Simplification of Surface Parametrizations

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ABSTRACT

Given a rational parametrization of an algebraic surface, we try to reduce the degree by a suitable reparametrization. We give an algorithm that produces a parametrization with a degree that is at most twice the minimal degree. The problem is closely related to the simplification of linear systems of plane curves by Cremona transformations.

1. INTRODUCTION

A rational surface is a surface that has a parametric representation by rational functions in two parameters. The parametrization is not unique. In this paper, we assume that we have given a parametrization and ask the question whether we can find a simpler parametrization for the same surface. By "simple", we mean that the degree of the polynomials in the numerator or denominator of the rational functions are small. There are several motivations for this goal: first, parametrizations of smaller degree can be represented by less data. Second, implicitization is easier when the degree is smaller. Third, a small parametrization makes it easier to find rational curves of small degree on the given surface.

Our main result is an algorithm that produces a reparametrization which is at most twice as large as the smallest possible reparametrization. The computational cost is a polynomial number of field operations and solutions of univariate equations of polynomial degree, where the measure for the input is the degree of the given parametrization. We believe that the computation of the smallest possible parametrization would be too expensive for most applications, e.g. in computer aided geometric design.

Unlike in other simplification algorithms (e.g. [14] or [6] considering improper curve parametrizations, or [5], considering the equivalent Lüroth problem), we do not attempt to turn an improper parametrization into a proper one. We assume that the given parametrization is already proper (and we also produce a proper reparametrization). We remark that our problem does not make sense in the context of rational curves, because all proper parametrizations of a rational curve have the same degree (see [15]).

We also do not attempt to simplify the coefficient field of the parametrization, as the authors in [1, 17, 2] do for the curve case.

Our problem is similar to the reduction of linear systems of plane curves by Cremona transformations. This problem has been considered by the many authors, see [3, 4, 8]. The main difference to our problem is that there, one attempts to do a reduction by quadratic Cremona transformation, and this gives in turn a proof of the classical result that the Cremona transformations are generated by the quadratic ones (see [16]). Unfortunately, the classical methods work only for linear systems with genus less than or equal to 4 (this corresponds to the simplification of parametrizations of surfaces of sectional genus less than or equal to 4.) Also, the reduction algorithms are quite complicated, and we think that the computational costs would be large.

For arbitrary genus, the first result bounding the degree of the reduced form was given in [13], formulated in the terminology of parametrizations. More precisely, theorem 4 in that paper gives an upper bound for the smallest possible parametrization in terms of the sectional genus. The same paper contains also techniques for finding nontrivial lower bounds for the degree of a parametrization (nontrivial means not the bound that follows immediately by Bezout's theorem), which will be essential for proving the main statement in this paper. However, the proofs in [13] are not constructive.

The main idea for the simplification algorithm in this paper is to simulate the parametrization algorithm [12]. Since we already have a parametrization available, we do not need to resolve the singularity of the surface, and this is the most expensive subtask in [12]. Essentially, the resolution of the singularities can be replaced by the analysis of the base points of the parametrization.

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2. PROBLEM DESCRIPTION

Throughout the paper, we assume that k is an algebraically closed field.

A parametrization of a projective algebraic surface is a map

$$p : \mathbb{P}^2 \to S \subset \mathbb{P}^n, (s : t : u) \mapsto (F_0 : \cdots : F_n),$$

where $F_0, \ldots, F_n$ are homogeneous polynomials in $k[s,t,u]$ of the same degree d. We may assume that $F_0, \ldots, F_n$ do not have a common divisor (otherwise we can cancel it without changing the ratio). The integer d is called the *degree* of the parametrization.

Let $t : \mathbb{P}^2 \to \mathbb{P}^2$ be a birational automorphism of the projective plane (also called *Cremona transformation*). Assume that the inverse of t is given by three polynomials G_0, G_1, G_2, homogeneous of the same degree, in the three variables s_1, t_1, u_1. The parameter change t gives rise to a new parametrization $p_1 := p \circ t^{-1} : \mathbb{P}^2 \to S$. It is represented by the $n+1$-tuple of polynomials arising from $(F_0(G_0, G_1, G_2), \ldots, F_n(G_0, G_1, G_2))$ after cancellation of common factors.

In the case of algebraic curves, the birational automorphisms of the parameter space $\mathbb{P}^1$ are precisely the Möbius transformations. They preserve the degree. In the surface case, the degree is not preserved by Cremona transformations. For any parametrization, we can find Cremona transformations such that the degree of the transformed parametrization is arbitrarily large. Our goal is to use Cremona transformations in order to reduce the degree of a given parametrization. See the discussion in the introduction for the motivation for this goal.

Theoretically, it is possible to find a parametrization of minimal degree by solving a big system of algebraic equations in the indeterminate coefficients of the parametrizing polynomials. This approach is, of course, in practice not feasible even for very small examples. Clearly, we look for a more efficient algorithm.

The algorithm that will be given here is fast: it is a polynomial algorithm, and it performs well on test examples of moderate degree. We could not prove that the degree of the computed parametrization is minimal. (In the examples considered, this seems to be the case.) We will prove a weaker result: the degree of the computed parametrization is at most twice as big as the minimal degree.

3. BASE POINT ANALYSIS

In this section, we explain the concept of infinitely near base points and give an algorithm for computing the forest (set of trees) of base points.

