Computation of blowing up centers*

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Abstract

Given a birational projective morphism of quasi-projective varieties $f : Z \to X$. We want to find the ideal sheaf $\mathcal{I}$ over X such that the blowing up of X along $\mathcal{I}$ corresponds to f. In this paper we approach the problem from two directions, solving two subcases. First we present a method that determines $\mathcal{I}$ when f is the composition of blowing ups along known centers, then by another method, we compute $\mathcal{I}$ directly from $i(Z) \subset \mathbf{P}^n_X$.

1 Introduction

Blowing ups are an important subclass of proper birational morphisms, in fact, birational projective morphisms of quasi-projective varieties are nothing more than blowing ups. This is clear from Theorem 7.17 on page 166 in [8] which provides the most important basis of this paper. The proof of the theorem given by Hartshorne is non-constructive and highly abstract. To my best knowledge algorithmic solutions have never been considered.

The problem is the following: Given a birational projective morphism of quasi-projective varieties $f : Z \to X$, that is, we have a closed immersion $i : Z \hookrightarrow \mathbf{P}^n_X$ for some n such that f factors as $p \circ i$, where p is the projection $\mathbf{P}^n_X \to X$. Then by Theorem 7.17 in [8] we have a coherent sheaf of ideals $\mathcal{I}$ over X such that Z is isomorphic to the blowing up $\tilde{X}$ of X along $\mathcal{I}$, and f corresponds to $\pi : \tilde{X} \to X$ under this isomorphism. Under the assumption that the base field k is computable, provided that Z and X are given by finitely represented sheaves of finitely generated k-algebras and f by k-algebra homomorphisms, we want to compute $\mathcal{I}$.

In this paper we approach the problem from two directions, solving two subcases which are general enough to be of interest. The following famous problem serves as the main motivation for the first case: Given Y a reduced closed subscheme of a nonsingular quasi-projective variety X, compute an ideal sheaf $\mathcal{I}$ over X, such that the blowing up of X along $\mathcal{I}$ induces an embedded resolution of the singularities of Y (for additional information see [9, 12]). In the classical approach [13, 9] the resolution is defined (in a non-constructive way) as the composition of finitely many well chosen blowing ups above the

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initial variety. Constructive solutions appeared in [12, 2, 6], followed by an implementation based on Villamayor's results in [4, 3].

In section 3 of this paper we take $f = \pi_1 \circ \cdots \circ \pi_k$, where $\pi_i : X_i \to X_{i-1}$ is a blowing up of X_{i-1} (with $X_0 = X$) along the ideal sheaf $\mathcal{I}_i$, e.g. we consider the sequence of blowing ups computed by the resolution algorithm of [4]. Then we present a straightforward method to compute an ideal sheaf $\mathcal{I}$ over X_0 whose blowing up is isomorphic to X_k. In section 4 we briefly describe the details of the representation of a sequence of blowing ups as it is in the algorithm of [4], demonstrating the method of section 3.

The second subcase of the original problem requires a more direct attack. In section 5 we take $i(Z) \subset \mathbf{P}^n_X$, given by a sheaf of finitely generated graded k-algebras, which is represented by finitely many quasi-projective sections. Then we present a simple method to determine an ideal sheaf $\mathcal{I}$ over X whose blowing up yields a variety isomorphic to Z.

The results might be interesting not only on their own right but also because of the connections to resolution of singularities; moreover the paper might be interesting for people working on factorization of morphisms.

The results are related to the topics discussed in the recent papers: [10] on a criterion for two ideals over X to give rise to isomorphic blowing ups; [11] on a criterion for an ideal to give rise to nonsingular blowing up (up to a Nash blowing up).

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2 Preliminaries

The notation comes from [8] and the context is as the one at Theorem 7.17. We additionally assume for the varieties we work with that they are noetherian, where the affine rings corresponding to open subsets of the variety are finitely generated k-algebras over some computable algebraically closed field k.

The blowing up of a quasi-projective variety X along an ideal sheaf $\mathcal{I}$ is defined as $\tilde{X} = \mathbf{Proj}(\bigoplus_{d \geq 0} \mathcal{I}^d)$, where $\mathcal{I}^0 = \mathcal{O}_X$ and π is the projection $\tilde{X} \to X$. On an affine subset $U \subseteq X$, where $U = \operatorname{Spec} A$, $A = k[x_1, \ldots, x_m]/\langle f_1, \ldots, f_l\rangle$ and $\mathcal{I}_U = \langle g_0, \ldots, g_n\rangle A$, the vanishing ideal J of the blowing up $\tilde{U}$ in $U \times \mathbf{P}^n_k$ is the kernel of the ring homomorphism $\varphi : A[y_0, \ldots, y_n] \to A[t]$ that fixes A and sends $y_i \mapsto tg_i$. This ideal can be computed using the method of Gröbner bases [5, 1] as

$$\texttt{gbasis}(\{f_1, \ldots, f_l, y_0 - tg_0, \ldots, y_n - tg_n\}, T),$$

where T is an elimination term order for t on the term-monoid over the variables $x_1, \ldots, x_m, y_0, \ldots, y_n, t$, and the generators of J are just the elements of the Gröbner basis which are free of t. The ideal J is homogeneous in the y_is giving rise to a grading on $\tilde{A} = A[y_0, \ldots, y_n]/J$: $\deg a = 0$ for $a \in A$, $\deg y_i = 1$.

The blowup $\tilde{U}$ admits a natural affine cover along the principal open subsets of $\mathbf{P}^n_k$: $\tilde{U}_i = \operatorname{Spec} A_{(i)}$, where $A_{(i)} = \tilde{A}/\langle y_i - 1\rangle$ is the ith dehomogenization of $\tilde{A}$ which is isomorphic to $A[g_0/g_i, \ldots, g_n/g_i]$. The gluing maps of the $\tilde{U}$s can be constructed from the local blowing up and blowing down maps composed with the gluing maps of X.

