Article Submitted to Journal of Symbolic Computation

Simplification of Surface Parametrizations – a Lattice Polygon Approach

JOSEF SCHICHO

RISC, Univ. Linz, A-4040 Linz, Austria.
email: schicho@risc.uni-linz.ac.at

Abstract

Given a convex lattice polygon, we compute a descending sequence of lattice polygons obtained by repeatedly passing to the convex hull of the interior lattice points. This process gives the idea for an algorithm that simplifies a given parametric surface by reparametrization.

Introduction

A rational surface is a surface that has a parametric representation by rational functions in two parameters. The parametrization is not unique. Given a parametric surface, can we find a simpler parametrization for the same surface? By "simple", we mean that the degree of the polynomials in the numerator or denominator of the rational functions are small. There are several motivations for this question: first, parametrizations of smaller degree can be represented by less data. Second, implicitization is easier when the degree is smaller. Third, a small parametrization makes it easier to find rational curves of small degree on the given surface.

In (Schicho, 2002), we gave an algorithm that produces a reparametrization which is at most twice as large as the smallest possible reparametrization. The input is assumed to be a proper parametrization. In this paper, we give an interpretation of that algorithm in terms of toric geometry and lattice polygons. More precisely, we specialize the algorithm to the case of toric surfaces, and describe it for this case by operations on lattice polygons. There are several motivations for such an interpretation: first, the lattice geometric algorithm and its correctness proof is much easier to understand and more elementary than the algebraic algorithm in (Schicho, 2002), and so the lattices make it possible to get a better understanding for the general case. In fact, it is surprising to observe that most ideas and difficulties in the general algorithm have a lattice-geometric counterpart. Second, the lattice geometric algorithm gives some ideas how to generalize the algorithm to three-folds. Third, the lattice geometric picture of the general algorithm is quite elegant and provides another nice connection between algebraic geometry and lattice geometry.

It is not very hard to give a lattice description of adjoints (Theorem 3.1), which is actually valid in any dimension: the m-adjoints correspond to the polygon obtained by moving m units inward in each direction. This result is related to the description of the canonical class of toric varieties (see Fulton (1993)). In the surface case, however, we have another more useful description (Theorem 3.3): taking m-adjoints corresponds to passing m times to the convex hull of the interior lattice points.

The paper is written as if the toric case was the stepping stone for solving the general case. In reality, it was the other way round: the general algorithm was devised first. The toric interpretation was found later, inspired by discussions with some colleagues. Among them, let me mention Rimas Krasauskas, who used toric surfaces for constructing multi-sided patches for computer aided geometric design (Krasauskas, 2001); and Gavin Brown, who used toric three-folds as main examples for investigations on pluricanonical divisors (Brown, 1999). The idea was presented at the ISSAC in Lille, but it was too late to mention it in the paper version.

The author has been supported by the Austrian Science Fund (FWF) in the frame of SFB-project F13.

1. The Problem

A parametric surface is given by a map

$$p : \mathbb{C}^2 \to S \subset \mathbb{P}^n, (s,t) \mapsto (F_0 : \cdots : F_n),$$

where $F_0, \ldots, F_n$ are polynomials in $\mathbb{C}[s,t]$ We assume that the parametrization is proper (i.e. generically injective). The *parametric degree* is defined as the maximum of the total degrees of $F_0, \ldots, F_n$.

Let $t : \mathbb{C}^2 \to \mathbb{C}^2$ be a birational automorphism of the plane (also called *Cremona transformation*). Assume that t is given by two rational functions S, T, in the variables s', t'. The parameter change t gives rise to a new parametrization $p_1 := p \circ t : \mathbb{C}^2 \to S$. It is represented by the $n+1$-tuple of polynomials arising from $(F_0(S,T), \ldots, F_n(S,T))$ after clearing denominators and cancellation of common factors.

In the case of algebraic curves, the birational automorphisms of the parameter space $\mathbb{C}^1$ are precisely the Möbius transformations. They preserve the degree. Hence the parametric degree is the same for all proper parametrizations; in fact, it coincides with the implicit degree. In the surface case, the parametric degree is not preserved by Cremona transformations. In general, we can find a Cremona transformation that reduces the parametric degree.

Example 1.1: Consider the parametric surface

$$(x,y,z) = \left(\frac{2s+2t^5}{s^2+2st^5+t^{10}+t^2+1}, \frac{2t}{s^2+2st^5+t^{10}+t^2+1}, \frac{s^2+2st^5+t^{10}+t^2-1}{s^2+2st^5+t^{10}+t^2+1}\right)$$

of parametric degree 10. Substituting the Cremona transformation

$$(s,t) = (s' - t'^5, t')$$

yields the parametrization

$$(x,y,z) = \left(\frac{2s'}{s'^2+t'^2+1}, \frac{2t'}{s'^2+t'^2+1}, \frac{s'^2+t'^2-1}{s'^2+t'^2+1}\right)$$

of degree 2.

In (Schicho, 2002), we presented an algorithm that takes a parametric surface and computes a Cremona transformation in order to reduce the degree. We could not prove that the parametric degree of the computed reparametrization is minimal. But it was shown that the parametric degree of the reparametrization is at most twice as big as the minimal one. The algebraic complexity of the algorithm is polynomial, and it performs well on test examples of moderate degree.

2. The Theory of Adjoints

The algorithm in (Schicho, 2002) uses the concept of adjoints. In this section, we recall the relevant facts from this theory.

Let $X \subset \mathbb{P}^n$ be a projective variety. We assume that $\pi : \tilde{X} \to X$ is a resolution of the singularities of X, i.e. π is a regular birational map and $\tilde{X}$ is projective and nonsingular. It is well-known that such a resolution exists, but it is not unique in general.

Recall that for any effective class of divisors D on $\tilde{X}$, we get an associated rational map $m_D : \tilde{X} \to \mathbb{P}^r$, where $r := \dim(|D|)$. The map m_D is determined up to projective transformations of the image.

Let $H \in \mathrm{Cl}(\tilde{X})$ be the pullback of the class of hyperplane sections. Let $K \in \mathrm{Cl}(\tilde{X})$ be the canonical class. Let $b_{n,m} : \tilde{X} \to \mathbb{P}^r$ be the map associated to the class $nH + mK$, if this class is effective. The adjoint map $a_{n,m} : X \to \mathbb{P}^r$ is defined as the composition $a_{n,m} \circ \pi^{-1}$. We define the adjoint numbers $v_{n,m} := \dim(|nH + mK|) + 1$. If $nH + mK$ is not effective, or equivalently if $v_{n,m} = 0$, then $a_{n,m}$ is undefined. It can be shown that $v_{n,m}$ and $a_{n,m}$ do not depend on the choice of the resolution π.

For instance, the map $a_{1,0}$ is the map associated to the complete linear system of hyperplane sections. This map is birational. In the case that this system is already complete (e.g. for nonsingular hypersurfaces), $a_{1,0}$ is the identity.

The maps $a_{0,m}$ are of special interest, because their image does not depend on the projective embedding of X in $\mathbb{P}^n$. Therefore, they are called “canonical” (or sometimes “pluricanonical”) maps.

Unfortunately, we have $v_{0,m} = 0$ for all m if X is a rational variety, hence we do not have canonical maps in this case. For rational surfaces, one can find another adjoint map which gives a simple birational model.

For an implicitly given rational surface, the parametrization algorithm (Schicho, 1998) computes adjoint maps in order to reduce the general case to one of the base cases in the classification above. In this situation, it is quite hard to compute the adjoints, because one needs to resolve the singularities of the given surface.