Let

$$p : \mathbb{P}^2 \to S \subset \mathbb{P}^n, (s : t : u) \mapsto (F_0 : \cdots : F_n)$$

be a surface parametrization. The set of common intersections of $F_0, \ldots, F_n$ is a finite subset of $\mathbb{P}^2$, called the *base locus* of p. The points in the base locus are called *plane base points*. If the minimum of the orders of $F_0, \ldots, F_n$ at the base point a is equal to m, then m is called the multiplicity of the base point a.

In order to get a nicer theory, infinitely near base points have been introduced. Let a be a base point of multiplicity m. Without loss of generality, assume that a has projective coordinates $(0 : 0 : 1)$. The blowing up of $\mathbb{P}^2$ at a can be described by three (overlapping) charts, namely:

1. the complement of $\{a\}$ in $\mathbb{P}^2$;
2. the affine plane with coordinates $s, \tilde{t}$ (the second corresponds to the ratio $t : s$). Any point with $s \neq 0$ is identified with the point $(s : s\tilde{t} : 1)$ in the first chart. The line $s = 0$ is also called the *exceptional line*;
3. the affine plane with coordinates $t, \tilde{s}$ (the second corresponds to the ratio $s : t$). Any point outside the exceptional line $t = 0$ is identified with the point $(\tilde{s}t : t : 1)$ in the first chart. Any point with $\tilde{s} \neq 0$ is identified with the point $(\tilde{s}t, \tilde{s}^{-1})$ in the second chart.

Since a is a base point of multiplicity m, the total transforms $\tilde{F}_i := F_i(s, st, 1)$ (or $F_i(st, t, 1)$ in the second chart) are divisible by s^m (or t^m in the second chart). The quotients $s^{-m}\tilde{F}_i$ (or $t^{-m}\tilde{F}_i$) are called reduced transforms, and are denoted by F_i'. The reduced transforms $F_0', \ldots, F_n'$ do not have a common divisor, but they may have common intersection points on the exceptional line. These are called *infinitely near base points.* The multiplicity of an infinitely near base point is defined as the minimum of the orders of the reduced transforms. This process can be iterated, so that we may have infinitely near points of the second generation, and so on.

Example 1. Let p be the parametrization

$$(s : t : u) \mapsto (s^{11} : t^3(t^2 + su)^4 : s^8t^3 : s^4u(su + t^2)^3)$$

of degree 11. The equations

$$s^{11} = t^3(t^2 + su)^4 = s^8t^3 = s^4u(su + t^2)^3 = 0$$

have only a single zero, namely $a_1 := (0 : 0 : 1)$. We have $\mathrm{ord}_{a_1}(F_0) = 11, \mathrm{ord}_{a_1}(F_1) = 7, \mathrm{ord}_{a_1}(F_2) = 11, \mathrm{ord}_{a_1}(F_3) = 7$. Therefore $m_1 = 7$.

The reduced transforms in the first chart C_1 are $s^{11}t^4, (s + t)^4, s^8t^4, s^4(s + t)^3$. The only zero on the exceptional line $t = 0$ is $a_2 := (0, 0)_{C_1}$. The smallest order is $\mathrm{ord}_{a_2}(F_2') = 4$, hence $m_2 = 4$. The reduced transforms in the second chart C_2 are $s^4, t^3(t^2s + 1)^4, s^4t^3, (t^2s + 1)^3$; there are no zeroes on the exceptional line $s = 0$, hence the second chart contributes nothing.

We blow up a_2 and get the reduced transforms $s^{11}t^{11}, (s + 1)^4, s^8t^8, s^4t^3(s + 1)^3$ in the first chart C_3. The only zero on the exceptional line is $a_3 := (-1, 0)_{C_3}$. Its multiplicity is $m_3 = \mathrm{ord}_{a_3}(F_2'') = 4$. In the second chart C_4, the reduced transforms read $s^{11}t^4, (1 + t)^4, s^8t^4, s^3(1 + t)^3$. The only zero on the exceptional line is $(0, -1)$, but we saw this point already in the first chart C_3. Hence the second chart contributes nothing new.

We move a_3 to the origin and blow it up. In the first chart C_5, we get the reduced transforms $(st - 1)^{11}t^7, s^4, (st - 1)^8t^4, s^3t^2(st - 1)^3$. The only zero on the exceptional line is $a_4 := (0, 0)_{C_5}$. Its multiplicity is $m_4 = \mathrm{ord}_{a_4}(F_2''') = 4$. The second chart contributes nothing.

The reduced transforms on the blowup of a_4 do not intersect. Therefore, we have the following base point forest:

$$(a_1, 7) \to (a_2, 4) \to (a_3, 4) \to (a_4, 4).$$

In our complexity model, we assume that the costs of field operations are constant, and the cost of factoring a univariate polynomial is polynomial in the degree. In order to compute the base points with polynomial cost, we need a new definition and a lemma.

By construction, the base points come in a forest (set of trees). The roots correspond to the plane base points. The children of a node corresponding to the base point a are the intersection points of the reduced transforms. The *multiplicity depth* of a plane base point a is defined as the largest sum of the multiplicities in a single branch of the tree of base points with root a.

LEMMA 1. *Let M be an upper bound for the multiplicity depth. Then the base points in the tree with root a and their multiplicities can be computed from the M-th order Taylor expansion of the polynomials $F_0, \dots, F_n$ around a.*

PROOF. We proceed by induction over the depth r of the forest of base points.

If $r = 1$, then we have a plane base point. Thus, we only need to compute the order of the F_i at a, and we know that this order is at most M. Obviously, this can be done using only the M-th order Taylor expansions.

