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A Collection of Denominator Bounds To Solve Parameterized Linear Difference Equations in ΠΣ-Fields*

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Abstract

An important application of solving parameterized linear difference equations in ΠΣ-fields, a very general class of difference fields, is simplifying and proving of nested multisum expressions and identities. This article provides essential algorithmic building blocks that enable to search for all solutions of such difference equations. More precisely, these algorithms allow to exploit a denominator elimination strategy which amounts to look for solutions in a polynomial ring instead of searching for rational function solutions.

1. Introduction

In [Kar81] M. Karr developed algorithms to solve parameterized first order linear difference equations in ΠΣ-fields, a very general class of difference fields. He observed that one can simplify indefinite nested multisums in this ΠΣ-field setting by eliminating sum quantifiers. Besides this I observed in [Sch00] that Zeilberger's creative telescoping trick [Zei90] is in the scope of Karr's algorithm which allows to compute a recurrence for a huge class of definite multisums. Moreover I generalized Karr's reduction techniques in [Sch02b] so that one is able to search for all solutions of parameterized linear difference equations with arbitrary order in a given ΠΣ-field. Hence one is able to solve those recurrences that are obtained by the creative telescoping method, and therefore one not only can prove but even discover a big variety of definite multisum identities. These algorithms are available in form of a summation package called `Sigma` [Sch00] in the computer algebra system Mathematica.

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In order to achieve these extensions, I first streamlined Karr's ideas to a simpler algorithm and second generalized this more compact algorithm in [Sch01, Sch02b]. Within these generalizations a denominator elimination technique plays an essential role on that I will focus in this article. ΠΣ-fields are constructed by a tower of transcendental elements, or in other words ΠΣ-fields are represented by a field of rational functions in several variables. This article contributes to reduce the problem from looking for solutions of a linear difference equation in the field of rational functions to searching for solutions in a polynomial ring. This denominator elimination strategy was originally introduced by Abramov in [Abr89b, Abr95] and varied in [vH98] for one of the most simplest cases of ΠΣ-fields with only one transcendental extension. M. Bronstein generalized this algorithm in [Bro00] to a class of difference field extensions that contains ΠΣ-fields. By this denominator bound algorithm one is able to reduce partially the problem from the field of rational functions to the polynomial ring. Together with results from [Kar81] this enables to solve the denominator bound problem for first order linear difference equations in full generality in a given ΠΣ-field.

Since we specialize Bronstein's more general results to ΠΣ-fields, we can streamline the essential results in our concrete situation. Moreover this article brings together the work of Karr and Bronstein and develops carefully an algorithm to solve the denominator bound problem for first order linear difference equations in ΠΣ-fields. In particular we analyze some important properties of that algorithm which are needed for further development in the theory of ΠΣ-fields and indefinite summation. Furthermore Karr's ideas are generalized which enables to solve the denominator bound problem for various cases of higher order linear difference equations.

In the next section the denominator bound problem is introduced in the general context of solving linear difference equations. After defining ΠΣ-fields in Section 3, the denominator elimination strategy is concretized in the ΠΣ-field setting in Section 4. Then the problem under discussion will be divided into two subproblems that are motivated in Section 5. Finally these subproblems are considered separately in Sections 6 and 7. Combining the two results in the last section results in an algorithm that solves the denominator bound problem for parameterized first order linear difference equations in ΠΣ-fields.

2. The Denominator Bound Problem

In this article we provide an essential reduction technique to solve parameterized linear difference equations in a very general subclass of difference fields, so called ΠΣ-fields. First we introduce the definition of difference fields.

Definition 2.1. A *difference field* (resp. ring) is a field (resp. ring) $\mathbb{F}$ together with a field (resp. ring) automorphism $\sigma : \mathbb{F} \to \mathbb{F}$. In the sequel a difference field (resp. ring) given by the field (resp. ring) $\mathbb{F}$ and automorphism σ is denoted by $(\mathbb{F}, \sigma)$. Moreover the subset $\mathbb{K} := \{k \in \mathbb{F} \,|\, \sigma(k) = k\}$ is called the *constant field* of the difference field $(\mathbb{F}, \sigma)$.

It is easy to see that the constant field $\mathbb{K}$ of a difference field $(\mathbb{F}, \sigma)$ is a subfield of $\mathbb{F}$. In the sequel we will assume that **all** fields are of characteristic 0. Then it is immediate that for any field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ we have $\sigma(q) = q$ for $q \in \mathbb{Q}$. Hence in any difference field, $\mathbb{Q}$ is a subfield of its constant field.

Example 2.1. Let $\mathbb{K}(t)$ be the field of rational function over the field $\mathbb{K}$, this means $\mathbb{K}(t)$ is the quotient field of the polynomial ring $\mathbb{K}[t]$. Then we can define uniquely the difference field $(\mathbb{K}(t), \sigma)$ with constant field $\mathbb{K}$ where the field automorphism $\sigma : \mathbb{K}(t) \to \mathbb{K}(t)$ is canonically defined by $\sigma(t) = t + 1$.

As illustrated in [Sch01, Sch02b] one is able to discover and prove a huge class of indefinite and definite multisum identities by solving parameterized linear difference equations in ΠΣ-fields; in particular one can carry out indefinite summation, Zeilberger's creative telescoping idea and solving recurrences.

Solving Parameterized Linear Difference Equations

- Given a difference field $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, $a_1, \dots, a_m \in \mathbb{F}$ with $m \geq 1$ and $(a_1 \dots a_m) \neq (0, \dots, 0) =: \mathbf{0}$ and $f_1, \dots, f_n \in \mathbb{F}$ with $n \geq 1$.
- Find all $g \in \mathbb{F}$ and all $c_1, \dots, c_n \in \mathbb{K}$ with $a_1\,\sigma^{m-1}(g) + \dots + a_m\,g = c_1\,f_1 + \dots + c_n\,f_n$.

The solutions of the above problem are described by a set. For its definition note that in the difference field $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, $\mathbb{F}$ can be interpreted as a vector space over $\mathbb{K}$.

Definition 2.2. Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$ and consider a subspace $\mathbb{V}$ of $\mathbb{F}$ as a vector space over $\mathbb{K}$. Let $\mathbf{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}^m$ and $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. We define the *solution space* for $\boldsymbol{a}, \boldsymbol{f}$ in $\mathbb{V}$ by

$$\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V}) = \{(c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{V} : a_1\,\sigma^{m-1}(g) + \dots + a_m\,g = c_1\,f_1 + \dots + c_n\,f_n\}.$$

It follows immediately that $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$ is a vector space over $\mathbb{K}$. Moreover in [Sch02b] based on [Coh65] it is proven that this vector space has finite dimension.

Proposition 2.1. *Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$ and assume $\boldsymbol{f} \in \mathbb{F}^n$ and $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^m$. Let $\mathbb{V}$ be a subspace of $\mathbb{F}$ as a vector space over $\mathbb{K}$. Then $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$ is a vector space over $\mathbb{K}$ with maximal dimension $m+n-1$.*

Finally some notations are introduced. Let $\mathbb{F}$ be a field and $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. For $h \in \mathbb{F}$ we write $h\,\boldsymbol{f} = (h\,f_1, \dots, h\,f_n) \in \mathbb{F}^n$ and $\boldsymbol{f} \wedge h = (f_1, \dots, f_m, h) \in \mathbb{F}^{n+1}$. If $\boldsymbol{c} \in \mathbb{F}^n$, we define the vector product $\boldsymbol{c}\,\boldsymbol{f} = \sum_{i=1}^{n} c_i\,f_i$. Moreover for a function $\sigma : \mathbb{F} \to \mathbb{F}$, $\boldsymbol{a} \in \mathbb{F}^m$ and $g \in \mathbb{F}$, we introduce $\sigma_{\boldsymbol{a}} g := a_1\,\sigma^{m-1}(g) + \dots + a_m\,g \in \mathbb{F}$. This yields to the compact description $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V}) = \{\boldsymbol{c} \wedge g \in \mathbb{K}^n \times \mathbb{V} \,|\, \sigma_{\boldsymbol{a}} g = \boldsymbol{c}\,\boldsymbol{f}\}$.

In [Sch02b] several reduction techniques are introduced in order to search for a basis of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$ in ΠΣ-fields. One of the main steps is the denominator elimination strategy which pops in [Abr89b, Abr95, vH98] for the rational case $(\mathbb{K}(t), \sigma)$ as it is defined in Example 2.1. Here one is concerned to find a basis of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{K}(t))$ for some $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{K}[t]^m$ and $\boldsymbol{f} \in \mathbb{K}[t]^1$ by the following strategy that will be explained in details in Section 4.

The Denominator Elimination Strategy

1. Compute a *denominator bound* $d \in \mathbb{K}[t]^*$ such that for all elements in the solution space $\boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{K}(t))$ we have $d\, g \in \mathbb{K}[t]$.
2. Compute a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{K}[t])$ for $\boldsymbol{a'} := (\,\frac{a_1}{\sigma^{m-1}(d)}, \dots, \frac{a_{m-1}}{\sigma(d)}, \frac{a_m}{d}\,) \in \mathbb{K}(t)^m$.
3. Reconstruct a basis $\{\boldsymbol{c_1} \wedge \frac{g_1}{d}, \dots, \boldsymbol{c_l} \wedge \frac{g_l}{d}\}$ of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{K}(t))$.

Then one is able to bound the polynomial solutions in $\mathbb{K}[t]$ by degree bounds, like in [Abr89a, Pet92]. This empowers to compute a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{K}[t])$ by solving a linear system of equations.
As will be introduced in the next section, a ΠΣ-field $(\mathbb{K}(t_1)\dots(t_{e-1})(t_e), \sigma)$ with constant field $\mathbb{K}$ is constructed by a tower of transcendental extensions t_i; for further considerations we set $\mathbb{F} := \mathbb{K}(t_1)\dots(t_{e-1})$ for such a ΠΣ-field. As already mentioned above, in [Sch02b] one is concerned in finding a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t_e)))$ with $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t_e]^m$ and $\boldsymbol{f} \in \mathbb{F}[t_e]^n$. As for the rational case $\mathbb{K}(t)$, one first bounds the denominators of possible solutions. This is the basic problem on that we focus in this article.

The denominator bound problem

- Given a ΠΣ-field $(\mathbb{F}(t_e), \sigma)$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t_e]^m$ and $\boldsymbol{f} \in \mathbb{F}[t_e]^n$.
- Find $d \in \mathbb{F}[t_e]^*$ such that for all $\boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t_e))$ we have $d\, g \in \mathbb{F}[t_e]$.

If one can compute such a denominator bound d, one can reduce the problem to find a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{F}[t_e])$ for a specific $\boldsymbol{a'} \in \mathbb{F}[t_e]^m$. This denominator elimination strategy is intensively analyzed in Section 4 for ΠΣ-fields. Finally by further reduction techniques carefully considered in [Sch02b, Sch02a] one aims to find a basis of the solution space $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{F}[t_e])$.

Based on [Kar81, Bro00] this article explains how one can find such a denominator bound d in the general setting of ΠΣ-fields for the first order case, i.e. $\mathbf{0} \neq \boldsymbol{a'} \in \mathbb{F}(t_e)^2$. Moreover these ideas are generalized to the higher order case for some special cases. Together with [Sch02a], this empowers to design algorithms in [Sch02b] that allow to search for a basis of parameterized linear difference equations in ΠΣ-fields. In particular for the first order case complete algorithms are developed that compute a basis of the solution space. All these algorithms are available in a summation package called `Sigma` that is implemented in the computer algebra system Mathematica.

3. ΠΣ-Fields and Some Important Properties

This work restricts to ΠΣ-fields that are introduced in [Kar81, Kar85] and further analyzed in [Bro00, Sch01, Sch02b]. In the following the basic definition and properties are introduced that are needed in the sequel.

3.1. The Definition of ΠΣ-Extensions

In order to define ΠΣ-fields, the notion of difference field extensions is needed.

Definition 3.1. Let $(\mathbb{E}, \sigma_{\mathbb{E}})$, $(\mathbb{F}, \sigma_{\mathbb{F}})$ be difference fields. $(\mathbb{E}, \sigma_{\mathbb{E}})$ is called a *difference field extension* of $(\mathbb{F}, \sigma_{\mathbb{F}})$, if $\mathbb{F} \subseteq \mathbb{E}$ and $\sigma_{\mathbb{F}}(f) = \sigma_{\mathbb{E}}(f)$ for all $f \in \mathbb{F}$.

Example 3.1. Let $(\mathbb{K}(t), \sigma)$ be the difference field defined in Example 2.1, refined by $\mathbb{K} := \mathbb{Q}$, and consider the the field extension $\mathbb{Q}(t)(z)$ of $\mathbb{Q}(z)$ where z is transcendental over $\mathbb{Q}(t)$. Then one can define uniquely the field automorphism $\sigma' : \mathbb{Q}(t)(z) \to \mathbb{Q}(t)(z)$ where the following holds: $\sigma'(f) = \sigma(f)$ for all $f \in \mathbb{Q}(t)$ and $\sigma(z) = \alpha\, z + \beta$ for some $\alpha \in \mathbb{Q}(t)^*$ and $\beta \in \mathbb{Q}(t)$. Clearly, $(\mathbb{Q}(t)(z), \sigma')$ is a difference field extension of $(\mathbb{Q}(t), \sigma)$.

If $(\mathbb{E}, \tilde{\sigma})$ is a difference field extension of $(\mathbb{F}, \sigma)$, we will not distinguish anymore that $\sigma : \mathbb{F} \to \mathbb{F}$ and $\tilde{\sigma} : \mathbb{E} \to \mathbb{E}$ are actually different automorphisms.

Definition 3.2. $(\mathbb{F}(t), \sigma)$ is a Π-*extension* of $(\mathbb{F}, \sigma)$ if $\sigma(t) = \alpha\, t$ with $\alpha \in \mathbb{F}^*$, t is transcendental over $\mathbb{F}$ and $\text{const}_\sigma \mathbb{F}(t) = \text{const}_\sigma \mathbb{F}$.

According to [Kar81] we introduce the notion of the homogeneous group which plays an essential role in the theory of ΠΣ-fields.

Definition 3.3. The *homogeneous group* of $(\mathbb{F}, \sigma)$ is $\mathrm{H}_{(\mathbb{F},\sigma)} := \{\frac{\sigma(g)}{g} \mid g \in \mathbb{F}^*\}$.

One can easily check that $\mathrm{H}_{(\mathbb{F},\sigma)}$ forms a multiplicative group. With this notion one obtains an equivalent description of a Π-extension. This result with its proof can be found in [Kar85, Theorem 2.2] or [Sch01, Theorem 2.2.2].

Theorem 3.1. *$(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t$ where $\alpha \in \mathbb{F}^*$. Then $(\mathbb{F}(t), \sigma)$ is a* Π-extension *of $(\mathbb{F}, \sigma)$ if and only if there does not exist an $n > 0$ such that $\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$.*

Next we define Σ-extensions according to Karr's notions.

Definition 3.4. $(\mathbb{F}(t), \sigma)$ is a Σ-*extension* of $(\mathbb{F}, \sigma)$ if

1. $\sigma(t) = \alpha\, t + \beta$ with $\alpha, \beta \in \mathbb{F}^*$ and $t \notin \mathbb{F}$,
2. there does not exist a $g \in \mathbb{F}(t) \setminus \mathbb{F}$ with $\frac{\sigma(g)}{g} \in \mathbb{F}$, and
3. for all $n \in \mathbb{Z}^*$ we have that $\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)} \Rightarrow \alpha \in \mathrm{H}_{(\mathbb{F},\sigma)}$.

Example 3.2. In Example 2.1 $(\mathbb{K}(t), \sigma)$ is a Σ-extension of $(\mathbb{K}, \sigma)$.

In particular the following properties in Σ-extensions hold that are essential for this work. This result is a direct consequence of [Sch01, Theorem 2.2.3] which is a corrected version of [Kar81, Theorem 3] or [Kar85, Theorem 2.3].

Theorem 3.2. *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$. Then $(\mathbb{F}(t), \sigma)$ is canonically defined by $\sigma(t) = \alpha\, t + \beta$ for some $\alpha, \beta \in \mathbb{F}^*$, t is transcendental over $\mathbb{F}$ and $\text{const}_\sigma \mathbb{F}(t) = \text{const}_\sigma \mathbb{F}$.*

Now we are ready to define ΠΣ-extension.