In our situation, we have given a parametric surface. By projectivization, we obtain a rational map $t : \mathbb{P}^2 \to S$. By successively blowing up base points, we obtain a map $\beta : \mathbb{Y} \to \mathbb{P}^2$ and a resolution $\tilde{t} : \mathbb{Y} \to S$, $\tilde{t} = t \circ \beta$. We can use this resolution to construct the maps $b_{n,m} : \mathbb{Y} \to \mathbb{P}^r$. In fact, the rational maps $p_{n,m} : \mathbb{P}^2 \to \mathbb{P}^r$, $p_{n,m} := b_{n,m} \circ \beta^{-1}$ may be considered as a parametric version of adjoint maps; we do not need an implicit description of S in order to compute them. An explicit algorithm for computing these maps is contained in (Schicho, 2002).

3. Toric Surfaces

A toric surface is a parametric surface parametrized by monomials:

$$(s,t) \mapsto (s^{a_0}t^{b_0} : \cdots : s^{a_n}t^{b_n}),$$

with $a_0, \ldots, b_n \in \mathbb{Z}$. We assume that the parametrization is proper, or equivalently that the lattice points (a_i, b_i), $i = 0, \ldots, n$, generate the whole lattice $\mathbb{Z}^2$ by integral affine linear combinations. The convex hull of the lattice points is called the lattice polygon of the toric surface.

Geometric properties of the surface correspond to combinatoric properties of the lattice points. For instance, the degree d of the surface is equal to twice the area of the lattice polygon; and the number p_1, the genus of a generic hyperplane section, is equal to the number of lattice points in the interior.

Remark: For arbitrary rational varieties with $p_1 > 0$, we have the inequality $d \leq 4p_1 + 5$ (Schicho, 1999). Of course, it holds especially for toric surfaces with $p_1 > 0$. In terms of convex lattice polygons, this is assertion is equivalent to the statement

$$2 \cdot \text{area} \leq 4 \cdot \text{number of interior points} + 5,$$

which has been proved in (Scott, 1976) by combinatorial methods.

Let Γ be the lattice polygon corresponding to a toric surface. The parametric degree $e(\Gamma)$ is equal to the mixed volume of the lattice polygon Γ and the triangle $\Delta_{\binom{0}{0},\binom{1}{0},\binom{0}{1}}$. Equivalently, it is the smallest number e, such that lattice polygon is contained in a triangle of the form $\Delta_{\binom{a}{b},\binom{a+e}{b},\binom{a}{b+e}}$. We also call such triangles normal. For instance, the toric surfaces in Figure 1 has degree 4.

A toric Cremona transformation is one of the form

$$(s,t) = (s^{a_1}t^{b_1}, s^{a_2}t^{b_2}),$$

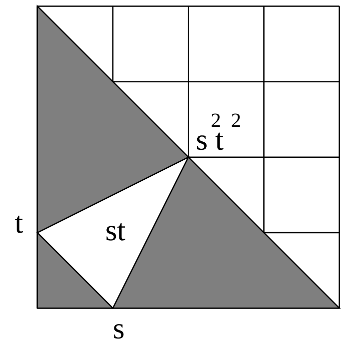

Figure 1: A toric surface with parametric degree 4.

where $a_1 b_2 - a_2 b_1 = \pm 1$. This corresponds to expressing the lattice points in another $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^2$. Translations of the lattice polygon correspond to passing to a different coordinate representation of the same map to projective space, so they are already absorbed in this setting.

The simplification problem for toric surfaces (using only toric Cremona transformations, in order to preserve the toric structure) is therefore equivalent to the following problem about lattice polygons.

Input: a convex lattice polygon Γ.

Output: a unimodular transformation τ, such that the transformed polygon $\tau(\Gamma)$ is contained in a normal triangle of smallest possible size.

A similar problem has been considered before in lattice geometry: in (Arnold, 1980; Bárány and Pach, 1992), the authors are interested to make the transformed polygon be contained in a parallelogram of controllable size, in order to give an estimate for the number of unimodular equivalence classes with certain constraints. However, their method does not give any idea what to do in the non-toric case. We like to think of another method that generalizes to arbitrary rational surfaces. To this end, we translate the concept of adjoints into the language of lattice polygons. We do this in two steps. The first step generalizes to toric varieties of arbitrary dimension, and the second step is specific to the surface case.

A nonzero vector $(u, v) \in \mathbb{Z}^2$ is called a direction iff $\gcd(u, v) = 1$. The set of all directions is denoted by R.

Let $\Gamma \subset \mathbb{R}^2$. Let r be a direction. We set

$$M(\Gamma, r) := \inf_{p \in \Gamma} \langle r, p \rangle,$$

where $\langle ., . \rangle$ is the usual scalar product. For the sake of convenience, we define the infimum of $\mathbb{R}$ to be $-\infty$ and the infimum of $\emptyset$ to $+\infty$. Of course, if Γ is compact, especially if Γ is a polygon, then the infimum is obtained; the points where the infimum is obtained are called extremal.

Theorem 3.1 (First Translation): *Let $p : \mathbb{C}^2 \to S$ be a toric surface with lattice polygon Γ. Let n, m be nonnegative integers. Then the parametric adjoint map $p_{n,m} : \mathbb{C}^2 \to \mathbb{P}^r$ – if exists – is toric, defined by all points in the convex set*

$$\Omega_{n,m}(\Gamma) := \bigcap_{r\in R}\{p \mid \langle r,p\rangle \geq nM(\Gamma,r)+m\}.$$

Proof: By definition, a function f is in the linear space defining $a_{n,m}$ if and only if it pulls back to a function in $\mathcal{L}(nH+mK)$ on some resolution. This is the case if and only if $f\cdot(\omega_0)^m$ is in the V-module $M_{n,m}$ generated by the products of n-th powers of the $s^{a_i}t^{b_i}$ and m-th powers of top degree differential forms of V, for every discrete valuation ring V. Here, ω_0 is the unique top degree differential form that is invariant under toric Cremona transformations, namely $\omega_0 := \frac{ds\wedge dt}{st}$. Because of the existence of toric resolutions, the linear space is generated by monomials $f = s^a t^b$, and we can restrict to toric valuations, i.e. valuations that are determined by the values on the monomials. These valuations are in one-to-one correspondence with the directions.

Let V be the discrete toric valuation corresponding to the direction $r = (u, v)$. The value of a monomial $f = s^a t^b$ is equal to $ua + vb$. Let p, q be integers such that $up + vq = 1$. The top degree differential forms are generated by

$$d(s^v t^{-u}) \wedge d(s^p t^q) = s^{p+v}t^{q-u}\omega_0.$$

The value of the scalar factor is $u(p+v)+v(q-u) = 1$. Hence $f\cdot(\omega_0)^m \in M_{n,m}$ if and only if its value is greater than or equal to $nM(\Gamma,r)+m$. □

Remark: From its definition, we only know the set $\Omega_{n,m}(\Gamma)$ is compact and convex. One can show that it suffices to take only a finite number of directions into a account. We do not use this fact here, so we omit the detailed proof.

Remark: By the preceding remark, $\Omega_{n,m}(\Gamma)$ is a polygon. We will see below that it is in fact a lattice polygon. For higher dimension, this need not be the case. For instance, let Γ be the tetrahedron with vertices $(0,0,0)^t$,$(3,3,-3)^t$, $(3,-3,3)^t$, $(-3,3,3)^t$. For $n = m = 1$, we have

$$\Omega_{1,1} = \{(x,y,z)^t \mid x+y \geq 1, y+z \geq 1, x+z \geq 1, -x-y-z \geq -2\},$$

and this is the tetrahedron with vertices $(\frac{1}{2},\frac{1}{2},\frac{1}{2})^t$, $(1,1,0)^t$, $(1,0,1)^t$, $(0,1,1)^t$.

