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Degree Bounds to Find Polynomial Solutions of Parameterized Linear Difference Equations in $\Pi\Sigma$-Fields*

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Abstract

An important application of solving parameterized linear difference equations in $\Pi\Sigma$-fields, a very general class of difference fields, is simplifying and proving of nested multisum expressions and identities. Together with other reduction techniques described elsewhere, the algorithms considered in this article can be used to search for all solutions of such difference equations. More precisely, within a typical reduction step one often is faced with subproblems to find all solutions of linear difference equations where the solutions live in a polynomial ring. The algorithms under consideration deliver degree bounds for these polynomial solutions.

1. Introduction

M. Karr defined in [Kar81, Kar85] a very general class of difference fields, so called $\Pi\Sigma$-fields, under two aspects. First $\Pi\Sigma$-fields allow to describe indefinite nested multisums in a formal way, and second one is capable of solving first order linear difference equations in this $\Pi\Sigma$-field setting; this amounts to simplify indefinite nested multisums by elimination of sum quantifiers. In [Sch01, Sch02b] I streamlined Karr's ideas based on [Bro00] to a compact algorithm and generalized the underlying reduction techniques which enables to search for all solutions of parameterized linear difference equations with arbitrary order in $\Pi\Sigma$-fields. By this general algorithm one is not only able to deal with indefinite summation, but also can prove a huge class of definite multisum identities by applying Zeilberger's creative telescoping trick [Zei90] in the setting of $\Pi\Sigma$-fields. Moreover by using our general algorithm one can solve recurrences, obtained by creative telescoping, in the $\Pi\Sigma$-field setting and hence one even can discover definite

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multisum identities. These algorithms are available in form of a package called `Sigma` [Sch00, Sch01] in the computer algebra system Mathematica.

In order to solve parameterized linear difference equations in ΠΣ-fields, one generates a reduction process that is introduced in [Sch02b]. In this reduction one is faced with subproblems to find all solutions of parameterized linear difference equations where the solutions live in a polynomial ring. Then one can apply further reduction techniques given in [Sch02b] to solve this difference equation, if one knows a degree bound of the solutions in that polynomial ring. As illustrated in Section 2 these reduction techniques are well known in one of the most simplest cases of ΠΣ-fields. In particular in [Abr89a, Pet92, SAA95, PWZ96] one computes these degree bounds for the polynomial solutions of a given difference equation with rational coefficients. In this work we try to develop further degree bounds of a given difference equation in the much more general setting of ΠΣ-fields. Based on the work of [Kar81] I develop algorithms to compute degree bounds for first order linear difference equations in ΠΣ-fields. Whereas in Karr's work theoretical and computational aspects are mixed, I try to separate his results in several parts to achieve more transparency. Furthermore all prove steps are carefully carried out, whereas in Karr's work the essential proves are omitted. Combining this result with [Sch02b, Sch02a] gives a complete algorithm to solve first order linear difference equations in ΠΣ-fields. Similarly to the first order case one needs degree bounds for linear difference equations of higher order. As it turns out, it is much harder to find such degree bounds in the general setting of ΠΣ-fields. In this work I generalized Karr's degree bounds to the higher order case which enables to treat at least some special cases of linear difference equations. In this sense the degree bounds under consideration contribute to important developments to solve linear difference equations in ΠΣ-fields.

First the degree bound problem is introduced in the general context of solving linear difference equations. After defining ΠΣ-fields in Section 3, some basic strategies for the degree bound problem are specified in Section 4. Finally in Sections 5 and 6 methods are developed that find several degree bounds in ΠΣ-fields. Especially in Section 7 this leads to an algorithm that solves the degree bound problem for first order linear difference equations in ΠΣ-fields. In particular we analyze some important properties of that algorithm which are needed for further development in the theory of ΠΣ-fields and indefinite summation. Moreover in Section 8 results from [Sch01] are introduced that enable to find degree bounds for linear difference equations in an important subclass of ΠΣ-fields.

2. The Degree Bound Problem

In [Abr89b, Abr89a, Abr95] S. Abromov is concerned in finding all solutions $g(t)$ in the field of rational functions $\mathbb{K}(t)$ with characteristic 0 that fulfill linear difference equations of the type

$$a_m(t)\,g(t+m)+\cdots+a_0(t)\,g(t)=f(t) \tag{1}$$

where $a_i(t)$ and $f(t)$ are polynomials in $\mathbb{K}[t]$. Looking closer at this problem, one immediately sees that this problem can be formalized in difference fields.

Definition 2.1. A *difference field* (resp. ring) is a field (resp. ring) $\mathbb{F}$ together with a field (resp. ring) automorphism $\sigma : \mathbb{F} \to \mathbb{F}$. In the sequel a difference field (resp. ring) given by the field (resp. ring) $\mathbb{F}$ and automorphism σ is denoted by $(\mathbb{F}, \sigma)$. Moreover the subset $\mathbb{K} := \{k \in \mathbb{F} \,|\, \sigma(k) = k\}$ is called the *constant field* of the difference field $(\mathbb{F}, \sigma)$.

It is easy to see that the constant field $\mathbb{K}$ of a difference field $(\mathbb{F}, \sigma)$ is a subfield of $\mathbb{F}$. In the sequel we will assume that **all** fields are of characteristic 0. Then it is immediate that for any field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ we have $\sigma(q) = q$ for $q \in \mathbb{Q}$. Hence in any difference field, $\mathbb{Q}$ is a subfield of its constant field.
Problem 1 can be described by difference equations in $(\mathbb{K}(t), \sigma)$:

Example 2.1. Let $\mathbb{K}(t)$ be the field of rational function over the field $\mathbb{K}$, this means $\mathbb{K}(t)$ is the quotient field of the polynomial ring $\mathbb{K}[t]$. Then we can define uniquely the difference field $(\mathbb{K}(t), \sigma)$ with constant field $\mathbb{K}$ where the field automorphism $\sigma : \mathbb{K}(t) \to \mathbb{K}(t)$ is canonically defined by $\sigma(t) = t + 1$.

As illustrated in [Sch01, Sch02b] one is able to discover and prove a huge class of indefinite and definite multisum identities by solving parameterized linear difference equations in ΠΣ-fields; in particular one can carry out indefinite summation, Zeilberger's creative telescoping idea and solving recurrences.

Solving Parameterized Linear Difference Equations

- Given a difference field $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, $a_1, \dots, a_m \in \mathbb{F}$ with $m \geq 1$ and $(a_1 \dots a_m) \neq (0, \dots, 0) =: \mathbf{0}$ and $f_1, \dots, f_n \in \mathbb{F}$ with $n \geq 1$.
- Find all $g \in \mathbb{F}$ and all $c_1, \dots, c_n \in \mathbb{K}$ with $a_1\, \sigma^{m-1}(g) + \dots + a_m\, g = c_1\, f_1 + \dots + c_n\, f_n$.

By the above remarks one can immediately see that problem (1) is contained in our general problem by choosing the difference field $(\mathbb{K}(t), \sigma)$ as defined in Example 2.1. Furthermore note that in any difference field $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, the field $\mathbb{F}$ can be interpreted as a vector space over $\mathbb{K}$. Hence the above problem can be described by the following set called solution space.