3 Unification of composed blowing ups

The elementary step of the process is to determine the unification of two consecutive blowing ups. Let us consider $X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2$, blowing ups along centers $\mathcal{I}_1, \mathcal{I}_2$ respectively. Let π_1 over an affine subset $U_0 \subseteq X_0$ be described as in section 2, and let the restriction of $\mathcal{I}_2$ to $U_1 = \tilde{U}_0$ be $I_2 \subseteq A_1 = \tilde{A}_0$ generated by homogeneous elements $h_0, \ldots, h_s$.

Let the quasi-projective variety $U_2 = \tilde{U}_1$ be the blowing up variety of U_1 along I_2. It also admits a natural affine cover via the affine cover of U_1. Using the construction of section 2 again, let the coordinate algebra of $\tilde{U}_{1(i)}$, the blowing up of the ith affine patch of U_1, be $\tilde{A}_{1(i)} = A_{1(i)}[z_0, \ldots, z_s]/\ker\varphi_2$, where φ_2 is the homomorphism corresponding to the blowing up defined by the ith dehomogenizations of the h_j in $A_{1(i)}$. Then $U_{ij} = \operatorname{Spec} A_{(i)(j)}$, $A_{(i)(j)} = \tilde{A}_{1(i)}/\langle z_j - 1\rangle$.

Let $\underline{\varphi}_1 : A_1 \to A_0$ send $y_l \mapsto g_l$, and let us denote $\underline{h}_k = \underline{\varphi}_1 h_k$. With this notation we have that $A_{(i)(j)}$ is isomorphic to

$$A_0[\frac{g_0}{g_i}, \ldots, \frac{g_n}{g_i}, \frac{\underline{h}_0}{\underline{h}_j}\frac{g_i^{\deg_1 h_j}}{g_i^{\deg_1 h_0}}, \ldots, \frac{\underline{h}_s}{\underline{h}_j}\frac{g_i^{\deg_1 h_j}}{g_i^{\deg_1 h_s}}].$$

Indeed the dehomogenization of h_k with respect to y_i yields $\underline{h}_k/g_i^{\deg_1 h_k} \in A_{1(i)}$.

In A_1 we want to consider only those blowing ups which can be defined along ideals generated by forms of the same degree. In this case the $g_i^{\deg_1 h_j}/g_i^{\deg_1 h_k}$ factors cancel in the above generators, which allows us to show an isomorphism between the blowing up varieties in proposition 3.3. Fortunately, we can always reduce to this case, according to remark 3.2, using the following definition.

Definition 3.1. Let $I = \bigoplus_{d\geq 0} I_d$ be a homogeneous ideal in a graded ring $S = \bigoplus_{d\geq 0} S_d$. For $i \geq 0$ the ith *inverse saturation* of I is $\bigoplus_{d\geq i} I_d$.

Remark 3.2. In context of the previous definition: (i) Let S be generated by the degree one elements $y_0, \ldots, y_n$ over some ring A and let I be generated by a finite set $F \subset S$. Then the ith inverse saturation of I is generated by the finite set

$$\Big(\bigcup_{d=0}^{i-1}\bigcup_{j=0}^{n}\{y_j^{i-d} f \mid f \in F \cap S_d\}\Big) \cup \Big(F \setminus \bigcup_{d=0}^{i-1} S_d\Big).$$

(ii) Any inverse saturation of an ideal I has the same saturation as of I by construction. (iii) Because of (ii) any inverse saturation of a homogeneous ideal I defines the same closed subscheme as I itself (Exercise 5.10(b) in [8]), in particular, it gives rise to the same blowing up as I.

Let I_2, the ideal of the blowing up over U_1, be generated in maximal degree d, that is, $d = \max_i \deg h_i$. Instead of I_2 we blow up U_1 along its dth inverse saturation I_2' generated by, say, $h_1', \ldots, h_r'$. Just as above, we construct the natural affine cover of the blowing up variety U_2', getting $U_{ij}' = \operatorname{Spec} A_{(i)(j)}'$, where $A'_{(i)(j)}$ is isomorphic to

$$A_0[\frac{g_0}{g_i},\ldots,\frac{g_n}{g_i},\frac{\underline{h}'_0}{\underline{h}'_j},\ldots,\frac{\underline{h}'_r}{\underline{h}'_j}].$$

Let U_3 be the blowing up of U_0 along the ideal in A_0 defined as the product of I_1, the ideal of the first blowing up center, and the image of I'_2 along $\underline{\varphi}_1$:

$$\langle g_0,\ldots,g_n\rangle\langle \underline{h}'_0,\ldots,\underline{h}'_r\rangle = \langle g_0\underline{h}'_0,\ldots,g_n\underline{h}'_r\rangle.$$

Using a suitable notation we describe the vanishing ideal J_3 of U_3 to be the kernel of the ring homomorphism $\varphi_3 : A_0[v_{00},\ldots,v_{nr}] \to A_0[t]$ that fixes A_0 and sends $v_{ij} \mapsto tg_i\underline{h}'_j$. The coordinate ring of U_3 is the graded ring $A_3 = A_0[v_{00},\ldots,v_{nr}]/J_3$.

