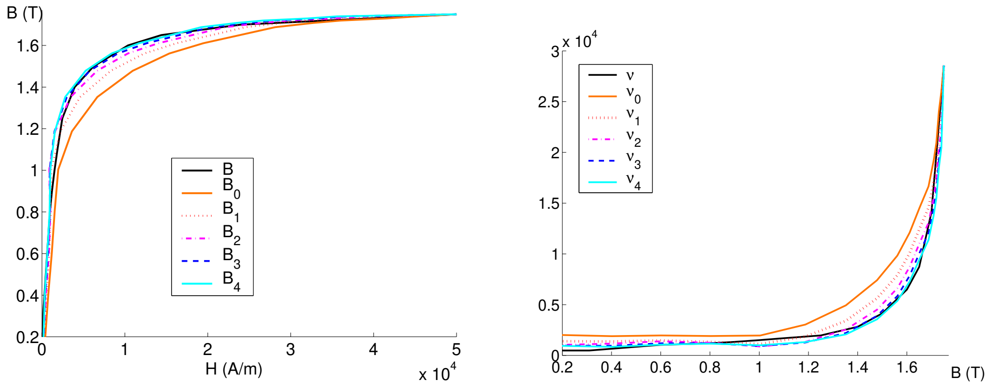

FIGURE 10. Convergence with exact data: $H - B$ curves (left) and $B - \nu$ curves (right).

To do tests with noisy data, we generate synthetic data corresponding to the prescribed exact curve by forward computations on a finer FE mesh and adding random noise of the respective noise level. Figures 11, 12, 13 show the first four Newton iterates with 1, 2, and 3 per cent noise, respectively. Note that in a realistic experimental setup, the measurement noise is between 1 and 2 per cent at most.

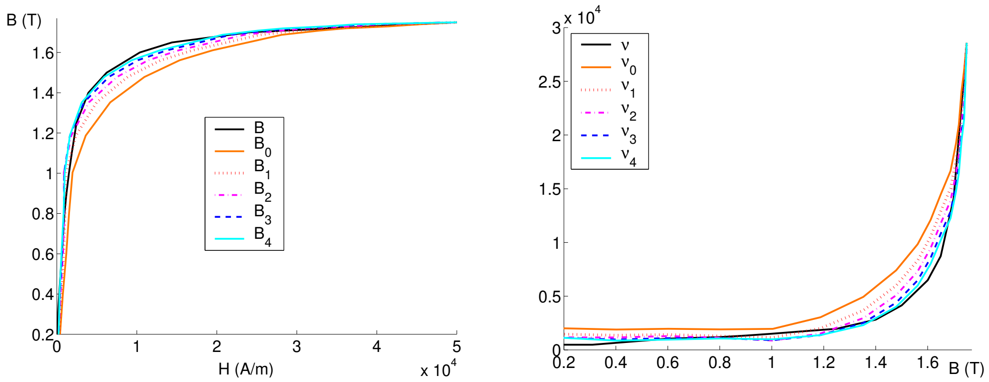

FIGURE 11. Convergence with noisy data ($\delta = 1$ % noise): $H - B$ curves (left) and $B - \nu$ curves (right).

Here the finest discretization level according to the rule (2.12) is only used in the last step according to (2.8), (i.e. the 2nd step in case $\delta = 2\%$ or 3% and the 3rd step in case $\delta = 1\%$,) so that the effort can be reduced to 60 per cent in case $\delta = 2\%, 3\%$ and even to 50 per cent in case $\delta = 1\%$, as compared to a full discretization in each Newton step.

A more precise look at the iteration error by plotting the relative error of the $B - \nu$ curves (see Figure 14) shows that indeed, due to the ill posedness, the iterates following the ones chosen according to (2.8) would have become worse again, and therefore confirms that (2.8) is an appropriate stopping criterion for the Newton inversion iteration.