Definition 2.2. Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$ and consider a subspace $\mathbb{V}$ of $\mathbb{F}$ as a vector space over $\mathbb{K}$. Let $\mathbf{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}^m$ and $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. We define the *solution space* for $\boldsymbol{a}, \boldsymbol{f}$ in $\mathbb{V}$ by

$$\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V}) = \{(c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{V} : a_1\, \sigma^{m-1}(g) + \dots + a_m\, g = c_1\, f_1 + \dots + c_n\, f_n\}.$$

It follows immediately that $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$ is a vector space over $\mathbb{K}$. Moreover in [Sch02b] based on [Coh65] it is proven that this vector space has finite dimension.

Proposition 2.1. *Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$ and assume $\boldsymbol{f} \in \mathbb{F}^n$ and $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^m$. Let $\mathbb{V}$ be a subspace of $\mathbb{F}$ as a vector space over $\mathbb{K}$. Then $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$ is a vector space over $\mathbb{K}$ with maximal dimension $m+n-1$.*

Finally some notations are introduced. Let $\mathbb{F}$ be a field and $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. For $h \in \mathbb{F}$ we write $h\,\boldsymbol{f} = (h\,f_1, \dots, h\,f_n) \in \mathbb{F}^n$ and $\boldsymbol{f} \wedge h = (f_1, \dots, f_m, h) \in \mathbb{F}^{n+1}$. If $\boldsymbol{c} \in \mathbb{F}^n$, we define the vector product $\boldsymbol{c}\,\boldsymbol{f} = \sum_{i=1}^{n} c_i\,f_i$. Moreover for a function $\sigma : \mathbb{F} \to \mathbb{F}$, $\boldsymbol{a} \in \mathbb{F}^m$ and $g \in \mathbb{F}$, we introduce $\sigma_{\boldsymbol{a}} g := a_1\,\sigma^{m-1}(g) + \dots + a_m\,g \in \mathbb{F}$. This leads to the compact description of

$$\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V}) = \{\boldsymbol{c} \wedge g \in \mathbb{K}^n \times \mathbb{V} \mid \sigma_{\boldsymbol{a}} g = \boldsymbol{c}\,\boldsymbol{f}\}.$$

In [Sch02b] several reduction techniques are introduced in order to compute a basis of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$ in ΠΣ-fields. An important step is the so called denominator bound method which popped up the first time in [Abr89b, Abr95] for the rational case (1). Here one is concerned to find a basis of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{K}(t))$ for some $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{K}[t]^m$ and $\boldsymbol{f} \in \mathbb{K}[t]^1$ in the difference field $(\mathbb{K}(t), \sigma)$ given in Example 2.1 by the following strategy.

The Denominator Elimination Strategy

1. Compute a *denominator bound* $d \in \mathbb{K}[t]^*$ such that for all elements in the solution space $\boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{K}(t))$ we have $d\,g \in \mathbb{K}[t]$.
2. Compute a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{K}[t])$ for $\boldsymbol{a'} := (\frac{a_1}{\sigma^{m-1}(d)}, \dots, \frac{a_{m-1}}{\sigma(d)}, \frac{a_m}{d}) \in \mathbb{K}(t)^m$.
3. Reconstruct a basis $\{\boldsymbol{c_1} \wedge \frac{g_1}{d}, \dots, \boldsymbol{c_l} \wedge \frac{g_l}{d}\}$ of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{K}(t))$.

Hence it remains to find a basis of the solution space $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{K}[t])$. Here an essential step is to restrict the solution range $\mathbb{K}[t]$ by $b \in \mathbb{N}_0$ to a finite dimensional subspace

$$\mathbb{K}[t]_b := \{f \in \mathbb{K}[t] \mid \deg(f) \leq b\}$$

of $\mathbb{K}[t]$ over $\mathbb{K}$. This means we have to find a *degree bound* b such that

$$\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{K}[t]) = \mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{K}[t]_b).$$

Then it is a matter of solving a linear system of equations to compute a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{K}[t]_b)$. In [Abr89a, Pet92, SAA95, PWZ96] several algorithms are introduced that allow to determine this degree bound of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{K}[t])$; as it turns out in [PW00], all these algorithms are equivalent and compute exactly the same degree bound of a specific solution space.

As will be introduced in the next section, a ΠΣ-field $(\mathbb{K}(t_1) \dots (t_{e-1})(t_e), \sigma)$ with constant field $\mathbb{K}$ is constructed by a tower of transcendental extensions t_i; for further considerations we set $\mathbb{F} := \mathbb{K}(t_1) \dots (t_{e-1})$ for such a ΠΣ-field. In [Sch02b] several reduction techniques are introduced that allow to search for a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t_e)))$ with $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}[t_e]^m$ and $\boldsymbol{f} \in \mathbb{F}[t_e]^n$. Similarly to the rational case $\mathbb{K}(t)$, one first bounds the denominator of the solutions in $\mathbb{F}(t_e)$ and reduces the problem to find a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{F}[t_e])$ for specific $\boldsymbol{a'} \in \mathbb{F}[t_e]^m$ and $\boldsymbol{f'} \in \mathbb{F}[t]^n$. This strategy is intensively analyzed in [Sch02a] for ΠΣ-fields which combines results from [Kar81, Bro00]. Then in a second reduction step one has to solve the degree bound problem.

The Degree Bound Problem

- Given a ΠΣ-field $(\mathbb{F}(t_e), \sigma)$, $\mathbf{0} \neq \boldsymbol{a'} \in \mathbb{F}[t_e]^m$ and $\boldsymbol{f'} \in \mathbb{F}[t_e]^n$.
- Find a degree bound $b \in \mathbb{N}_0$ such that $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}[t_e]) = \mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}[t_e]_b)$ where $\mathbb{F}[t_e]_b := \{f \in \mathbb{F}[t_e] \mid \deg(f) \leq b\}$.

Finally by further reduction techniques carefully considered in [Sch02b] one tries to find a basis of the solution space $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}[t_e]_b)$.

This article will explain how one can find such a degree bound b in the general setting of ΠΣ-fields for the first order case, i.e. $\mathbf{0} \neq \boldsymbol{a'} \in \mathbb{F}[t_e]^2$. Starting from results of [Kar81], these bounds are developed step by step from theoretical and computational point of views. Moreover these ideas are then generalized to the higher order case for some special cases.

3. ΠΣ-Fields and Some Important Properties

As already described in previous sections, this work restricts to so called ΠΣ-fields that are introduced in [Kar81, Kar85] and further analyzed in [Bro00, Sch01, Sch02a]. In the following the basic definition and properties are introduced that are needed in the sequel.

3.1. The Definition of ΠΣ-Extensions

In order to define ΠΣ-fields, the notion of difference field extensions is needed.

Definition 3.1. Let $(\mathbb{E}, \sigma_{\mathbb{E}})$, $(\mathbb{F}, \sigma_{\mathbb{F}})$ be difference fields. $(\mathbb{E}, \sigma_{\mathbb{E}})$ is called a *difference field extension* of $(\mathbb{F}, \sigma_{\mathbb{F}})$, if $\mathbb{F} \subseteq \mathbb{E}$ and $\sigma_{\mathbb{F}}(f) = \sigma_{\mathbb{E}}(f)$ for all $f \in \mathbb{F}$.