Definition 3.5. $(\mathbb{F}(t), \sigma)$ is called a ΠΣ-extension of $(\mathbb{F}, \sigma)$, if $(\mathbb{F}(t), \sigma)$ is a Π- or a Σ-extension of $(\mathbb{F}, \sigma)$.

Clearly if $(\mathbb{F}, \sigma)$ is a difference field, also $(\mathbb{F}, \sigma^k)$ is a difference field for any $k \in \mathbb{Z}$. Moreover if $(\mathbb{F}(t), \sigma)$ is a ΠΣ-extension of $(\mathbb{F}, \sigma)$, also $(\mathbb{F}(t), \sigma^k)$ is a difference field extension of $(\mathbb{F}, \sigma^k)$.

Example 3.3. Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t$ with $\alpha \in \mathbb{F}^*$ and $k \in \mathbb{Z}$. Then for $k \geq 0$ we have $\sigma^k(t) = t \prod_{i=0}^{k-1} \sigma^i(\alpha)$, whereas for $k < 0$ we have $\sigma^k(t) = t \prod_{i=1}^{k} \sigma^{-i}(1/\alpha)$.

This motivates us to the following definition.

Definition 3.6. Let $(\mathbb{F}, \sigma)$ be a difference field, $f \in \mathbb{F}^*$ and $k \in \mathbb{Z}$. The *σ-factorial* $(f)_k$ is defined by $\prod_{i=0}^{k-1} \sigma^i(f)$, if $k \geq 0$, and by $\prod_{i=1}^{k} \sigma^{-i}(1/f)$, if $k < 0$.

The next lemma will be used over and over again; it gives the link between ΠΣ-extensions and its domain of rational functions. The proof is straightforward.

Lemma 3.1. *Let* $(\mathbb{F}(t), \sigma)$ *be a* ΠΣ*-extension of* $(\mathbb{F}, \sigma)$. *Then* $\mathbb{F}(t)$ *is a field of rational functions over* $\mathbb{K}$. *Furthermore,* σ *is an automorphism of the polynomial ring* $\mathbb{F}[t]$, *i.e.* $(\mathbb{F}[t], \sigma)$ *is a difference ring extension of* $(\mathbb{F}, \sigma)$. *Additionally, for all* $f, g \in \mathbb{F}[t]$ *we have that* $\deg(\sigma(f)) = \deg(f)$ *and* $\gcd(\sigma(f), \sigma(g)) = \sigma(\gcd(f, g))$.

We need the following notations for such a polynomial ring $\mathbb{F}[t]$ and its quotient field $\mathbb{F}(t)$. By $\deg(f)$ we denote the degree of $f \in \mathbb{F}[t]$; by convention we set $\deg(0) := -\infty$. Furthermore, if $f = \sum_{i=0}^{n} f_i\, t_i \in \mathbb{F}[t]$, the i-th *coefficient* f_i of f is denoted by $[f]_i$, i.e. $[f]_i = f_i$. If $i > n$, we have $[f]_i = 0$. We define the *order* of $f \in \mathbb{F}[t]$, $\operatorname{ord}(f)$, as the maximal $m \geq 0$ such that $t^m \mid q$. For the zero-polynomial we define $\operatorname{ord}(0) := -1$. Moreover we say that $\frac{p}{q} \in \mathbb{F}(t)$ is in *reduced representation* if $p, q \in \mathbb{F}[t]$, $\gcd(p, q) = 1$ and q is monic. Let $f = \frac{p}{q}$ be in reduced representation. Then we define the denominator of f by $\operatorname{den}(f) = q$.

3.2. ΠΣ-Fields and Some Properties

For the definition of ΠΣ-fields properties on the constant field are required.

Definition 3.7. A field $\mathbb{K}$ is called *computable*, if

- for any $k \in \mathbb{K}$ one is able to decide, if $k \in \mathbb{Z}$,
- polynomials in the polynomial ring $\mathbb{K}[t_1, \dots, t_n]$ can be factored over $\mathbb{K}$ and
- one knows how to compute a basis of $\left\{(n_1, \dots, n_k) \in \mathbb{Z}^k \mid c_1^{n_1} \cdots c_k^{n_k} = 1\right\}$ which is a submodule of $\mathbb{Z}^k$ over $\mathbb{Z}$ for any $(c_1, \dots, c_n) \in \mathbb{K}^n$.

Lemma 3.2. *Any field of rational functions* $\mathbb{Q}(n_1, \dots, n_r)$ *is computable.*

Finally ΠΣ-fields are essentially defined by ΠΣ-extensions. Unlike Karr's definition in this work we force *additionally* that the constant field is computable.

Definition 3.8. Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$. $(\mathbb{F}, \sigma)$ is called a ΠΣ-*field* over $\mathbb{K}$, if $\mathbb{K}$ is computable, $\mathbb{F} := \mathbb{K}(t_1)\dots(t_n)$ for $n \geq 0$ and $(\mathbb{F}(t_1, \dots, t_{i-1})(t_i), \sigma)$ is a ΠΣ-extension† of $(\mathbb{F}(t_1, \dots, t_{i-1}), \sigma)$ for all $1 \leq i \leq n$.

Example 3.4. Note that the difference field $(\mathbb{Q}(t), \sigma)$, defined in Example 2.1 with $\mathbb{K} := \mathbb{Q}$, is a ΠΣ-field over $\mathbb{Q}$. Now consider the difference field extension $(\mathbb{Q}(t)(z), \sigma)$ of $(\mathbb{Q}(t), \sigma)$ as it is constructed in Example 3.1 with $\sigma(z) = \alpha\, z + \beta$ for some $\alpha \in \mathbb{Q}(t)^*$ and $\beta \in \mathbb{Q}(t)$. Then one can show that $(\mathbb{Q}(t)(z), \sigma)$ is a ΠΣ-extension of $(\mathbb{Q}(t), \sigma)$, if one chooses $(\alpha, \beta) = (t + 1, 0)$ or $(\alpha, \beta) = \left(1, \frac{1}{t+1}\right)$. Hence in both instances $(\mathbb{Q}(t)(z), \sigma)$ are ΠΣ-fields over $\mathbb{Q}$.

ΠΣ-fields are designed in such a way that the following problem, stated in form of a theorem, can be solved. Its proof follows from [Kar81, Theorem 9].

Theorem 3.3. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-field and assume $(f_1, \dots, f_n) \in \mathbb{F}(t)^n$. Then there exists an algorithm that computes a finite basis of the submodule $\left\{(z_1, \dots, z_n) \in \mathbb{Z}^n \mid f_1^{z_1} \dots f_n^{z_n} \in \mathrm{H}_{(\mathbb{F},\sigma)}\right\}$ of $\mathbb{Z}^n$.*

This computational aspect is essential in Section 7 in order to solve the denominator bound problem for first order linear difference equations in ΠΣ-fields. The next theorem is taken from [Kar85, Theorem 4]. This result allows to generalize the denominator bound for first order linear difference equations to higher order linear difference equations for some specific cases in Subsection 7.4.

Theorem 3.4. *If $(\mathbb{F}, \sigma)$ be a ΠΣ-field, $(\mathbb{F}, \sigma^k)$ is a ΠΣ-field for all $k \in \mathbb{Z}^*$.*

3.3. Permutation Isomorphisms in ΠΣ-Fields

In Section 8 we provide an algorithm that solves the denominator bound problem for parameterized first order linear difference equations in a given ΠΣ-field. In particular some properties of this algorithm will be shown that are needed for further investigations in the theory of ΠΣ-fields and indefinite summation. These properties are based on isomorphisms that are introduced in the following.

Definition 3.9. The difference fields $(\mathbb{F}, \sigma)$, $(\tilde{\mathbb{F}}, \tilde{\sigma})$ are *isomorph* if there is a field isomorphism $\tau : \mathbb{F} \to \tilde{\mathbb{F}}$ with $\tau\, \sigma = \tilde{\sigma}\, \tau$. τ is called *difference field isomorphism.*

The following lemma follows immediately by the commutativity of $\tau\, \sigma = \tilde{\sigma}\, \tau$.

Lemma 3.3. *Let $(\mathbb{F}, \sigma)$, $(\tilde{\mathbb{F}}, \tilde{\sigma})$ be difference fields and $\tau : \mathbb{F} \to \tilde{\mathbb{F}}$ be a difference field isomorphism. Then $g \in \mathrm{H}_{(\mathbb{F},\sigma)}$ if and only if $\tau(g) \in \mathrm{H}_{(\tilde{\mathbb{F}},\tilde{\sigma})}$ for any $g \in \mathbb{F}$.*

In this work we consider the following almost trivial difference field isomorphism of ΠΣ-fields which basically permutates the extensions in the tower of extensions.

†For the case $i = 0$ this means that $(\mathbb{F}(t_1), \sigma)$ is a ΠΣ-extension of $(\mathbb{F}, \sigma)$.

Definition 3.10. Let $(\mathbb{F}(s_1)\dots(s_e),\sigma)$ and $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be ΠΣ-fields and $\tau:\mathbb{F}(s_1)\dots(s_e)\to\mathbb{F}(t_1)\dots(t_e)$ be a difference field isomorphism. If for all $f\in\mathbb{F}$ we have $\tau(f)=f$ and if there is a bijective map $\phi:X\to X$ with $X:=\{1,\dots,e\}$ such that $\tau(s_i)=s_i=t_{\phi(i)}$ for all $1\leq i\leq e$, we say that $(\mathbb{F}(s_1,\dots,s_e),\sigma)$ and $(\mathbb{F}(t_1,\dots,t_e),\sigma)$ are *isomorph by a permutation.*

If we assume that $(\mathbb{G}),\sigma)$ and $(\mathbb{H},\sigma)$ are ΠΣ-fields which are isomorph by a permutation, we can write $\mathbb{G}$ and $\mathbb{H}$ as fields of rational functions, say $\mathbb{G}=\mathbb{F}(s_1,\dots,s_e)$ and $\mathbb{H}=\mathbb{F}(t_1,\dots,t_e)$ for some $e\geq 0$. Moreover there is a difference field isomorphism $\tau:\mathbb{G}\to\mathbb{H}$ defined in the following way: $\tau(f)=f$ for all $f\in\mathbb{F}$ and $\tau(s_i)=s_i=t_{\phi(i)}$ for some permutation ϕ. This means that we can reorder the extensions in $\mathbb{G}$ by the permutation ϕ which yields to the ΠΣ-field $(\mathbb{H},\sigma)$. Since for any $f\in\mathbb{G}$ we have $f=\tau(f)\in\mathbb{H}$, we will ignore the difference field automorphism τ and interpret any f in $\mathbb{H}$ also as an element in $\mathbb{G}$ and vice versa.

4. The Denominator Elimination Strategy

In this section the denominator elimination strategy, that was sketched in Section 2 for the rational case $(\mathbb{K}(t),\sigma)$, will be generalized for a ΠΣ-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$. Let $\mathbf{0}\neq\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}(t)^m$ with $a_1\,a_m\neq 0$ and $\boldsymbol{f}\in\mathbb{F}(t)^n$. Here I will give the main idea how one can achieve the reduction

$$\begin{tikzcd} \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t)) \arrow[d, shift right, "\text{denominator elimination}"'] \\ \mathrm{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{F}[t]) \arrow[u, shift right, "\text{by denominator bounding}"'] \end{tikzcd} \tag{1}$$

for a specific $\boldsymbol{a'}\in\mathbb{F}[t]^m$. Then with [Sch02b, Sch02a] one can apply further reduction techniques in order to find a basis of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{F}[t])$ in the polynomial ring $\mathbb{F}[t]$. Given this basis, one finally reconstructs a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$. In this reduction the simple Lemma 4.1 gives the main idea.

Lemma 4.1. *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$ and $\mathbb{F}=\mathbb{W}\oplus\mathbb{V}$ be a direct sum of subspaces $\mathbb{V}$ and $\mathbb{W}$ of $\mathbb{F}$ over $\mathbb{K}$. Let $\mathbf{0}\neq\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}^m$, $\boldsymbol{f}\in\mathbb{F}^n$, and let $d\in\mathbb{F}^*$ be such that for all $\boldsymbol{c}\wedge g\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$ we have $d\,g\in\mathbb{W}$; set $\boldsymbol{a'}:=\left(\frac{a_1}{\sigma^{m-1}(d)},\dots,\frac{a_{m-1}}{\sigma(d)},\frac{a_m}{d}\right)$. Then $\{\boldsymbol{c_1}\wedge\frac{g_1}{d},\dots,\boldsymbol{c_l}\wedge\frac{g_l}{d}\}$ is a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$ if and only if $\{\boldsymbol{c_1}\wedge g_1,\dots,\boldsymbol{c_l}\wedge g_l\}$ is a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$.*

Proof: We have

$$\begin{aligned}\sigma_{\boldsymbol{a}}g&=a_1\,\frac{\sigma^{m-1}(d)}{\sigma^{m-1}(d)}\,\sigma^{m-1}(g)+\dots+a_{m-1}\,\frac{\sigma(d)}{\sigma(d)}\,\sigma(g)+\frac{d}{d}\,a_m\,g\\&=\frac{a_1}{\sigma^{m-1}(d)}\,\sigma^{m-1}(g\,d)+\dots+\frac{a_{m-1}}{\sigma(d)}\,\sigma(g\,d)+\frac{a_m}{d}\,d\,g=\sigma_{\boldsymbol{a'}}(g\,d)\end{aligned}$$

and thus $\boldsymbol{c}\wedge\frac{g}{d}\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$ if and only if $\boldsymbol{c}\wedge g\in\mathrm{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{W})$. Since any set $\{\boldsymbol{c_1}\wedge\frac{g_1}{d},\dots,\boldsymbol{c_l}\wedge\frac{g_l}{d}\}\subseteq\mathbb{K}^n\times\mathbb{F}$ is linearly independent over $\mathbb{K}$ if and only if $\{\boldsymbol{c_1}\wedge g_1,\dots,\boldsymbol{c_l}\wedge g_l\}\subseteq\mathbb{K}^n\times\mathbb{F}$ is linearly independent, the lemma follows. □

Next we introduce the subset $\mathbb{F}(t)^{(frac)}$ of $\mathbb{F}(t)$ as

$$\mathbb{F}(t)^{(frac)} := \{\frac{p}{q} \in \mathbb{F}(t) \mid \tfrac{p}{q} \text{ is in reduced representation and} \deg(p) < deg(q)\}.$$

It follows immediately that $\mathbb{F}[t]$ and $\mathbb{F}(t)^{(frac)}$ are subspaces of $\mathbb{F}(t)$ over $\mathbb{K}$. In particular by polynomial division with remainder the next statement holds.

Lemma 4.2. *Let $\mathbb{F}(t)$ be a field of rational functions and consider $\mathbb{F}[t]$ and $\mathbb{F}(t)^{(frac)}$ as subspaces of $\mathbb{F}(t)$ over $\mathbb{K}$. Then we have $\mathbb{F}(t) = \mathbb{F}[t] \oplus \mathbb{F}(t)^{(frac)}$.*

Then looking at Lemma 4.1, the basic idea is to compute a particular $d \in \mathbb{F}[t]^*$ such that

$$\forall \boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t] \oplus \mathbb{F}(t)^{(frac)}) : \; d\, g \in \mathbb{F}[t]. \tag{2}$$

It is immediate that such a specific $d \in \mathbb{F}[t]^*$ bounds the denominator.

Definition 4.1. Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$, $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. Then $d \in \mathbb{F}[t]^*$ is a *denominator bound* of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ if (2) holds.

By combining Lemmas 4.1 and 4.2 one immediately obtains the following theorem which explains how one can achieve the reduction sketched in (1).