Proposition 3.3. *With the notation introduced so far, we have that U_3 is isomorphic to U'_2 which is isomorphic to U_2.*

Proof. By remark 3.2(iii) it is clear that U_2 and U'_2 are isomorphic. We show an isomorphism between U_3 and U'_2 by "assembling" both from isomorphic affine varieties in an isomorphic way. Let us cover U_3 naturally by affine subsets $U_{ij} = \operatorname{Spec} A_{(ij)}$, where $A_{(ij)} = A_3/\langle v_{ij}-1\rangle$ which is isomorphic to

$$A_0[\frac{g_0\underline{h}'_0}{g_i\underline{h}'_j},\ldots,\frac{g_n\underline{h}'_r}{g_i\underline{h}'_j}].$$

Now we observe that the generators contain

$$\frac{g_k}{g_i} = \frac{g_k\underline{h}'_j}{g_i\underline{h}'_j}, \qquad \frac{\underline{h}'_l}{\underline{h}'_j} = \frac{g_i\underline{h}'_l}{g_i\underline{h}'_j}$$

and that all the other generators can be eliminated by the equations

$$\frac{g_k\underline{h}'_l}{g_i\underline{h}'_j} = \Big(\frac{g_k}{g_i}\Big)\Big(\frac{\underline{h}'_l}{\underline{h}'_j}\Big) \quad \text{for } k\neq i, l\neq j.$$

This makes $U_{ij} \subseteq U_3$ isomorphic to $U'_{ij} \subseteq U'_2$. Moreover, since the corresponding affine coordinate rings $A_{(ij)}$ and $A'_{(i)(j)}$ are exactly the same subrings of the function field over U_0, they give rise to the same gluing maps. □

For a simple example that goes through all the computations we refer to the first half of section 6.

Proposition 3.4. *The method of proposition 3.3 can be generalized to unify two composed blowing ups over a quasi-projective variety.*

Proof. Let X_0 be covered by affine subsets $U_0^{(i)} = \operatorname{Spec} A_0^{(i)}$, $i = 1,\ldots,k$. Let X_0 be blown up along $\mathcal{I}_1$ yielding $X_1 = \tilde{X}_0$ which can be covered by quasi-projective varieties $U_1^{(i)} = \tilde{U}_0^{(i)}$ with homogeneous coordinate rings $A_1^{(i)} = \tilde{A}_0^{(i)}$. Let then X_1 be blown up along $\mathcal{I}_2$ which restricts to $\mathcal{I}_2|_{U_1^{(i)}} = I_2^{(i)} \subseteq A_1^{(i)}$, homogeneous ideals generated in maximal degree $d^{(i)}$ respectively.

We consider now the ideal sheaf $\mathcal{I}_2'$ given by its restrictions over $U_1^{(i)}$ as the dth inverse saturation of $I_2^{(i)}$, where $d = \max_i d^{(i)}$. We can replace $\mathcal{I}_2$ by $\mathcal{I}_2'$ in the second blowing up; the gluing maps of the quasi-projective parts $U_2^{(i)\prime}$ of the blowing up variety arise naturally from the gluing maps of the $U_2^{(i)}$ and the isomorphisms $U_2^{(i)} \cong U_2^{(i)\prime}$.

Then according to proposition 3.3 we can compute ideals $I_3^{(i)} \subseteq A_0^{(i)}$ that unify the two blowing ups above $U_0^{(i)}$. We need to extend the results to X_0.

The image of $\mathcal{I}_2'$ in $\mathcal{O}_{X_0}$ induced by the maps $\underline{\varphi}_1^{(i)}$ defines a sheaf over X_0 which we denote by $\underline{\mathcal{I}}_2'$; in other terms $\underline{\mathcal{I}}_2' = \underline{\varphi}_1(\pi_{1*}\mathcal{I}_2')$, where $\underline{\varphi}_1$ is a morphism of sheaves over X_0 induced by the $\underline{\varphi}_1^{(i)}$ maps along the proper morphism π_1. The unified blowing up center is then $\mathcal{I}_1 \cdot \underline{\mathcal{I}}_2'$. $\square$

For a given sequence $X_0 \xleftarrow{\pi_1} X_1 \cdots \xleftarrow{\pi_k} X_k$ we proceed by unifying blowing ups "at the end". That is, first we unify $X_{k-2} \xleftarrow{\pi_{k-1}} X_{k-1} \xleftarrow{\pi_k} X_k$ obtaining $X_{k-2} \xleftarrow{\pi_{k-1}'} X_{k+1}$, where X_{k+1} is a isomorphic to X_k. This way we have a sequence of blowing ups with length one less than the original sequence, thus in $k-1$ steps we reach the blowing up $X_0 \xleftarrow{\pi_1'} X_{2k-1}$ along an ideal sheaf $\mathcal{I}_{2k-1}$ in $\mathcal{O}_{X_0}$, where X_{2k-1} is isomorphic to X_k.

4 An application in resolution of singularities

For more than ten years now, there are constructive proofs available for resolution of singularities over fields of zero characteristic [12, 2, 6]. In [4, 3] we report on an implementation of Villamayor's algorithm which computes embedded desingularizations of hypersurfaces (X, H), where X is a nonsingular ambient variety and H is a variety in X of codimension one. (Since then there is a straightforward generalization of the algorithm to the non-hypersurface case [7].) An embedded resolution is then constructed by computing a suitable sequence of blowing ups $X = X_0 \leftarrow \cdots \leftarrow X_k$ such that the preimage $H_k \subset X_k$ of H has normal crossings while the strict transform of H is nonsingular.

In this section we shortly describe a data structure that represent such a sequence of blowing ups and apply the results of section 3 to compute the unified blowing up that resolves the singularities "in one step".

We represent a quasi-projective variety X via affine covering, where an affine chart U is described by its finitely generated k-algebra, which is given by generators $x_1, \ldots, x_m$ and relations $f_1, \ldots, f_l \in k[x_1, \ldots, x_m]$. The affine charts are glued together by birational maps $\phi_{UV} : U \to V$, which are given by k-algebra homomorphisms between coordinate algebras of open subsets of U and V. For instance, ϕ_{UV} can be given by a map $k[V] \to k[U_V]$, $y_i \mapsto g_i/h_i$ for $y_1, \ldots y_k$ generators of $k[V]$, where all $g_i, h_i \in k[U]$ and $U_V = U \setminus Z(h_1, \ldots, h_k)$. Such a representation is called an *atlas* of affine charts for X.