Assume $r > 1$. The multiplicity at a can be computed as in the case $r = 1$; call it m. For each child base point a_1, the N-th order Taylor expansion of the reduced transforms $F_i{}'$ around a_1 depend only on the $N+m$-th order Taylor expansion of the F_i. Thus, we may compute the Taylor expansions of the $F_i{}'$ around each child base up to order $M - m$. The parametrization given by the reduced transforms has multiplicity depth at most $M - m$, thus we can compute the base point trees with the children nodes as roots by induction hypothesis. □

Algorithm 1 BasePointAnalysis

```
P := {(p, d²+1, {F_0, ..., F_n}) | F_0(p) = ··· = F_n(p) = 0};
while P ≠ ∅ do
  translate p to the origin;
  throw away all terms of the F_i of order greater than M;
  remove a (p, M, {G_0, ..., G_n}) from P;
  m := min_{i=1}^n ord_p(G_i);
  make new node (p, m) to the output forest;
  blowup p, and compute the reduced transforms;
  compute common zeroes at exceptional line
  for all q in the zero set do
    add (q, M − m, set of reduced transforms) to P;
  end for
end while
```

The algorithm 1 computes the base points and their multiplicities for a given parametrization $F_0, \dots, F_r$. The correctness follows from Bezout's theorem, which implies that the multiplicity depth is bound by $d^2 + 1$. The computational cost is polynomial in the degree d of the parametrization.

We need to do the base point analysis in order to simplify the parametrization. But it is also useful to compute other interesting things:

- the degree of the surface S can be computed by the Bezout formula

$$\deg(S) = d^2 - \sum_{i=1}^{r} m_i^2,$$

  where the sum ranges over all base point indices;
- the genus of the generic plane section can be computed by the genus formula

$$p_1(S) = \frac{(d-1)(d-2)}{2} - \sum_{i=1}^{r} \frac{m_i(m_i - 1)}{2}.$$

For instance, the surface in example 1 has degree 24, and the genus of a generic section is 6.

See [4] for proofs and further discussions.

4. THE ADJOINT VECTORSPACES

The main idea for the simplification algorithm is to simulate the parametrization algorithm [12] on S. There, the implicit equation is used as an input. However, this implicit equation is mainly used to compute the *adjoint vectorspaces* of S, and this can be done also – and, in fact, much faster – if we have not the implicit equation but a proper parametrization.

Let p be a proper surface parametrization of degree d. Let $a_1, \dots, a_r$ be the base points and $m_1, \dots, m_r$ the corresponding multiplicities. Let n, m be non-negative integers. Then we define the *adjoint vectorspace* $V_{n,m}(p)$ as the k-space of all homogeneous polynomials of degree $nd - 3m$ having order at least $nm_i - m$ at each base point a_i (if $nm_i - m \le 0$, no condition is imposed for this base point). Here, the reduced transforms need to be computed by dividing out the $(nm_i - m)$-th power of the exceptional divisor.

The *adjoint numbers* are defined as

$$v_{n,m}(p) := \dim(V_{n,m}(p)).$$

If $v_{n,m} > 0$, then we define the *adjoint map* as the rational map

$$f_{n,m}(p) : \mathbb{P}^2 \to \mathbb{P}^{v_{n,m}-1}, (s:t:u) \mapsto (G_0 : \dots : G_{v_{n,m}-1}),$$

where $\{G_0, \dots, G_{v_{n,m}-1}\}$ is a basis of $V_{n,m}$.

Example 2. We compute $V_{1,1}(p)$, for p is in example 1. This is the vectorspace of all polynomials of degree 8, vanishing with order 6 at a_1 and with order 3 at a_2, a_3, a_4. A general polynomial with order at least 6 at a_0 is $H = a_0 s^6 u^2 + \dots + a_6 t^6 u^2 + a_7 s^7 u + \dots + a_{14} t^7 u + a_{15} s^8 + \dots + a_{23} t^8$.

The reduced transform in C_1 is equal to $H' = a_0 s^6 + \dots + a_6 + a_7 s^7 t + \dots + a_{14} t + a_{15} s^8 t^2 + \dots + a_{23} t^2$. The order at a_2 must be at least 3, hence $a_4 = a_5 = a_6 = a_{13} = a_{14} = a_{23} = 0$. The reduced transform in C_3 is equal to $H'' = a_0 s^6 t^3 + \cdots + a_3 s^3 + a_7 s^7 t^5 + \cdots + a_{12} s^2 + a_{15} s^8 t^7 + \cdots + a_{22} s$. The order at a_3 must be at least 3, hence $a_3 = a_{12} = a_{22} = 2a_2 - a_{11} = a_2 - a_{21} = a_{10} - a_{20} - a_1 = 0$. The reduced transform in C_5 is too long to be displayed; the condition that its order at a_4 is at least 3 leads to the equations $a_2 = a_{17} - a_7 = a_{10} - 2a_1 = a_{18} - a_8 = a_{19} - a_0 = a_9 - 2a_0 = 0$.

We solve this system of linear equations and plug the solution into H, obtaining $H = a_0 s^4 (su + t^2)^2 + a_8 s^5 t(su + t^2) + a_1 s^3 t(su + t^2)^2 + a_7 s^6 (su + t^2) + a_{15} s^8 + a_{16} s^7 t$. Therefore, $V_{1,1}$ is the vectorspace generated by $s^4(su+t^2)^2$, $s^5 t(su+t^2)$, $s^3 t(su+t^2)^2$, $s^6(su+t^2)$, s^8, $s^7 t$. There is a common divisor s^3, which can be divided out if we are only interested in the map $f_{1,1}$.