Theorem 4.1. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$, $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. Let $d \in \mathbb{F}[t]^*$ be a denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ and define $\boldsymbol{a}' := (\frac{a_1}{\sigma^{m-1}(d)}, \dots, \frac{a_{m-1}}{\sigma(d)}, \frac{a_m}{d})$. Then $\{\boldsymbol{c_1} \wedge \frac{g_1}{d}, \dots, \boldsymbol{c_l} \wedge \frac{g_l}{d}\}$ is a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ if and only if $\{\boldsymbol{c_1} \wedge g_1, \dots, \boldsymbol{c_l} \wedge g_l\}$ is a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$.*

Example 4.1. Construct the ΠΣ-field $(\mathbb{Q}(t)(z), \sigma)$ as in Example 3.4 with $\sigma(z) = (t+1)\, z$ and take $\boldsymbol{a} := (a_1, a_2)$ with $a_1 := (\, t^5\, (1+t)^5\, (1+z)\, (1+(1+t)\, z)\,)$ and $a_2 := (\, -(2+t)(3+t)^2\, (t+z)\, (-t+t^2+z)\,)$. As will be illustrated later, we find the denominator bound $d := z^4\, (1+z)\, (t+z)^2\, (-t+t^2+z)$ of $\mathrm{V}(\boldsymbol{a}, (0)\, , \mathbb{Q}(t)(z))$. Then by further reduction techniques we can compute the basis $\{(\, 0, \frac{t^3\, (1+t)^3\, (2+t)^2}{t-1}\,), (1,0)\}$ of $\mathrm{V}(\boldsymbol{a'}, (0)\, , \mathbb{Q}(t)[z])$ with $\boldsymbol{a'} := (\, \frac{a_1}{\sigma(d)}, \frac{a_2}{d}\,)$. Hence we obtain finally the basis $\{(\, 0 \frac{t^5\, (1+t)^3\, (2+t)^2}{d}\,), (1,0)\}$ of $\mathrm{V}(\boldsymbol{a}, (0)\, , \mathbb{Q}(t)(z))$ by Theorem 4.1.

In the remaining article we deal with the problem to compute a denominator bound $d \in \mathbb{F}[t]^*$ of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$. In this work this problem splits into two subproblems that are motivated in the next section.

5. Two Subproblems for the Denominator Elimination

Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$. In the following we refine the decomposition of $\mathbb{F}(t) = \mathbb{F}[t] \oplus \mathbb{F}(t)^{(frac)}$ (Proposition 5.1) by splitting $\mathbb{F}(t)^{(frac)}$ further into a direct sum of two subspaces. Then the denominator bound problem can be divided into two subproblems which amounts to eliminate the different parts of the direct sum components of the solution range $\mathbb{F}(t)$.

5.1. A Refined Decomposition of the Solution Range

Let $\mathbb{F}(t)$ be the field of rational functions, i.e. t is transcendental over a field $\mathbb{F}$. Furthermore let $\mathbb{K}$ be a subfield of $\mathbb{F}$ and consider $\mathbb{F}(t)$ as a vector space over $\mathbb{K}$. In this subsection we refine the decomposition $\mathbb{F}(t) = \mathbb{F}[t] \oplus \mathbb{F}(t)^{(frac)}$. Here we use techniques [vzGG99] that are needed to compute partial fraction decompositions. First by a refined version of the extended Euclidean algorithm we obtain the following decomposition.

Lemma 5.1. *Any $f \in \mathbb{F}(t)^{(frac)}$ can be uniquely represented in the form*

$$f = \frac{g}{t^k} + \frac{p}{q}$$

where $g \in \mathbb{F}[t]$, $\frac{p}{q} \in \mathbb{F}(t)$ is in reduced representation and $k > 0$ such that $\deg(g) < k$ *with $t \nmid g$, and* $\deg(p) < \deg(q)$ *with $t \nmid q$.*

Furthermore the following result follows by polynomial division with remainder.

Lemma 5.2. *Let $f \in \mathbb{F}[t]$ and $d \geq 0$ such that* $\deg(f) < d$ *and $d \nmid f$. Then $\frac{f}{t^d}$ can be uniquely represented in the form $\frac{f}{t^d} = \sum_{i=1}^{d} \frac{f_i}{t^i}$ where $f_i \in \mathbb{F}$.*

Next we define

$$\mathbb{F}(t)^{(fracpart)} := \{\frac{p}{q} \in \mathbb{F}(t)^{(frac)};\ t \nmid q\}$$

and obtain a refined decomposition of $\mathbb{F}(t)$.

Proposition 5.1. *Consider $\mathbb{F}[t]$, $\mathbb{F}[1/t] \setminus \mathbb{F}^*$ and $\mathbb{F}(t)^{(fracpart)}$ as subspaces of $\mathbb{F}(t)$ over $\mathbb{K}$. Then we have $\mathbb{F}(t)^{(frac)} = \mathbb{F}[1/t] \setminus \mathbb{F}^*) \oplus \mathbb{F}(t)^{(fracpart)}$ and hence $\mathbb{F}(t) = \mathbb{F}[t] \oplus (\mathbb{F}[1/t] \setminus \mathbb{F}^*) \oplus \mathbb{F}(t)^{(fracpart)}$.*

Proof: This follows immediately by Lemmas 4.2, 5.1 and 5.2. □

5.2. Decomposition of the Solution Range $\mathbb{F}(t)$ in a ΠΣ-extension

The final goal is to divide the denominator bound problem into two subproblems that will be motivated by the following results in the theory of ΠΣ-fields [Kar81]. First we introduce the period in a difference field extension.

Definition 5.1. Let $(\mathbb{E}, \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$. The *period* of $f \in \mathbb{E}^*$ is defined by

$$\mathrm{per}_{(\mathbb{F},\sigma)}(f) := \begin{cases} 0 & \text{if } \forall p > 0 :\ \frac{\sigma^p(f)}{f} \notin \mathbb{F} \\ \min\{p > 0 \,|\, \sigma^p(f)/f \in \mathbb{F}\} & \text{otherwise.} \end{cases}$$

Example 5.1. First consider the Σ-extension $(\mathbb{Q}(t), \sigma)$ of $(\mathbb{Q}, \sigma)$ canonically defined by $\sigma(t) = t + 1$ as in Example 2.1. Since for all $f \in \mathbb{Q}(t) \setminus \mathbb{Q}$ and for all $p > 0$ we have $\frac{\sigma^p(f)}{f} \notin \mathbb{Q}$, it follows that $\mathrm{per}_{(\mathbb{Q},\sigma)}(f) = 0$ for all $f \in \mathbb{Q}(t)$. Second consider the Π-extension $(\mathbb{Q}(t), \sigma)$ of $(\mathbb{Q}, \sigma)$ canonically defined by $\sigma(t) = 2\,t$. Then for all $i \in \mathbb{Z}$ we have $\frac{\sigma(t^i)}{t^i} = 2^i \in \mathbb{Q}$, and hence $\mathrm{per}_{(\mathbb{Q},\sigma)}(t^i) = 1$ for all $i \in \mathbb{Z}$.

The main observation in this section is Theorem 5.1 which states that all elements in a difference field have period 0 or 1. This theorem is included in [Kar81, Theorem 4] and is essentially the same as [Kar85, Lemma 3.2]. Compact proofs of this result can be found in [Bro00, Corollary 1 and 2] and are combined in the notions of ΠΣ-extensions in [Sch01, Theorem 2.2.4].

Theorem 5.1. *If $(\mathbb{F}(t),\sigma)$ is a Σ-extension of $(\mathbb{F},\sigma)$ then for all $f\in\mathbb{F}(t)\setminus\mathbb{F}$ we have $\operatorname{per}_{(\mathbb{F},\sigma)}(f)=0$ and $\big\{f\in\mathbb{F}(t)^*\,|\,\operatorname{per}_{(\mathbb{F},\sigma)}(f)=1\big\}=\mathbb{F}^*$. If $(\mathbb{F}(t),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$ then for all $f\in\mathbb{F}(t)^*$ we have $\operatorname{per}_{(\mathbb{F},\sigma)}(f)\in\{0,1\}$ and $\big\{f\in\mathbb{F}(t)^*\,|\,\operatorname{per}_{(\mathbb{F},\sigma)}(f)=1\big\}=\{h\,t^m\,|\,h\in\mathbb{F}^*\ \&\ m\in\mathbb{Z}\}$.*

In the sequel let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$. In this setting we consider a refined notion of the period.

Definition 5.2. Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$. Then $h\in\mathbb{F}[t]^*$ has *pure period* $m\in\{0,1\}$, if for all $f\in\mathbb{F}[t]\setminus\mathbb{F}$ with $f|h$ we have $\operatorname{per}_{(\mathbb{F},\sigma)}(f)=m$.

By Theorem 5.1 the following corollary follows immediately.

Corollary 5.1. *Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F}(t),\sigma)$. Any $f\in\mathbb{F}^*$ has pure period 0 and pure period 1. If $(\mathbb{F}(t),\sigma)$ is a Σ-extension, all $f\in\mathbb{F}[t]^*$ have pure period 0. Otherwise, if $(\mathbb{F}(t),\sigma)$ is a Π-extension, $f\in\mathbb{F}[t]^*$ has pure period 0 if and only if $t\nmid f$.*

In particular in a Π-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ we have that $f\in\mathbb{F}[t]^*$ has pure period 1 if and only if it has period 1. Finally we define two subsets of $\mathbb{F}(t)^{(frac)}$:

Definition 5.3. Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$. We define

$$\mathbb{F}(t)^{(0)}:=\left\{\frac{p}{q}\in\mathbb{F}(t)^{(frac)}\,|\,q\text{ has pure period }0\right\}\text{ and}$$
$$\mathbb{F}(t)^{(1)}:=\left\{\frac{p}{q}\in\mathbb{F}(t)^{(frac)}\,|\,\operatorname{per}_{(\mathbb{F},\sigma)}(q)=1\right\}.$$

As one can easily see, these two subsets are subspaces of $\mathbb{F}(t)^{(frac)}$ over $\mathbb{K}$. More precisely those sets are specified in the following way by Theorem 5.1.

Corollary 5.2. *If $(\mathbb{F}(t),\sigma)$ is a Σ-extension of $(\mathbb{F},\sigma)$ then $\mathbb{F}(t)^{(1)}=\{0\}$ and $\mathbb{F}(t)^{(0)}=\mathbb{F}(t)^{(frac)}$; if $(\mathbb{F}(t),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$ then $\mathbb{F}(t)^{(1)}=\mathbb{F}[1/t]\setminus\mathbb{F}^*$ and $\mathbb{F}(t)^{(0)}=\mathbb{F}(t)^{(fracpart)}$.*

Moreover $\mathbb{F}(t)^{(frac)}$ can be decomposed in a direct sum of $\mathbb{F}(t)^{(0)}$ and $\mathbb{F}(t)^{(1)}$ over $\mathbb{K}$. This is a direct consequence of Proposition 5.1 and Corollary 5.2.

Corollary 5.3. *Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ and consider $\mathbb{F}[t]$, $\mathbb{F}(t)^{(1)}$ and $\mathbb{F}(t)^{(0)}$ as subspaces of $\mathbb{F}(t)$ over $\mathbb{K}$. Then we have $\mathbb{F}(t)=\mathbb{F}[t]\oplus\mathbb{F}(t)^{(1)}\oplus\mathbb{F}(t)^{(0)}$.*

Finally this decomposition allows us to split an element $g \in \mathbb{F}(t)$ into three parts.

Definition 5.4. Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ decompose $h \in \mathbb{F}(t)$ by $h_p + h_1 + h_0 \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)} \oplus \mathbb{F}(t)^{(0)}$. h_p is called *polynomial part*, h_1 is called *fractional part with period 1* and h_0 is called *fractional part with pure period 0.*

Example 5.2. Construct the ΠΣ-field $(\mathbb{Q}(t)(z), \sigma)$ as in Example 3.4 with $\sigma(z) = (t+1)\,z$. Then the upper braces indicate how f splits into its polynomial part and fractional parts with pure period 0 and 1.

$$f = \frac{t + z + t\,z + 2\,z^2 + t\,z^3 + t\,z^4}{z^2(t+z)} = \underbrace{\overbrace{z\,t}^{\mathbb{Q}(t)[z]}}_{\mathbb{Q}(t)[z]} + \underbrace{\overbrace{\frac{1}{z} + \frac{t}{z^2}}^{\mathbb{Q}(t)(z)^{(1)}} + \overbrace{\frac{1}{z+1}}^{\mathbb{Q}(t)(z)^{(0)}}}_{\mathbb{Q}(t)(z)^{(0)}} \in \mathbb{Q}(t)(z)$$

Now construct the ΠΣ-field $(\mathbb{Q}(t)(z), \sigma)$ as in Example 3.4 with $\sigma(z) = z + \frac{1}{t+1}$. Then the lower braces indicate how the rational function splits into its polynomial part and fractional parts with pure period 0 only.

5.3. Period 0 and 1 Denominator Bounds

As already motivated in Section 4, in this article we try to use the denominator elimination strategy (1) by finding a denominator bound $d \in \mathbb{F}[t]^*$ of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ for given $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$ in a ΠΣ-field $(\mathbb{F}(t), \sigma)$. Within this reduction we will eliminate separately the fractional parts with pure period 0 and with period 1 as one can see in the reductions (3).

$$\begin{array}{ccc} \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)} \oplus \mathbb{F}(t)^{(0)}) & & \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t] \oplus \mathbb{F}(t)^{(0)} \oplus \mathbb{F}(t)^{(1)}) \\ \Big\downarrow \begin{array}{c}\text{period } 0\\ \text{elimination}\end{array} \Big\uparrow & & \Big\downarrow \begin{array}{c}\text{period } 1\\ \text{elimination}\end{array} \Big\uparrow \\ \mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)}) & & \mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}[t] \oplus \mathbb{F}(t)^{(0)}) \end{array} \tag{3}$$

More precisely, in the following two sections we are interested in finding a $d_0 \in \mathbb{F}[t]^*$ and a $d_1 \in \mathbb{F}[t]^*$ such that

$$\forall \boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)} \oplus \mathbb{F}(t)^{(0)}) : \; d_0\, g \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)} \text{ and} \tag{4}$$

$$\forall \boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t] \oplus \mathbb{F}(t)^{(0)} \oplus \mathbb{F}(t)^{(1)}) : \; d_1\, g \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(0)}. \tag{5}$$

Given such d_0 and d_1, one can apply Lemma 4.1 which leads to the reductions as sketched in (3). In the sequel such a d_0 will be called *period* 0 and d_1 will be called *period* 1 *denominator bound* of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$. Moreover, one can solve the denominator bound problem by those period 0 and 1 denominator bounds which will be shown in the following.

Lemma 5.3. *Let* $\mathbb{F}(t)$ *be a field of rational functions over* $\mathbb{F}$ *and decompose* $f = f_1 + f_2 + f_3 \in \mathbb{F}[1/t] \setminus \mathbb{F}^* \oplus \mathbb{F}[t] \oplus \mathbb{F}(t)^{(fracpart)}$. *Then* $\mathrm{ord}(\mathrm{den}(f)) = \mathrm{ord}(\mathrm{den}(f_1))$.