A blowing up of X along an ideal sheaf $\mathcal{I}$ is computed by blowing up the charts of X along the corresponding sections of $\mathcal{I}$. Say, over U with $\langle p_1, \ldots, p_r \rangle = \mathcal{I}|_U$ the blowing up variety $\tilde{U}$ is represented via its natural covering, where, for instance, the generators of the rth chart $\tilde{U}_r$ can be $x_1, \ldots, x_m$, $x_{m+1}, \ldots, x_{m+r-1}$ and the relations can be computed as in section 2, making $k[\tilde{U}_r]$ isomorphic to $k[U][p_1/p_r, \ldots, p_{r-1}/p_r]$. In fact, the blowing up centers in the resolution algorithm are always defined by regular parameters over U, so that the relations of $k[\tilde{U}_r]$ are $f_1, \ldots, f_l, p_r x_{m+1} - p_1, \ldots, p_r x_{m+r-1} - p_{r-1}$.

The blowing up map $\pi|_{\tilde{U}_r} : \tilde{U}_r \to U$ can be represented by the natural ring inclusion $k[U] \to k[\tilde{U}_r]$, and the blowing down map $\pi|_{\tilde{U}_r}^{-1} : U \to \tilde{U}_r$ by $k[\tilde{U}_r] \to k[U_r]$, $x_i \mapsto x_i$ for $i = 1, \ldots, m$ and $x_{m+i} \mapsto p_i/p_r$ for $i = 1, \ldots, r-1$, where $U_r = U \setminus Z(p_r)$.

The chart change maps between $\tilde{U}_i$ and $\tilde{V}_j$ arise naturally as $\pi_{\tilde{V}_j}^{-1} \circ \phi_{UV} \circ \pi|_{\tilde{U}_i} : \tilde{U}_i \to \tilde{V}_j$ (the inverse is computed analogously).

In this representation a sequence of blowing ups is given by trees of affine charts above the charts $U_{0,i}$ of X_0:

$$
\begin{array}{ccccccccc}
X_0 & & & & & U_{0,i} & & & \\
\pi_1 \uparrow & \mathcal{I}_1 & & & \nearrow & & \nwarrow & & \\
X_1 & & & U_{1,i}^{(1)} & & \cdots & & U_{1,i}^{(r_1)} & \\
\vdots & & & & & & & & \\
\pi_k \uparrow & \mathcal{I}_k & \nearrow & & & \uparrow & & & \nwarrow \\
X_k & & U_{k,i}^{(1,\ldots,1)} & \cdots & & U_{k,i}^{(i_1,\ldots,i_k)} & \cdots & & U_{k,i}^{(r_1,\ldots,r_k)}
\end{array}
$$

In order to compute an ideal sheaf over X that defines a blowing up variety isomorphic to X_k, we proceed as suggested at the end of section 3. That is, we compute an ideal sheaf over X_{k-2} that unifies the last two blowing ups $X_{k-2} \leftarrow X_{k-1} \leftarrow X_k$.

First of all, we have to compute the sections of $\mathcal{I}_k$ over the quasi-projective patches $\tilde{U}_i$, where $U_i = U_{k-2,i}^{(i_1,\ldots,i_{k-2})}$ for simpler notation. This can be done easily: Let $\tilde{A} = k[\tilde{U}_i]$, which is a homogeneous k-algebra over $A = k[U_i]$ of the form $A[y_0, \ldots, y_r]/J$, so that $U_{k-1,i}^{(i_1,\ldots,i_{k-2},j)}$ (denoted by U_{ij} for simplicity) is isomorphic to $\tilde{U}_i \setminus Z(y_j)$ with coordinate algebra $A_j = \tilde{A}/\langle y_j - 1 \rangle$.

We have $\mathcal{I}_k|_{U_{ij}}$ given by a finite generating set, say $f_{j1}, \ldots, f_{jr_j}$. We compute $\mathcal{I}_k|_{\tilde{U}_i}$ by computing the ideal intersection $\bigcap_{j=0}^{r} \langle \beta_j(f_{j1}), \ldots, \beta_j(f_{jr_j}) \rangle$, where β_j is the homogenization with respect to y_j.

Then, as in proposition 3.4, we compute $d = \max_i d_i$, where d_i is the maximal degree of the generators of $\mathcal{I}_k|_{\tilde{U}_i}$, and replace $\mathcal{I}_k$ by its dth inverse saturation $\mathcal{I}_k'$. We push $\mathcal{I}_k'$ along π_{k-1} obtaining $\underline{\mathcal{I}}_k'$ which is given by finite generating sets over the U_i. The product $\mathcal{I}_{k-1}' = \mathcal{I}_{k-1} \cdot \underline{\mathcal{I}}_k'$ can then easily be computed. We repeat the whole procedure now for $X_{k-3} \xleftarrow[\mathcal{I}_{k-2}]{\pi_{k-2}} X_{k-2} \xleftarrow[\mathcal{I}_{k-1}']{\pi_{k-1}'} X_k' \cong X_k$.

At last we should note that although the unification process does not require Gröbner basis computation, to compute the ideal intersections in general we need Gröbner bases.

5 A direct approach

In this section we intend to find an ideal sheaf $\mathcal{I}$ as in Theorem 7.17 of [8] assuming that we know $i(Z) \subset \mathbf{P}_X^n$ for some n. As in section 3, first we consider the problem only above affine subsets $U \subseteq X$, with $U = \operatorname{Spec} A$, $A = k[x_1, \ldots, x_m]/J$, so that $\tilde{U} \cong \pi^{-1} U$ lives in $\mathbf{A}_k^m \times \mathbf{P}_k^n$. The next lemma generalizes the observation that the blowing up variety can be naturally covered by affine subsets whose coordinate algebras are subrings of the quotient field of the affine variety. The information on the structure of these rings will help us to say something about the saturated homogeneous ideal of the blowing up variety.