Algorithm 2 AdjointVectorspace

```
H := a general polynomial of degree nd − 3m;
while base point forest is not empty do
   remove a root (p, l) from the base point forest;
   translate p to the origin;
   force all coefficients of H of order less than nl − m to be
   zero {this imposes linear conditions on the coefficients};
   compute the reduced transform of H;
end while
```

Using algorithm 2, we can compute a basis for the the adjoint vectorspace $V_{n,m}$. Since we only need to expand H up to order m times the multiplicity depth, the computational cost is polynomial in d, m, n.

Obviously, the adjoint number $v_{1,m}$ is zero for $m > 3d$. Also, $v_{1,0} \geq 3 > 0$ because $V(p)$ is contained in $V_{1,0}(p)$. The smallest number μ such that $v_{1,\mu+1} = 0$ is called the *adjoint depth* of S.

4.1 Adjoints and Divisors

In this subsection, we develop a "proving environment", which will be used in most of the proofs. The setup is similar as in [12] and [13]. The main purpose is to relate the adjoint vectorspaces defined above to the adjoint vectorspaces in [12].

Let S be a rational surface, and let p be a proper parametrization of degree d. Let $a_1, \ldots, a_r$ be the base points, and let $m_1, \ldots, m_r$ be the corresponding multiplicities. Let $\pi : \mathbb{Y} \to \mathbb{P}^2$ be the blowing up at all base points. Then $p \circ \pi : \mathbb{Y} \to S$ is a desingularization of S. The pullback of a plane section is $H := dL - \sum_{i=1}^r m_i A_i$, where L is the pullback of a line in $\mathbb{P}^2$ and A_i is the exceptional divisor corresponding to a_i. Because the canonical divisor is equal to $K = -3L + A_1 + \cdots + A_r$, the polynomials in the vectorspace $V_{n,m}$ pull back to equations in the linear system $|nH + mK|$ on $\mathbb{Y}$. In other words, $f_{n,m}(p) \circ \pi : \mathbb{Y} \to \mathbb{P}^{v_{n,m}-1}$ is the map associated by the linear system $|nH + mK|$. By theorem A.1 in [12], it follows that $v_{n,m}$ is equal to the adjoint number defined in [12]; especially, it is independent of the parametrization. The adjoint map $f_{n,m}$ is equal to $g_{n,m} \circ p$, where $g_{n,m} : S \to \mathbb{P}^{v_{n,m}-1}$ is the adjoint map defined in [12].

If we blow down all exceptional curves orthogonal to H, then we get a surface X_0. Let $D_0 \in \mathrm{Div}(X_0)$ be the direct image of H. Then (X_0, D_0) is a minimal pair in the sense of [12], i.e. X_0 has no exceptional curve that is orthogonal to D_0.

For $i > 1$, let X_i be the blowing down of all exceptional curves orthogonal to $D_{i-1} + K(X_{i-1})$ on X_{i-1}. Let D_i the direct image of $D_{i-1} + K(X_{i-1})$. Then (X_i, D_i) is again a minimal pair. By theorem A.4 in [12], the adjoint map $f_{1,i}$ is equal to the composition of the map associated to D_i with the minimalization maps and the parametrization.

This allows us to figure out the fixed components of $V_{1,i}$ in many cases. Any such fixed component is also a fixed component of $|H + iK|$. If it is known that $|D_i|$ has no fixed components, then all the fixed components of $|H + iK|$ are exceptional divisors that are blown down in one of the minimalization maps $\mathbb{Y} \to X_0$, $X_0 \to X_1$, $\ldots$, $X_{i-2} \to X_{i-1}$. The multiplicity of the fixed component E in $|H + iK|$ is at most i (see the proof of theorem A.4 in [12]).

5. SOME SPECIAL CASES

The algorithms described in this section produce reparametrizations of special surfaces: surfaces with rational sections, conical surfaces, and Del Pezzo surfaces. This special cases are important because the general algorithm in the next section uses these cases as subalgorithms. This explains why we do not compute directly the simpler parametrization p', but the Cremona transformation t such that $p' = p \circ t^{-1}$.

5.1 Surfaces with Rational Sections

We say that a rational surface has *rational sections* iff $v_{1,1} = 0$. One can show that this is equivalent to the vanishing of the sectional genus (see [9]). More general, it holds that $v_{1,1}$ is equal to the sectional genus. However, we will not use these facts here.

The following classification theorem is useful for simplifying the parametrization of a surface with rational sections.

THEOREM 1. *Let $S \subset \mathbb{P}^n$ be a rational surface with rational sections. Then one of the following is true.*

1. *$v_{2,1} = 0$, and $S = \mathbb{P}^2$.*
2. *$v_{2,1} = 1$, and S is a quadric in $\mathbb{P}^3$.*
3. *$v_{2,1} \geq 2$, $v_{2,2} = 0$, S is a ruled surface, the image of $f_{2,1}$ is a rational normal curve, and the lines of the ruling are the fibers of $f_{2,1}$.*
4. *$v_{2,1} = 3$, $v_{2,2} = 0$, and $f_{2,1}$ is a Cremona transformation.*

PROOF. See [12] (applied for the case $\mu = 0$). □

In subcase (1), the smallest parametrization is the identity $(s : t : u)$, which has degree 1.

In subcase (2), the smallest parametrization is the inverse of a stereographic projection, and it has degree 2.

In subcase (4), the smallest parametrization is obtained by applying the Cremona transformation $f_{2,1}$. Let p' be the transformed parametrization. Then the map $f_{2,1}(p')$ is the identity. No curve gets contracted, therefore $V_{2,1}(p')$ has no common factors. Therefore $V_{2,0}(p')$ is equal to the space of all homogeneous polynomials of degree 4. Moreover, $V_{2,0}(p')$ has no double base points. It follows that p' has degree 2 and no base points.