Example 3.1. Let $(\mathbb{K}(t), \sigma)$ be the difference field defined in Example 2.1, refined by $\mathbb{K} := \mathbb{Q}$, and consider the the field extension $\mathbb{Q}(t)(z)$ of $\mathbb{Q}(z)$ where z is transcendental over $\mathbb{Q}(t)$. Then one can define uniquely the field automorphism $\sigma' : \mathbb{Q}(t)(z) \to \mathbb{Q}(t)(z)$ where the following holds: $\sigma'(f) = \sigma(f)$ for all $f \in \mathbb{Q}(t)$ and $\sigma(z) = \alpha\, z + \beta$ for some $\alpha \in \mathbb{Q}(t)^*$ and $\beta \in \mathbb{Q}(t)$. Clearly, $(\mathbb{Q}(t)(z), \sigma')$ is a difference field extension of $(\mathbb{Q}(t), \sigma)$.

If $(\mathbb{E}, \tilde{\sigma})$ is a difference field extension of $(\mathbb{F}, \sigma)$, we will not distinguish anymore that $\sigma : \mathbb{F} \to \mathbb{F}$ and $\tilde{\sigma} : \mathbb{E} \to \mathbb{E}$ are actually different automorphisms.

Definition 3.2. $(\mathbb{F}(t), \sigma)$ is a Π-*extension* of $(\mathbb{F}, \sigma)$ if $\sigma(t) = \alpha\, t$ with $\alpha \in \mathbb{F}^*$, t is transcendental over $\mathbb{F}$ and $\mathrm{const}_\sigma \mathbb{F}(t) = \mathrm{const}_\sigma \mathbb{F}$.

According to [Kar81] we introduce the notion of the homogeneous group which plays an essential role in the theory of ΠΣ-fields.

Definition 3.3. The *homogeneous group* of $(\mathbb{F}, \sigma)$ is $\mathrm{H}_{(\mathbb{F},\sigma)} := \{\frac{\sigma(g)}{g} \mid g \in \mathbb{F}^*\}$.

One can easily check that $\mathrm{H}_{(\mathbb{F},\sigma)}$ forms a multiplicative group. With this notion one obtains an equivalent description of a Π-extension. This result and its proof can be found in [Kar85, Theorem 2.2] or [Sch01, Theorem 2.2.2].

Theorem 3.1. *$(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\, t$ where $\alpha\in\mathbb{F}^*$. Then $(\mathbb{F}(t),\sigma)$ is a* Π-extension *of $(\mathbb{F},\sigma)$ if and only if there does not exist an $n>0$ such that $\alpha^n\in\mathrm{H}_{(\mathbb{F},\sigma)}$.*

Next we define Σ-extensions according to Karr's notions.

Definition 3.4. $(\mathbb{F}(t),\sigma)$ is a *Σ-extension* of $(\mathbb{F},\sigma)$ if

1. $\sigma(t)=\alpha\, t+\beta$ with $\alpha,\beta\in\mathbb{F}^*$ and $t\notin\mathbb{F}$,
2. there does not exist a $g\in\mathbb{F}(t)\setminus\mathbb{F}$ with $\frac{\sigma(g)}{g}\in\mathbb{F}$, and
3. for all $n\in\mathbb{Z}^*$ we have that $\alpha^n\in\mathrm{H}_{(\mathbb{F},\sigma)}\Rightarrow\alpha\in\mathrm{H}_{(\mathbb{F},\sigma)}$.

In particular we have two special cases that are of interest in this article.

Definition 3.5. Let $(\mathbb{F}(t),\sigma)$ be a Σ-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\, t+\beta$. It is called a *simple* *Σ-extension*, if for all $n>0$ we have $\alpha^n\notin\mathrm{H}_{(\mathbb{F},\sigma)}$. If we have $\alpha=1$, it is called a *proper sum extension*.

Example 3.2. In Example 2.1 $(\mathbb{K}(t),\sigma)$ is a proper sum extension of $(\mathbb{K},\sigma)$.

Actually we are basically interested in extensions, similarly to Π-extensions, where $\sigma(t)=\alpha\, t+\beta$, $(\alpha,\beta\notin\mathbb{F}^*)$, t transcendental and $\mathrm{const}_\sigma\mathbb{F}(t)=\mathrm{const}_\sigma\mathbb{F}$. Under these considerations property (1.) with $t\notin\mathbb{F}$ fits to the desired goal. Unfortunately condition (3.) seems to be quite technical, and indeed is needed for computational aspects for instance in Theorem 3.4. In particular if we deal with simple Σ-extensions, we have even stronger properties which gives more flexibility to compute degree bounds of a solution space as can be seen in Section 5.1. On the other side in many cases one is interested in proper sum extensions, this means $\alpha=1$, and hence condition (3.) is obsolete since $1\in\mathrm{H}_{(\mathbb{F},\sigma)}$. Moreover the next result states that in a Σ-extension t is transcendental and $\mathrm{const}_\sigma\mathbb{F}(t)=\mathrm{const}_\sigma\mathbb{F}$. This statement is a direct consequence of [Sch01, Theorem 2.2.3] which is a corrected version of [Kar81, Theorem 3] or [Kar85, Theorem 2.3].

Theorem 3.2. *Let $(\mathbb{F}(t),\sigma)$ be a Σ-extension of $(\mathbb{F},\sigma)$. Then $(\mathbb{F}(t),\sigma)$ is canonically defined by $\sigma(t)=\alpha\, t+\beta$ for some $\alpha,\beta\in\mathbb{F}^*$, t is transcendental over $\mathbb{F}$ and* $\mathrm{const}_\sigma\mathbb{F}(t)=\mathrm{const}_\sigma\mathbb{F}$.

Similarly to Π-extensions an alternative description of Σ-extensions can be given. This result follows from [Kar81, Theorem 1] or [Kar85, Theorem 3] and is essentially the same as [Sch01, Corollary 2.2.3].

Theorem 3.3. *Let $(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\, t+\beta$ where $\alpha,\beta\in\mathbb{F}^*$. Then $(\mathbb{F}(t),\sigma)$ is a* Σ-extension *of $(\mathbb{F},\sigma)$ if and only if there does not exist a $g\in\mathbb{F}$ with $\sigma(g)-\alpha\, g=\beta$, and property (3.) from Definition 3.4 holds.*

Now we are ready to define ΠΣ-extension.

Definition 3.6. $(\mathbb{F}(t), \sigma)$ is called a ΠΣ-extension of $(\mathbb{F}, \sigma)$, if $(\mathbb{F}(t), \sigma)$ is a Π- or a Σ-extension of $(\mathbb{F}, \sigma)$.

Clearly if $(\mathbb{F}, \sigma)$ is a difference field, also $(\mathbb{F}, \sigma^k)$ is a difference field for any $k \in \mathbb{Z}$. Moreover if $(\mathbb{F}(t), \sigma)$ is a ΠΣ-extension of $(\mathbb{F}, \sigma)$, also $(\mathbb{F}(t), \sigma^k)$ is a difference field extension of $(\mathbb{F}, \sigma^k)$. More precisely automorphisms of such difference extensions are defined in the following way.