Now, we will give a simpler geometric construction of $\Omega_{n,m}$. We start by observing that $\Omega_{1,0}(\Gamma) = \Gamma$, and $\Omega_{n,0}(\Gamma) = n\Gamma$, the n-th multiple of Γ in the sense of Minkowski sums.

If Γ is a lattice polygon and r is a direction, then we say that r is an edge direction of Γ iff there are at least two extremal points with respect to r. The set of all edge directions of Γ is denoted by $E(\Gamma)$. Obviously, Γ is the intersection of the half planes $\{p \mid \langle r,p\rangle \geq M(\Gamma,r)\}$ corresponding to its edge directions.

We define the repeated interiors of the lattice polygon Γ recursively by $\Gamma^0 = \Gamma$ and Γ^{m+1} is the convex hull of the interior of the lattice points of Γ^m. We will also use Γ' as a shorthand for Γ^1.

LEMMA 3.1: *Let Γ be a lattice polygon with at least one interior lattice point. Let r be a direction satisfying one of the following conditions:*

a) $M(\Gamma^m, r) = M(\Gamma', r) + m - 1$ for some $m > 1$;

b) r is an edge direction of Γ';

c) r is an edge direction of Γ.

Then we have

$$M(\Gamma', r) = M(\Gamma, r) + 1.$$

Proof: Clearly, $M(\Gamma', r) \geq M(\Gamma, r) + 1$ because all points p in the interior of Γ satisfy $\langle r, p\rangle > M(\Gamma, r)$. We want to show that equality holds.

Assume, indirectly, that there is a lattice point p_0 in Γ such that $\langle r, p_0\rangle \leq M(\Gamma', r) - 2$. Let L be the line $\{p \mid \langle r, p\rangle = M(\Gamma', r) - 1$. Then L intersects the interior of Γ in a non-empty open line segment S, which must not contain any lattice points.

A standard technique in geometry is to choose a convenient coordinate system. We are restricted because we need to respect the lattice, so that we can only change the coordinate frame by unimodular transformations and integral translations. The group of unimodular transformations acts transitively on the directions, and the fixed group for the direction $(1, 0)$ is the set of matrices $\begin{pmatrix} 1 & 0 \\ a & \pm 1 \end{pmatrix}$, where $a \in \mathbb{Z}$. But by a suitable unimodular transformation and translation, we may reduce to the following situation:

1. the line L has equation $x = 1$;
2. the line segment S is contained in $\{\binom{1}{y} \mid 0 < y < 1\}$;
3. the point p_0 has coordinates $\binom{x_0}{y_0}$, with $x_0 \leq y_0 \leq 0$.

For any point $p_1 \in \Gamma'$, the line segment connecting p_0 and p_1 intersects L in a unique point p_2. Since p_0 is a point in Γ and p_1 is a point in the interior of Γ, p_2 is also in the interior of S. Therefore, $p_2 = \binom{1}{y_2}$, with $0 < y_2 < 1$. It follows that $p_1 = \binom{x_1}{y_1}$ with $0 < y_1 < x_1$ (see Figure 2). This shows that $M(\Gamma', r_1) \geq 1$ and $M(\Gamma', r_2) \geq 1$, where $r_1 := (0, 1)$ and $r_2 := (1, -1)$. In other words, Γ' is contained in the closed convex set

$$\Lambda = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid 1 \leq y \leq x - 1 \right\}.$$

We distinguish the two cases (a) and (b).

Case (a): r is a direction satisfying $M(\Gamma^m, r) = M(\Gamma', r) + m - 1 = m + 1$ for some $m > 1$. Since $\Gamma' \subset \Lambda$, Γ^m is contained in the set

$$\Lambda^{m-1} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid m \leq y \leq x - m \right\}.$$

On the other hand, it contains a point $\binom{m+1}{y_3}$. We get $m \leq y_3 \leq 1$, a contradiction.

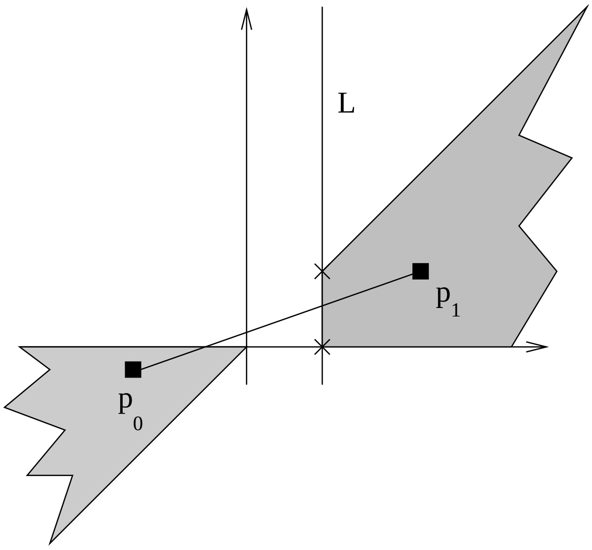

Figure 2: The line p_0p_1 must meet the line L between the marked points.

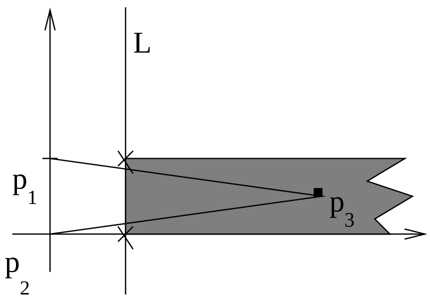

Figure 3: The lines p_1p_3 and p_2p_3 must meet the line L between the marked points.

Case (b): r is a tangent direction of Γ'. Then we must have two lattice points $\binom{2}{y_3}$, with $1 \leq y_3 \leq 1$, which is also a contradiction.

In order to prove case (c), we define the line L as $\{p \mid \langle r, p\rangle = M(\Gamma, r) + 1\}$. Let p_1, p_2 be extremal points of Γ with respect to r such that the line segment between them contains no other lattice point. Again, let S be the intersection of L with the interior of Γ, which is a non-empty open line segment without lattice points. We choose coordinates such that

1. the line L has equation $x = 1$;
2. the line segment S is contained in $\{\binom{1}{y} \mid 0 < y < 1\}$;
3. $p_1 = \binom{0}{0}$ and $p_2 = \binom{0}{1}$.

For any point $p_3 = \binom{x_3}{y_3}$ with $x_3 \geq 1$, the line segment connecting p_1 and p_3 intersects L in an interior point of Γ, hence in a point in S. Therefore, $y_3 > 0$.

Also, the line segment connecting p_2 and p_3 intersects S, which implies $y_3 < 1$. It follows that there are no lattice points with $x_3 \geq 1$ (see Figure 3). Hence $\Gamma' = \emptyset$, contradicting our assumption. $\square$

LEMMA 3.2: *Let Γ be a lattice polygon, $m \geq 0$. Let r be an edge direction of Γ^m. Then we have*

$$M(\Gamma^m, r) = M(\Gamma, r) + m.$$

Proof: We prove the statement by induction on m. For $m = 0$, the statement is trivial. For $m = 1$, the statement follows from Lemma 3.1 (a).

Assume $m > 1$. By induction hypothesis, we have $M(\Gamma^m, r) = M(\Gamma', r) + m - 1$. Therefore, we can apply Lemma 3.1 (b), and get $M(\Gamma', r) = M(\Gamma, r) + 1$. $\square$

We introduce some notation. For $l \geq 1$, $m, n \geq 0$, either m or n positive, $\square_{l,m,n}$ denotes the polygon with vertices $\binom{0}{0}, \binom{0}{l}, \binom{lm+n}{0}, \binom{n}{l}$. For $m \geq 1$, Δ_m denotes the normal triangle of size m (with vertices $\binom{0}{0}, \binom{m}{0}, \binom{0}{m}$). Note that $\square_{1,1,0} = \Delta_1$.