The subcase (3) is more complicated. In analogy with the parametrization algorithm [12], subalgorithm *FiberIsLine*, we can produce a parametrization of degree m with a base point of multiplicity $m-1$ (and maybe other base points), for some $m \geq 2$. Note that this is the largest possible multiplicity. We need a little lemma.

LEMMA 2. *Let V be a vectorspace of polynomials of degree d, defining a rational map $f : \mathbb{P}^2 \to C \subset \mathbb{P}^n$, such that the image is a rational normal curve. Assume that the polynomials in V do not have a common factor. Let $a_1, \ldots, a_r$ be the base points of f, and let $m_1, \ldots, m_r$ be the corresponding multiplicities.*

Then $d = nd'$, and $m_i = nm_i'$ for all i, where d', m_1', $\ldots$, m_r' are suitable integers; the vectorspace V' of polynomials of degree d' with order at least m_i' at each base point a_i has dimension 2 and defines a map $f' : \mathbb{P}^2 \to \mathbb{P}^1$, and f is the composition of f with the n-uple embedding.

PROOF. Let $s : C \to \mathbb{P}^1$ be an isomorphism of C to the projective line. Let V'' be the defining vectorspace of the map $s \circ f : \mathbb{P}^2 \to \mathbb{P}^1$. The composition g with the n-uple embedding of $\mathbb{P}^1$ is defined by V''^n. Because f and g have the same image and the set of fibers is the same, they differ only by a projective automorphism of the image. It follows that the defining vectorspaces are equal, i.e. $V''^n = V$. Therefore $V'' = V'$. □

Let $W := V_{2,1}$ (maybe after removing common factors), and let W' be vectorspace constructed as in lemma 2. Let $\{H_0, H_1\}$ be a basis of W'. The fibers of $f_{2,1}$ are the plane curves with equation $\lambda_1 H_0 - \lambda_0 H_1$, $(\lambda_0 : \lambda_1) \in \mathbb{P}^1$. We intersect the generic fiber with the preimage of the hyperplane $\alpha F_0(s,t,u) - F_1(s,t,u)$ in $\mathbb{P}^n$, where $(F_0 : \cdots : F_n)$ is the parametrization and α is a generic constant. Because the fibers are mapped to lines, there is precisely one intersection point outside the base locus of W'. (There could be a degenerate case, in which this point lies in the base locus of $\{F_0, F_1\}$; this can be avoided by replacing F_1 by F_i, $i > 1$ – see [12] for a more details.) In general, there is a unique common intersection point is outside the base locus of both vectorspaces. This implies that the rational map

$$t_0 : \mathbb{P}^2 \to \mathbb{A}^2, (s:t:u) \mapsto \left(\frac{F_1(s,t,u)}{F_0(s,t,u)}, \frac{H_1(s,t,u)}{H_0(s,t,u)} \right)$$

is birational (see [10]). The homogenization of t_0 is the Cremona transformation $(F_1 H_0 : F_0 H_1 : F_0 H_0)$. We apply it to p; let p' be the transform. The map $f_{2,1}(p')$ is equal to the map $(t' : u')$ composed with the N-uple embedding (where $N = v_{2,1} - 1$). By theorem 7 in [13], the linear system $|2D_i + K|$ has no fixed components. So, any common divisor of $V_{2,1}(p')$ must be a line through $a := (1:0:0)$. The degree of $V_{2,0}(p')$ is equal to the multiplicity of $V_{2,0}(p')$ at a plus 2. It follows that the degree of $V_{1,0}(p')$ is equal to the multiplicity of $V_{1,0}(p')$ at a plus 1, as desired.

Example 3. The map $f_{1,1}$ from example 2 is a parametrization of a surface with rational sections in $\mathbb{P}^5$. After reducing the common divisor, its degree is 5. The base point forest is

$$(a_1, 3) \to (a_2, 2) \to (a_3, 2) \to (a_4, 2).$$

The vectorspace $V_{2,1}$ is the vectorspace of all polynomials of degree 7 with order 5 at a_1 and order 3 at a_2, a_3, a_4. Using algorithm AdjointVectorspace, we can compute the basis $\{s^3(su+t^2)^2, s^5(su+t^2), s^7\}$. Thus, $v_{1,1} = 3$, and we have either subcase (3) or subcase (4).

We cancel the common factor s^3 and get degree 4 and base point forest

$$(a_1, 2) \to (a_2, 2) \to (a_3, 2) \to (a_4, 2).$$

The self-intersection number is zero, therefore the image is a quadric curve in $\mathbb{P}^2$, and we have subcase (3). The vectorspace V' is the space of all polynomials of degree 2 passing simply through the base points. A basis is $\{H_0, H_1\} = \{s^2, su+t^2\}$.

Finally, we apply the Cremona transformation $(F_1 H_0 {:} F_0 H_1 : F_0 H_0) = (s^4 t(su+t^2) : s(su+t^2)^3 : s^3(su+t^2)^2) = (s^3 t : (su+t^2)^2 : s^2(su+t^2))$. The result is the parametrization

$$(s' : t' : u') \mapsto (t'^2 u'^2 : s' t'^3 : t' u'^3 : s' t'^2 u' : u'^4 : s' t' u'^2)$$

of degree 4. It has a base point $(1:0:0)$ of maximal multiplicity 3.

The parametrization constructed above can still have quite large degree. We need to simplify it further.