Lemma 5.1. *The coordinate ring of any affine subset of the blowing up variety $\tilde{U}$ is isomorphic to a subring of the quotient field $k(U)$.*

Proof. Let the coordinate ring of $\tilde{U}$ be $\tilde{A}$ and let the affine patch be defined as the complement of the zero set of $0 \neq f \in \tilde{A}$ with positive degree. We want to show that the ring $\tilde{A}[y]/\langle fy-1\rangle$ is isomorphic to a subring of $k(U)$.

We know that $\tilde{U}$ is isomorphic to some $\tilde{U}' = \pi^{-1}U \subset \mathbf{P}^r_U$, for some r, with vanishing ideal $\tilde{J}'$ and coordinate ring $\tilde{A}'$ which is the blowup algebra of an ideal $I \subseteq A$. This isomorphism pulls f to some $0 \neq f' \in \tilde{A}'$. Let $d = \deg f'$ (> 0) and let us apply d-uple embedding for $\tilde{U}'$ induced by $\mathbf{A}^m_k \times \mathbf{P}^r_k \hookrightarrow \mathbf{A}^m_k \times \mathbf{P}^N_k$, $N = \binom{r+d}{r} - 1$, obtaining $\tilde{U}''$. Then f' pushes to some linear element $f'' = g_0v_0 + \cdots + g_Nv_N$ $(g_i \in A)$ in the coordinate ring of $\tilde{U}''$.

By remark 5.4(iii), $\tilde{U}''$ is the blowing up variety of U along I^d, generated by, say, $f_0, \ldots, f_N$. The polynomial $\underline{f}'' = g_0f_0 + \cdots + g_Nf_N \in I^d$ pulls back to f''. Now, it is well known that the coordinate ring of the $N+1$st principal affine subset of the blowing up variety, using the generating set $f_0, \ldots, f_N, \underline{f}''$, is isomorphic to $A[f_0/\underline{f}'', \ldots, f_N/\underline{f}'']$, while this affine subset is isomorphic to the chosen affine patch of $\tilde{U}$. $\square$

We try to find suitable generators for the ideal of the center by examining the relations of the coefficients of the homogeneous variables in the generators of the defining ideal of $\tilde{U}$. Here is the method we use.

Let $\tilde{J} \subseteq A[y_0, \ldots, y_n]$ be the defining ideal of $\tilde{U}$; it must contain elements with positive degree (except if the blowing up is an isomorphism over U and there are no homogeneous variables present). We compute the saturation of $\tilde{J}$ and if it contains no linear equations, let d be the maximal degree of elements in a fixed generating set of $\tilde{J}$, and let us apply d-uple embedding for $\tilde{U}$ (as in the proof of lemma 5.1). We obtain linear elements in the vanishing ideal of the embedded variety; therefore, without the loss of generality, we can assume that $\tilde{J}$ contains linear equations.

Let the degree one generators of a finite generating set of $\tilde{J}$ be $g_1, \ldots, g_k$. Let $g_{ij} \in A$ be the coefficient of y_j in g_i and let us consider the module of syzygies defined by the homogeneous equation system

$$\begin{pmatrix} g_{10} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{k0} & \cdots & g_{kn} \end{pmatrix} \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let $(h_0, \ldots, h_n) \in A^{n+1}$ be a generator of the solution module. The next lemma and proposition shows that these A-elements are generators of an ideal which is suitable for the definition of the blowing up.

Lemma 5.2. *Let $\tilde{J} \subset A[y_0, \ldots, y_n]$ be the saturated defining ideal of a blowing up variety $\tilde{U}$ with linear equations. Then there are $f_0, \ldots f_n \in A$ such that $f_jy_i - f_iy_j \in \tilde{J}$ for all $0 \leq i, j \leq n$.*

Proof. Let $h_0, \ldots, h_n \in A$ be the components of a generator of the module of syzygies of the linear equations of $\tilde{J}$.

For $i \neq j$, let us take the jth principal affine subset of $\tilde{U}$, whose coordinate ring must be isomorphic to a subring of $k(U)$ by lemma 5.1. Under this isomorphism y_i is mapped to some p_{ij}/q_{ij}, $p_{ij}, q_{ij} \in A$, $q_{ij} \neq 0$, which forces the jth dehomogenization of $\tilde{J}$ to contain an element $q_{ij}y_i - p_{ij}$.

Homogenizing the equation we get that $q_{ij}y_i - p_{ij}y_j$ must be in (the saturated) $\tilde{J}$. But by construction we must have $q_{ij}h_i = p_{ij}h_j$, hence $s_{ij}q_{ij} = r_{ij}h_j$, $s_{ij}p_{ij} = r_{ij}h_i$, for some $r_{ij}, s_{ij} \in A$, $r_{ij} \neq 0$.

Let $r = \prod_{0 \leq i,j \leq n, i \neq j} r_{ij}$, and let $f_i = rh_i$, $i = 0, \ldots, n$. We have that $f_iy_j - f_jy_i = s_{ij}r/r_{ij}(p_{ij}y_j - q_{ij}y_i) \in \tilde{J}$ when $i \neq j$. If $i = j$ the containment is obvious. □

Proposition 5.3. *Let I be generated by $h_0, \ldots, h_n$, the components of a generator of the module of syzygies, obtained by the method of this section. The blowing up of U defined by I is isomorphic to $\tilde{U}$.*

Proof. By lemma 5.2, we have a $0 \neq r \in A$, such that $r(h_iy_j - h_jy_i) \in \tilde{J}$ for all $0 \leq i, j \leq n$. The blowing up variety of U defined by I is isomorphic to $\tilde{U}$ because the coordinate rings of their principal affine subsets are the same subrings of $k(U)$.