THEOREM 3.2: *Any lattice polygon without interior lattice points is unimodular equivalent to $\square_{1,m,n}$ or to Δ_2.*

Proof: We proceed similar as in the proof of case (c) in Lemma 3.1. Let r be an edge direction of Γ. Let p_1, p_2 be the two end points of the side of Γ corresponding to r. Let L be the line $\{p \mid \langle r, p\rangle = M(\Gamma, r) + 1\}$. If L does not contain interior points of Γ, then Γ fits between two parallel lines with distance one unit, and it is easy to show that it can be transformed to a $\square_{1,m,n}$. Otherwise, let S be the set of all interior points of Γ lying on L. This is an open line segment without any lattice points. We choose coordinates such that

1. the line L has equation $x = 1$;
2. the line segment S is contained in $\{\binom{1}{y} \mid 0 < y < 1\}$;
3. $p_1 = \binom{0}{0}$ and $p_2 = \binom{0}{m}$, for some $m \geq 1$.

For any point lattice point $p_3 = \binom{x_3}{y_3} \in \Gamma$ with $x_3 \geq 2$, the line segment connecting p_1 and p_3 intersects L in a point in the closure of S. Therefore, $y_3 \geq 0$. Also, the line segment connecting p_2 and p_3 intersects the closure of S, which implies $(m-1)x_3 + y_3 \leq m$. We distinguish three cases.

If $m = 1$, then $0 \leq y_3 \leq 1$. Hence Γ is contained in the set $\{\binom{x}{y} \mid 0 \leq y \leq 1\}$. Then Γ can be transformed to a $\square_{1,m',n}$.

If $m \geq 2$, then

$$0 \leq y_3 \leq m - x_3(m-1) \leq m - 2(m-1) \leq 2 - m,$$

which is only possible if $m = 2$, $x_3 = 2$, and $y_3 = 0$. Then Γ is equal to Δ_2. $\square$

LEMMA 3.3: *Let Γ be a lattice polygon, $m \geq 0$. Then $\Omega_{1,m}(\Gamma) = \Gamma^m$.*

Proof: For any direction r, the extremal points p are on the boundary. This shows that $M(\Lambda', r) \geq M(\Lambda, r) + 1$ for any lattice polygon Δ. Consequently, $M(\Gamma^m, r) \geq M(\Gamma) + r$, and therefore $\Gamma^m \subseteq \Omega_{1,m}$.

Assume that Γ^m is a proper polygon with non-empty interior. By Lemma 3.2, we have

$$\begin{aligned} \Gamma^m &= \bigcap_{r \in E(\Gamma^m)} \{p \mid \langle r, p\rangle \geq M(\Gamma^m, r)\} \\ &= \bigcap_{r \in E(\Gamma^m)} \{p \mid \langle r, p\rangle \geq M(\Gamma^m, r) + m\} \\ &\supseteq \bigcap_{r \in R} \{p \mid \langle r, p\rangle \geq M(\Gamma^m, r) + m\} = \Omega_{1,m}. \end{aligned}$$

Now, we have to treat the three degenerate cases.

Assume that Γ^m is a line segment, say connecting the points q_1 and q_2. The above argument shows that $\Omega_{1,m}$ is contained in the line supporting through q_1, q_2. There are two edge directions r_1, r_2 of Γ^{m-1}, such that q_i is the only extremal endpoint of $\Omega_{1,m}$ with respect to r_i. By Lemma 3.2 and Lemma 3.1 (c), we have

$$M(\Omega_{1,m}, r) = M(\Gamma, r) + m = M(\Gamma^{m-1}, r) + 1 = M(\Gamma^m, r)$$

for $r = r_1, r_2$. Hence the same points are extremal on Γ^m, which proves that $\Gamma^m = \Omega_{1,m}$.

Assume that Γ^m is a single point p. The same computation as in the line segment case shows that p is extremal with respect to any edge direction of Γ^{m-1}. This shows that p is the only point of $\Omega_{1,m}$.

The case that Γ^m is empty splits into several subcases.

Assume that Γ^{m-1} is a proper polytope. By Theorem 3.2, Γ^{m-1} is unimodular equivalent to $\Box_{1,a,b}$ or to Δ_2. By Lemma 3.2, the set Γ^m is contained in the set

$$\bigcap_{r \in E(\Gamma^{m-1})} \{p \mid \langle r, p\rangle \geq M(\Gamma^{m-1}, r) + 1\}.$$

But this set is empty by a case by case check.

The case where Γ^{m-1} is a line segment can be treated in the same way as the previous case.

Assume that Γ^{m-1} is a point p. In the proof of the case where Γ^m is a point, we saw that there are directions r with

$$\langle r, p\rangle = M(\Gamma^{m-1}, r) = M(\Gamma, r) + m - 1.$$

Hence p is not a point of $\Omega_{1,m}$. Since $\Omega_{1,m}$ is a subset of $\Omega_{1,m-1} = \{p\}$, it follows that p is empty.

Finally, assume that Γ^{m-1} is empty. If $n < m$ is the smallest integer such that Γ^l is not empty, then $\Gamma^{n+1} = \Omega_{1,n+1}$ by one of the previous cases. Hence both sets are empty, and $\Omega_{1,m}$ is also empty because it is a subset of $\Omega_{1,n+1}$. $\Box$

THEOREM 3.3: *Let Γ be a lattice polygon. Then $\Omega_{n,m}(\Gamma) = (n\Gamma)^m$.*

Proof: From the definition of $\Omega_{n,m}$ and by Lemma 3.3, we get

$$\Omega_{n,m}(\Gamma) = \Omega_{1,m}(n\Gamma) = (n\Gamma)^m.$$

□

4. Toric Simplification

The idea for our toric simplification algorithm is the following: instead of Γ (which may have a very complicated shape) we simplify Γ^m, where m is as large as possible. As the process of adjunction strips away points in the various directions in a uniform way, there is hope that the same unimodular transformation also simplifies Γ.

Passing from Γ to Γ' is not a reversible process, as the repeated interior operation is neither surjective nor injective. But we will use an operation that is close to an inverse operation. If Γ is a convex polygon, then we set

$$\Upsilon^m(\Gamma) := \bigcap_{r \in E(\Gamma)} \{p \mid \langle r, p \rangle \geq M(\Gamma, r) - m\}.$$

By definition, this is a convex polygon, but its vertices need not be lattice points in general.

LEMMA 4.1: *Let Γ be a lattice polygon, and assume $m \geq 0$.*

a) If Λ is a lattice polygon such that $\Lambda^m = \Gamma$, then $\Lambda \subseteq \Upsilon^m(\Gamma)$.

b) If $M(\Upsilon^m(\Gamma), r) > M(\Gamma, r) - m$ for some direction r, then there is no Λ such that $\Lambda^m = \Gamma$.

Proof: Let r be an edge direction of Γ. Let Λ be such that $\Lambda^m = \Gamma$. Then $M(\Gamma, r) = M(\Lambda, r) + m$ by Lemma 3.2. Hence $\Lambda \subseteq \{p \mid \langle r, p \rangle \geq M(\Gamma, r) - m\}$, and (a) follows because this subset relation holds for all $r \in E(\Gamma)$.