Proof: Let $f_3 = \frac{p}{q}$ be in reduced representation, in particular we may assume that $t \nmid q$. If $\text{ord}(\text{den}(f_1)) = 0$ then $f_1 = 0$, hence $f = f_2 + f_3 = f_2 + \frac{p}{q} = \frac{f_2\,q+p}{q}$ and thus $\text{ord}(\text{den}(f)) = 0$. Otherwise, if $\text{ord}(\text{den}(f_1)) > 0$, let $f_1 = \frac{u}{t^d}$ be in reduced representation, in particular $d \geq 1$ and $t \nmid u \neq 0$. Then we may write $f = f_1+f_2+f_3 = \frac{u}{t^d}+f_2+\frac{p}{q} = \frac{u\,q+f_2\,t^d\,q+p\,t^d}{t^d\,q}$ where $t \nmid u\,q$, thus $t \nmid u\,q+f_2\,t^d\,q+p\,t^d$ and hence $\text{ord}(\text{den}(f)) = d = \text{ord}(\text{den}(f_1))$. □

Lemma 5.4. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$, $\boldsymbol{f} \in \mathbb{F}[t]^n$ and $d_0 \in \mathbb{F}[t]^*$ be a period 0 denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$. Then for any $h \in \mathbb{F}[t]^*$ and any $\boldsymbol{c}\wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ we have $h\,d_1\,g \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)}$.*

Proof: If $(\mathbb{F}(t), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$ then $\mathbb{F}(t)^{(1)} = \{0\}$ by Corollary 5.2, hence $\mathbb{F}(t) = \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)}$ and thus the lemma holds. Otherwise assume the Π-extension case. Then by Corollary 5.2 we have that $\mathbb{F}(t)^{(1)} = \mathbb{F}[1/t] \setminus \mathbb{F}^*$. Hence by (4) and Lemma 5.3 it follows that $d_1\,g = \frac{a}{t^k}$ for some $a \in \mathbb{F}[t]^*$ with $t \nmid a$ and $k \in \mathbb{Z}$. Hence for any $h \in \mathbb{F}[t]^*$ we have $h\,d_1\,g = \frac{a'}{t^l}$ for some $a' \in \mathbb{F}[t]$ with $t \nmid a'$ and $l \leq k$. Thus by polynomial division with remainder $h\,d_1\,g \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)}$ which proves the lemma. □

Lemma 5.5. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$, $\boldsymbol{f} \in \mathbb{F}[t]^n$ and $d_0 \in \mathbb{F}[t]^*$ be a period 0 denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$. Then for any $h \in \mathbb{F}[t]^*$ and any $\boldsymbol{c}\wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ we have $h\,d_0\,g \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(0)}$.*

Proof: If $(\mathbb{F}(t), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$ then $\mathbb{F}(t)^{(1)} = \{0\}$ by Corollary 5.2 and hence the lemma holds. Otherwise suppose the Π-extension case and let $d_0\,g = \frac{a}{b}$ be in reduced representation. Since $\mathbb{F}(t)^{(1)} = \mathbb{F}[1/t] \setminus \mathbb{F}^*$ by Corollary 5.2, it follows together with (5) and Lemma 5.3 that $t \nmid b$. Now an arbitrary $h \in \mathbb{F}[t]^*$ and write $h\,d_0\,g = \frac{a'}{b'}$ in reduced representation. Then $t \nmid b'$. Hence by polynomial division with remainder $h\,d_0\,g \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(0)}$ which proves the lemma. □
Combining these results gives a recipe how a denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ can be obtained.

Corollary 5.4. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. Furthermore let $d_0, d_1 \in \mathbb{F}[t]^*$ be period 0 and 1 denominator bounds of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ respectively. Then $d_0\,d_1$ is a denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$.*

Proof: Let $d := d_0\,d_1$ and $\boldsymbol{c}\wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$. By Lemmas 5.4 and 5.5 we have $d\,g \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(0)}$ and $d\,g \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)}$. Hence $d\,g \in \mathbb{F}[t]$ by Corollary 5.3. □

Example 5.3. In Example 4.1 the denominator bound $d = d_0\,d_1$ splits into its period 0 denominator bound $d_0 := (1+z)\,(t+z)^2\,(-t+t^2+z)$ with pure period 0 and its period 1 denominator bound $d_1 := z^4$ with period 1.

In the next section an algorithm is presented that computes a period 0 denominator bound $d_0 \in \mathbb{F}[t]^*$ in a ΠΣ-field $(\mathbb{F}(t), \sigma)$. In particular it will turn out that this d_0 not only bounds the fractional part of the solutions with pure period 0, but itself has pure period 0. Moreover in Section 7 we develop an algorithm that computes a period 1 denominator bound $d_1 \in \mathbb{F}[t]^*$ with period 1 in the ΠΣ-field $(\mathbb{F}(t),\sigma)$ for the special case $\boldsymbol{a} \in \mathbb{F}[t]^2$. In addition we will find a period 1 denominator bound for several special cases of $\boldsymbol{a} \in \mathbb{F}[t]^m$, namely Situations 7.1 and 7.3, which enables us to solve the denominator bound problem for those solution spaces $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$.

6. An Algorithm for the Period 0 Denominator Bound

In the sequel an algorithm is developed that finds a period 0 denominator bound. More precisely one is concerned to compute a polynomial $d_0 \in \mathbb{F}[t]^*$ such that (4) holds in a ΠΣ-field $(\mathbb{F}(t),\sigma)$. Here we restrict to the case $\boldsymbol{a} = (a_1,\dots,a_m) \in \mathbb{F}^m$ with $m \geq 2$, $a_1\,a_m \neq 0$ and $\boldsymbol{f} \in \mathbb{F}^n$.

6.1. The Spread and Specification

In the following let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$. As it will turn out the spread function will play an essential role in order to compute a period 0 denominator bound. Here we follow the definition of [Bro00] restricting to the special case of ΠΣ-extensions.

Definition 6.1. Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$ and $f,g \in \mathbb{F}[t]^*$. We define the *spread* of f and g w.r.t. σ as

$$\mathrm{spread}_{(\mathbb{F},\sigma)}(f,g) = \{m \geq 0 \mid \deg(\gcd(f,\sigma^m(g))) > 0\}.$$

The next proposition states when the set $\mathrm{spread}_{(\mathbb{F},\sigma)}(f,g)$ for $f,g \in \mathbb{F}[t]$ is finite.

Proposition 6.1. *Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$ and $f,g \in \mathbb{F}[t]^*$. Then $\mathrm{spread}_{(\mathbb{F},\sigma)}(f,g)$ is finite if and only if $(\mathbb{F}(t),\sigma)$ is a Σ-extension of $(\mathbb{F},\sigma)$ or $t \nmid \gcd(f,g)$.*

Proof: By [Bro00, Theorem 6] it follows that $\mathrm{spread}_{(\mathbb{F},\sigma)}(f,g)$ is an infinite set if and only if g has a nontrivial factor $p \in \mathbb{F}[t] \setminus \mathbb{F}$ with $\mathrm{per}_{(\mathbb{F},\sigma)}(p) \neq 0$ such that $\sigma^n(p) \mid f$ for some $n \geq 0$. By Theorem 5.1 this is possible if and only if $(\mathbb{F}(t),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$ and $t \mid \gcd(f,g)$. □

Remark 6.1. Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-field. Then one can write $p \in \mathbb{F}[t]^*$ uniquely as $p = p_0\,p_1$ where p_1 is monic and has period 1 and p_0 has pure period 0. By Corollary 5.1 we have the following facts. If $(\mathbb{F}(t),\sigma)$ is a Σ-extension of $(\mathbb{F},\sigma)$, $p_0 := p$ and $p_1 := 1$; otherwise, $p_1 = t^i$ for some $i \in \mathbb{N}_0$ and $p_0 \in \mathbb{F}[t]^*$ with $t \nmid p_0$. Hence for any irreducible factor f of p_0, $\mathrm{spread}_{(\mathbb{F},\sigma)}(f,f)$ is finite and for any irreducible factor f of p_1, $\mathrm{spread}_{(\mathbb{F},\sigma)}(f,f)$ is infinite. This gives in [Bro00, Definition 11] the motivation to call p_0 the *finite* and p_1 the *infinite part* of p. Going back to Subsection 5.3 in this section we try to find a d_0 with (4), that has pure period 0, which means that we search for the finite part of a denominator bound. Whereas in Section 7 we are dealing with the problem to find a d_1 with (5), that has period 1, which means to look for the infinite part of the denominator bound.

We have the following simple fact that follows from Proposition 6.1.

Corollary 6.1. *Let* $(\mathbb{F}(t), \sigma)$ *be a* ΠΣ*-extension of* $(\mathbb{F}, \sigma)$ *and* $f, g \in \mathbb{F}[t]^*$. *If* $f \in \mathbb{F}^*$ *or* $g \in \mathbb{F}^*$ *then* $\mathrm{spread}_{(\mathbb{F},\sigma)}(f, g) = \emptyset$.

Now the question arises if one can compute the spread of f and g, if it is finite. In order to give an answer to this question, we first define the following relation.

Definition 6.2. Let $(\mathbb{E}, \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ and $f, g \in \mathbb{E}^*$. Then f is $(\mathbb{F}, \sigma)$-equivalent to g, $f \sim_{(\mathbb{F},\sigma)} g$, if there exists a $k \in \mathbb{Z}$ such that

$$\frac{\sigma^k(f)}{g} \in \mathbb{F}. \tag{6}$$

Lemma 6.1. *Let* $(\mathbb{F}, \sigma)$ *be a difference field and* $f, g \in \mathbb{F}^*$ *with* $f \sim_{(\mathbb{F},\sigma)} g$. *If* $\mathrm{per}_{(\mathbb{F},\sigma)}(f) = 0$, *there is a unique* $k \in \mathbb{Z}$ *such that (6).*

Proof: Assume there are $k, l \in \mathbb{Z}$ with $k > l$ such that $\frac{\sigma^k(f)}{g} \in \mathbb{F}$ and $\frac{\sigma^l(f)}{g} \in \mathbb{F}$. We have $f, g \neq 0$, and thus $\sigma^l(f), \sigma^k(f) \neq 0$. Hence $\frac{\sigma^k(f)}{\sigma^l(f)} = \frac{\sigma^k(f)}{g} \frac{g}{\sigma^l(f)} \in \mathbb{F}^*$ and thus $\frac{\sigma^{k-l}(f)}{f} \in \mathbb{F}^*$ with $k - l > 0$. Therefore $\mathrm{per}_{(\mathbb{F},\sigma)}(f) > 0$, a contradiction. □
By that lemma we can define the function $\mathrm{spec} : \mathbb{F}(t)^* \times \mathbb{F}(t)^* \to \mathbb{Z} \cup \{\perp\}$ with an extra symbol $\perp$ in the following way.

Definition 6.3. Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ and $f, g \in \mathbb{F}(t)^*$. If $f \sim_{(\mathbb{F},\sigma)} g$ then we define the specification of f, g, in symbols $\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f}, \mathrm{g})$, by

$$\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f}, \mathrm{g}) = \begin{cases} k & \text{if } k \text{ fulfills (6) and } \mathrm{per}_{(\mathbb{F},\sigma)}(f) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if $f \nsim_{(\mathbb{F},\sigma)} g$, $\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f}, \mathrm{g}) = \perp$.

In [Kar81] the specification of f and g is defined in a general difference field extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$. In this work we focus for simplicity only on ΠΣ-extensions. As will be shown in the next lemma the specification $\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f}, \mathrm{g})$ for $f, g \in \mathbb{F}(t)^*$ characterizes if we have $f \sim_{(\mathbb{F},\sigma)} g$. Furthermore, if $f \sim_{(\mathbb{F},\sigma)} g$, it gives the certificate for this relation:

Lemma 6.2. *Let* $(\mathbb{F}(t), \sigma)$ *be a* ΠΣ*-extension of* $(\mathbb{F}, \sigma)$ *and* $f, g \in \mathbb{F}(t)^*$. *Then* $f \sim_{(\mathbb{F},\sigma)} g$ *if and only if* $k := \mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f}, \mathrm{g}) \in \mathbb{Z}$. *If* $k \in \mathbb{Z}$ *then* $\frac{\sigma^k(f)}{g} \in \mathbb{F}$.

Proof: The first statement follows by definition of spec. If $\mathrm{per}_{(\mathbb{F},\sigma)}(f) = 0$, the second statement also follows by definition. Now assume $\mathrm{per}_{(\mathbb{F},\sigma)}(f) = 1$ and $k := \mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f},\mathrm{g}) \in \mathbb{Z}$. Then $k = 0$ by definition. By Corollary 5.2 $(\mathbb{F}(t),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$ and $f = u\,t^l$ for some $u \in \mathbb{F}^*$ with $l \in \mathbb{Z}$. Furthermore by the first statement it follows $f \sim_{(\mathbb{F},\sigma)} g$. Hence there is an $r \in \mathbb{Z}$ with $\frac{\sigma^r(f)}{g} = \frac{\sigma^r(u)\,(\alpha)_r^l\,t^l}{g} \in \mathbb{F}$. But this means that $g = t^l\,v$ for some $v \in \mathbb{F}^*$. Hence $\frac{\sigma^0(f)}{g} = \frac{u}{v} \in \mathbb{F}$ and the lemma follows. □

In [Kar81, Chapter 2] M. Karr develops algorithms to find the specification in ΠΣ-fields. This results in Theorem 9 that contains the following statement.

Theorem 6.1. *Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-field and $f,g \in \mathbb{F}(t)^*$. Then there exists an algorithm that computes* $\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f},\mathrm{g})$.

From now on we assume that $(\mathbb{F}(t),\sigma)$ is a ΠΣ-field and we take $f,g \in \mathbb{F}[t]^*$. In order to guarantee that $\mathrm{spread}_{(\mathbb{F},\sigma)}(f,g)$ is finite (Proposition 6.1), we suppose additionally that $(\mathbb{F}(t),\sigma)$ is a Σ-extension of $(\mathbb{F},\sigma)$ or $t \nmid \gcd(f,g)$. In [Bro00] M. Bronstein observed that one is capable of computing the finite set $\mathrm{spread}_{(\mathbb{F},\sigma)}(f,g)$ by using Karr's spec-algorithm. In this work we want to summarize these results. In particular by restricting to the ΠΣ-field case and using additional notions of Karr, we are able to streamline these aspects.

We have the following properties of the spec-function.

Lemma 6.3. *Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$ and $f,g,h \in \mathbb{F}(t)^*$. Then* $\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f},\mathrm{f}) = 0$*. If $f \sim_{(\mathbb{F},\sigma)} g$ then* $\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f},\mathrm{g}) = -\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{g},\mathrm{f})$*. If in addition $g \sim_{(\mathbb{F},\sigma)} h$ then* $\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f},\mathrm{h}) = \mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f},\mathrm{g}) + \mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{g},\mathrm{h})$.

Proof: The first two statements follow immediately. Now assume $f \sim_{(\mathbb{F},\sigma)} g$ and $g \sim_{(\mathbb{F},\sigma)} h$, i.e. there are $k,l \in \mathbb{Z}$ and $u,v \in \mathbb{F}$ such that $\frac{\sigma^k(f)}{g} = u$ and $\frac{\sigma^l(g)}{h} = v$. Then it follows $\frac{\sigma^{k+l}(f)}{\sigma^l(g)} = \sigma^l(u) \in \mathbb{F}$ and hence $\frac{\sigma^{k+l}(f)}{h} = \sigma^l(u)\,v \in \mathbb{F}$. Thus $\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f},\mathrm{h}) = \mathrm{k}+\mathrm{l} = \mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f},\mathrm{g}) + \mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{g},\mathrm{h})$. □

As a direct consequence of the previous lemma we have the following statement.

Corollary 6.2. *Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$. Then $\sim_{(\mathbb{F},\sigma)}$ is an equivalence relation.*

The next result enables to compute the spread by using the spec function.

Lemma 6.4. *Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$, and $f = f_1^{m_1}\dots f_r^{m_r} \in \mathbb{F}[t]^*$, $g = g_1^{n_1}\dots g_s^{n_s} \in \mathbb{F}[t]^*$ be complete factorizations over $\mathbb{F}$. Assume further that $(\mathbb{F}(t),\sigma)$ is a Σ-extension of $(\mathbb{F},\sigma)$ or $t \nmid \gcd(f,g)$. Then we have $\deg(\gcd(f,\sigma^k(g))) > 0$ if and only if there exists i and j with $1 \le i \le r$ and $1 \le j \le s$ such that* $\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{g_j},\mathrm{f_i}) = \mathrm{k}$.