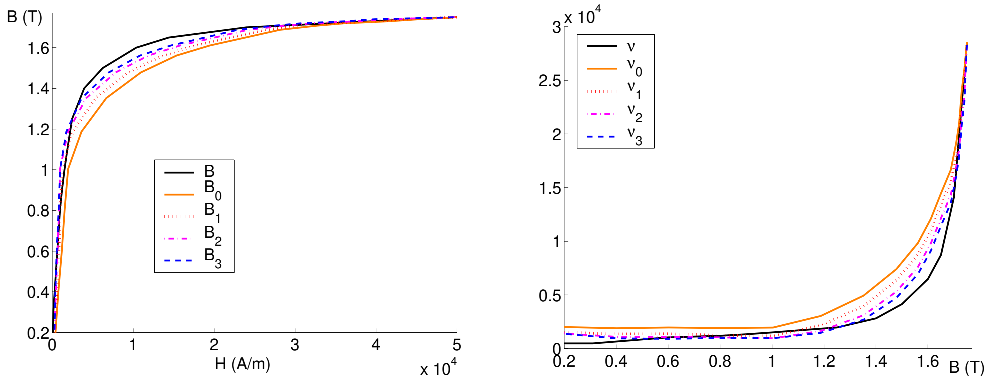

FIGURE 12. Convergence with noisy data ($\delta = 2$ % noise): $H - B$ curves (left) and $B - \nu$ curves (right).

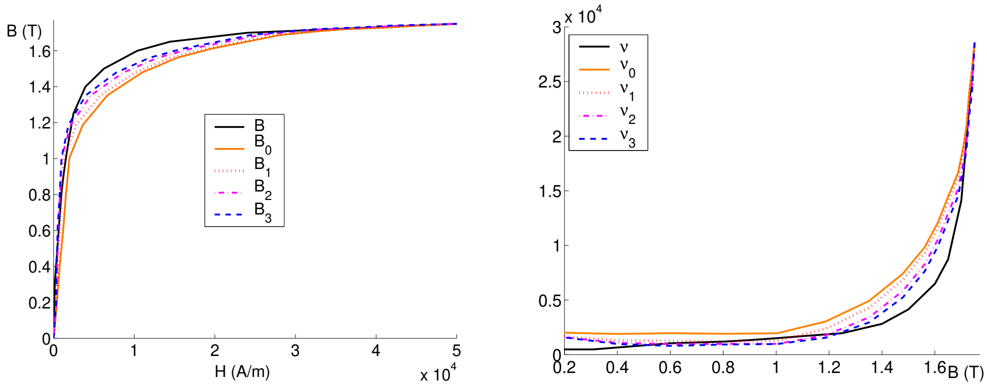

FIGURE 13. Convergence with noisy data ($\delta = 3$ % noise): $H - B$ curves (left) and $B - \nu$ curves (right).

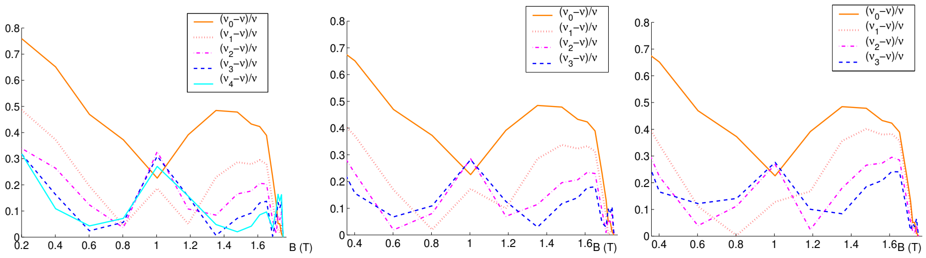

FIGURE 14. Relative error $B \to \frac{\nu_i(B)-\nu(B)}{\nu(B)}$ for 1, 2, and 3 % noise.

5 Conclusions and Remarks

In the context of magnetism with large excitation amplitudes we often will not only encounter nonlinearity but also hysteresis effects. Note that the identification of hysteresis curves is possible in a principally similar way to our proposed reconstruction scheme of $B - H$ curves as real functions. For instance, the identification of a Preisach model can be seen as the reconstruction of function of two variables (the weight function) and two constants (the polarization values).

The proposed method is also applicable to several other parameter identification problems or even to other inverse problems. Note that the inversion principles presented here are quite general.