In order to prove (b), assume that $M(\Upsilon^m, r) > M(\Gamma, r) - m$ for some arbitrary edge direction r, and assume indirectly that $\Lambda^m = \Gamma$ for some Λ. By the definition of repeated interiors, we have $M(\Gamma, r) \geq M(\Lambda, r) + m$. On the other hand, we have $M(\Lambda, r) \geq M(\Upsilon^m, r)$, because $\Lambda \subseteq \Upsilon^m$ by (a). This contradicts our assumption $M(\Upsilon^m, r) > M(\Gamma, r) - m$. □

It is necessary to classify the polygons that can appear at the end of the adjunction process. For polygons without interior lattice polygons, we already have the classification in Theorem 3.2.

We want to show that Γ is already simplified if Γ^m is. We start with the case of Δ_2, which is the simplest.

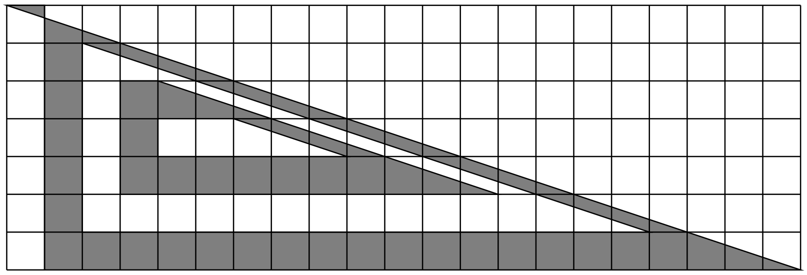

Figure 4: The inverse interiors of $\square_{1,3,2}$.

THEOREM 4.1: *Let Γ be a lattice polygon, and let $m \in \mathbb{N}$. Assume that $\Gamma^m = \Delta_2$. Then the following are true.*

a) $e(\Gamma) = 3m + 2$.

b) For any lattice Λ unimodular equivalent to Γ, we have $e(\Lambda) \geq 3m + 2$.

Proof: (a): Obviously, $\Upsilon^m(\Delta_2) = \Delta_{3m+2}$. Hence the statement follows immediately from Lemma 4.1 (a).

(b): Let Λ be unimodular equivalent to Γ. Then $\Lambda \subseteq \Delta_{e(\Lambda)}$ by the definition of the symbol e. Therefore, $\Lambda^m \subseteq (\Delta_{e(\Lambda)})^m = \Delta_{e(\Lambda)-3m}$. On the other hand, Λ^m is unimodular equivalent to Δ_2, therefore $e(\Lambda) - 3m \geq 2$. $\square$

In the case of $\square_{1,a,b}$, we can only prove some slightly weaker statement: the polygon Γ may not be of smallest possible one, but it is not much larger than the smallest.

THEOREM 4.2: *Let Γ be a lattice polygon, and let $m, a, b \in \mathbb{N}$. Assume that $\Gamma^m = \square_{1,a,b}$. Then the following are true.*

a) If $a \geq 1$, then $e(\Gamma) \leq (a+2)m + a + b$; otherwise, $e(\Gamma) \leq 4m + b + 1$.

b) For any lattice Λ unimodular equivalent to Γ, we have

$$e(\Lambda) \geq 3m + \max(a, 1) + b \geq \max\left(\frac{(a+2)m + a + b}{2}, \frac{4m + b + 1}{2}\right).$$

c) $b \geq (a-2)m$.

Proof: We begin with the first inequality of (b). As in the proof of Theorem 4.1 (b), we have $\Lambda^m \subseteq \Delta_{e(\Lambda)-3m}$. On the other hand, Λ^m is unimodular equivalent to $\square_{1,a,b}$. We conclude that $e(\Lambda) - 3m \geq a + b$. If $a = 0$, then we also conclude $e(\Lambda) - 3m \geq b + 1$, because Δ_b does not contain anything unimodular equivalent to the rectangle $\square_{1,0,b}$.

Let

$$p_1 := \begin{pmatrix} -m \\ -m \end{pmatrix}, p_2 := \begin{pmatrix} b + m + a + am \\ -m \end{pmatrix}, p_3 := \begin{pmatrix} b + m - am \\ m + 1 \end{pmatrix},$$
$$p_4 := \begin{pmatrix} -m \\ m + 1 \end{pmatrix}, p_5 := \begin{pmatrix} -m \\ \frac{a+b+2m}{a} \end{pmatrix};$$

these are the intersections of the support lines of Γ^m along the edge directions, offset by m units. If $b \geq (a-2)m$, then Υ^m is the trapezoid $p_1p_2p_3p_4$, a translation of $\Box_{1+2m,a,b+(2-a)m}$. (In the subcase $b = (a-2)m$, this trapezoid degenerates into a triangle because $p_3 = p_4 = p_5$). If $b < (a-2)m$, then Υ^m is the triangle $p_1p_2p_5$. In this case, let $r := (0, -1)\}$. We have

$$M(\Upsilon^m, r) = \langle r, p_5 \rangle = -\frac{a+b+2m}{a} > -m-1 = M(\Gamma, r) - m,$$

and by Lemma 4.1 (b), this is not possible. The situation is graphically explained in Figure 4. Hence (c) holds. The second inequality of (b) is a numerical consequence of (c).

If $a \geq 1$, then $\Upsilon = \Box_{1+2m,a,b+(2-a)m}$ is contained in a normal triangle Δ_n, where

$$n := a(1+2m) + b + (2-a)m = (a+2)m + a + b.$$

By Lemma 4.1 (a), Γ is also contained in Δ_n. If $a = 0$, then we need Υ is a rectangle with side length $1 + 2m$ and $b + 2m$, which is contained in a Δ_{4m+b+1}. By Lemma 4.1 (a), Γ is also contained in Δ_{4m+b+1}. This shows (a). $\Box$

Here is the classification of the lattice polygons with a line segment as the first interior.

THEOREM 4.3: *Let Λ be a lattice polygon such that Λ' is a line segment with n points (of course, $n \geq 2$). Then Λ is unimodular equivalent to a polygon with vertices*

$$\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ 2 \end{pmatrix}, \begin{pmatrix} d \\ 2 \end{pmatrix}, \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} n+1 \\ 1 \end{pmatrix} \right],$$

where $0 \leq a \leq b$, $0 \leq c \leq d$, $b + d \leq 2n + 2$, $d \leq b$, and the two points in square brackets may or may not be there.

Proof: We transform the line segment Γ' to be the line segment between $\binom{1}{1}$ and $\binom{n}{1}$. By Lemma 3.1 (b), Γ is contained in the set $\{\binom{x}{y} \mid 0 \leq y \leq 2\}$. There can be at most two vertices on each of the lines $y = 0$, $y = 1$, and $y = 2$. Straightforward case by case analysis shows that we can always reduce to the above situation. $\Box$

THEOREM 4.4: *Let Γ be a lattice polygon, and let $m \geq 0, n \geq 2$. Assume that Γ^{m+1} is a line segment with n points, and that Γ^m is a lattice polygon of the type described in Theorem 4.3. Then the following are true.*

a) $e(\Gamma) \leq \max(4m + n + 3, 2n + 2m + 2)$.

b) For any lattice Λ unimodular equivalent to Γ, we have $e(\Lambda) \geq 3m + n + 2$.