Example 3.3. Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$, $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Furthermore assume $k \in \mathbb{Z}$. Then for $k \geq 0$ we have $\sigma^k(t) = t \prod_{i=0}^{k-1} \sigma^i(\alpha) + \gamma$ for some $\gamma \in \mathbb{F}$; whereas for $k < 0$ we have $\sigma^k(t) = t \prod_{i=1}^{k} \sigma^{-i}(1/\alpha) + \gamma$ for some $\gamma \in \mathbb{F}$. If $\beta = 0$, $\gamma = 0$ for all $k \in \mathbb{Z}$.

This motivates us to the following definition.

Definition 3.7. Let $(\mathbb{F}, \sigma)$ be a difference field, $f \in \mathbb{F}^*$ and $k \in \mathbb{Z}$. The *σ-factorial* $(f)_k$ is defined by $\prod_{i=0}^{k-1} \sigma^i(f)$, if $k \geq 0$, and by $\prod_{i=1}^{k} \sigma^{-i}(1/f)$, if $k < 0$.

Example 3.4. Continuing the previous example we have $\sigma^k(t) = (\alpha)_k\, t + \gamma$ with $\gamma \in \mathbb{F}$ for all $k \in \mathbb{Z}$. In particular if $\beta = 0$, we have $\sigma^k(t) = (\alpha)_k\, t$.

3.2. ΠΣ-Extensions and the Field of Rational Functions

The next lemma will be used over and over again; it gives the link between ΠΣ-extensions and its domain of rational functions. The proof is straightforward.

Lemma 3.1. *Let* $(\mathbb{F}(t), \sigma)$ *be a ΠΣ-extension of* $(\mathbb{F}, \sigma)$*. Then* $\mathbb{F}(t)$ *is a field of rational functions over* $\mathbb{K}$*. Furthermore,* σ *is an automorphism of the polynomial ring* $\mathbb{F}[t]$*, i.e.* $(\mathbb{F}[t], \sigma)$ *is a difference ring extension of* $(\mathbb{F}, \sigma)$*. Additionally, for all* $f, g \in \mathbb{F}[t]$ *we have* $\deg(\sigma(f)) = \deg(f)$ *and* $\gcd(\sigma(f), \sigma(g)) = \sigma(\gcd(f, g))$*.*

Some notions are needed for such a polynomial ring $\mathbb{F}[t]$ and its quotient field $\mathbb{F}(t)$. By convention the *zero-polynomial* 0 has degree $-\infty$. Furthermore, if $f = \sum_{i=0}^{n} f_i\, t_i \in \mathbb{F}[t]$, the i-th *coefficient* f_i of f will be denoted by $[f]_i$, i.e. $[f]_i = f_i$. If $i > n$, we have $[f]_i = 0$. Moreover we define the rank function $\| \ \|$ of $\mathbb{F}[t]$ by

$$\|f\| := \begin{cases} -1 & \text{if } f = 0 \\ \deg(f) & \text{otherwise.} \end{cases}$$

Now we will generalize these notions from $\mathbb{F}[t]$ to its quotient field $\mathbb{F}(t)$. For this we consider the subspace $\mathbb{F}(t)^{(frac)}$ of $\mathbb{F}[t]$ over $\mathbb{K}$ defined by

$$\mathbb{F}(t)^{(frac)} := \left\{ \frac{p}{q} \in \mathbb{F}(t) \,|\, p \in \mathbb{F}[t], q \in \mathbb{F}[t]^* \text{ and } \deg(p) < deg(q) \right\}.$$

Then by polynomial division with remainder the next statement holds.

Lemma 3.2. *Let $\mathbb{F}(t)$ be a field of rational functions and consider $\mathbb{F}[t]$ and $\mathbb{F}(t)^{(frac)}$ as subspaces of $\mathbb{F}(t)$ over $\mathbb{K}$. Then we have $\mathbb{F}(t) = \mathbb{F}[t] \oplus \mathbb{F}(t)^{(frac)}$.*

Decompose $f \in \mathbb{F}(t)$ by $f = f_p + f_r \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(frac)}$. Then we generalize the coefficient of f by $[f]_i := [f_p]_i$ and the rank of f by $\|f\| := \|f_p\|$. For $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}(t)^n$ we define $[\boldsymbol{f}]_i := ([f_1]_i, \dots, [f_n]_i) \in \mathbb{F}^n$ and $\|\boldsymbol{f}\| := \max_i \|f_i\|$. The next two lemmas will be used over and over again; the proof is straightforward by using Lemma 3.2 and properties of the degree function.

Lemma 3.3. *Let $f, g \in \mathbb{F}(t)$. Then $\|f + g\| \leq \max(\|f\|, \|g\|)$. Furthermore if $\|f\|, \|g\| \neq -1$, we have $\|f\, g\| = \|f\| + \|g\|$. Moreover, we have $\|\boldsymbol{c}\, \boldsymbol{f}\| \leq \|\boldsymbol{f}\|$ for any $\boldsymbol{f} \in \mathbb{F}^n$ and $\boldsymbol{c} \in \mathbb{F}$.*

Lemma 3.4. *Let $\mathbb{F}(t)$ be a field of rational functions and $f, g \in \mathbb{F}(t)$. Then $[f + g]_d = [f]_d + [g]_d$ for any $d \in \mathbb{N}_0$. If $d := \|f\| \geq 0$ and $e := \|g\| \geq 0$ then $[f\, g]_r = \sum_{i+j=r} [f]_i\, [g]_j$ for any r with $\max(d, e) \leq r \leq d + e$. Furthermore if $\sigma : \mathbb{F}(t) \to \mathbb{F}(t)$ is a field automorphism then $[\sigma(f)]_d = \sigma([f]_d)$ for any $d \in \mathbb{N}_0$.*

With this notations one can formulate a simple fact that is needed later. In particular we take into account the following property of $\boldsymbol{a}$ that will be important for later considerations.

Situatation 3.1. Assume $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ with $\|a_r\| = \|\boldsymbol{a}\|$ for some $r \in \{1, \dots, m\}$ and $\|a_i\| < \|\boldsymbol{a}\|$ for all i with $i \neq r$

Lemma 3.5. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$, $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $f, g \in \mathbb{F}[t]$ such that $\sigma_{\boldsymbol{a}} g = f$. Then $\|f\| \leq \|\boldsymbol{a}\| + \|g\|$. Furthermore, if $\boldsymbol{a}$ is as in Situation 3.1 and $g \in \mathbb{F}[t]^*$ then $\|f\| = \|\boldsymbol{a}\| + \|g\|$.*

Proof: If $g = 0$, we have $f = \sigma_{\boldsymbol{a}} g = 0$ and hence $-1 = \|f\| \leq \|\boldsymbol{a}\| + \|g\|$ holds by $\|g\| = -1$ and $\|\boldsymbol{a}\| \geq 0$. Otherwise assume that $g \neq 0$, i.e. $\|g\| \geq 0$. By Lemma 3.3 it follows that

$$\|f\| = \|\sigma_{\boldsymbol{a}} g\| = \|a_1\, \sigma^{m-1}(g) + \dots + a_m\, g\| \leq \max(\|a_1\, \sigma^{m-1}(g)\|, \dots, \|a_m\, g\|).$$