A parameter identification problem that is in many aspects similar to the one considered here, is the reconstruction of the material tensors $\mathbf{c}^E = \mathbf{c}^E(|\mathrm{DIV}^T\vec{d}|, |\mathrm{grad}\phi|)$, $\mathbf{e} = \mathbf{e}(|\mathrm{DIV}^T\vec{d}|, |\mathrm{grad}\phi|)$, $\varepsilon^S = \varepsilon^S(|\mathrm{DIV}^T\vec{d}|, |\mathrm{grad}\phi|)$ of nonlinear piezoelectric media in

$$\begin{aligned} \rho\omega^2\vec{d} + \mathrm{DIV}\left(\mathbf{c}^E\mathrm{DIV}^T\vec{d} + \mathbf{e}^T\mathrm{grad}\phi\right) &= 0 \quad \text{in } \Omega \\ -\bigtriangledown\left(\mathbf{e}\mathrm{DIV}^T\vec{d} - \varepsilon^S\mathrm{grad}\phi\right) &= 0 \quad \text{in } \Omega\ , \end{aligned} \tag{5.1}$$

from electrical and mechanical measurements for different frequencies ω. Here, $\vec{d} = (d_x, d_y, d_z)$ are the mechanical displacements, ϕ the electric potential, ρ the density, ω the frequency, and DIV denotes the divergence of a dyadic. Analogies to the $B - H$ curve identification problem in magnetics are:

- One forward problem consists of solving several BVPs namely (5.1) for different frequencies ω.
- The searched for parameter depends nonlinearly on the field quantities, more precisely on first derivatives of the state variables.
- The differential operator is of second order wrt. the state variable and of first order wrt. the searched for parameter.
- Hysteresis effects may appear.

The piezoelectric parameter identification problem has to be expected to be more complicated, though: More parameters and more field quantities are involved and the underlying system of PDEs is indefinite instead of elliptic.

Appendix

Remark 1 *(well-posedness of F as defined in Subsection 2.1)*
For being allowed to make use of an appropriate trace theorem, that enables us to establish well-definedness of the measurements $\oint_c \vec{A}\cdot\vec{ds} = \int_{\Omega_c}\vec{B}\cdot\vec{n}d\sigma$, we need $\bigtriangledown\times\vec{A} = \vec{B} \in H^{1/2}(D)$ for a solution $\vec{A}$ of (2.1). Since the right hand side of (2.1)

$$\vec{A}' \mapsto I\frac{1}{|\Omega_c|}\int_{D_c}\vec{A}'\cdot\vec{e}_J\ dV$$

is not only in $H_0(curl)'$ but even in L^2 and all quantities appearing in (1.2) (especially the domain and the reluctivity) are sufficiently smooth, we can guarantee this by proceeding along the lines of a regularity proof for nonlinear PDEs (given e.g. in Section 8.3 of [4]), making use of the linear regularity result

$$\vec{A}\in H_0(curl)\ \wedge\ \int_D \bigtriangledown\times\vec{A}'\cdot\tilde{\nu}\ \ \bigtriangledown\times\vec{A}\ dV = \int_D \vec{J}\cdot\vec{A}'\ dV\ \forall\vec{A}'\in H_0(curl)\quad \text{with } \vec{J}\in L^2(D)^3$$
$$\Longrightarrow \bigtriangledown\times\vec{A}\in H^1(D)^3\ , \tag{5.1}$$

for $\tilde\nu \in W^1_\infty(D)$. The latter can be seen as follows: Using the uniquely defined decomposition of any $H_0(curl)$ function into a divergence free $H^1_0(D)^3$ function and the gradient of an $H^1_0(D)$ function:

$$\vec{A} = \vec{w} + \nabla p \quad \textit{where } \vec{w} \in H^1_0(D)^3 \ , \ \mathrm{div}\vec{w} \equiv 0 \ , \quad p \in H^1_0(D) \ ,$$

we get (with div, grad *of a vector valued function to be understood component wise)*

$$\begin{aligned} \mathrm{div}(\tilde\nu \mathrm{grad}\vec{w}) &= \Big(\mathrm{div}(\tilde\nu\mathrm{grad}) - \mathrm{grad}(\tilde\nu\mathrm{div})\Big)\vec{w} \\ &= \nabla \times (\tilde\nu \nabla \times \vec{w}) = \nabla \times (\tilde\nu \nabla \times \vec{A}) = J \in L^2(D)^3 \ , \end{aligned}$$

so that by component wise application of standard regularity results for solutions of $\mathrm{div}(\tilde\nu\mathrm{grad})$ *equations, we obtain that $\vec{w} \in H^2(D)^3$, and hence $\nabla \times \vec{A} = \nabla \times \vec{w} \in H^1(D)^3$.*