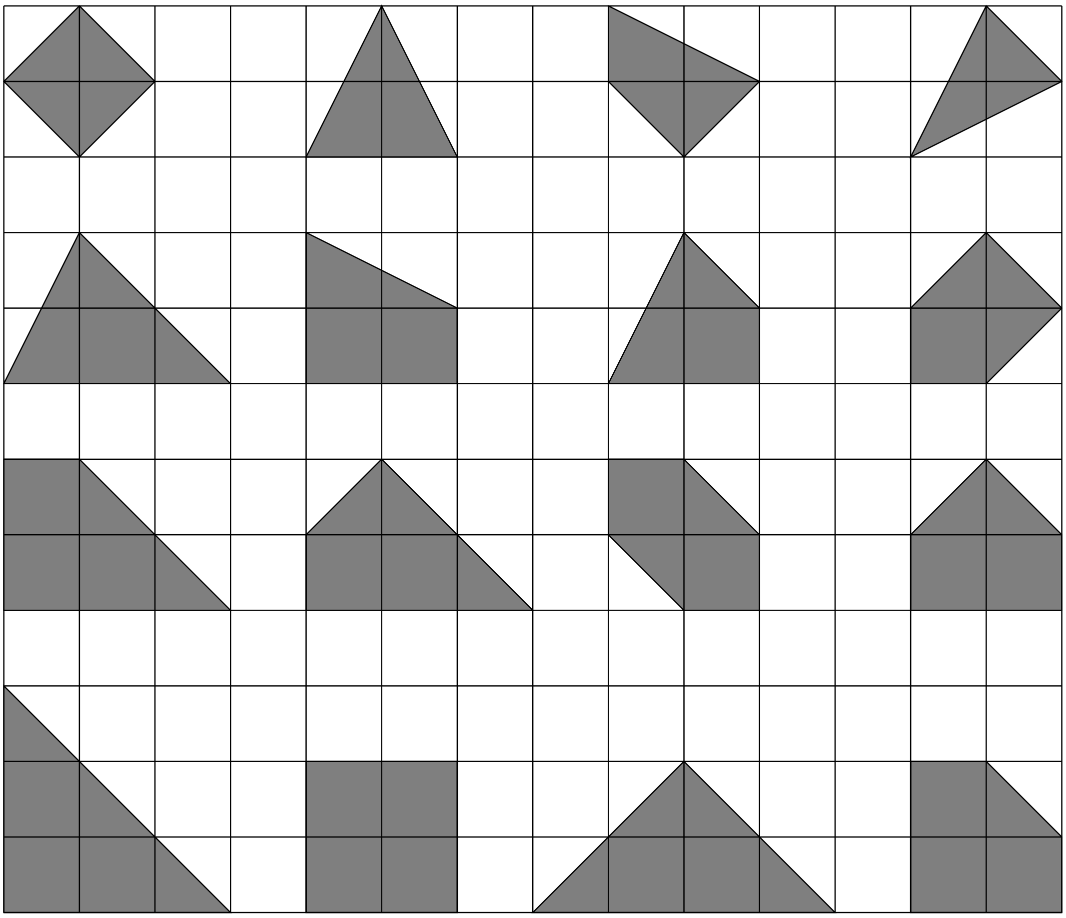

Figure 5: The lattice polygons with one interior point.

Proof: We begin with the first inequality of (b). As in the proof of the theorems 4.1 and 4.2, we have $\Lambda^{m+1} \subseteq \Delta_{e(\Lambda)-3m-3}$. Hence $e(\Lambda) - 3m - 3 \geq n - 1$.

For the proof of (a), we have to compute $\Upsilon^m(\Gamma^m)$. A straightforward computation gives that $\Upsilon^m \subseteq \Box_{2m+2,0,2m+n+1}$ for $b \leq n+1$ – in this case, $\Upsilon^m \subseteq \Delta_{4m+n+3}$ – and $\Upsilon^m \subseteq \Box_{2m+2,b-n-1,mn+2n+2m+2-mb-b}$ for $b \geq n+2$. In this case, we know also that $mn + 2n + 2m + 2 - mb - b \geq 0$. It follows that $\Upsilon^m \subseteq \Delta_{mb+b-mn}$, and $mb + b - mn \leq 2n + 2m + 2$ because of the above inequality. □

The classification of lattice polygons with one interior point has been done in Rabinowitz (1989). Here is the result.

Theorem 4.5: *Let Γ be a lattice polygon with one interior point. Then Γ is unimodular equivalent to one of the polygons in Figure 5.*

Theorem 4.6: *Let Γ be a lattice polygon, and let $m \geq 0$. Assume that Γ^{m+1} is a single point, and that Γ^m is one of the lattice polygons in Figure 5. Then the following are true.*

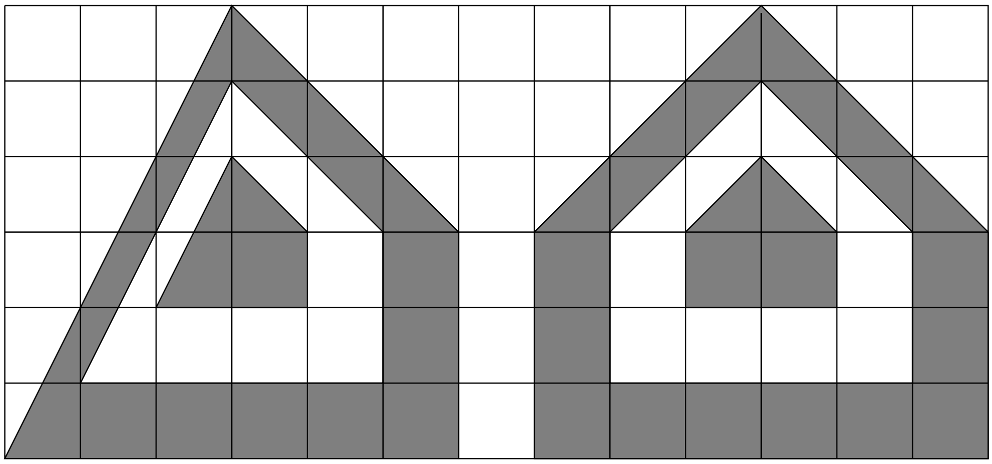

Figure 6: In case of one interior point, the inverse interior is similar.

a) If Γ^m *is the polygon number 2 or 3, then* $e(\Gamma) \leq 4m + 4$*; otherwise,* $e(\Gamma) \leq 3m + 3$.

b) For any lattice Λ *unimodular equivalent to* Γ*, we have* $e(\Lambda) \geq 3m + 3$.

Proof: For any Λ unimodular equivalent to Γ, we have $\Lambda^{m+1} \subseteq \Delta_{e(\Lambda)-3m-3}$. Hence $e(\Lambda) - 3m - 3 \geq 0$, which shows (b).

For all polygons Π in Figure 5, we have $\Upsilon^m(\Pi) = (m+1)\Pi$ (see Figure 6). (In fact, this can be shown from Lemma 3.1 (c) directly, without using Theorem 4.5). Two polygons in Figure 5 are contained in Δ_4, and all the other are contained in Δ_3. Hence we get that Υ^m is contained in Δ_{4m+4} in case Γ^m is the polygon number 2 or 3, and it is contained in Δ_{3m+3} in all other cases. By Lemma 4.1, (a) follows. $\square$

For a polygon without interior lattice points, we say that it is in good position iff it is equal to $\square_{1,m,n}$ or Δ_2 (up to translation). For a polygon with several interior lattice points, but all on a line, we say that it is in good position iff it is equal to a polygon as described in Theorem 4.3. For a polygon with exactly one interior lattice point, we say that it is in good position iff it is equal to one of the polygons in Figure 5.

Here is our toric simplification algorithm.

1. Compute the smallest m such that Γ^{m+1} is not a proper polygon.
2. Bring Γ^m into good position by a suitable unimodular transformation τ.
3. Apply τ to Γ.

Example 4.1: The polygon Γ in Figure 7 has parametric degree $e(\Gamma) = 17$. Since Γ^3 is a line segment, we get $m = 2$, and we have to bring the white polygon Γ^2 into good position. As it can be seen in Figure 8, the transformed Γ has parametric degree 8.