First, we observe that a parametrization with a base point a_1 of maximal multiplicity $d-1$ has no other base points of multiplicity greater than 1. Otherwise there would be a line intersecting a generic curve in the defining vectorspace in more than d points, contradicting Bezout's theorem.

Second, we observe that the number of base points on the exceptional line of the blowing up at a_1 is at most $d-1$. Otherwise, all polynomials in the defining vectorspace would have order larger than $d-1$ at the point a_1, which contradicts the definition of multiplicity.

Here is a lemma that allows to control the degree of a parametrization with a base point of maximal multiplicity.

LEMMA 3. *Let p be a parametrization of degree d, with a $d-1$-fold base point a_1 and simple base points $a_2, \ldots, a_r$. Let d' be the smallest number such that the vectorspace V' of all polynomials of degree d' with order at least $d'-1$ at a_1 and order at least one at $a_2, \ldots, a_r$ defines a parametrization. Let $b_1, \ldots, b_{2d'-r-1}$ be generic points in $\mathbb{P}^2$. Let W be the vectorspace of all polynomials in V vanishing at $b_1, \ldots, b_{2d-r-1}$.*

Then W defines a Cremona transformation. The transformed parametrization has degree $d+d'-r$, a base point b_1 of multiplicity $d+d'-r-1$, and $2d'-r$ simple base points. Either $2d'-r=1$ or all simple base points lie on the exceptional line of the blowup at b_1.

PROOF. The degree of the transformed parametrization is equal to the intersection product of the divisors defining p and W, and it is therefore $dd'-(d-1)(d'-1)-(r-1) = d+d'-r$. The Cremona transformation transforms the lines through a_1 to a pencil of lines, and the common point of these lines is b_1. Because these lines intersect the transformed linear system in a single point, b_1 must have maximal multiplicity $d+d'-r-1$. The number of simple points can be inferred from the fact that Cremona transformations do not change the self-intersection number.

The surface X_0 (see subsection 4.1) is a ruled surface S_e for some $e \geq 0$. If $e \geq 1$, the vectorspace W corresponds to the map contracting the unique cross section with self-intersection $-e$. All other irreducible curves have nonnegative self-intersection. Therefore, all divisors on $\mathbb{Y}$ with negative self-intersection are going to be contracted by the transformed parametrization, and this is only possible if all simple base points lie on the exceptional line. If $e=0$, then the second ruling is transformed to another pencil of lines intersecting the transform of V in $d+d'-r-1$ points, so we get a second plane base point. $\square$

Remark 1. The generic points can be chosen randomly, but by bad luck we could pick points that leads to a map which is not birational. In this case, the image of the map is a rational normal curve. To be on the safe side, we can recompute the base points and see if the system W has self-intersection zero. If this is not the case, then W defines a Cremona transformation, and the above lemma holds.

Example 4. Let p be the parametrization

$$(s:t:u) \mapsto (t^2u^2 : st^3 : tu^3 : st^2u : u^4 : stu^2)$$

from example 3. The base point forest looks like

$$(a_1,3) \to (a_2,1) \qquad (a_3,1) \to (a_4,1).$$

There is no line passing through the three simple base points. The conics through all four base points do not define a map to a curve. The cubics with double point at a_1 and simple points at a_2, a_3, a_4 define a parametrization, so $d'=3$. We assign another simple base point b_1, infinitely near to a_4 on the line through a_3, a_4. This defines the Cremona transformation

$$(s:t:u) \mapsto (st^2 : stu : u^3).$$

We apply it and get the parametrization

$$(s':t':u') \mapsto ({s'}^2u' : {s'}^2t' : s't'u' : s'{t'}^2 : {t'}^2u' : {t'}^3)$$

of degree 3, with a double base point and a simple base point.

Algorithm 3 ReparametrizeRationalSections

compute the base point forest with BasePointAnalysis;
compute $W := V_{2,1}$ and $v_{2,1}$ with AdjointVectorspace;
if $v_{2,1}=0$ **then**
  return the input p;
else if $v_{2,1}=1$ **then**
  π:= a stereographic projection;
  return $\pi \circ p$;
else if $v_{2,1}=3$ and $f_{2,1}$ is birational **then**
  return $f_{2,1}$;
else
  compute the base point forest of W;
  divide each multiplicity and the degree by $v_{2,1}-1$;
  H_0, H_1:= a basis for the result vectorspace;
  $t_0 :=$ the map $(s:t:u) \mapsto (F_1H_0 : F_0H_1 : F_0H_0)$;
  $p := p \circ t_0^{-1}$;
  compute the new base point forest {there is a point a_1 of maximal multiplicity and some simple base points};
  $d:=1$; $q := id_{\mathbb{P}^2}$;
  while Image(q) is two-dimensional **do**
    $q_{old} := q$; $d := d+1$;
    W:= the vectorspace of polynomials of degree d with order $d-1$ at a_1 passing through the simple base points;
    q:= the map defined by W;
  end while
  compose q_{old} with projections from generic points, until the image is $\mathbb{P}^2$;
  return $q_{old} \circ t_0$;
end if

Algorithm 3 takes a parametrization of a surface with rational sections, and computes a reparametrizing Cremona transformation. The computational cost of algorithm 3 is polynomial in the degree of the given parametrization.

5.2 Conical Surfaces

We say that a rational surface is *conical* iff the image of $f_{1,1}(p)$ is a rational normal curve. It follows from [12], lemma 5.4, that p maps the fibers of $f_{1,1}$ to conics.