Proof (Proposition 1)
Fix $\sigma \in (1, \frac{3}{2})$. For any $z \in L^2([\underline{I}, \overline{I}])$ we have

$$\begin{aligned} &\|T_k^* z\|_{H^2([\underline{B},\overline{B}])} \\ &= \int_{\mathbb{R}} (1+\xi^2)^2 \left| \int_{[\underline{I},\overline{I}]} z(I) \int_D e^{-iB_k^I\xi} (\mathcal{F}\eta)(\xi) \vec{B}_k^I \cdot (\nabla \times \vec{Z}^I) \ dV \ dI \right|^2 d\xi \\ &= \int_{\mathbb{R}} \left| \int_{[\underline{I},\overline{I}]} z(I) \int_D e^{-iB_k^I\xi} \tfrac{1}{1+\xi^2} \vec{B}_k^I \cdot (\nabla \times \vec{Z}^I) \ dV \ dI \right|^2 d\xi \\ &\leq \|z\|_{(H^\sigma([\underline{I},\overline{I}]))^*} \int_{\mathbb{R}} \left(\int_{[\underline{I},\overline{I}]} \left(|\vec{W}^I|^2 + |\tfrac{d}{dI}\vec{W}^I|^2 \right) \ dI \right)^{2-\sigma} \\ &\qquad \left(\int_{[\underline{I},\overline{I}]} \left(|\vec{W}^I|^2 + |\tfrac{d}{dI}\vec{W}^I|^2 + |\tfrac{d^2}{dI^2}\vec{W}^I|^2 \right) \ dI \right)^{\sigma-1} \ d\xi \end{aligned} \tag{5.2}$$

where

$$W^I := \int_D e^{-iB_k^I\xi} \frac{1}{1+\xi^2} \vec{B}_k^I \cdot (\nabla \times \vec{Z}^I) \ dV$$

and we have used the interpolation inequality

$$\forall w \in H^2([\underline{I}, \overline{I}]) \ : \quad \|w\|_{H^\sigma([\underline{I},\overline{I}])} \leq \|w\|^{2-\sigma}_{H^1([\underline{I},\overline{I}])} \|w\|^{\sigma-1}_{H^2([\underline{I},\overline{I}])} \ .$$

The derivatives of W^I with respect to I are

$$\begin{aligned} \tfrac{d}{dI}\vec{W}^I &= \int_D e^{-iB_k^I\xi} \Big(\ \frac{-i\vec{e}_{B_k^I} \cdot \vec{B}^I_{k\ I}\xi}{1+\xi^2} \vec{B}_k^I \cdot (\nabla \times \vec{Z}^I) \\ &\qquad + \tfrac{1}{1+\xi^2} \left(\vec{B}^I_{k\ I} \cdot (\nabla \times \vec{Z}^I) + \vec{B}_k^I \cdot (\nabla \times \vec{Z}^I_I) \right) \Big) \ dV \end{aligned}$$

$$\begin{aligned} \tfrac{d^2}{dI^2}\vec{W}^I &= \int_D e^{-iB_k^I\xi} \Big(\ \frac{-(\vec{e}_{B_k^I} \cdot \vec{B}^I_{k\ I})^2\xi^2}{1+\xi^2} \vec{B}_k^I \cdot (\nabla \times \vec{Z}^I) \\ &\quad + \frac{-i(\vec{e}_{B_k^I} \cdot \vec{B}^I_{k\ II} - |(id - proj_{\vec{e}_{B_k^I}})\vec{B}^I_{k\ I}|^2 / B_k^I)\xi}{1+\xi^2} \vec{B}_k^I \cdot (\nabla \times \vec{Z}^I) \\ &\quad + 2\frac{-i\vec{e}_{B_k^I} \cdot \vec{B}^I_{k\ I}\xi}{1+\xi^2} \left(\vec{B}^I_{k\ I} \cdot (\nabla \times \vec{Z}^I) + \vec{B}_k^I \cdot (\nabla \times \vec{Z}^I_I) \right) \\ &\quad + \tfrac{1}{1+\xi^2} \left(\vec{B}^I_{k\ II} \cdot (\nabla \times \vec{Z}^I) + \vec{B}_k^I \cdot (\nabla \times \vec{Z}^I_{II}) + 2\vec{B}^I_{k\ I} \cdot (\nabla \times \vec{Z}^I_I) \right) \Big) \ dV \end{aligned}$$