Please note that we have $\|a_i\, \sigma^{m-i}(g)\| \leq \|a_i\| + \|\sigma^{m-i}(g)\|$, if $a_i = 0$; otherwise, if $a_i \neq 0$, we even have equality. Moreover if $a_i = 0$ and $a_j \neq 0$ then $\|a_i\| + \|\sigma^{m-i}(g)\| < \|a_j\| + \|\sigma^{m-j}(g)\|$. Since there exists an j with $a_j \neq 0$, it follows that

$$\max(\|a_1\, \sigma^{m-1}(g)\|, \dots, \|a_m\, g\|) = \max(\|a_1\| + \|\sigma^{m-1}(g)\|, \dots, \|a_m\| + \|g\|).$$

By Lemma 3.1 we have $\|\sigma^i(g)\| = \|g\|$ for all $i \in \mathbb{Z}$ and thus

$$\max(\|a_1\| + \|\sigma^{m-1}(g)\|, \dots, \|a_m\| + \|g\|) = \max(\|a_1\|, \dots, \|a_m\|) + \|g\| = \|\boldsymbol{a}\| + \|g\|$$

which proves the first statement of the lemma. If there exists additionally an r as in Situation 3.1, we have

$$\|a_1\, \sigma^{m-1}(g) + \dots + a_m\, g\| = \|a_r\, \sigma^{m-r}(g)\| = \max(\|a_1\, \sigma^{m-1}(g)\|, \dots, \|a_m\, g\|)$$

and by the same argumentations as for the first statement the second result follows immediately. □

3.3. ΠΣ-Fields and Some Properties

For the definition of ΠΣ-fields properties on the constant field are required.

Definition 3.8. A field $\mathbb{K}$ is called *computable*, if

- for any $k \in \mathbb{K}$ one is able to decide, if $k \in \mathbb{Z}$,
- polynomials in the polynomial ring $\mathbb{K}[t_1, \dots, t_n]$ can be factored over $\mathbb{K}$ and
- one knows how to compute a basis of $\left\{(n_1, \dots, n_k) \in \mathbb{Z}^k \mid c_1^{n_1} \cdots c_k^{n_k} = 1\right\}$ which is a submodule of $\mathbb{Z}^k$ over $\mathbb{Z}$ for any $(c_1, \dots, c_n) \in \mathbb{K}^n$.

Lemma 3.6. *Any field of rational functions $\mathbb{Q}(n_1, \dots, n_r)$ is computable.*

Finally ΠΣ-fields are essentially defined by ΠΣ-extensions. Unlike Karr's definition in this work we force *additionally* that the constant field is computable.

Definition 3.9. Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$. $(\mathbb{F}, \sigma)$ is called a ΠΣ-*field* over $\mathbb{K}$, if $\mathbb{K}$ is computable, $\mathbb{F} := \mathbb{K}(t_1) \dots (t_n)$ for $n \geq 0$ and $(\mathbb{F}(t_1, \dots, t_{i-1})(t_i), \sigma)$ is a ΠΣ-extension[†] of $(\mathbb{F}(t_1, \dots, t_{i-1}), \sigma)$ for all $1 \leq i \leq n$.

Example 3.5. Note that the difference field $(\mathbb{Q}(t), \sigma)$, defined in Example 2.1 with $\mathbb{K} := \mathbb{Q}$, is a ΠΣ-field over $\mathbb{Q}$. Now consider the difference field extension $(\mathbb{Q}(t)(z), \sigma)$ of $(\mathbb{Q}(t), \sigma)$ as it is constructed in Example 3.1 with $\sigma(z) = \alpha\, z + \beta$ for some $\alpha \in \mathbb{Q}(t)^*$ and $\beta \in \mathbb{Q}(t)$. Then one can show that $(\mathbb{Q}(t)(z), \sigma)$ is a ΠΣ-extension of $(\mathbb{Q}(t), \sigma)$, if one chooses $(\alpha, \beta) = (t+1, 0)$ or $(\alpha, \beta) = \left(1, \frac{1}{t+1}\right)$. Hence in both instances $(\mathbb{Q}(t)(z), \sigma)$ are ΠΣ-fields over $\mathbb{Q}$.

ΠΣ-fields are designed in such a way that the following problem, stated in form of a theorem, can be solved. Its proof follows from [Kar81, Theorem 9].

Theorem 3.4. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-field and assume $(f_1, \dots, f_n) \in \mathbb{F}(t)^n$. Then there exists an algorithm that computes a finite basis of the submodule $\left\{(z_1, \dots, z_n) \in \mathbb{Z}^n \mid f_1^{z_1} \dots f_n^{z_n} \in \mathrm{H}_{(\mathbb{F},\sigma)}\right\}$ of $\mathbb{Z}^n$.*

The next theorem is taken from [Kar85, Theorem 4]. This result allows to generalize the denominator bound for first order linear difference equations to higher order linear difference equations for some specific cases in Subsection 5.2.

Theorem 3.5. *If $(\mathbb{F}, \sigma)$ be a ΠΣ-field, $(\mathbb{F}, \sigma^k)$ is a ΠΣ-field for all $k \in \mathbb{Z}^*$.*

In particular this theorem can be refined by the following result.

Corollary 3.1. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-field and $k \in \mathbb{Z}^*$. Then $(\mathbb{F}(t), \sigma)$ is a Π-extension (resp. Σ-extension) of $(\mathbb{F}, \sigma)$ if and only if $(\mathbb{F}(t), \sigma^k)$ is a Π-extension (resp. Σ-extension) of $(\mathbb{F}, \sigma)$. Moreover, $(\mathbb{F}(t), \sigma)$ is a simple Σ-extension of $(\mathbb{F}, \sigma)$ if and only if $(\mathbb{F}(t), \sigma^k)$ is a simple Σ-extension of $(\mathbb{F}, \sigma^k)$.*

[†]For the case $i = 0$ this means that $(\mathbb{F}(t_1), \sigma)$ is a ΠΣ-extension of $(\mathbb{F}, \sigma)$.

Proof: Let $(\mathbb{F}(t),\sigma)$ be a Π-extension of $(\mathbb{F},\sigma)$ with $\sigma(t) = \alpha\, t$ and $\alpha \in \mathbb{F}^*$. Then by Theorem 3.5 $(\mathbb{F}(t),\sigma^k)$ is a Π-extension of $(\mathbb{F},\sigma^k)$ with $\sigma^k(t) = (\alpha)_k\, t$. Conversely, if $(\mathbb{F}(t),\sigma^k)$ is a Π-extension of $(\mathbb{F},\sigma)$ with $\sigma^k(t) = \alpha'\, t$ and $\alpha' \in \mathbb{F}^*$, by Theorem 3.5 $(\mathbb{F}(t),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$ with $\sigma(t) = (\alpha')_{-k}$. Hence Π-extensions are in both directions transformed to Π-extension. But then the same must be valid for Σ-extensions by Theorem 3.5. Now let $(\mathbb{F}(t),\sigma)$ be a simple Σ-extension of $(\mathbb{F},\sigma)$ with $\sigma(t) = \alpha\, t + \beta$. Then by the first statement of this corollary $(\mathbb{F}(t),\sigma^k)$ is a Σ-extension of $(\mathbb{F},\sigma)$ with $\sigma^k(t) = (\alpha)_k\, t + \gamma$ for some $\gamma \in \mathbb{F}^*$. What remains to show is that there does not exist an $n > 0$ with $(\alpha)_k^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$. For this let x be transcendental over $\mathbb{F}$ and consider the difference field extension $(\mathbb{F}(x),\sigma)$ of $(\mathbb{F},\sigma)$ canonically defined by $\sigma(x) = \alpha\, x$. Since $(\mathbb{F}(t),\sigma)$ is a simple Σ-extension of $(\mathbb{F},\sigma)$, there does not exist an $n > 0$ such that $\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$ and hence $(\mathbb{F}(x),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$ by Theorem 3.1. Therefore by the first statement of this corollary $(\mathbb{F}(x),\sigma^k)$ is a Π-extension of $\sigma(x) = (\alpha)_k\, x$. But this means that there does not exist an $n > 0$ such that $(\alpha)_k^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$ by Theorem 3.1. The reverse direction is analogous. □

3.4. Permutation Isomorphisms in ΠΣ-Fields

In Section 7 we provide an algorithm that solves the degree bound problem for parameterized first order linear difference equations in a given ΠΣ-field. In particular some properties of this algorithm will be shown that are needed for further investigations in the theory of ΠΣ-fields and indefinite summation. These properties are based on isomorphisms that are introduced in the following.