In analogy with the parametrization algorithm [12], subalgorithm *FiberIsConic*, we can produce a parametrization of degree m with a base point a_1 of multiplicity $m-2$, for some $m \geq 4$. There may be other base points, but their multiplicities is at most 2. If there is more than one double base point, then all double base points lie on the exceptional divisor of the blowup at a_1.

The method is similar to the one used in subcase (3) for surfaces with rational sections. Because of space restriction, we cannot present the algorithm completely in this paper. The main ideas have already appeared in [12] or in the previous section. In brief, the method works as follows:

1. factor $f_{1,1}$ into a map $p: \mathbb{P}^2 \to \mathbb{P}^1$ followed by an N-uple embedding;
2. let $q: \mathbb{P}^2 \to \mathbb{P}^2$ be defined by three suitable components of p, such that the map $(p,q): \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^2$ is birational;

3. the image of (p,q) is a of degree 2 in the second block of coordinates. Compute its equation;
4. use the algorithm FindPoint in [12] to construct a solution of this equation, depending on the first block of coordinates;
5. parametrize the generic conic section of the image by projecting from this solution (see [12] or [11]);
6. together with p, the above parametrization gives rise to a Cremona transformation, similar to subcase (3) in the previous subsection. Apply it. The result is a degree d parametrization with a base point of multiplicity $d-2$.
7. reduce the double base points with a method analoguous to the one described in lemma 3.

Full details will appear in a forthcoming paper.

5.3 Del Pezzo Surfaces

We say that a rational surface is *Del Pezzo* iff $v_{1,1}=1$. It is possible to adapt the parametrization algorithm in [9] for the Del Pezzo case to the situation where we have no implicit equation but a parametric representation given. The main idea there is to project from a tangent plane, and the tangent plane can also be computed when the surface is given parametrically. The details will appear in the forthcoming paper announced above.

Alternatively, one can also achieve polynomial complexity by the idea to implicitize and compute a parametrization from the implicit equations. The reason is that the degree of Del Pezzo surfaces is bounded (at most 9), and the degree of a minimal parametrization is also bounded (at most 4). For a proof of these bounds, see [4, 7].

6. THE GENERAL CASE

In this section, we present the main algorithm for simplifying arbitrary proper parametrizations, and state the main result: the computed parametrization is at most twice as big as the smallest possible one.

Recall that μ is the smallest number such that $v_{1,\mu+1}=0$. the *adjoint depth* of S. According to [12], lemmas 5.2-5.7, we have one of the following cases.

1. The map $f_{1,\mu}$ is a parametrization of a surface with rational sections.
2. The image of the map $f_{1,\mu}$ is a rational normal curve, and the map $f_{2,2\mu-1}$ is a parametrization of a conical surface.
3. We have $v_{1,\mu}=1$, and the map $f_{1,\mu}$ degenerates to the constant function to $\mathbb{P}^0$ (the one-point space). The map $f_{1,\mu-1}$ (or $f_{2,2\mu-2}$ in case $v_{1,\mu-1}=3$, or $f_{1,3\mu-3}$ in case $v_{1,\mu-1}=2$) is a parametrization of a Del Pezzo surface.

In order to tell case (2) from case (3), it is not needed to compute the image of $f_{1,\mu}$. According to lemmas 5.2-5.5 in [12], we have case (1) iff $v_{2,2\mu+1}>0$ or ($v_{1,\mu}=3$ and $v_{2,2\mu}=6$).

Algorithm 4 SimplifyParametrization

compute the base point forest with BasePointAnalysis;
$\mu:=0$; $W:=V_{0,0}$;
repeat
  $V:=W$;
  $\mu:=\mu+1$;
  $W:=V_{1,\mu}$;
until $W=\{0\}$
$\mu:=\mu-1$;
if $v_{1,\mu}=1$ **then**
  if $v_{1,\mu-1}\geq 4$ **then**
    $V:=V_{1,\mu-1}$;
  else if $v_{1,\mu-1}=3$ **then**
    $V:=V_{2,2\mu-2}$;
  else if $v_{1,\mu-1}=2$ **then**
    $V:=V_{3,3\mu-3}$;
  end if
  compute t with ReparametrizeDelPezzo applied to V;
else if $v_{2,2\mu+1}=0$ and ($v_{1,\mu}\neq 3$ or $v_{2,2\mu}\neq 6$) **then**
  compute t with ReparametrizeConical applied to $V_{2,2\mu-1}$;
else
  compute t with ReparametrizeRationalSections applied to $V_{1,\mu}$;
end if
return $p\circ t^{-1}$;

We propose algorithm 4 for simplifying a given parametrization. Its complexity is polynomial in the degree of the parametrization. The above classification shows that the algorithm computes indeed a reparametrization. Its degree is estimated in the following theorem.

THEOREM 2. *Let d be the degree of the parametrization computed by algorithm 4.*

In case (1), we have $d\leq 4\mu+v_{1,\mu}-2$.

In cases (2) and (3), we have $d\leq 4\mu+2v_{1,\mu}-2$.

PROOF. Let p' be the parametrization computed by algorithm 4.

Case (1): By lemma A.8 in [12], $|D_\mu|$ has no fixed components and no base points. We distinguish the four subcases arising in theorem 1.

Subcase (1): $f_{1,\mu}(p')$ is the identity. No curve gets contracted, so $V_{1,\mu}(p')$ has no fixed components. It follows $d=3\mu+1\leq 4\mu+1=4\mu+v_{1,\mu}-2$.