Proof: Assume there are i and j such that $\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{g_j},\mathrm{f_i}) = \mathrm{k} \in \mathbb{Z}$. Then by Lemma 6.2 $\frac{\sigma^k(g_j)}{f_i} \in \mathbb{F}$. Hence $\sigma^k(g_j) \mid \gcd(\sigma^k(g), f)$ with $\deg(f_i) > 0$. Contrary, assume that there is a $k \in \mathbb{Z}$ such that $\deg(\gcd(f, \sigma^k(g))) > 0$. Then there is an irreducible $h \in \mathbb{F}[t]$ with $\deg(h) > 0$, $h \mid f$ and $h \mid \sigma^k(g)$. Therefore one can take $c, d \in \mathbb{F}^*$ with $h = c\, f_i$ and $h = d\, \sigma^k(g_j)$ for some $1 \leq i \leq r$, $1 \leq j \leq s$. Thus $\sigma^k(g_j)/f_i \in \mathbb{F}$. If $\mathrm{per}_{(\mathbb{F},\sigma)}(g_j) = 0$, by Lemma 6.1 k is uniquely determined and therefore $\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{g_j}, \mathrm{f_i}) = \mathrm{k}$. Now assume that $\mathrm{per}_{(\mathbb{F},\sigma)}(g_j) = 1$. Then by Theorem 5.1 we have $t \mid h$ and thus $t \mid \gcd(f, g)$, a contradiction. ☐

Given f and g in complete factorizations as in the above lemma, one obtains

$$\mathrm{spread}_{(\mathbb{F},\sigma)}(f, g) = \left\{\mathrm{spec}_{(\mathbb{F},\sigma)}(\mathrm{f_i}, \mathrm{g_j}) \in \mathbb{Z} \,|\, 1 \leq \mathrm{i} \leq \mathrm{r}, 1 \leq \mathrm{j} \leq \mathrm{s}\right\} \tag{7}$$

as a direct consequence. Therefore one can compute the set $\mathrm{spread}_{(\mathbb{F},\sigma)}(f, g)$, since one can compute the spec-function in the ΠΣ-field $(\mathbb{F}(t), \sigma)$. As side remark note that in [Bro00, Lemma 1] one obtains the same result in terms of the spread-function instead of the spec-function. These observations lead to the following theorem which is included in [Bro00, Theorem 7].

Theorem 6.2. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-field and $f, g \in \mathbb{F}[t]^*$. Assume further that $(\mathbb{F}(t), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$ or $t \nmid \gcd(f, g)$. Then there exists an algorithm that computes the finite set $\mathrm{spread}_{(\mathbb{F},\sigma)}(f, g)$.*

Example 6.1. Consider the ΠΣ-field $(\mathbb{Q}(t)(z), \sigma)$ as in Example 3.4 with $\sigma(z) = (t + 1)\, z$, and take $f := (-1 + t)^5\, t^4\, (1 + z)\, (t + z)$ and $g := -(2 + t)(3 + t)^2(t + z)(-t + t^2 + z)$. Since $\sigma(h) = (1 + t)\, (1 + z)$ and $\sigma^2(h) = (2 + t)(1 + (1 + t)\, z)$ for $h := t + z$, it follows that $\mathrm{spread}_{(\mathbb{Q}(t),\sigma)}(f, g) = \{0, 1, 2\}$ by (7).

Finally we want to give a better strategy than just using (7) for the computation of the set $\mathrm{spread}_{(\mathbb{F},\sigma)}(f, g)$. Here we exploit the fact that $\sim_{(\mathbb{F},\sigma)}$ is an equivalence relation. Namely one first factorizes f and g and groups these factors of f and g respectively into equivalence classes under $\sim_{(\mathbb{F},\sigma)}$, say $C_1, \dots, C_p$ for f and $D_1, \dots, D_q$ for g. Within this construction pick out a representant α_i for each class C_i and β_i for D_i such that for all $x \in C_i$ one knows $\mathrm{spec}_{(\mathbb{F},\sigma)}(\alpha_\mathrm{i}, \mathrm{x})$, and for all $y \in D_i$ one knows $\mathrm{spec}_{(\mathbb{F},\sigma)}(\beta_\mathrm{i}, \mathrm{y})$. Then one just has to compare its representants and obtains the set $\mathrm{spread}_{(\mathbb{F},\sigma)}(f, g)$ by using Lemmas 6.3 and 6.4.

Remark 6.2. In [Kar85] the notion of σ-factorizations are introduced. Given the *σ-factorization* of f, one immediately obtains those sets C_i with their representants α_i which are needed in the algorithm suggested in the proof of Theorem 6.2. M. Bronstein reinvented this notion by calling it *orbital decomposition* of f. In order to prove [Bro00, Theorem 7], which contains our Theorem 6.2, M. Bronstein just applies the strategy sketched above by using the representation of f and g in form of its orbit decompositions.

In the end some further properties of the spread function are mentioned which are needed for later considerations. More precisely Lemma 6.5 and Proposition 6.2 are used in the proofs of Lemmas 6.8 and 6.9 that are relevant to prove important properties stated in Proposition 8.1 and Theorem 8.2. These results are quite obvious and are omitted to the reader.

Lemma 6.5. *Let* $(\mathbb{F}(t),\sigma)$ *and* $(\mathbb{G}(t),\sigma)$ *be* ΠΣ*-fields which are isomorph by a permutation and let* $f,g\in\mathbb{F}[t]^*$. *Then* $\operatorname{spread}_{(\mathbb{F},\sigma)}(f,g)=\operatorname{spread}_{(\mathbb{G},\sigma)}(f,g)$.

Lemma 6.6. *Let* $\mathbb{K}$ *be a subfield of* $\mathbb{F}$ *and let* $\mathbb{K}[t]$ *and* $\mathbb{F}[t]$ *be polynomial rings with coefficients in* $\mathbb{K}$ *and* $\mathbb{F}$ *respectively. Then there exists an embedding of* $\mathbb{K}[t]$ *in* $\mathbb{F}[t]$, *i.e.* $\mathbb{K}[t]\subseteq\mathbb{F}[t]$. *Let* $f,g\in\mathbb{K}[t]^*$ *with* $u=\gcd_{\mathbb{K}[t]}(f,g)$ *and* $v=\gcd_{\mathbb{F}[t]}(f,g)$. *Then there exists a* $c\in\mathbb{F}^*$ *such that* $u=v\,c$.

Due to the previous lemma the next proposition follows.

Proposition 6.2. *Let* $(\mathbb{F}(x_1,\dots,x_e)(t)(s),\sigma)$ *and* $(\mathbb{F}(s)(x_1,\dots,x_e)(t),\sigma)$ *be two* ΠΣ*-fields which are isomorph by a permutation. Let* $f,g\in\mathbb{F}(x_1,\dots,x_e)[t]^*$ *and consider them also as elements in the polynomial ring* $\mathbb{F}(s)(x_1,\dots,x_e)[t]$. *Then* $\operatorname{spread}_{(\mathbb{F}(x_1,\dots,x_e),\sigma)}(f,g)=\operatorname{spread}_{(\mathbb{F}(s)(x_1,\dots,x_e),\sigma)}(f,g)$.

6.2. M. Bronstein's Period 0 Denominator Bound

Based on results of the previous subsection we can introduce an algorithm that computes a period 0 denominator bound of a solution space. Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$ and $f,g\in\mathbb{F}[t]^*$. Moreover suppose that $(\mathbb{F}(t),\sigma)$ is a Σ-extension of $(\mathbb{F},\sigma)$ or $t\nmid\gcd(f,g)$. Hence by Proposition 6.1 we may assume that $\operatorname{spread}_{(\mathbb{F},\sigma)}(f,g)$ is a finite set. In particular we can define the following sequence.

Definition 6.4. Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$. Let $f,g\in\mathbb{F}[t]^*$ and assume that $t\nmid\gcd(f,g)$ or $(\mathbb{F}(t),\sigma)$ is a Σ-extension of $(\mathbb{F},\sigma)$. Furthermore consider $\operatorname{spread}_{(\mathbb{F},\sigma)}(f,g)=\{m_1>m_2>\dots>m_s\}$. Then the sequence $\langle(p_i,q_i,u_i)\,|\,1\leq i\leq s+1\rangle$ is called *bounding sequence* of f and g if

1. $p_1:=f$, $q_1:=g$, $u_1:=1$ and
2. for $1\leq i\leq s$ we have with $d_i:=\gcd(p_i,\sigma^{m_i}(q_i))$ that

$$p_{i+1}:=\frac{p_i}{d_i},\qquad q_{i+1}:=\frac{q_i}{\sigma^{-m_i}(d_i)}\qquad\text{and } u_{i+1}:=u_i\prod_{j=0}^{m_i}\sigma^{-j}(d_i).$$

Example 6.2. Consider the ΠΣ-field $(\mathbb{Q}(t)(z),\sigma)$, $f\in\mathbb{Q}(t)[z]$ and $g\in\mathbb{Q}(t)[z]$ as in Example 6.1 where we have $\operatorname{spread}_{(\mathbb{Q}(t),\sigma)}(f,g)=\{0,1,2\}$. Following the computation rules for the bounding sequence $\langle(p_i,q_i,u_i)\,|\,1\leq i\leq 4\rangle$ of f and g, we obtain $u_4=(1+z)\,(t+z)^2\,(-t+t^2+z)$ that is from special interest.

First we show a lemma that is heavily used in Section 7.1, but also will be needed in Proposition 6.3.

Lemma 6.7. *Let* $(\mathbb{F}(t),\sigma)$ *be a* Π*-extension of* $(\mathbb{F},\sigma)$ *and* $q\in\mathbb{F}[t]$. *Then for all* $k\in\mathbb{Z}$ *we have* $\operatorname{ord}(q)=\operatorname{ord}(\sigma^k(q))$.

Proof: Let $d := \operatorname{ord}(q)$. If $d = -1$, $q = 0$ and the lemma clearly holds. Now assume that $d \geq 0$ and that there exists a $k \in \mathbb{Z}$ with $\operatorname{ord}(q) \neq \operatorname{ord}(\sigma^k(q))$. We can write $q = t^d p$ for some $p \in \mathbb{F}[t]^*$ with $t \nmid p$. Since $\sigma^k(q) = \sigma^k(t^d)\,\sigma^k(p) = (\alpha)_k\, t^d\, \sigma^d(p)$ where $\sigma^d(p) \in \mathbb{F}[t]$ and $(\alpha)_k \in \mathbb{F}^*$, it follows by our proof assumption $\operatorname{ord}(q) \neq \operatorname{ord}(\sigma^k(q))$ that $t \mid \sigma^k(p)$ and thus $\sigma^k(p)/t \in \mathbb{F}[t]$. Then $\sigma^{-k}(\frac{\sigma^k(p)}{t}) = \frac{p}{\sigma^{-k}(t)} = \frac{p}{(\alpha)_{-k}\, t} \in \mathbb{F}[t]$ with $(\alpha)_{-k} \in \mathbb{F}^*$ and therefore $t \mid p$, a contradiction. □

Proposition 6.3. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$. Let $f, g \in \mathbb{F}[t]^*$ with $t \nmid g$ and consider the bounding sequence $\langle (p_i, q_i, u_i) \,|\, 1 \leq i \leq s+1 \rangle$ of f and g. Then u_{s+1} has pure period 0.*

Proof: If $(\mathbb{F}(t), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$, all elements in $\mathbb{F}(t)$ have pure period 0 by Corollary 5.1, and hence the proposition follows. Now assume that we have a Π-extension. We will show that $t \nmid u_i$ and $t \nmid q_i$ for all $1 \leq i \leq s+1$. Clearly $u_1 := 1$ and $q_1 := g$ fulfill this condition. Now assume that $t \nmid u_i$ and $t \nmid q_i$ for all $1 \leq i \leq e$ for some $1 \leq e \leq s$. Hence by Lemma 6.7, we have $\operatorname{ord}(\sigma^{m_e}(q_e)) = 0$, thus $t \nmid \operatorname{ord}(\sigma^{m_e}(q_e)) = 0$, and therefore $t \nmid d_e := \gcd(p_e, \sigma^{m_e}(q_e))$. Similarly, by Lemma 6.7 we have for all $0 \leq j \leq m_e$ that $t \nmid \sigma^{-j}(d_e)$, and hence by $t \nmid u_e$ that $t \nmid u_{e+1} := u_e \prod_{j=0}^{m_e} \sigma^{-j}(d_e)$. Clearly $t \nmid q_{e+1} := q_e/\sigma^{-m_e}(d_e)$ which finishes the induction step. In particular u_{s+1} has pure period 0 by Corollary 5.1. □

The next theorem yields to an algorithm that computes a period 0 denominator bound of a solution space.

Theorem 6.3. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ and $\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ with $a_1 \neq 0 \neq a_m$. Let $f := \sigma^{m-1}(a_1)$ and $g := \frac{a_m}{t^{\operatorname{ord}(a_m)}}$, and consider the bounding sequence $\langle (p_i, q_i, u_i) \,|\, 1 \leq i \leq s+1 \rangle$ of f and g. If there is an $h \in \mathbb{F}(t)$ with $\sigma_{\boldsymbol{a}} h \in \mathbb{F}[t]$ and $h = p \oplus h_1 \oplus h_0 \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)} \oplus \mathbb{F}(t)^{(0)}$ then $\operatorname{den}(h_0) \mid u_{s+1}$. Moreover u_{s+1} has pure period 0.*

Proof: The part of the theorem is a direct consequence of [Bro00, Theorems 8 and 10]. The second part follows by Proposition 6.3, since $t \nmid g$. □

Since one can compute the finite set $\operatorname{spread}_{(\mathbb{F},\sigma)}(f, g)$ by Theorem 6.2, one can compute the bounding sequence of f and g by simple gcd-computations. This is the essential step why Theorem 6.3 directly leads to Algorithm 6.1 that solves the period 0 denominator bound problem. This algorithm is a generalization of Abramov's denominator bound algorithm [Abr95] from the ΠΣ-field $(\mathbb{K}(t), \sigma)$ as defined in Example 2.1 to the general case of ΠΣ-fields.

Algorithm 6.1. Compute a period 0 denominator bound.

d_0=`Den0Bound(`$(\mathbb{F}(t), \sigma), \boldsymbol{a}, \boldsymbol{f}$`)`

Input: A ΠΣ-field $(\mathbb{F}(t), \sigma)$, $\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ with $a_1\, a_m \neq 0$, $m \geq 2$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$.

Output: A period 0 denominator bound $d_0 \in \mathbb{F}[t]^*$ that has pure period 0. Moreover, if $(\mathbb{F}(t), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$, d_0 is a denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$; otherwise there exists an $x \in \mathbb{N}_0$ such that $d_0\, t^x$ is a denominator bound.

(1) Set $f := \sigma^{m-1}(a_1) \in \mathbb{F}[t]^*$ and $g := \frac{a_m}{t^{\operatorname{ord}(a_m)}} \in \mathbb{F}[t]^*$ and compute $\operatorname{spread}_{(\mathbb{F},\sigma)}(f, g)$.

(2) Compute the bounding sequence $\langle(p_i, q_i, u_i) \,|\, 1 \leq i \leq s+1\rangle$ of f and g.
(3) RETURN u_{s+1}

Theorem 6.4. *Algorithm 6.1 is correct.*

Proof: Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-field, $\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}^m$ with $m \geq 2$, $a_1\, a_m \neq 0$ and $\boldsymbol{f} \in \mathbb{F}^n$. Let $d_0 \in \mathbb{F}[t]^*$ be the result of $\texttt{DenOBound(}(\mathbb{F}(t), \sigma), \boldsymbol{a}, \boldsymbol{f}\texttt{)}$. By Theorem 6.3 $d_0 \in \mathbb{F}[t]^*$ has pure period 0 and fulfills (4). Hence d_0 is a period 0 denominator bound. If $(\mathbb{F}(t), \sigma)$ is a Σ-extension, by Corollary 5.2 $\mathbb{F}(t)^{(1)} = \{0\}$, and hence by Theorem 6.3 d_0 is a denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$. Otherwise let $(\mathbb{F}(t), \sigma)$ be a Π-extension and take a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$, say $\{\boldsymbol{c_1} \wedge \frac{x_1}{y_1}, \dots, \boldsymbol{c_l} \wedge \frac{x_l}{y_l}\}$ where $\frac{x_i}{y_i}$ is in reduced representation. Set $b := \max_i(\mathrm{ord}(y_i))$ and define $d_1 := t^b$. Then by Lemma 5.3 d_1 fulfills (5). Since d_0 also fulfills (4), by Corollary 5.4 $d_0\, d_1$ is a denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$. □

Example 6.3. Consider the $\Pi\Sigma$-field $(\mathbb{Q}(t)(z), \sigma)$ and $\boldsymbol{a} = (a_1, a_2)$ as in Example 4.1. Then $f := \sigma^{-1}(a_1)$ and $g := \frac{a_2}{t^{\mathrm{ord}(a_2)}} = a_2$ are exactly those f and g as in Example 6.2, and hence $u_4 = (1+z)\,(t+z)^2\,(-t+t^2+z)$ is the result of Algorithm 6.1 with input $\texttt{DenOBound}((\mathbb{Q}(\texttt{t})(\texttt{z}), \sigma), \boldsymbol{a}, (0))$. Moreover u_4 is the period 0 denominator bound of $\mathrm{V}(\boldsymbol{a}, (0), \mathbb{Q}(t)(z))$ from Example 5.3.