Definition 3.10. The difference fields $(\mathbb{F},\sigma)$, $(\tilde{\mathbb{F}},\tilde{\sigma})$ are *isomorph* if there is a field isomorphism $\tau : \mathbb{F} \to \tilde{\mathbb{F}}$ with $\tau\,\sigma = \tilde{\sigma}\,\tau$. τ is called *difference field isomorphism.*

The following lemma follows immediately by the commutativity of $\tau\,\sigma = \tilde{\sigma}\,\tau$.

Lemma 3.7. *Let $(\mathbb{F},\sigma)$, $(\tilde{\mathbb{F}},\tilde{\sigma})$ be difference fields, $\tau : \mathbb{F} \to \tilde{\mathbb{F}}$ be a difference field isomorphism, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^m$ and $f \in \mathbb{F}$. Then $\sigma_{\boldsymbol{a}} g = f$ if and only if $\tilde{\sigma}_{\boldsymbol{\tau(a)}}\tau(g) = \tau(g)$ for any $g \in \mathbb{F}$. Moreover for all $g \in \mathbb{F}^*$ we have $g \in \mathrm{H}_{(\mathbb{F},\sigma)}$ if and only if $\tau(g) \in \mathrm{H}_{(\tilde{\mathbb{F}},\tilde{\sigma})}$.*

In this work we consider the following almost trivial difference field isomorphism of ΠΣ-fields which basically permutates the extensions in the tower of extensions.

Definition 3.11. Let $(\mathbb{F}(s_1)\dots(s_e),\sigma)$ and $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be ΠΣ-fields and $\tau : \mathbb{F}(s_1)\dots(s_e) \to \mathbb{F}(t_1)\dots(t_e)$ be a difference field isomorphism. If for all $f \in \mathbb{F}$ we have $\tau(f) = f$ and if there is a bijective map $\phi : X \to X$ with $X := \{1,\dots,e\}$ such that $\tau(s_i) = s_i = t_{\phi(i)}$ for all $1 \leq i \leq e$, we say that $(\mathbb{F}(s_1,\dots,s_e),\sigma)$ and $(\mathbb{F}(t_1,\dots,t_e),\sigma)$ are *isomorph by a permutation.*

If we assume that $(\mathbb{G}),\sigma)$ and $(\mathbb{H},\sigma)$ are ΠΣ-fields which are isomorph by a permutation, we can write $\mathbb{G}$ and $\mathbb{H}$ as fields of rational functions, say $\mathbb{G} = \mathbb{F}(s_1, \dots, s_e)$ and $\mathbb{H} = \mathbb{F}(t_1, \dots, t_e)$ for some $e \geq 0$. Moreover there is a difference field isomorphism $\tau : \mathbb{G} \to \mathbb{H}$ defined in the following way: $\tau(f) = f$ for all $f \in \mathbb{F}$ and $\tau(s_i) = s_i = t_{\phi(i)}$ for some permutation ϕ. This means that we can reorder the extensions in $\mathbb{G}$ by the permutation ϕ which yields to the $\Pi\Sigma$-field $(\mathbb{H}, \sigma)$. Since for any $f \in \mathbb{G}$ we have $f = \tau(f) \in \mathbb{H}$, we will ignore the difference field automorphism τ and interpret any f in $\mathbb{H}$ also as an element in $\mathbb{G}$ and vice versa.

4. The Degree Bound Problem in ΠΣ-fields

As pointed out in the introduction, the main goal is to find a basis of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ in a $\Pi\Sigma$-field $(\mathbb{F}(t), \sigma)$ over $\mathbb{K}$ with $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. In [Sch02b] several reduction techniques are introduced that enables to search for such a basis. In particular in [Bro00, Sch02a], as sketched for the rational case $(\mathbb{Q}(t), \sigma)$ in Section 2, the denominator elimination strategy allows to reduce –at least partially– the problem from finding a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ to computing a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}[t])$ for some $\boldsymbol{a'} \in \mathbb{F}[t]^m$ and $\boldsymbol{f'} \in \mathbb{F}[t]^n$. Then having a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}[t])$ one easily can reconstruct a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ as it is shown in details in [Sch02b, Sch02a]. So from now on we focus on finding a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$. For this subproblem an essential step is to determine a degree bound $b \in \mathbb{N}_0 \cup \{-1\}$ such that for all $\boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$ one has $\deg(g) \leq b$. Of course,

$$\mathbb{F}[t]_d := \{f \in \mathbb{F}[t] \mid \deg(f) \leq d\} = \{f \in \mathbb{F}[t] \mid \|g\| \leq d\}$$

is a finite subspace of $\mathbb{F}(t)$ over $\mathbb{K}$ for any $d \in \mathbb{N}_0 \cup \{-1\}$. In particular we have $\mathbb{F}[t]_{-1} = \{0\}$. In other words, we try to find a $b \in \mathbb{N}_0 \cup \{-1\}$ such that

$$\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]) = \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_b) \tag{2}$$

which is exactly the *degree bound problem* specified in Section 2. Now assume that we can find such a b for $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$, and moreover suppose that

$$b \geq \max(-1, \|\boldsymbol{f}\| - \|\boldsymbol{a}\|). \tag{3}$$

This basically guarantees that $\boldsymbol{f} \in \mathbb{F}[t]_{\|\boldsymbol{a}\|+b}$ by Lemma 3.5. If one adapts this bound b with property (2) such that it also by fulfills property (3), one can apply further reduction techniques in order to search for a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_b)$. These reductions are carefully analyzed in [Sch02b].

Motivated by this remarks we focus on determining a bound b of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$ with (2), which finally allows to search for a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_b)$ and hence of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$. We consider a slightly more general situation in order to allow more flexibility to different reduction techniques as they are applied in [Kar81].

Definition 4.1. Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. By $\mathbb{W}_d$ we denote $\{f \in \mathbb{W} \mid \|f\| \leq d\}$ which clearly is a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. Let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. $b \in \mathbb{N}_0 \cup \{-1\}$ is called *degree bound* of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ if $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W}) = \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W}_b)$.