Note that the elementary linear relations already determine the solution of the syzygy up to multiplication by nonzero A-elements, which is sufficient for us, because such tuples define isomorphic blowing ups (Exercise 7.11(b) in [8]). □

Remark 5.4. (i) Since the generators of the solution module can differ only by nonzero factors, we can compute one which componentwise divides all the others. We choose this module generator to define I by default. (ii) A generating set for the solution syzygy module can be obtained via extended Gröbner basis computation. The saturation of an ideal can also be computed by means of Gröbner bases. (iii) Passing to d-uple embedding of the blowing up variety corresponds to replacing the ideal I of the blowing up center by I^d. This corresponds with the fact that I and I^d define isomorphic blowing ups (Exercise 7.11(a) in [8]). The homogeneous vanishing ideal of the embedded variety can be computed by means of Gröbner bases, or if d is at least as big as the maximal degree of a given generating set of the vanishing ideal of the blowing up variety, by simply taking the dth inverse saturation (see definition 3.1) of this vanishing ideal and replacing degree-d power products by their corresponding new variables. This yields linear relations in the vanishing ideal of the embedded variety; obviously, the higher degree equations of the embedding of the ambient projective space must also be added to get a complete generating set.

Proposition 5.5. *The method of proposition 5.3 can be generalized to compute an ideal sheaf over X whose blowing up is isomorphic to $i(Z) \subset \mathbf{P}^n_X$.*

Proof. Let X be given by its affine subsets $U^{(i)} = \operatorname{Spec} A^{(i)}$, $i = 1, \ldots, k$. By proposition 5.3, we can compute ideals $I^{(i)} \subseteq A^{(i)}$ whose blowing ups are isomorphic to quasi-projective patches $\tilde{U}^{(i)}$ of $i(Z)$.

To make sure that $I^{(i)}$ define a sheaf of ideals over X, we act similarly to the approach of proposition 3.4. If we have to compute d-uple embedding when we apply the method for one of the $\tilde{U}^{(i)}$, let $d^{(i)}$ be the calculated value of the map. Then instead of $d^{(i)}$, we compute d-uple embedding for all $\tilde{U}^{(i)}$, where $d = \max_i d^{(i)}$. Thus at any stage of the computation we use sections of a sheaf of defining ideals for $i(Z)$ in some, possibly higher dimensional, $\mathbf{P}^N_X$. Finally, the solutions define an ideal sheaf $\mathcal{I}$ over X because remark 5.4(i) makes the choice of the module generator unique. □

In the rest of this section we deal with a representational issue, whose solution can be reduced to the one above. When X, quasi-projective, is given by a finitely generated k-algebra and Z is computed via a sequence of blowing ups $X \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_l} Z$, so that $Z \subset X \times \mathbf{P}^{n_1}_k \times \cdots \times \mathbf{P}^{n_l}_k$, Z can be represented by a single multi-homogeneous k-algebra. Proposition 5.6 tells us when this construction arises naturally.

Proposition 5.6. *Let X be a quasi-projective variety, described by its coordinate k-algebra A with grading* deg. *Let the blowing up center be defined by the* deg*-homogeneous ideal I. The blowup algebra, $\bigoplus_{d\geq 0} I^d$ viewed as a graded A-algebra, is a finitely generated k-algebra iff I is finitely generated by A elements of the same degree.*

Proof. Let us assume that $I = \langle h_0, \ldots, h_n\rangle$ with $d = \deg h_0 = \cdots = \deg h_n$. The construction of section 2 gives us the blowup algebra as a k-algebra: Let $\varphi : A[z_0, \ldots, z_n] \to A[t]$ fix A and send $z_i \mapsto th_i$, and let $\tilde{A} = A[z_0, \ldots, z_n]/\ker\varphi$. Observe that $\tilde{A}$ is bigraded with $\deg z_k = d$ and $\deg' a = 0$ for $a \in A$, $\deg' z_k = 1$. That is, because $h_0, \ldots, h_n$ are of the same degree, $\ker\varphi$ is not only $\deg'$- but also deg-homogeneous.

On the other hand, let us assume that I cannot be generated by A elements of the same degree. Let $\{h_0, \ldots, h_n\}$ be a fixed set of generators of I. Without the loss of generality we can assume that $h_0, \ldots, h_r$, $r < n$, are the generators with minimal degree, and that $h_n \notin \langle h_0, \ldots, h_r\rangle$. If we try to apply the above construction, we find that no deg-homogeneous equations exist that relate z_i and z_n, for any $i \in \{1, \ldots, r\}$, which are homogeneous also in the z_ks. In general, a bigraded finitely generated k-algebra cannot be isomorphic to $\bigoplus_{d\geq 0} I^d$, because no bihomogeneous relations can be written down for generators of different degrees. □

Remark 5.7. In context of proposition 5.6, if I cannot be generated by elements of the same degree, we can always replace it by its dth inverse saturation, where d is greater than or equal to the maximal degree of elements in a finite generating set.

Proposition 5.6 implies that if Z is given by a multi homogeneous k-algebra, the (unknown) blowing up centers were generated in uniform degrees for all blowing ups. Let us choose i to be the l-fold Segre-embedding, so that $i(Z) \subset \mathbf{P}_X^{\prod_i (n_i+1)-1}$. In this case, since X can be described by a finitely generated k-algebra A, we have that $i(Z) \subset \mathbf{P}_X^{\prod_i (n_i+1)-1} = \mathrm{Proj}(A')$ is defined by a homogeneous ideal J, where $A' = A[z_{i_1,\ldots,i_l} \,|\, 0 \leq i_j \leq n_j, j = 1, \ldots, l]$. By the construction of the Segre-embedding and the fact that the coefficients of the z-variables are from A, we get that the k-algebra homomorphism $\varphi : A' \to A$, for which the z-homogeneous part of $\ker\varphi$ is just J, sends $z_{i_1,\ldots,i_l}$ to the product

$$f_{1,i_1}(\underline{\varphi}_2 f_{2,i_2}) \cdots (\underline{\varphi}_2 \cdots \underline{\varphi}_l f_{l,i_l}) \in A,$$

where f_{j,i_j} is the i_jth generator of the jth blowing up center and $\underline{\varphi}_j : A_j \to A_{j-1}$ is the k-algebra homomorphism from the coordinate algebra of the jth blowing up variety to the $j-1$st, sending the newly introduced homogeneous variable $y_{j,i}$ of A_j to $f_{j,i} \in A_{j-1}$, for $0 \le i \le n_j$ (we assume $A_0 = A$).