By Theorem 6.3 one obtains some important consequences which play in [Sch01, Chapter 4] an important role in the theory of d'Alembertian solutions. In particular the following corollary is included in [Bro00, Corollary 3].

Corollary 6.3. *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$, $\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}^m$ with $a_1\, a_m \neq 0$ and $h \in \mathbb{F}(t)$ with $\sigma_{\boldsymbol{a}} h \in \mathbb{F}[t]$. Then $h \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)}$. More precisely, if $(\mathbb{F}(t), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$ then $h \in \mathbb{F}[t]$. Otherwise, if $(\mathbb{F}(t), \sigma)$ is a Π-extension of $(\mathbb{F}, \sigma)$ then $h = \frac{p}{t^k}$ for some $p \in \mathbb{F}[t]$ and $k \geq 0$.*

Proof: Let $h \in \mathbb{F}(t)$ with $\sigma_{\boldsymbol{h}} g \in \mathbb{F}[t]$ and define f, g as in Theorem 6.3. It follows that $f, g \in \mathbb{F}^*$ and thus by Corollary 6.1 $\mathrm{spread}_{(\mathbb{F},\sigma)}(f, g) = \emptyset$. Therefore the bounding sequence of f and g is $\langle(f, g, 1)\rangle$, in particular $u_1 = 1$. By Theorem 6.3 it follows that for any $h \in \mathbb{F}(t)$ with $h = p + h_1 + h_0 \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)} \oplus \mathbb{F}(t)^{(0)}$ we have $\mathrm{den}(h_0) \mid u_1 = 1$ and thus $h_0 = 0$. Consequently $h \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(1)}$. By Corollary 6.3 one can refine this result further as stated in the corollary. □
In the end we consider two important properties of bounding sequences that are needed later in Proposition 8.1 and Theorem 8.2.

Lemma 6.8. *Let $(\mathbb{F}(t), \sigma)$ and $(\mathbb{G}(t), \sigma)$ be $\Pi\Sigma$-fields which are isomorph by a permutation. Furthermore let $f, g \in \mathbb{F}[t]^*$ where either $t \nmid \gcd(f, g)$ or $(\mathbb{F}(t), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$. Let*

$$\langle(p_i, q_i, u_i) \,|\, 1 \leq i \leq s+1\rangle \qquad \text{and} \qquad \langle(p'_i, q'_i, u'_i) \,|\, 1 \leq i \leq s'+1\rangle \tag{8}$$

be the bounding sequences of f and g in $\mathbb{F}(t)$ and $\mathbb{G}(t)$ respectively. Then $s = s'$ and $u_{s+1} = c\, u'_{s+1}$ for some $c \in \mathbb{F}^*$.

Proof: By Lemma 6.5 the spread of f and g is in both domains the same. Consequently $s = s'$. Furthermore by the construction of the bounding sequences p_i and p'_i, q_i and q'_i, and r_i and r'_i are the same up to constants in $\mathbb{F}^*$. □

Lemma 6.9. *Let* $(\mathbb{F}(x_1,\dots,x_e)(t)(s),\sigma)$ *and* $(\mathbb{F}(s)(x_1,\dots,x_e)(t),\sigma)$ *be two* ΠΣ-*fields which are isomorph by a permutation and consider* $\mathbb{F}(s)(x_1,\dots,x_e)[t]$ *as a polynomial ring extension of* $\mathbb{F}(x_1,\dots,x_e)[t]$*; let* $f,g \in \mathbb{F}(x_1,\dots,x_e)[t]^*$ *where* $(\mathbb{F}(x_1,\dots,x_e)(t),\sigma)$ *is a* Σ-*extension of* $(\mathbb{F}(x_1,\dots,x_e),\sigma)$ *or* $t \nmid \gcd(f,g)$*. Let (8) be the bounding sequences of* f,g *in* $\mathbb{F}(x_1,\dots,x_e)[t]$ *and* $\mathbb{F}(s)(x_1,\dots,x_e)[t]$ *respectively. Then* $s = s'$ *and* $u_{s+1} = c\,u'_{s+1}$ *for some* $c \in \mathbb{F}(s)(x_1,\dots,x_e)^*$.

Proof: In both domains the spread of f and g is the same by Proposition 6.2. Hence $s = s'$. Furthermore by Lemma 6.6 one can easily verify in the same fashion as in the proof of Proposition 6.3 that the bounding sequences can differ only by a constant in $\mathbb{F}(s)(x_1,\dots,x_e)^*$. □

7. Methods for the Period 1 Denominator Bound

We focus on computing a period 1 denominator bound of a given solution space $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ for a ΠΣ-field $(\mathbb{F}(t),\sigma)$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. Given such a d_1, the denominator bound problem is solved for $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$. In deed, one can compute a period 0 denominator bound $d_0 \in \mathbb{F}[t]^*$ by Theorem 6.4, and hence $d_0\, d_1$ delivers a denominator bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ by Corollary 5.4.
If $(\mathbb{F}(t),\sigma)$ is a Σ-extension of $(\mathbb{F},\sigma)$ then $\mathbb{F}(t)^{(1)} = \{0\}$ by Corollary 5.2 and therefore we can choose $d_1 = 1$. Hence we only have to deal with the case that $(\mathbb{F}(t),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$. In the sequel we reduce the above problem to computing a $b \in \mathbb{N}_0$ for several cases $\boldsymbol{a}$ such the following condition holds:

$$\forall g \in \mathbb{F}(t) : \sigma_{\boldsymbol{a}} g \in \mathbb{F}[t] \Rightarrow b \geq \operatorname{ord}(\operatorname{den}(g)). \tag{9}$$

Then t^b is a period 1 denominator bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ by the following lemma.

Lemma 7.1. *Let* $(\mathbb{F}(t),\sigma)$ *be a* Π-*extension of* $(\mathbb{F},\sigma)$*,* $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ *and* $\boldsymbol{f} \in \mathbb{F}[t]^n$*. If* $b \in \mathbb{N}_0$ *fulfills (9),* t^b *is a period* 1 *denominator bound of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$.

Proof: We have $\mathbb{F}(t)^{(1)} = \mathbb{F}[1/t] \setminus \mathbb{F}^*$ by Corollary 5.2. Then by choosing $d_1 := t^b$ it follows immediately (5) by Lemma 5.3 which proves the lemma. □
We solve problem (9) for the first order case, i.e. $\boldsymbol{a} \in \mathbb{F}[t]^2$, which is based on the work of [Kar81]. In addition we extend these bound techniques from the first order to the higher order case which allows to solve problem (9) for a special class of linear difference equations (see Situation 7.1 and Situation 7.3).

7.1. Some Properties of the Order and Denominator Function

Proposition 7.1 is needed in the proofs of Theorems 7.1 and 7.5 that enable to find bounds b as in (9) for Situations 7.1 and 7.3. All lemmas contribute to the proof of this proposition or to the proofs of Theorems 7.1 and 7.5.

Lemma 7.2. *Let t be transcendental over $\mathbb{F}$, $d \geq 1$ and $f = \sum_{i=1}^{d} \frac{f_i}{t^i} \in \mathbb{F}[1/t] \backslash \mathbb{F}^*$. Then* $\operatorname{ord}(\operatorname{den}(f)) = d$ *if and only if* $f_d \neq 0$. *Furthermore,* $\operatorname{ord}(\operatorname{den}(f)) \leq d$.

Proof: If $f_d \neq 0$ then $f = \sum_{i=1}^{d} \frac{f_i}{t^i} = \frac{f_d + t\, f_{d-1} + \dots + t^{d-1} f_1}{t^d} =: \frac{u}{t^d}$ where $u \in \mathbb{F}[t]$ with $t \nmid u$. Hence $\operatorname{ord}(\operatorname{den}(f)) = d$. Contrary, suppose $f_d = 0$. If $f = 0$ then clearly $\operatorname{ord}(\operatorname{den}(f)) = \operatorname{ord}(1) = 0 < d$. In particular, $d \neq \operatorname{ord}(\operatorname{den}(f))$. Otherwise, if $f \neq 0$, let $l < d$ be maximal such that $f_l \neq 0$. Then $\operatorname{ord}(\operatorname{den}(f)) = l < d$ by the first part of the proof and thus $\operatorname{ord}(\operatorname{den}(f)) \neq d$. Moreover we have $\operatorname{ord}(\operatorname{den}(f)) \leq d$ in any case which follows immediately by the above considerations. □

Lemma 7.3. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-field of $(\mathbb{F}, \sigma)$ and $f \in \mathbb{F}(t)$. Then we have* $\sigma(\operatorname{den}(f)) = u \operatorname{den}(\sigma(f))$ *for some $u \in \mathbb{F}^*$.*

Proof: For $f = 0$ the lemma holds. Let $f = \frac{a}{b} \in \mathbb{F}[t]^*$ and $\sigma(f) = \frac{a'}{b'}$ be in reduced representation. Then $\frac{a'}{b'} = \sigma(f) = \frac{\sigma(a)}{\sigma(b)}$. Since $\gcd(a, b) = 1$, we have $\gcd(\sigma(a), \sigma(b)) = 1$ by Lemma 3.1. As $\gcd(a', b') = 1$, there is a $u \in \mathbb{F}$ with $\sigma(\operatorname{den}(f)) = \sigma(b) = u\, b' = \operatorname{den}(\sigma(f))$. □

Lemma 7.4. *Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$, $a \in \mathbb{F}[t]^*$ and $g \in \mathbb{F}(t)^*$ with* $\operatorname{ord}(\operatorname{den}(g)) > 0$. *Then* $\operatorname{ord}(\operatorname{den}(a\, \sigma^i(g))) = \max(0, \operatorname{ord}(\operatorname{den}(g)) - \operatorname{ord}(a))$ *for all $i \geq 0$.*

Proof: Let $d := \operatorname{ord}(\operatorname{den}(g)) > 0$ and $g = \frac{u}{v\, t^d}$ for some $u, v \in \mathbb{F}[t]^*$ with $t \nmid u, v$ and let $a = t^p\, b$ for some $p \geq 0$ and $b \in \mathbb{F}[t]^*$ with $t \nmid b$. Then

$$a\, \sigma^i(g) = t^p\, b\, \sigma^i\!\left(\frac{u}{v\, t^d}\right) = \frac{b}{t^{d-p}}\, \frac{\sigma^i(u)}{\sigma^i(v)\ (\alpha)_i^d} \tag{10}$$

with $d - p \in \mathbb{Z}$. Clearly we have $\sigma^i(u) \in \mathbb{F}[t]^*$ and $\sigma^i(v)\ (\alpha)_i^d \in \mathbb{F}[t]^*$. Since $\operatorname{ord}(u) = \operatorname{ord}(v) = 0$, it follows that $t \nmid \sigma^i(u)$ and $t \nmid \sigma^i(v)\ (\alpha)_i^d$ by Lemma 6.7. Together with (10) the lemma follows: Namely, if $d - p \geq 0$ then $\operatorname{den}(\operatorname{ord}(a\, \sigma^i(g))) = d - p = \max(0, d - p)$, otherwise $\operatorname{den}(\operatorname{ord}(a\, \sigma^i(g))) = 0 = \max(0, d - p)$ □

Proposition 7.1. *Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$, assume $\mathbf{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ and take $p := \min_i \{\operatorname{ord}(a_i) \mid a_i \neq 0\}$. Let $g \in \mathbb{F}(t)$ with $d :=$* $\operatorname{ord}(\operatorname{den}(g)) > p$ *and define $S := \{a_i \mid \operatorname{ord}(a_i) = p\}$. Then* $\operatorname{ord}(\operatorname{den}(\sigma_{\boldsymbol{a}} g)) < d - p$ *if and only if* $\operatorname{ord}(\operatorname{den}(\sum_{i \in S} a_i\, \sigma^{m-i}(g))) < d - p$.

Proof: Take $h_i := a_i\, \sigma^{m-i}(g)$ for all $1 \leq i \leq m$ and write

$$h_i = h_{i1} + h_{i2} + h_{i3} \in \mathbb{F}[1/t] \setminus \mathbb{F}^* \oplus \mathbb{F}[t] \oplus \mathbb{F}(t)^{(fracpart)}.$$

First we show that one can represent h_{i1} as

$$h_{i1} = \sum_{j=1}^{o_i} \frac{\tilde{h}_{ij}}{t^j} \tag{11}$$

for some $0 \leq o_i < d-p$ and $\tilde{h}_{ij} \in \mathbb{F}$. If $a_i = 0$ then $h_i = 0$. Hence with $o_i := \operatorname{ord}(\operatorname{den}(h_i)) = \operatorname{ord}(1) = 0 < d-p$ we can write h_{i1} as in (11). Otherwise, if $a_i \neq 0$, take $o_i := \operatorname{ord}(\operatorname{den}(h_i))$. Then by Lemma 7.4 it follows that $o_i = \max(0, d-p_i) \leq d-p$ and therefore by Lemma 7.2 we may write (11) with $h_{ij} \in \mathbb{F}$. Now split $\sigma_{\boldsymbol{a}} g$ via $\sigma_{\boldsymbol{a}} g = f_1 + f_2 + f_3 \in \mathbb{F}[1/t] \setminus \mathbb{F}^* \oplus \mathbb{F}[t] \oplus \mathbb{F}(t)^{(fracpart)}$. Then we have $f_1 = h_{11} + \cdots + h_{m1} = \sum_{j=1}^{o_1} \frac{\tilde{h}_{1j}}{t^j} + \cdots + \sum_{j=1}^{o_m} \frac{\tilde{h}_{mj}}{t^j}$ and hence

$$\operatorname{ord}(\operatorname{den}(f)) < d-p \overset{\text{Lemma (5.3)}}{\Leftrightarrow} \operatorname{ord}(\operatorname{den}(\sum_{j=1}^{o_1} \frac{\tilde{h}_{1j}}{t^j} + \cdots + \sum_{j=1}^{o_m} \frac{\tilde{h}_{mj}}{t^j})) < d-p$$

$$\overset{\text{Lemma (7.2)}}{\Leftrightarrow} \sum_{i \in S} \tilde{h}_{io_i} = 0 \overset{\text{Lemma (7.2)}}{\Leftrightarrow} \operatorname{ord}(\operatorname{den}(\sum_{i \in S} \sum_{j=1}^{o_i} \frac{\tilde{h}_{ij}}{t^j})) < d-p.$$

By $\sum_{i \in S} \sum_{j=1}^{o_i} \frac{\tilde{h}_{ij}}{t^j} = \sum_{i \in S} a_i \, \sigma^{m-i}(g)$ the proposition follows. □

7.2. A Simple Case

Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. The next theorem delivers a bound $b \in \mathbb{N}_0$ with (9) for the following case.

Situatation 7.1. Assume $\mathbf{0} \neq \boldsymbol{a} = (a_1, \ldots, a_m) \in \mathbb{F}[t]^m$ with $\operatorname{ord}(a_r) = p$ for some $r \in \{1, \ldots, m\}$ and $\operatorname{ord}(a_i) > p$ for all $a_i \neq 0$ with $i \neq r$

Theorem 7.1. *Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ and $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ as in Situation 7.1. If $g \in \mathbb{F}(t)$ with $\sigma_{\boldsymbol{a}} g \in \mathbb{F}[t]$ then $\operatorname{ord}(\operatorname{den}(g)) \leq p$.*

Proof: Let $g \in \mathbb{F}(t)$ with $\sigma_{\boldsymbol{a}} g \in \mathbb{F}[t]$ and $\operatorname{ord}(\operatorname{den}(g)) > p$. Then $\operatorname{ord}(\operatorname{den}(\sigma_{\boldsymbol{a}} g)) = \operatorname{ord}(1) = 0$ and thus $\operatorname{ord}(\operatorname{den}(a_r \, \sigma^{m-r}(g))) < \operatorname{ord}(\operatorname{den}(g)) - p$ by Proposition 7.1. But $\operatorname{ord}(\operatorname{den}(a_r \, \sigma^{m-r}(g))) = \max(0, \operatorname{ord}(\operatorname{den}(g)) - p) = \operatorname{ord}(\operatorname{den}(g)) - p$ by Lemma 7.4, a contradiction. □

7.3. Period 1 Denominators of First Order Linear Difference Equations

Let $(\mathbb{F}(t), \sigma)$ be ΠΣ-field where $(\mathbb{F}(t), \sigma)$ a Π-extension of $(\mathbb{F}, \sigma)$, $\boldsymbol{f} \in \mathbb{F}[t]^n$ and $\mathbf{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}[t]^2$. In this section we will deal with the problem to find a bound b as in (9). If $\operatorname{ord}(a_1) \neq \operatorname{ord}(a_2)$, Theorem 7.1 provides a bound b. What remains is the case $\operatorname{ord}(a_1) = \operatorname{ord}(a_2)$. More precisely we focus on finding a bound b for Situation 7.2.