with

$$\vec{B}^I_{k\ I} := \frac{d}{dI}\vec{B}_k^I \ , \quad \vec{B}^I_{k\ II} := \frac{d^2}{dI^2}\vec{B}_k^I \ , \quad \vec{Z}^I_I := \frac{d}{dI}\vec{Z}^I \ , \quad \vec{Z}^I_{II} := \frac{d^2}{dI^2}\vec{Z}^I \ ,$$

so that, by our assumptions, we can further continue our estimate in the last line of (5.2) according to

$$\|T_k^* z\|_{H^2([\underline{B},\overline{B}])} \leq C \|z\|_{(H^\sigma([\underline{I},\overline{I}]))^*} \int_{\mathbb{R}} \left(\frac{1}{1+\xi^2} \right)^{2-\sigma} d\xi \ ,$$

where the integral over $\mathbb{R}$ is finite as long as $2(2-\sigma) > 1$, i.e., $\sigma < \frac{3}{2}$. Since $L^2([\underline{I},\overline{I}])$ is dense in $(H^\sigma([\underline{I},\overline{I}]))^*$, the assertion follows. □

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References

[1] A. Brandt, *Multilevel adaptive solutions to boundary value problems* Math. Comp. 31 (1977), 333-390.
[2] C. de Boor, C., *A Practical Guide to Splines*, Springer, Berlin, 1978.
[3] H.W. Engl, M. Hanke, A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, 1996
[4] L.C. Evans, *Partial Differential Equations*, AMS, Providence, 1998.
[5] C.W. Groetsch, *Inverse Problems in Mathematical Sciences*, Vieweg, Braunschweig, 1993.
[6] W. Hackbusch, *Multi-Grid Methods and Applications*, Springer, Berlin–Heidelberg–New York, 1985.
[7] V.E. Henson, M.A. Limber, S.F. McCormick, B.T. Robinson, *Multilevel Image Reconstruction with Natural Pixels*, SIAM J.Sci.Comp. 17, 1996, 193-216.
[8] N. Ida, J.P. Bastos, *Electromagnetics and calculation of fields*, Springer, Berlin–Heidelberg–New York, 1992.
[9] D. Ioan, M. Rebican, A. Gasparics, *$B-H$ characteristic extraction using devices with non-uniform magnetic field*, COMPEL 18 (1999),469-482.
[10] B. Kaltenbacher, *Regularization by projection with a posteriori discretization level choice for linear and nonlinear ill-posed problems*, Inverse Problems 16 (2000), 1523-1539.
[11] B. Kaltenbacher, *On the regularizing properties of a full multigrid method for ill-posed problems*, to appear in Inverse Problems 17 (2001).
[12] B. Kaltenbacher, *A multi-grid method with a priori and a posteriori level choice for the regularization of nonlinear ill-posed problems*, SFB013-report No.00-3, University of Linz, 2000.
[13] B. Kaltenbacher, *V-cycle convergence of some multigrid methods for ill-posed problems*, SFB013-report, University of Linz, 2000.
[14] M. Kaltenbacher, S. Reitzinger, M. Schinnerl, J. Schöberl, H. Landes, *Multigrid methods for the computation of 3D electromagnetic field problems*, COMPEL 20 (2001).
[15] J.T. King, *Multilevel algorithms for ill-posed problems*, Numer. Math. 61 (1992), 311–334.
[16] A. K. Louis, *Inverse und schlecht gestellte Probleme*, Teubner, Stuttgart, 1989.
[17] V.A. Morozov, *Regularization Methods for Ill-Posed Problems*, CRC Press, Boca Raton, 1993.
[18] F. Natterer, *The Mathematics of Computerized Tomography*, Teubner, Stuttgart, 1986.
[19] A.N. Tikhonov, V.A. Arsenin, *Methods for Solving Ill-Posed Problems*, Nauka, Moskau, 1979.
[20] G.M. Vainikko, A.Y. Veterennikov, *Iteration Procedures in Ill–Posed Problems*, Nauka, Moscow, 1986. In Russian.