More precisely we are concerned in the more general problem to find degree bounds of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ for several special cases $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$. Please note that our original problem (2) is included in the problem under consideration by choosing $\mathbb{W} := \mathbb{F}[t]$ which is a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. In particular we will solve the degree bound problem for the first order case, i.e. for any $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^2$. These bounds under consideration allow to design a complete algorithm to solve parameterized first order linear difference equations in ΠΣ-fields in [Sch01, Sch02b]. Moreover for the higher order case we are able to deal with the Situations 3.1, 5.2 and 6.3 in the corresponding settings of Π-, simple Σ- and Σ-extensions.

4.1. A Key Property to Determine Degree Bounds in ΠΣ-Fields

The following lemma gives the key idea to find degree bounds of parameterized first order linear difference equation in ΠΣ-fields. In particular it is applied in Lemma 6.1, Proposition 4.1 and Theorems 5.1 and 6.1.

Lemma 4.1. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ where $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ with $l := \|\boldsymbol{a}\|$, $\boldsymbol{f} \in \mathbb{F}[t]^n$, and $\boldsymbol{b} := \big([a_1]_l\, (\alpha)^d_{m-1}, \dots, [a_m]_l\, (\alpha)^d_0\big) \in \mathbb{F}^m$ for some $d \in \mathbb{N}_0$. If there exists a $g = w\, t^d + r \in \mathbb{F}(t)$ with $w \in \mathbb{F}^*$, $\|r\| < d$ and $\|\sigma_{\boldsymbol{a}} g\| < \|\boldsymbol{a}\| + d$ then $\sigma_{\boldsymbol{b}} w = 0$.*

Proof: Let $g = w\, t^d + r$ with $w \in \mathbb{F}^*$, $\|r\| < d$, $l := \|\boldsymbol{a}\|$ and $\|\sigma_{\boldsymbol{a}} g\| < l + d$. Then by Lemma 3.4 it follows that

$$\begin{aligned}
0 &= [\sigma_{\boldsymbol{a}} g]_{l+d} \\
&= \big[a_1\, \sigma^{m-1}(w\, t^d + r) + \dots + a_m\, (w\, t^d + r)\big]_{l+d} \\
&= \big[a_1\, \sigma^{m-1}(w\, t_d) + \dots + a_m\, w\, t^d\big]_{l+d} + \underbrace{\big[a_1\, \sigma^{m-1}(r) + \dots + a_m\, r\big]_{l+d}}_{=0} \\
&= \Big[a_1\, \sigma^{m-1}(w)\ (\alpha)^d_{m-1}\ t^d + \dots + a_m\, w\ (\alpha)^d_0\ t^d\Big]_{d+l} \\
&= [a_1\,]_l\ (\alpha)^d_{m-1}\ \sigma^{m-1}(w) + \dots + [a_m]_l\ (\alpha)^d_0\ w = 0
\end{aligned}$$

and therefore $\sigma_{\boldsymbol{b}} w = 0$. □

The following proposition is a direct consequence of Lemma 4.1.

Proposition 4.1. *Let $(\mathbb{F}(t), \sigma)$ be a proper sum extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. Let $\boldsymbol{f} \in \mathbb{F}[t]^n$ and $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and define $\boldsymbol{b} := [\boldsymbol{a}]_{\|a\|}$. If there does not exist a $w \in \mathbb{F}^*$ such that $\sigma_{\boldsymbol{b}} w = 0$ then $\max(\|\boldsymbol{f}\| - \|\boldsymbol{a}\|, -1)$ is a degree bound of $\mathrm{V}(\boldsymbol{b}, \boldsymbol{f}, \mathbb{W})$.*

Proof: Suppose there are a $g \in \mathbb{F}[t]$ and a $\boldsymbol{c} \in \mathbb{K}^n$ with $d := \|g\| > \max(\|\boldsymbol{f}\| - \|\boldsymbol{a}\|, -1)$ and $\sigma_{\boldsymbol{a}} g = \boldsymbol{c}\, \boldsymbol{f}$. Take such a g with $w := [g]_d \in \mathbb{F}^*$. Since $\|\boldsymbol{a}\| \geq 0$, we have $\|\sigma_{\boldsymbol{a}} g\| = \boldsymbol{c}\, \boldsymbol{f} \leq \|\boldsymbol{f}\| < \|g\| + \|\boldsymbol{a}\|$. Hence by Lemma 4.1 it follows $\sigma_{\boldsymbol{b}} w = 0$. □

Let $(\mathbb{F}, \sigma)$ be a difference field, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. Then this proposition states that there cannot exist a proper sum extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$

with $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}) \subsetneq \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$, if there does not exist already a $w \in \mathbb{F}^*$ with $\sigma_{\boldsymbol{a}} w = 0$. This criterium plays an important role in the theory of proper sum solutions, a subclass of d'Alembertian solutions and Liouvillian solutions, that is considered in details in the ΠΣ-field setting in [Sch02a].

4.2. A Bound Criterion and a Special Case of the Degree Bound Problem

Next we introduce a bound criterion, namely Corollary 4.1, which gives a proof strategy to decide if a b is a degree bound of a given solution space. First we state a lemma that follows immediately by the definition of degree bounds.

Lemma 4.2. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. Let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$, $b \in \mathbb{N}_0 \cup \{-1\}$ and $f \in \mathbb{F}[t]$. If b is a degree bound of $\mathrm{V}(\boldsymbol{a}, (f), \mathbb{W})$, for all $g \in \mathbb{W}$ with $\sigma_{\boldsymbol{a}} g = f$ we have $\|g\| \leq b$.*

Then we obtain the following result.

Theorem 4.1. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over $\mathbb{K}$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $b \in \mathbb{N}_0 \cup \{-1\}$. If for all $f \in \mathbb{F}[t]$ with $\|f\| \leq \|\boldsymbol{f}\|$ it follows that b is a degree bound of $\mathrm{V}(\boldsymbol{a}, (f), \mathbb{W})$ then b is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.*

Proof: Assume b is a degree bound of $\mathrm{V}(\boldsymbol{a}, (f), \mathbb{W})$ for all $f \in \mathbb{F}[t]$ with $\|f\| \leq \|\boldsymbol{f}\|$. Let $\boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$, i.e.

$$\sigma_{\boldsymbol{a}} g = \boldsymbol{c}\, \boldsymbol{f}. \tag{4}$$

Take $f := \boldsymbol{c}\, \boldsymbol{f}$. By $\|f\| = \|\boldsymbol{c}\, \boldsymbol{f}\| \leq \|\boldsymbol{f}\|$ and (4) we may conclude that b is a degree bound of $\mathrm{V}(\boldsymbol{a}, (f), \mathbb{W})$ and it follows that $\|g\| \leq b$ by Lemma 4.2. Consequently for all $\boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ we have $\|g\| \leq b$ and thus $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W}) = \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W}_b)$ which proves the theorem. □

In the next sections the following Corollary 4.1 will be **heavily** used in proofs for checking if a particular b is a degree bound of a given solution space. The corollary follows immediately by Lemma 4.2 and Theorem 4.1.

Corollary 4.1. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over $\mathbb{K}$ and let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$. Let $b \in \mathbb{N}_0 \cup \{-1\}$ be such that for all $f \in \mathbb{F}[t]$ and $g \in \mathbb{W}$ with $\|f\| \leq \|\boldsymbol{f}\|$ and $\sigma_{\boldsymbol{a}} g = f$ we have $\|g\| \leq b$. Then b is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.*

We apply this corollary and obtain the following result which gives a degree bound for linear difference equations that are specified in Situation 3.1.