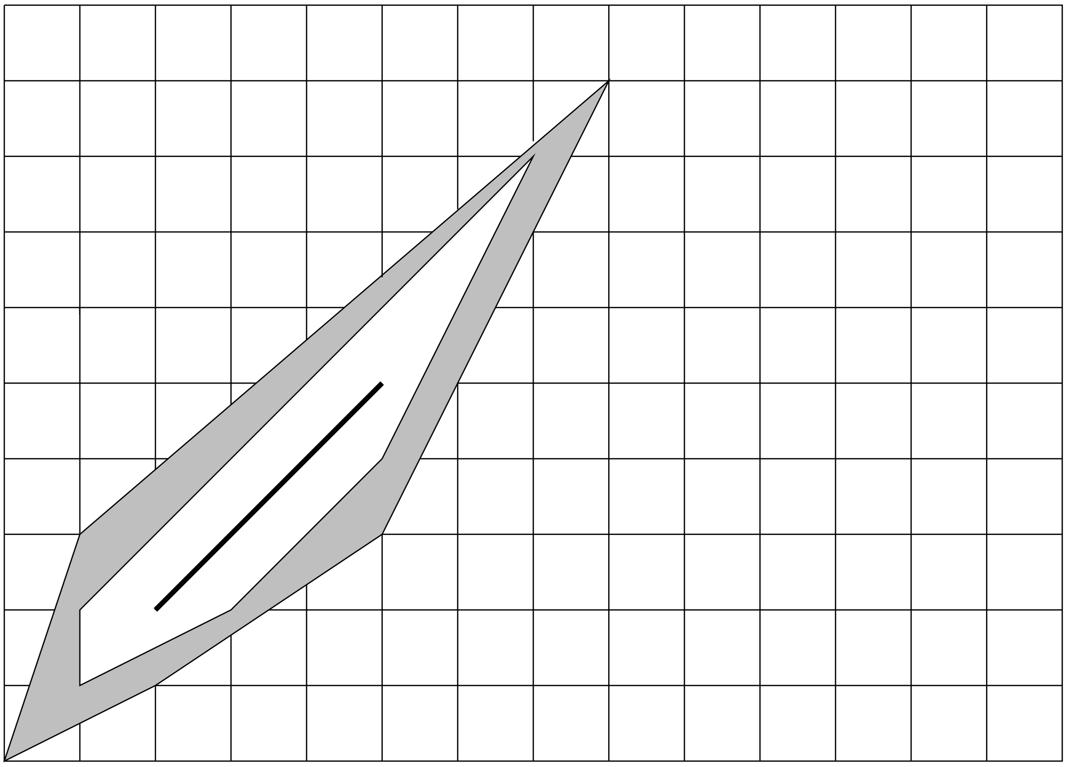

Figure 7: Simplification of a lattice polygon.

THEOREM 4.7: *Let* Γ *be a lattice polygon,* Σ *the result of the simplification algorithm applied to* Γ*, and* Λ *another lattice polygon unimodular equivalent to* Γ*. Then* $e(\Lambda) \geq e(\Sigma)/2$.

Proof: Because the operation "repeated interior" is respected by unimodular transformations, Σ fulfills the assumption of Theorem 4.2, Theorem 4.4, or Theorem 4.6. The statement follows then from these three theorems. □

5. Arbitrary Parametric Surfaces

The toric simplification algorithm can be easily generalized to arbitrary parametric surfaces, because it uses only operations that generalize. Here is a straightforward generalization.

1. Given a parametric surface $p : \mathbb{C}^2 \to S$, compute the smallest m such that the parametric adjoint map $p_{1,m+1}$ either does not exist, or its image is not a surface.
2. Find a Cremona transformation t such that $p_{1,m} \circ t : \mathbb{C}^2 \to \mathbb{P}^r$ has smallest possible parametric degree.
3. Return $p \circ t$.

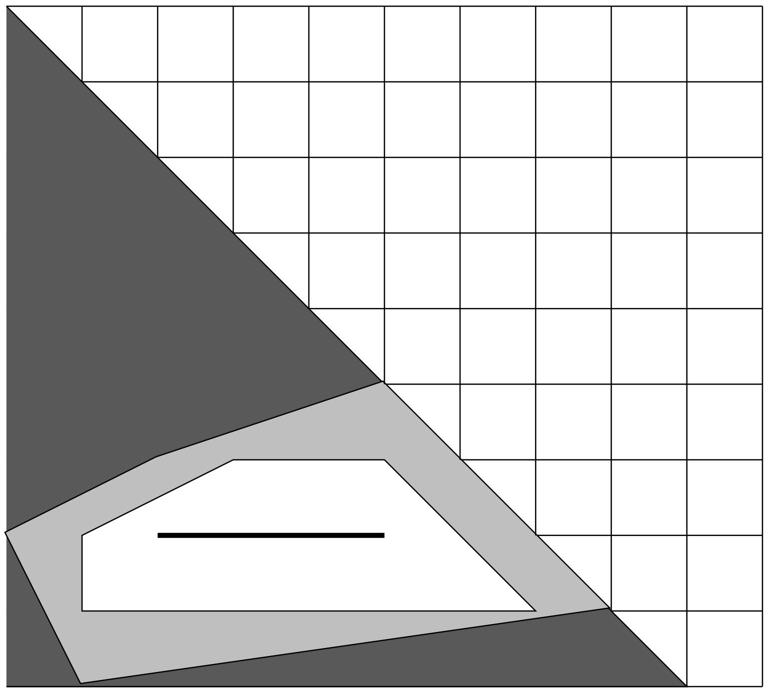

Figure 8: The result of the simplification.

The straightforward generalization does not work in general, because the general theory of adjoints does not give enough structural information about $p_{1,m}$, with m chosen as above. What is known (Schicho, 1998) is the following.

THEOREM 5.1: *Let $p : \mathbb{C}^2 \to S$ be a parametric surface, properly parametrized. Let m be the largest number such that $v_{1,m} > 0$, or equivalently such that the adjoint map $p_{1,m}$ does exist. Then one of the following cases is true.*

1. *$p_{1,m}$ is birational onto the image. This image is either the projective plane, or a quadric surface in $\mathbb{P}^3$, or a rational scroll, or a Veronese surface.*
2. *$p_{1,m}$ maps to a rational normal curve. Then $p_{2,2m-1}$ is birational onto a conical surface, i.e. a surface generated by a pencil of conics.*
3. *$p_{1,m}$ maps to a point, or equivalently $v_{1,m} = 1$. If $v_{1,m-1} \geq 4$, then $p_{1,m-1}$ is birational to a Del Pezzo surface. If $v_{1,m-1} = 3$, then $p_{2,2m-2}$ is birational to a Del Pezzo surface. Otherwise, $v_{1,m-1} = 2$, and $p_{3,3m-3}$ is birational to a Del Pezzo surface.*

Remark: In case (1), $p_{1,m}$ is toric: the projective plane corresponds to Δ_1, the nonsingular quadric corresponds to $\square_{1,0,1}$, the singular quadric corresponds to $\square_{1,2,0}$, the rational scrolls to $\square_{1,a,b}$, and the Veronese surface to Δ_2.

In case (2), $p_{1,m}$ is also toric, corresponding to a line segment. The toric conical surfaces are precisely those with a line segment as the first interior.

In case (3), $p_{1,m}$ is trivially toric, corresponding to a point. The toric Del Pezzo surfaces correspond to the lattice polygons with one interior lattice points in Figure 5. There are also non-toric Del Pezzo surfaces, but their implicit degree is bounded by 9, as in the toric case.

We already know how to parametrize rational scrolls and the Veronese surface minimally. For conical surfaces and Del Pezzo surfaces, one can also find a minimal parametrization (see Schicho (2002)). Therefore, the above theorem suggests the following modified generalization to the non-toric case.