Summarizing the observations of the previous paragraph, we have that A'/J is the blowup algebra of A that belongs to the center which is obtained by applying the method of section 3. That is, if we choose i be the l-fold Segre-embedding, such that $i(Z)$ is represented by A'/J, we can apply the method of proposition 5.5 to obtain the center in A that unifies the sequence of blowing ups $X \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_l} Z$.

6 An extended example

Let $X_0 = \mathbf{A}_k^3 = \operatorname{Spec} R_0$, where $R_0 = k[x_1,x_2,x_3]$ and k is, for instance, the algebraic closure of the rationals. Let us blow up X_0 along the ideal $I_1 = \langle x_1x_2, x_3\rangle$, getting $X_1 \subset \mathbf{A}_k^3 \times \mathbf{P}_k^1$ with coordinate k-algebra $R_1 = k[x_1,x_2,x_3,y_0,y_1]/\langle x_1x_2y_1 - x_3y_0\rangle$, which can be covered by affine subsets naturally: $X_{10} = \operatorname{Spec} R_{10}$, $R_{10} = R_1/\langle y_0 - 1\rangle \cong k[x_1,x_2,x_3/(x_1x_2)]$, $X_{11} = \operatorname{Spec} R_{11}$, $R_{11} = R_1/\langle y_1 - 1\rangle \cong k[x_1,x_2,x_3,x_1x_2/x_3]$.

We have that X_{11} is singular at the point $I_2 = \langle x_1,x_2,x_3,y_0\rangle$, which is outside X_{10}. We blow up X_{11} along this point and represent the blowing up variety by its affine subsets $X_{20}, X_{21}, X_{22}, X_{23}$ with corresponding k-algebras:

$$R_{20} \cong k[x_1,x_2,x_3,\tfrac{x_1x_2}{x_3},\tfrac{x_2}{x_1},\tfrac{x_3}{x_1},\tfrac{x_2}{x_3}] \cong k[x_1,\tfrac{x_2}{x_1},\tfrac{x_3}{x_1},\tfrac{x_2}{x_3}],$$

$$R_{21} \cong k[x_1,x_2,x_3,\tfrac{x_1x_2}{x_3},\tfrac{x_1}{x_2},\tfrac{x_3}{x_2},\tfrac{x_1}{x_3}] \cong k[x_2,\tfrac{x_1}{x_2},\tfrac{x_3}{x_2},\tfrac{x_1}{x_3}],$$

$$R_{22} \cong k[x_1,x_2,x_3,\tfrac{x_1x_2}{x_3},\tfrac{x_1}{x_3},\tfrac{x_2}{x_3},\tfrac{x_1x_2}{x_3^2}] \cong k[x_3,\tfrac{x_1}{x_3},\tfrac{x_2}{x_3},\tfrac{x_1x_2}{x_3^2}],$$

$$R_{23} \cong k[x_1,x_2,x_3,\tfrac{x_1x_2}{x_3},\tfrac{x_3}{x_2},\tfrac{x_3}{x_1},\tfrac{x_3^2}{x_1x_2}] \cong k[x_1,x_2,\tfrac{x_1x_2}{x_3},\tfrac{x_3}{x_2},\tfrac{x_3}{x_1}].$$

Now we have that $X_{10}, X_{20}, X_{21}, X_{22}, X_{23}$ can be glued together to form X_2 the blowing up variety of X_1 along $\mathcal{I}_2$ defined by I_2 over X_{11} and 1 over X_{10}. Note that we also have $\mathcal{I}_2 = \langle x_1,x_2,x_3,y_0\rangle R_1$.

Let us use the method of section 3 to find the blowing up that results a variety isomorphic to X_2. The homomorphism $\underline{\varphi}_1 : R_1 \to R_0$ is defined as $a \mapsto a$ for $a \in R_0$, $y_0 \mapsto x_1x_2$, $y_1 \mapsto x_3$. A generating set for the first inverse saturation of $\mathcal{I}_2$ is $\{x_1y_0, x_2y_0, x_3y_0, x_1y_1, x_2y_1, x_3y_1, y_0\}$, which in this case can be easily simplified to $\mathcal{I}_2' = \{x_1y_1, x_2y_1, x_3y_1, y_0\}$.

The blowing up center is then defined by $\langle x_1x_2, x_3\rangle \cdot \langle x_1x_3, x_2, x_3, x_3^2, x_1x_2\rangle$ in R_0, which results $I = \langle x_1^2x_2^2, x_1x_2x_3, x_1x_2^2, x_1x_3^2, x_3^3\rangle$ after some straightforward simplification.

Let us blow up X_0 along I and compute the natural affine cover for the blowing up variety $X_{30}, X_{31}, X_{32}, X_{33}, X_{34}$ with corresponding k-algebras:

$$R_{30} \cong k[x_1,x_2,x_3,\tfrac{x_3}{x_1x_2},\tfrac{x_3^2}{x_1x_2^2},\tfrac{x_3^2}{x_1^2x_2},\tfrac{x_3^3}{x_1^2x_2^2}] \cong k[x_1,x_2,\tfrac{x_3}{x_1x_2}] \cong R_{10},$$

$$R_{31} \cong k[x_1,x_2,x_3,\tfrac{x_1x_2}{x_3},\tfrac{x_3}{x_2},\tfrac{x_3}{x_1},\tfrac{x_3^2}{x_1x_2}] \cong k[x_1,x_2,\tfrac{x_1x_2}{x_3},\tfrac{x_3}{x_2},\tfrac{x_3}{x_1}] \cong R_{23},$$