Subcase (2): $f_{1,\mu}(p')$ is a degree 2 parametrization of a quadric surface. Then $f_{1,\mu}(p')$ has two simple base points, call them b_1,b_2. There is a unique contracted curve, namely the line L through b_1,b_2. This line is the only fixed component of $V_{1,\mu}(p')$, and its exponent in the common divisor is less than or equal to μ. It follows $d\leq 4\mu+2=4\mu+v_{1,\mu}-2$.

Subcase (3): $f_{1,\mu}(p')$ is of degree d_0 with a base point b_1 of multiplicity d_0-1 and simple base points $b_2,\ldots,b_{r_0}$, where

$r_0 = 2d_0 - v_{1,\mu} + 2$; and we have either $r_0 = 2$ or $b_2, \dots, b_{r_0}$ lie on the exceptional divisor of the blowup at b_1.

The only curves that are contracted by $f_{1,\mu}(p')$ are the lines through b_1 and b_i, $i = 2, \dots, r_0$. These lines are the possible fixed components of $V_{1,\mu}(p')$. Their exponent is less than or equal to μ. It follows that $d = 3\mu + d_0 + f$, where f is the number of fixed components counted with multiplicity, and $f \le (r_0 - 1)\mu$. Moreover, p' has b_1 as a base point of multiplicity $d_0 - 1 + \mu + f$, and the sum of the multiplicities at $b_2, \dots, b_{r_0}$ is equal to $(r_0 - 1)\mu + r_0 - 1 + f$. Because the multiplicity at b_1 is greater than or equal to the sum of the multiplicities at $b_2, \dots, b_{r_0}$, we have $(r_0 - 1)\mu \le d_0 + \mu - r_0$. Hence $d \le 4\mu + 2d_0 - r_0 = 4\mu + v_{1,\mu} - 2$.

Subcase (4): $f_{1,\mu}(p')$ is the 2-uple embedding of $\mathbb{P}^2$. No curve gets contracted, so $V_{1,\mu}(p')$ has no fixed components. It follows $d = 3\mu + 2 \le 4\mu + 4 = 4\mu + v_{1,\mu} - 2$.

Case (2): $f_{1,\mu}(p')$ is the projection from the point a_1 followed by the $(v_{1,\mu} - 1)$-uple embedding of $\mathbb{P}^1$, and $f_{2,2\mu-1}(p')$ is a parametrization of degree d_0, with a_1 being a $d_0 - 2$-fold base point, double base points $a_2, \dots, a_{r_0}$, where $r_0 = d_0 - v_{1,\mu} - 1$, and maybe simple base points; if $r_0 \ge 3$, the double base points are on the exceptional divisor of the blowpup at a_1. The system $|D_\mu|$ does not have base points or fixed components. Therefore the possible fixed components of $V_{1,\mu}(p')$ are the lines through a_1 and a_i, $i = 2, \dots, r_0$, with exponent at most μ. It follows that $d = 3\mu + v_{1,\mu} - 1 + f$, where f is the number of fixed components, counted with multiplicity, $f \le (r_0 - 1)\mu$. Moreover p' has b_1 as a base point of multiplicity $\mu + v_{1,\mu} - 1 + f$, and this number must be greater then or equal to the sum of the multiplicities at $b_2, \dots, b_{r_0}$, which is $(r_0 - 1)\mu + f$. Therefore $f \le (r_0 - 1)\mu \le \mu + v_{1,\mu} - 1$ and $d \le 4\mu + 2v_{1,\mu} - 2$.

Case (3): $f_{1,\mu-1}(p')$ has degree 3 and only simple base points, or degree 4 and two double base points. In the first subcase, there is no -1-curve contracted by $f_{1,\mu-1}(p')$. Therefore $d = 3\mu$. In the second case, there is a unique such curve, namely the line through the two base points. Its exponent is μ, and therefore $d = 4\mu$. In both cases, we have $d \le 4\mu + 2v_{1,\mu} - 2$, because $v_{1,\mu} = 1$. ☐

In order to show that the degree of the computed parametrization is at most twice as large as the minimal degree, we recall the following well-known bounds [13].

THEOREM 3. *Let d_0 be the minimal degree of a proper parametrization.*

In case (1), we have $d_0 \ge 3\mu + \frac{1}{2}(v_{1,\mu} - 2)$.

In cases (2) and (3), we have $d_0 \ge 3\mu + v_{1,\mu} - 1$.

Especially, we have $d \le 2d_0$ in every case.

Example 5. Let p be the parametrization from example 1, of degree 11 with a 7-fold base point and three 4-fold base points. We have already computed $V_{1,1}$ in example 2. The vectorspace $V_{1,2}$ is the zero space, hence we have $\mu = 1$, and case (1) holds. The reparametrization of $f_{1,1}$ has been done in examples 3 and 4. The reparametrizing Cremona transformation is the composite of the Cremona transformations in example 3 and example 4, namely

$$(s : t : u) \mapsto (t(su + t^2) : s^2 t : s^3).$$

The result is the parametrization

$$(s' : t' : u') \mapsto (t'^4 u'^3 : s'^4 t' u'^2 : t'^7 : (s' u'^2 - t'^3) s'^3 u')$$

of degree 7. As $v_{1,\mu} = 6$, there is no parametrization of degree less than 5.

Remark 2. In principle, algorithm 4 can also be applied to improper parametrizations. However, the result is then not necessarily close to the smallest possible reparametrization. Since we reparametrize by composing with Cremona transformations, the computed parametrization will then again be improper, while the smallest possible parametrization is likely to be proper. What can be positively said is that the computed parametrization is at most twice as large as the smallest reparametrization by a Cremona transformation.

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