1. Compute the smallest m such that the parametric adjoint map $p_{1,m+1}$ either does not exist.
2. If $p_{1,m}$ is birational, set $p' := p_{1,m}$.
   If $p_{1,m}$ maps to a rational normal curve, set $p' := p_{2,2m-1}$.
   If $v_{1,m} = 1$ and $v_{1,m-1} \geq 4$, set $p' := p_{1,m-1}$.
   If $v_{1,m} = 1$ and $v_{1,m-1} = 3$, set $p' := p_{2,2m-2}$.
   Otherwise (namely if $v_{1,m} = 1$ and $v_{1,m-1} = 2$), set $p' := p_{3,3m-3}$.
3. Find a Cremona transformation t such that $p' \circ t : \mathbb{C}^2 \to \mathbb{P}^r$ has smallest possible parametric degree.
4. Return $p \circ t$.

This is precisely the algorithm described in (Schicho, 2002).

Remark: The above algorithm may be restricted to toric surfaces and translated again to the language of lattice polygons. Let us compare this second toric algorithm with the first one in the previous section.

In the cases where $p_{1,m}$ is birational or ($v_{1,m} = 1$ and $v_{1,m-1} \geq 4$), we have to bring the same polygon into good position (namely Γ^m or Γ^{m-1}, depending on the case). The case $v_{1,m} = 1$ and $v_{1,m-1} \leq 4$ does not occur for toric surfaces, because every polygon with one interior lattice point has at least 4 lattice points. In the case where Γ^m is a line segment, the polygons really differ. The first algorithm simplifies $\Lambda_1 := \Gamma^{m-1}$, and the second algorithm simplifies $\Lambda_2 := (2\Gamma)^{2m-1}$. We claim that $\Lambda_2 = (2\Lambda_1)'$. It suffices to show $(2\Gamma)^{2m-2} = 2\Lambda_1$. To do this, we compute

$$(2\Gamma)^{2m-2} = \Omega_{2,2m-2}(\Gamma) = \bigcap_{r\in R}\{p \mid \langle r,p\rangle \geq 2M(\Gamma,r) + 2m-2\}$$
$$= \bigcap_{r\in R}\{2p \mid \langle r,p\rangle \geq M(\Gamma,r) + m-1\} = 2\Omega_{1,m-1}(\Gamma) = 2\Lambda_1.$$

A geometric analysis shows that $E(\Lambda_2) = E(\Lambda_1) \cup E(\Gamma^m)$ (see Figure 9). It follows that Λ_1 and Λ_2 are brought into good position by the same unimodular transformations.

The parametric degree of the result can be estimated using the following quantitative result from (Schicho, 1999).

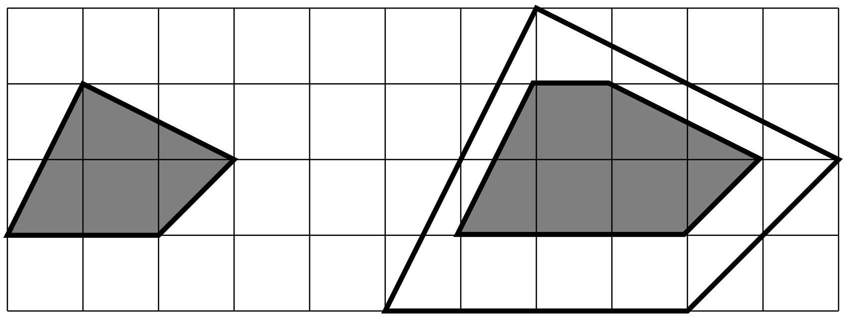

Figure 9: Comparison of the two toric simplification algorithms.

THEOREM 5.2: *Let S be a rational surface. Let m be the largest number such that $v_{1,m} > 0$. such that the adjoint map $a_{1,m}$ does exist. Let d_p be the "intrinsic parametric degree", i.e. the smallest possible parametric degree of a proper parametrization. Then the following are true.*

1. *If $a_{1,m}$ is birational, then we distinguish three cases, depending on its image.*
   (a) *If the image is a Veronese surface, then $d_p = 3m + 2$.*
   (b) *If the image is the projective plane, then $d_p = 3m + 1$.*
   (c) *If the image is a quadric surface or a rational scroll, then* $3m + \frac{v_{1,m}-1}{2} \leq d_p \leq 4m + v_{1,m} - 2$.
2. *If $a_{1,m}$ maps to a rational normal curves, then* $3m + v_{1,m} - 1 \leq d_p \leq 4m + 2v_{1,m} - 2$.
3. *If $v_{1,m} = 1$, then $3m \leq d_p \leq 4m$.*

Proof: See (Schicho, 1999), Theorem 8, Lemma 9, Theorem 9, and Proposition 1 (the cases (2) and (3) above are subsumed to one case there). □

Theorem 5.2 contains an implicit existence statement, namely it states the existence of a parametrization of degree less than or equal to the given upper bound for d_p. It is important to note that such a parametrization is actually constructed by the simplification algorithm (see Schicho (2002) for a proof). As in the toric case, we can conclude that the computed parametrization is at most twice as big as the smallest possible parametrization.

Remark: In (Schicho, 1999), we can also find some statements bounding d_p in terms of the implicit degree d or the sectional genus $p_1 = v_{1,1}$. In the toric case, these bounds can be improved significantly, because we have $v_{a,b+1} \leq v_{a,b} - 3$ for all $a, b \in \mathbb{N}$ such that $v_{a,b+1} > 0$. The reason is that $v_{a,b} - v_{a,b+1}$ is equal to the number of all lattice points in the boundary of $\Omega_{a,b}(\Gamma)$, and this number is at least 3. It follows that $d_p \leq 6m + 2v_{1,m} \leq v_{1,0}$, and $v_{1,0}$ (the number of all lattice points in Γ) is equal to $d - p_1 + 2$ by Pick's theorem.

Some of the statements in Lemma 3.1 can also be given an algebraic interpretation. For instance, the statement of case (c) is equivalent to stating that if the vectorspace of 1-adjoints is not zero, then it generates the sheaf of 1-adjoints. This statement turns out to be wrong for some non-toric examples (e.g. the classical case of a Del Pezzo surface of degree 1).

For toric surfaces, we have seen that "passing to the m-adjoint surface" is the same as "m times passing to the 1-adjoint surface". This also fails in the non-toric case.

References

Arnold, V. I. (1980). Statistics of integral convex polygons. *Funkt. Anal. i Pril.*, **14**:1–3.

Bárány, I., Pach, J. (1992). On the number of convex lattice polygons. *Combin. Probab. Comput.*, **1**:295–302.

Brown, G. (1999). Pluricanonical cohomology across flips. *Bull. London Math. Soc.*, **31**:513–522.

Fulton, W. (1993). *Introduction to toric geometry.* Princeton University Press.

Krasauskas, R. (2001). The shape of toric surfaces. In *Spring conference on Computer Graphics.* CS Press.

Rabinowitz, S. (1989). A census of convex lattice polygons with at most one interior lattice point. *Ars Combin.*, **28**:83–96.

Schicho, J. (1998). Rational parametrization of surfaces. *J. Symb. Comp.*, **26**(1):1–30.

Schicho, J. (1999). A degree bound for the parameterization of a rational surface. *J. Pure Appl. Alg.*, **145**:91–105.

Schicho, J. (2002). Simplification of surface parametrizations. In *Proc. ISSAC 2002.* ACM Press. to appear.

Scott, P. (1976). On convex lattice polygons. *Bull. Austral. Math. Soc.*, **15**:395–399.