$$R_{32} \cong k[x_1,x_2,x_3,\tfrac{x_1x_2^2}{x_3^2},\tfrac{x_2}{x_3},\tfrac{x_2}{x_1},\tfrac{x_3}{x_1}] \cong k[x_1,\tfrac{x_2}{x_3},\tfrac{x_2}{x_1},\tfrac{x_3}{x_1}] \cong R_{20},$$

$$R_{33} \cong k[x_1,x_2,x_3,\tfrac{x_1^2x_2}{x_3^2},\tfrac{x_1}{x_3},\tfrac{x_1}{x_2},\tfrac{x_3}{x_2}] \cong k[x_2,\tfrac{x_1}{x_3},\tfrac{x_1}{x_2},\tfrac{x_3}{x_2}] \cong R_{21},$$

$$R_{34} \cong k[x_1, x_2, x_3, \tfrac{x_1^2 x_2^2}{x_3^3}, \tfrac{x_1 x_2}{x_3^2}, \tfrac{x_1}{x_3}, \tfrac{x_2}{x_3}] \cong k[x_3, \tfrac{x_1 x_2}{x_3^2}, \tfrac{x_1}{x_3}, \tfrac{x_2}{x_3}] \cong R_{22}.$$

We got $X_{30} \cong X_{10}, X_{31} \cong X_{23}, X_{32} \cong X_{20}, X_{33} \cong X_{21}, X_{34} \cong X_{22}$, and they are glued together in an isomorphic way, where the gluing maps can be easily reconstructed from the inclusions of the R_{ij}s into $\mathcal{O}_{X_0}$.

Let us compute now the Segre-embedding $i(X_2')$ of $X_2' \subset \mathbf{A}_k^2 \times \mathbf{P}_k^1 \times \mathbf{P}_k^3$, obtained by blowing up X_1 along $\mathcal{I}_2' \subset R_1$, into $\mathbf{A}_k^3 \times \mathbf{P}_k^7$. The bigraded k-algebra of X_2' is $k[x_1, x_2, x_3, y_0, y_1, z_0, z_1, z_2, z_3]/\langle J \rangle$, where $J = \{z_1 z_0 - z_2 z_3, z_1 x_3 - x_2 z_2, x_1 z_1 - x_3 z_3, x_3 z_0 - x_1 z_2, z_0 x_2 - x_3 z_3, y_1 x_3 z_3 - y_0 z_2, x_2 y_1 z_3 - y_0 z_1, x_1 y_1 z_3 - y_0 z_0, x_2 x_1 z_2 - x_3^2 z_3, x_1 x_2 y_1 - x_3 y_0\}$.

The equations of $i(X_2')$ in $k[x_1, x_2, x_3, v_{00}, \ldots, v_{13}]$ can be determined by Gröbner basis computation:

$$\texttt{gbasis}(\{v_{ij} - y_i z_j \,|\, i = 0, 1, j = 0, 1, 2, 3\} \cup J, T),$$

where T is an elimination term order for the variables y_i, z_j, and the desired equations are the elements of the Gröbner basis which are free of the y_is and z_js. In this example they are:

$$\begin{aligned}
&\{x_3 v_{13} - v_{02}, x_2 v_{13} - v_{01}, x_1 v_{13} - v_{00}, v_{12} v_{03} - v_{02} v_{13}, v_{10} v_{11} - v_{12} v_{13}, v_{11} \\
&\ v_{03} - v_{13} v_{01}, v_{02} v_{11} - v_{01} v_{12}, v_{11} v_{00} - v_{02} v_{13}, v_{11} x_3 - x_2 v_{12}, v_{11} x_1 - v_{02}, \\
&\ v_{10} v_{03} - v_{13} v_{00}, v_{02} v_{10} - v_{12} v_{00}, v_{10} v_{01} - v_{02} v_{13}, v_{10} x_3 - x_1 v_{12}, v_{10} x_2 - v_{02}, \\
&\ v_{01} v_{00} - v_{02} v_{03}, x_3 v_{01} - x_2 v_{02}, x_1 v_{01} - x_3 v_{03}, x_3 v_{00} - x_1 v_{02}, x_2 v_{00} - x_3 v_{03}, \\
&\ x_2 v_{12} x_1 - x_3 v_{02}, x_1 x_2 v_{02} - x_3^2 v_{03}\}.
\end{aligned}$$

Next we apply the method of section 5 to find a blowing up which results a variety isomorphic to $i(X_2')$ (and hence to X_2' and X_2 too). We set up the homogeneous linear equation system from the linear equations of $i(X_2')$; the matrix is:

$$\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & x_3 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & x_2 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \\
0 & 0 & 0 & 0 & 0 & x_3 & -x_2 & 0 \\
0 & 0 & -1 & 0 & 0 & x_1 & 0 & 0 \\
0 & 0 & 0 & 0 & x_3 & 0 & -x_1 & 0 \\
0 & 0 & -1 & 0 & x_2 & 0 & 0 & 0 \\
0 & x_3 & -x_2 & 0 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & -x_3 & 0 & 0 & 0 & 0 \\
x_3 & 0 & -x_1 & 0 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & -x_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -x_3 & 0 & 0 & 0 & x_1 x_2 & 0 \\
0 & 0 & x_1 x_2 & -x_3^2 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

The syzygy solver returns a single generator for the solution submodule: $(-x_3 x_1^2 x_2, -x_3 x_1 x_2^2, -x_3^2 x_1 x_2, -x_1^2 x_2^2, -x_3^2 x_1, -x_3^2 x_2, -x_3^3, -x_3 x_1 x_2)$ which simplifies to the ideal generator set $\{x_1^2 x_2^2, x_1 x_2 x_3, x_1 x_2^2, x_1 x_3^2, x_3^3\}$ (generating I).

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