Situatation 7.2. Assume that $(a_1, a_1) \in \mathbb{F}[t]^2$ with $a_1 = t^p(1 + r_1)$ and $a_2 = t^p(-c + r_2)$ where $c \in \mathbb{F}^*$ and $r_1, r_2 \in \mathbb{F}[t]$ with $\operatorname{ord}(r_i) > 0$.

As will be seen later, we must be able to decide, if there exists a $d \geq 0$ for any $c, \alpha \in \mathbb{F}^*$ such that $c \, \alpha^d \in \mathrm{H}_{(\mathbb{F}, \sigma)}$. Furthermore, we must be able to compute such a d, if there exists one. By Theorem 3.3 all these problems can be solved.

The main idea of the following section is taken from Theorem 18 of [Kar81]. Whereas in Karr's version theoretical and computational aspects are mixed, I tried to separate his theorem in several parts to achieve more transparency.

Theorem 7.2. *Let* $(\mathbb{F}(t),\sigma)$ *be a* Π*-extension of* $(\mathbb{F},\sigma)$, $f\in\mathbb{F}[t]$ *and assume* $a_1,a_2\in\mathbb{F}[t]$ *as in Situation 7.2. If there exists a* $g\in\mathbb{F}(t)$ *with* $d:=\operatorname{ord}(\operatorname{den}(g))>p$ *such that* $\operatorname{ord}(\operatorname{den}(a_1\,\sigma(g)-a_2\,g))<d-p$ *then* $c\,\alpha^d\in\mathrm{H}_{(\mathbb{F},\sigma)}$.

Proof: Let $g\in\mathbb{F}(t)$ with $d:=\operatorname{ord}(\operatorname{den}(g))>p$, i.e. $g=\frac{u}{v\,t^d}$ for some $u,v\in\mathbb{F}[t]^*$ with $\gcd(u,v)=1$ and $t\nmid u,v$. We have

$$\begin{aligned}a_1\,\sigma(g)-a_2\,g&=(1+r_1)\,\frac{\sigma(u)}{\sigma(v)}\,\frac{1}{\alpha^d\,t^{d-p}}-(c-r_2)\frac{u}{v}\frac{1}{t^{d-p}}\\&=\frac{(1+r_1)\,\sigma(u)\,v-(c-r_2)\,u\,\sigma(v)\,\alpha^d}{\sigma(v)\,v}\,\frac{1}{\alpha^d\,t^{d-p}}.\end{aligned}\tag{12}$$

As $\operatorname{ord}(\operatorname{den}(a_1\,\sigma(g)-a_2\,g))<d-p$, we have $t\mid((1+r_1)\,\sigma(u)\,v-(c-r_2)\,u\,\sigma(v)\,\alpha^d)$ and hence

$$\big[(1+r_1)\,\sigma(u)\,v-(c-r_2)\,u\,\sigma(v)\,\alpha^d\big]_0=0.$$

Let $u_0:=[u]_0\in\mathbb{F}^*$ and $v_0:=[v]_0\in\mathbb{F}^*$. Since $t\mid r_1$ and $t\mid r_2$, we get

$$\sigma(u_0)\,v_0-c\,u_0\,\sigma(v_0)\,\alpha^d=0\Leftrightarrow\frac{\sigma(u_0)\,v_0}{u_0\,\sigma(v_0)}=c\,\alpha^d\Leftrightarrow\frac{\sigma(h)}{h}=c\,\alpha^d$$

for $h:=\frac{u_0}{v_0}\in\mathbb{F}^*$ and thus $c\,\alpha^d\in\mathrm{H}_{(\mathbb{F},\sigma)}$. □

In the proof of the previous theorem we just required that in the difference field extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ the element t is transcendental over $\mathbb{F}$ and $\sigma(t)=\alpha\,t$ holds for some $\alpha\in\mathbb{F}^*$. Only in the next lemma all properties of Π-extensions are really exploited. This result finally enables to find a b with (9) for Situation 7.2.

Lemma 7.5. *Let* $(\mathbb{F}(t),\sigma)$ *be a* Π*-extension of* $(\mathbb{F},\sigma)$ *with* $\sigma(t)=\alpha\,t$, $\alpha\in\mathbb{F}^*$ *and* $c\in\mathbb{F}^*$. *If there exists a* $d\in\mathbb{Z}$ *with* $c\,\alpha^d\in\mathrm{H}_{(\mathbb{F},\sigma)}$ *then* d *is uniquely determined.*

Proof: Assume there are $d_1,d_2\in\mathbb{Z}$ with $d_1<d_2$ and $c\,\alpha^{d_1}\in\mathrm{H}_{(\mathbb{F},\sigma)}$, $c\,\alpha^{d_2}\in\mathrm{H}_{(\mathbb{F},\sigma)}$, i.e. there are $g_1,g_2\in\mathbb{F}^*$ such that $\frac{\sigma(g_1)}{g_1}=c\,\alpha^{d_1}$, $\frac{\sigma(g_2)}{g_2}=c\,\alpha^{d_2}$. Since $d_2-d_1>0$, it follows that $\alpha^{d_2-d_1}=\frac{\sigma(g_2)/g_2}{\sigma(g_1)/g_1}=\frac{\sigma(g_2/g_1)}{g_2/g_1}$ and thus $\alpha^{d_2-d_1}\in\mathrm{H}_{(\mathbb{F},\sigma)}$. By Theorem 3.1 $(\mathbb{F}(t),\sigma)$ is not a Π-extension of $(\mathbb{F},\sigma)$, a contradiction. □

Combining the previous results leads to a recipe how to compute the desired b.

Theorem 7.3. *Let* $(\mathbb{F}(t),\sigma)$ *be a* Π*-extension of* $(\mathbb{F},\sigma)$ *and* $a_1,a_2\in\mathbb{F}[t]$ *as in Situation 7.2. Let* $g\in\mathbb{F}(t)$ *with* $a_1\,\sigma(g)+a_2\,g\in\mathbb{F}[t]$. *If there exists a* $d\in\mathbb{N}_0$ *such that* $c\,\alpha^d\in\mathrm{H}_{(\mathbb{F},\sigma)}$ *then* d *is uniquely determined and we have* $\operatorname{ord}(\operatorname{den}(g))\leq\max(d,p)$. *If there does not exist such a* d *then* $\operatorname{ord}(\operatorname{den}(g))\leq p$.

Proof: Let $g\in\mathbb{F}(t)$ with $f:=a_1\,\sigma(g)-a_2\,g\in\mathbb{F}[t]$. We have

$$\operatorname{ord}(\operatorname{den}(f))=\operatorname{ord}(1)=0.\tag{13}$$

1. Assume there exists a $d \geq 0$ with $c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$. Then by Lemma 7.5 d is uniquely determined. Assume $\operatorname{ord}(\operatorname{den}(g)) > p$. Since (13), by Theorem 7.2 it follows that $\operatorname{ord}(\operatorname{den}(g)) = d$ and therefore $\operatorname{ord}(\operatorname{den}(g)) = d = \max(p, d)$. Otherwise, if $\operatorname{ord}(\operatorname{den}(g)) \leq p$ then we have $\operatorname{ord}(\operatorname{den}(g)) \leq \max(p, d)$.
2. Assume there does not exist such a d. Since (13), by Theorem 7.2 it follows that $\operatorname{ord}(\operatorname{den}(g)) \leq p$.

□

By the previous considerations one immediately obtains an algorithm that computes a bound $b \in \mathbb{N}_0$ as in (9) for a ΠΣ-field $(\mathbb{F}(t), \sigma)$.

Algorithm 7.1. Compute the power of a period 1 denominator bound.

b=`Den1Bound(`$(\mathbb{F}(t), \sigma), \boldsymbol{a}$`)`

Input: A ΠΣ-field $(\mathbb{F}(t), \sigma)$, $(\mathbb{F}(t), \sigma)$ is a Π-extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}[t]^2$.

Output: A $b \in \mathbb{N}_0$ that fulfills (9)

(1) IF $\operatorname{ord}(a_1) \neq \operatorname{ord}(a_2)$ THEN RETURN $\min(\operatorname{ord}(a_1), \operatorname{ord}(a_2))$

(2) Set $p := \operatorname{ord}(a_1)$ and define $c := -\frac{[a_2]_p}{[a_1]_p}$

(3) If there exists a $d \in \mathbb{N}_0$ such that $c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$

(4) THEN take such a d and RETURN $\max(d, p)$ ELSE RETURN p

Theorem 7.4. *Algorithm 7.1 is correct.*

Proof: Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-field where $(\mathbb{F}(t), \sigma)$ is a Π-extension of $(\mathbb{F}, \sigma)$, and let $\mathbf{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}[t]^2$. Furthermore let $b \in \mathbb{N}_0$ be the result of `Den1Bound(`$(\mathbb{F}(t), \sigma), \boldsymbol{a}$`)`. By Theorem 7.1 b fulfills (9), if one exits in line (1). Otherwise we compute $c \in \mathbb{F}^*$ and obtain $\frac{\boldsymbol{a}}{[a_1]_p} = (t^p\,(1 + r_1), t^p\,(-c + r_2)$ as it is assumed in Situation 7.2. By Theorem 3.3 there exists an algorithm that decides, if there exists a $d \in \mathbb{N}_0$ with $c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$, and to compute such a d if it exists. Hence by Theorem 7.3 the result b fulfills property (9) in the line (4). □

Example 7.1. Consider the ΠΣ-field $(\mathbb{Q}(t)(z), \sigma)$ and $\boldsymbol{a} = (a_1, a_2)$ with $a_1, a_2 \in \mathbb{Q}(t)[z]$ as in Example 4.1. Then one obtains $c := -\frac{[a_2]_p}{[a_1]_p} = \frac{(-1+t)\,(2+t)\,(3+t)^2}{t^3\,(1+t)^5}$. By Karr's algorithms (Theorem 3.3) we can compute $h := \frac{t^5\,(1+t)^3\,(2+t)^2}{t^3\,(-1+t)}$ and $x := 4$ such that $\frac{\sigma(h)}{h} = c\,(t+1)^x$ holds. Hence $b := \max(x, \operatorname{ord}(a_1)) = 4$ is the result of Algorithm 7.1 with input `Den1Bound(`$(\mathbb{Q}(\mathtt{t})(\mathtt{z}), \sigma), \boldsymbol{a}$`)`. By Theorem 7.4 b fulfills condition (9) and thus $d_1 := z^4$ is a period 1 denominator bound of $\mathrm{V}(\boldsymbol{a}, (0), \mathbb{Q}(t)(z))$ by Lemma 7.1. This is exactly the bound from Example 5.3.

Finally we investigate some properties of this algorithm that are needed in the proofs of Proposition 8.1 and Theorem 8.2.

Proposition 7.2. *Let $(\mathbb{F}(t), \sigma)$ and $(\mathbb{G}(t), \sigma)$ be ΠΣ-fields which are isomorph by a permutation. Then for any $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^2$ we have*

$$\texttt{Den1Bound}((\mathbb{F}(\mathtt{t}), \sigma), \boldsymbol{a}) = \texttt{Den1Bound}((\mathbb{G}(\mathtt{t}), \sigma), \boldsymbol{a}).$$

Proof: Looking closer at Algorithm 7.4, for the inputs $\mathtt{Den1Bound}((\mathbb{F}(\mathtt{t}),\sigma),\boldsymbol{a})$ and $\mathtt{Den1Bound}((\mathbb{G}(\mathtt{t}),\sigma),\boldsymbol{a})$ there might be only line (3) where the result can change. We have that $\frac{c}{\alpha^d}\in \mathrm{H}_{(\mathbb{F},\sigma)}$ if and only if $\frac{c}{\alpha^d}\in \mathrm{H}_{(\mathbb{G},\sigma)}$ by Lemma 3.3. Hence there does not exist a $d\in\mathbb{N}_0$ with $\frac{c}{\alpha^d}\in \mathrm{H}_{(\mathbb{F},\sigma)}$ if and only if $\frac{c}{\alpha^d}\in \mathrm{H}_{(\mathbb{G},\sigma)}$. Furthermore, if there exists such a d, it is unique by Lemma 7.5 and thus we obtain the same d in both computations. Hence the result must be the same. □

Lemma 7.6. *Let $(\mathbb{F}(t),\sigma)$ be a Π-extension of $(\mathbb{F},\sigma)$ and $\boldsymbol{a}\in\mathbb{F}[t]^2$ as in Situation 7.2. Furthermore assume that there exists a $g\in\mathbb{F}(t)^*$ such that $\sigma_{\boldsymbol{a}}g=0$. Then there exists a $d\in\mathbb{Z}$ such that $\frac{c}{\alpha^d}\in \mathrm{H}_{(\mathbb{F},\sigma)}$.*

Proof: Let $g\in\mathbb{F}(t)$ with $\sigma_{\boldsymbol{a}}g=0$ where $g=\frac{u}{v\,t^d}$ for $d\in\mathbb{Z}$ and $u,v\in\mathbb{F}[t]^*$ with $\gcd(u,v)=1$ and $t\nmid u,v$. By (12) in the proof of Theorem 7.2 it follows that

$$\frac{(1+r_1)\,\sigma(u)\,v-(c-r_2)\,u\,\sigma(v)\,\alpha^d}{\sigma(v)\,v}\,\frac{1}{\alpha^d\,t^{d-p}}=0,$$

and therefore $\big[(1+r_1)\,\sigma(u)\,v-(c-r_2)\,u\,\sigma(v)\,\alpha^d\big]_0=0$. Let $u_0:=[u]_0\in\mathbb{F}^*$ and $v_0:=[v]_0\in\mathbb{F}^*$. Since $t\mid r_1$ and $t\mid r_2$, it follows as in the proof of Theorem 7.2 that $\sigma(u_0)\,v_0-c\,u_0\,\sigma(v_0)\,\alpha^d=0$ and hence $\frac{\sigma(u_0)\,v_0}{u_0\,\sigma(v_0)}=c\,\alpha^d$. Consequently $\frac{\sigma(h)}{h}=c\,\alpha^d$ for $h:=\frac{u_0}{v_0}\in\mathbb{F}^*$ and hence $c\,\alpha^d\in \mathrm{H}_{(\mathbb{F},\sigma)}$ which proves the lemma. □

Proposition 7.3. *Let $(\mathbb{F}(x_1,\dots,x_e)(t)(s),\sigma)$ and $(\mathbb{F}(s)(x_1,\dots,x_e)(t),\sigma)$ be ΠΣ-fields which are isomorph by a permutation. Furthermore let $\boldsymbol{f}\in\mathbb{F}(x_1,\dots,x_e)[t]^n$ and $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{F}(x_1,\dots,x_e)[t]^2$ such that there exists an $h\in\mathbb{F}(x_1,\dots,x_e)^*$ with $\sigma_{\boldsymbol{a}}h=0$. Then we have*

$$\mathtt{Den1Bound}((\mathbb{F}(\mathtt{x_1},\dots,\mathtt{x_e})(\mathtt{t}),\sigma),\boldsymbol{a})=\mathtt{Den1Bound}((\mathbb{F}(\mathtt{s})(\mathtt{x_1},\dots,\mathtt{x_e})(\mathtt{t}),\sigma),\boldsymbol{a}).$$

Proof: We consider the computation steps for both ΠΣ-fields $(\mathbb{F}(x_1,\dots,x_e)(t),\sigma)$ and $(\mathbb{F}(s)(x_1,\dots,x_e)(t),\sigma)$ and will prove that the output is always the same. Assume that $\sigma(t)=\alpha\,t+\beta$. If we have $\mathrm{ord}(a_1)\neq\mathrm{ord}(a_2)$, then in both cases the output will be the same in line (1). Otherwise assume equality. By Lemma 7.6 we find a $d\in\mathbb{Z}$ such that $c\,\alpha^d\in \mathrm{H}_{(\mathbb{F}(x_1,\dots,x_e),\sigma)}$. For this d we also have $c\,\alpha^d\in \mathrm{H}_{(\mathbb{F}(s)(x_1,\dots,x_e),\sigma)}$ by Lemma 3.3. Since d is unique by Lemma 7.5, it follows that for both cases we find the same d, and hence the result is the same in any case. □

7.4. A Generalization for Higher Order Linear Difference Equations

Let $(\mathbb{F}(t),\sigma)$ be ΠΣ-field where $(\mathbb{F}(t),\sigma)$ a Π-extension of $(\mathbb{F},\sigma)$, $\boldsymbol{f}\in\mathbb{F}[t]^n$ and $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{F}[t]^m$. In this section we will deal with the problem to find a bound $b\in\mathbb{N}_0$ with (9) for the more general Situation 7.3 that contains Situation 7.2.

Situatation 7.3. Assume $\mathbf{0}\neq\boldsymbol{a}=(a_1,\dots,a_\lambda,\dots,a_\mu\,\dots,a_m)\in\mathbb{F}[t]^m$ with $\lambda<\mu$, $\mathrm{ord}(a_\lambda)=\mathrm{ord}(a_\mu)=p$ and

$$\mathrm{ord}(a_i)>\mathrm{ord}(a_\lambda)\text{ or }a_i=0\ \ \forall i\neq\lambda,\mu.$$

In particular suppose that $a_\lambda=t^p+r_1$ and $a_\mu=-c\,t^p+r_2$ for $c\in\mathbb{F}^*$ and $r_1,r_2\in\mathbb{F}[t]$ with $\mathrm{ord}(r_1),\mathrm{ord}(r_2)>0$.

First we generalize Theorem 7.2.

Theorem 7.5. *Let* $(\mathbb{F}(t),\sigma)$ *be a* Π*-extensions of* $(\mathbb{F},\sigma)$*,* $\boldsymbol{a}\in\mathbb{F}[t]^m$ *as in Situation 7.3 and assume that* $(\mathbb{F}(t),\sigma^{\mu-\lambda})$ *is a* Π*-extension of* $(\mathbb{F}^{\mu-\lambda},\sigma)$*. If there exists a* $g\in\mathbb{F}(t)$ *with* $d:=\operatorname{ord}(\operatorname{den}(g))>p$ *such that* $\operatorname{ord}(\operatorname{den}(\sigma_{\boldsymbol{a}}g))<d-p$ *then* $\sigma^{\mu-m}(c)\ (\alpha)_{\mu-\lambda}^{d}\in\mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$*.*

Proof: Let $g\in\mathbb{F}(t)$ with $d:=\operatorname{ord}(\operatorname{den}(g))\geq p$ and assume $\operatorname{ord}(\operatorname{den}(\sigma_{\boldsymbol{a}}g))<d-p$. Then by Proposition 7.1 and Situation 7.3 it follows that

$$d-p>\operatorname{ord}(\operatorname{den}(a_\lambda\,\sigma^{m-\lambda}(g)+a_\mu\,\sigma^{m-\mu}(g)))$$

and thus by Lemmas 7.3 and 6.7 we have

$$\begin{aligned}d-p&>\operatorname{ord}(\sigma^{\mu-m}(\operatorname{den}(a_\lambda\,\sigma^{m-\lambda}(g)+a_\mu\,\sigma^{m-\mu}(g))))\\&=\operatorname{ord}(\operatorname{den}(\sigma^{\mu-m}(a_\lambda)\,\sigma^{\mu-\lambda}(g)+\sigma^{\mu-m}(a_\mu)\,g).\end{aligned}$$

By $\sigma^{\mu-m}(a_\lambda)=(\alpha)_{\mu-m}^{p}\ t^p+\sigma^{\mu-m}(r_1)$ and $\sigma^{\mu-m}(a_\mu)=-\sigma^{\mu-m}(c)\ (\alpha)_{\mu-m}^{p}\ t^p+\sigma^{\mu-m}(r_2)$ it follows that

$$\operatorname{ord}(\operatorname{den}(b_1\,\sigma^{\mu-\lambda}(g)+b_2\,g))<d-p$$

for $b_1:=t^p+\sigma^{\mu-m}(r_1)/\,(\alpha)_{\mu-m}^{p}$ and $b_2:=-\sigma^{\mu-m}(c)\,t^p+\sigma^{\mu-m}(r_2)/\,(\alpha)_{\mu-m}^{p}$. As $(\mathbb{F}(t),\sigma^{\mu-\lambda})$ is a Π-extension of $(\mathbb{F},\sigma^{\mu-\lambda})$, we may apply Theorem 7.2 and thus we obtain $\sigma^{\mu-m}(c)\,\alpha_{\mu-\lambda}^{d}\in\mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$. □

Finally we obtain a degree bound method for Situation 7.3.

Theorem 7.6. *Let* $(\mathbb{F}(t),\sigma)$ *be a* Π*-extension of* $(\mathbb{F},\sigma)$*,* $\boldsymbol{a}\in\mathbb{F}[t]^m$ *as in Situation 7.3 and suppose that* $(\mathbb{F}(t),\sigma^{\mu-\lambda})$ *is a* Π*-extension of* $(\mathbb{F},\sigma^{\mu-\lambda})$*. Let* $g\in\mathbb{F}(t)$ *such that* $\sigma_{\boldsymbol{a}}g=\mathbb{F}[t]$*. If there exists a* $d\in\mathbb{N}_0$ *such that* $\sigma^{(\mu-m)}(c)\,\alpha_{\mu-\lambda}^{d}\in\mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$ *then* d *is uniquely determined and* $\operatorname{ord}(\operatorname{den}(g))\leq\max(d,p)$*. Otherwise, if there does not exist such a* d *then* $\operatorname{ord}(\operatorname{den}(g))\leq p$*.*

Proof: Let $g\in\mathbb{F}(t)$ and $\boldsymbol{c}\in\mathbb{K}^n$ with $\sigma_{\boldsymbol{a}}g=\boldsymbol{c}\,\boldsymbol{f}=:f$. Since $f\in\mathbb{F}[t]$, we have

$$\operatorname{ord}(\operatorname{den}(f))=\operatorname{ord}(1)=0. \tag{14}$$

1. Assume there exists a $d\geq 0$ with $\sigma^{\mu-m}(c)\ (\alpha)_{\mu-\lambda}^{d}\in\mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$. Then by Lemma 7.5 d is uniquely determined. Suppose $\operatorname{ord}(\operatorname{den}(g))>p$. Since (14), by Theorem 7.5 it follows that $\operatorname{ord}(\operatorname{den}(g))=d=\max(p,d)$. Otherwise, if $\operatorname{ord}(\operatorname{den}(g))\leq p$, we have $\operatorname{ord}(\operatorname{den}(g))\leq\max(p,d)$.
2. Assume there does not exist such a d. Since (14), by Theorem 7.5 it follows that $\operatorname{ord}(\operatorname{den}(g))\leq p$.

□

If $(\mathbb{F}(t),\sigma)$ is a ΠΣ-field and $k\neq 0$, $(\mathbb{F}(t),\sigma^k)$ is a ΠΣ-field by Theorem 7.6. Hence one can decide if $\sigma^{\mu-m}(c)\ (\alpha)_{\mu-\lambda}^{d}\in\mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$ for some d and find such a d in case of existence by Theorem 3.3. Therefore we can apply Theorem 7.6 to compute a bound $b\in\mathbb{N}_0$ that fulfills (9) for the special case described in Situation 7.3. But then $d_1:=t^b$ is a period 1 denominator bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ by Lemma 7.1. Therefore by Corollary 5.4 and Algorithm 6.1 we obtain an algorithm that solves the degree bound problem for Situation 7.3.

7.5. Reducing the Degree of the Period 1 Denominator Bound

By the following simplification of a given solution space one might find a period 1 denominator bound with lower degree. Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$, $\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ and $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}[t]^n$. Furthermore define

$$k := \min(\operatorname{ord}(a_1), \dots, \operatorname{ord}(a_m), \operatorname{ord}(f_1), \dots, \operatorname{ord}(f_n)).$$

Then clearly we have $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t)) = \mathrm{V}(\frac{\boldsymbol{a}}{t^k}, \frac{\boldsymbol{f}}{t^k}, \mathbb{F}(t))$ for $\frac{\boldsymbol{a}}{t^k} \in \mathbb{F}[t]^m$ and $\frac{\boldsymbol{f}}{t^k} \in \mathbb{F}[t]^n$. Hence without less of generality, we can cancel out the common factor t^k in $\boldsymbol{a}$ and $\boldsymbol{f}$ which may decrease p in Situations 7.1, 7.2 and 7.3. Looking closer at Theorems 7.1, 7.3 and 7.6, one can obtain a smaller bound b that fulfills (5) and hence can find a period 1 denominator bound t^b with lower degree.

8. Some Properties for the First Order Case

Combining Algorithms 6.1 and 6.1 in Subsections 6.2 and 7.3 we obtain an algorithm that solves the denominator bound problem for parameterized first order linear difference equations in ΠΣ-fields.

Algorithm 8.1. Compute a denominator bound for the first order case.

d=`DenBound`$((\mathbb{F}(t), \sigma), \boldsymbol{a}, \boldsymbol{f})$

Input: A ΠΣ-field $(\mathbb{F}(t), \sigma)$ with $\boldsymbol{a} \in (\mathbb{F}[t]^*)^2$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$.

Output: A denominator bound $d \in \mathbb{F}[t]^*$ for $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$

(1) Let $d \in \mathbb{F}[t]^*$ be the result of `Den0Bound`$((\mathbb{F}(t), \sigma), \boldsymbol{a}, \boldsymbol{f})$ in Algorithm 6.1

(2) If $(\mathbb{F}, \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$ THEN $b := 0$

ELSE let $b \in \mathbb{N}_0$ be given by `Den1Bound`$((\mathbb{F}(t), \sigma), \boldsymbol{a})$ of Algorithm 7.1.

(3) RETURN $d\,t^b$

The correctness of this algorithm follows by Corollary 5.4 and by the correctness of Algorithms 6.1 and 6.1 which are stated in Theorems 6.4 and 7.4.

Theorem 8.1. *Algorithm 8.1 is correct.*

Example 8.1. Looking at Examples 4.1, 6.3 and 7.1, we compute the denominator bound $d := z^4\,(1+z)\,(t+z)^2\,(-t+t^2+z)$ of $\mathrm{V}(\boldsymbol{a}, (0), \mathbb{Q}(t)(z))$ with input `DenBound`$((\mathbb{Q}(\mathtt{t})(\mathtt{z}), \sigma), \boldsymbol{a}, (0))$.

As already pointed out in the introduction, ΠΣ-fields are a powerful setting to eliminate sum quantifiers in indefinite nested multisums. In [Sch01, Section 1.2.4] it turns out that one has to construct a ΠΣ-field in a very subtle way such that the nested level of a given multisum can be really reduced. In work under development these aspects are carefully analyzed and algorithms are developed that enable to reduce the nested level of a given indefinite multisum. For those developments several properties of Algorithm 8.1 are needed, namely Proposition 8.1 and Theorem 8.2, that will be shown in the sequel. In order to obtain these properties, we first modify the above algorithm by a slight modification that is explained in the following remark.

Remark 8.1. Additionally we normalize the output $d \in \mathbb{F}[t]^*$ of Algorithm 8.1. One possibility is to return $d \in \mathbb{F}[t]^*$ by forcing the leading coefficient of d to 1. In case this bound is needed for further computations one should choose an other possibility: Since $(\mathbb{F}(t), \sigma)$ is a $\Pi\Sigma$-field, $\mathbb{F}$ is given as a field of rational functions $\mathbb{K}(x_1, \dots, x_e)$ over a constant field $\mathbb{K}$. Then one can clear the denominator, i.e. one obtains $d \in \mathbb{K}[t_1, \dots, t_e][t]$ and takes the primitive part of d. Then d is unique up to a constant in $\mathbb{K}$. Moreover one can define a term ordering on $\mathbb{K}[t_1, \dots, t_e][t]$ and can pick out a uniquely defined term that can be normalized further.

Proposition 8.1. *Let $(\mathbb{F}(t), \sigma)$ and $(\mathbb{G}(t), \sigma)$ be $\Pi\Sigma$-fields which are isomorph by a permutation. Then for any $\boldsymbol{a} \in (\mathbb{F}[t]^*)^2$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$ we have*

$$\texttt{DenBound}((\mathbb{F}(\texttt{t}), \sigma), \boldsymbol{a}, \boldsymbol{f}) = \texttt{DenBound}((\mathbb{G}(\texttt{t}), \sigma), \boldsymbol{a}, \boldsymbol{f})$$

where the result of `DenBound` *is normalized as stated in Remark 8.1.*

Proof: First we look closer at Algorithm 6.1. By Lemma 6.8 the bounding sequences are the same up to a constant in $\mathbb{F}^*$, and therefore the results of the computations $\texttt{Den0Bound}((\mathbb{F}(t), \sigma), \boldsymbol{a}, \boldsymbol{f})$ and $\texttt{Den0Bound}((\mathbb{G}(t), \sigma), \boldsymbol{a}, \boldsymbol{f})$ in line (1) differ only up to a constant in $\mathbb{F}^*$. Hence after a normalization (Remark 8.1) they must be the same. Thus by Proposition 7.2 the same value b is obtained in line (2), and thus the final result must be the same. □

Theorem 8.2. *Let $(\mathbb{F}(x_1, \dots, x_e)(t)(s), \sigma)$ and $(\mathbb{F}(s)(x_1, \dots, x_e)(t), \sigma)$ be $\Pi\Sigma$-fields which are isomorph by a permutation and let $\boldsymbol{a} \in \mathbb{F}(x_1, \dots, x_e)[t]^2$ such that there exists an $h \in \mathbb{F}(x_1, \dots, x_e)^*$ with $\sigma_{\boldsymbol{a}} h = 0$; let $\boldsymbol{f} \in \mathbb{F}(x_1, \dots, x_e)[t]^n$. Then we have*

$$\texttt{DenBound}((\mathbb{F}(\texttt{x}_1, \dots, \texttt{x}_\texttt{e})(\texttt{t}), \sigma), \boldsymbol{a}, \boldsymbol{f}) = \texttt{DenBound}((\mathbb{F}(\texttt{s})(\texttt{x}_1, \dots, \texttt{x}_\texttt{e})(\texttt{t}), \sigma), \boldsymbol{a}, \boldsymbol{f})$$

where the result of `DenBound` *is normalized as stated in Remark 8.1.*

Proof: First we look closer at Algorithm 6.1. By Lemma 6.9 the bounding sequences in the computations given by $\texttt{Den0Bound}((\mathbb{F}(x_1, \dots, x_e)(t), \sigma), \boldsymbol{a}, \boldsymbol{f})$ and $\texttt{Den0Bound}((\mathbb{F}(s)(x_1, \dots, x_e)(t), \sigma), \boldsymbol{a}, \boldsymbol{f})$ are the same up to a constant in $\mathbb{F}(s)(x_1, \dots, x_e)^*$, and therefore the results in line (1) differ only up to a constant in $\mathbb{F}(s)(x_1, \dots, x_e)^*$. Hence after a normalization (Remark 8.1) they must be the same. Moreover Algorithm 7.1 delivers for $\texttt{Den1Bound}((\mathbb{F}(x_1, \dots, x_e)(t), \sigma), \boldsymbol{a}, \boldsymbol{f})$ and $\texttt{Den1Bound}((\mathbb{F}(s)(x_1, \dots, x_e)(t), \sigma), \boldsymbol{a}, \boldsymbol{f})$ the same value $b \in \mathbb{N}_0$ as a consequence of Proposition 7.3, and thus the final result must be the same. □

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