Theorem 4.2. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. Let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ as in Situation 3.1 and $\boldsymbol{f} \in \mathbb{F}[t]^n$. Then $\max(\|\boldsymbol{f}\| - \|\boldsymbol{a}\|, -1)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.*

Proof: We will proof the theorem by Corollary 4.1. Let $f \in \mathbb{F}[t]$ and $g \in \mathbb{W}$ be arbitrary but fixed with $\sigma_{\boldsymbol{a}} g = f$ and $\|f\| \leq \|\boldsymbol{f}\|$. We will show by case distinction that for an appropriate $b \in \mathbb{N}_0 \cup \{-1\}$ it follows that $\|g\| \leq b$ which will prove that b for the particular case is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. If $g \neq 0$ then by Lemma 3.5 it follows that $\|\boldsymbol{f}\| \geq \|f\| = \|\sigma_{\boldsymbol{a}} g\| = \|\boldsymbol{a}\| + \|g\|$ and therefore $\|g\| \leq \|\boldsymbol{f}\| - \|\boldsymbol{a}\|$. Otherwise, if $g = 0$ then $\|g\| = -1$. Altogether we have $\|g\| \leq \max(\|\boldsymbol{f}\| - \|\boldsymbol{a}\|, -1)$ and hence by Corollary 4.1 $\max(\|\boldsymbol{f}\| - \|\boldsymbol{a}\|, -1)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. □

5. Degree Bounds for Π- and Simple Σ-Extensions

Based on the work of [Kar81] we solve the degree bound problem for first order linear difference equations in the Π-extension setting. Here we noticed that these degree bound techniques cannot only be applied to Π-extension but also to simple Σ-extensions. Finally we extend these techniques from the first order to the higher order case which enables to solve the degree bound problem for a special class of linear difference equations.

In this section let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-field where $(\mathbb{F}(t), \sigma)$ is a Π- or a simple Σ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and $\sigma(t) = \alpha\, t + \beta$. Hence for all $n > 0$ we have $\alpha^n \notin \mathrm{H}_{(\mathbb{F},\sigma)}$. Furthermore let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$.

5.1. Degree Bounds of First Order Linear Difference Equations

In the sequel we solve the degree bound problem for $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ where $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}[t]^2$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. If $\|a_1\| \neq \|a_2\|$, Theorem 4.2 provides a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. Hence what remains is $\|a_1\| = \|a_2\| \geq 0$. More precisely we deal with the following case.

Situatation 5.1. Assume $(a_1, a_2) \in \mathbb{F}[t]^2$ with $a_1 = t^p + r_1$ and $a_2 = -c\, t^p + r_2$ for $c \in \mathbb{F}^*$, $p \geq 0$ and $r_1, r_2 \in \mathbb{F}[t]$ with $\|r_1\|, \|r_2\| < p$.

As will be seen later, we must be able to decide, if there exists a $d \geq 0$ for any $c, \alpha \in \mathbb{F}^*$ such that $c\, \alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$. Furthermore, we must be able to compute such a d, if there exists one. By Theorem 3.4 all these problems can be solved.

The main idea of the following section is taken from Theorem 15 of [Kar81]. Whereas in Karr's version theoretical and computational aspects are mixed, I tried to separate his theorem in several parts to achieve more transparency.

Theorem 5.1. *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ where $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Let $a_1, a_2 \in \mathbb{F}[t]$ as in Situation 5.1. If there exists a $g \in \mathbb{F}(t)$ with $\|g\| \geq 0$ such that*

$$\|a_1\, \sigma(g) + a_2\, g\| < \|g\| + p \tag{5}$$

then $\frac{c}{\alpha^{\|g\|}} \in \mathrm{H}_{(\mathbb{F},\sigma)}$.

Proof: Let $g = w\,t^d + r \in \mathbb{F}(t)$ with $w \in \mathbb{F}^*$ and $\|r\| < d$. By Lemma 4.1 we have $\sigma(w)\,\alpha^d - c\,w = 0$, hence $\frac{c}{\alpha^d} = \frac{\sigma(w)}{w}$ and consequently $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$. □

Example 5.1. Take the ΠΣ-field $(\mathbb{Q}(t)(z), \sigma)$ canonically defined by $\sigma(t) = t+1$ and $\sigma(z) = (t+1)\,z$ as in Example 3.5 and consider

$$t_2\,\sigma(g) - \underbrace{(t_1+1)^4}_{c}\,t_2\,g = -t_1\,(2+t_1)\,t_2\,(2+t_2^2+2\,t_1\,(1+t_2^2)+t_1^2\,(1+t_2^2)).$$

There is the solution $g = t_2^4 + t_2^2 + 1$; therefore inequality (5) is satisfied and it follows by Theorem 5.1 that $\frac{c}{\alpha^4} = \frac{(t_1+1)^4}{(t_1+1)^4} = 1 \in \mathrm{H}_{(\mathbb{Q}(t_1),\sigma)}$.

In the proof of the previous theorem we just required that in the difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ we have $\sigma(t) = \alpha\,t + \beta$ for some $\alpha, \beta \in \mathbb{F}$ where t is transcendental over $\mathbb{F}$. Only in the next lemma all properties of the supposed extensions are really exploited. This result finally enables to solve the degree bound problem for Situation 5.1.

Lemma 5.1. *Let $(\mathbb{F}(t), \sigma)$ be a Π- or a simple Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\,t + \beta$, $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Assume there exists a $d \in \mathbb{Z}$ for $c \in \mathbb{F}^*$ such that $c\,\alpha^d \in \mathrm{H}_{(\mathbb{F},\sigma)}$. Then d is uniquely determined.*

Proof: Assume there are $d_1, d_2 \in \mathbb{Z}$ with $d_1 < d_2$ and $c\,\alpha^{d_1} \in \mathrm{H}_{(\mathbb{F},\sigma)}$, $c\,\alpha^{d_2} \in \mathrm{H}_{(\mathbb{F},\sigma)}$ i.e. there are $g_1, g_2 \in \mathbb{F}^*$ such that $\frac{\sigma(g_1)}{g_1} = c\,\alpha^{d_1}$ and $\frac{\sigma(g_2)}{g_2} = c\,\alpha^{d_2}$. Since $d_2 - d_1 > 0$, it follows that $\alpha^{d_2-d_1} = \frac{\sigma(g_2)/g_2}{\sigma(g_1)/g_1} = \frac{\sigma(g_2/g_1)}{g_2/g_1}$ and thus $\alpha^{d_2-d_1} \in \mathrm{H}_{(\mathbb{F},\sigma)}$. By Theorem 3.1 $(\mathbb{F}(t), \sigma)$ is not a Π-extension of $(\mathbb{F}, \sigma)$, a contradiction. □
Combining the previous results leads to a recipe how to compute the desired degree bound b.

Theorem 5.2. *Let $(\mathbb{F}(t), \sigma)$ be a Π- or a simple Σ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and $\sigma(t) = \alpha\,t + \beta$ with $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Let $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over $\mathbb{K}$, let $\boldsymbol{f} \in \mathbb{F}[t]^n$ and assume $a_1, a_2 \in \mathbb{F}[t]$ as in Situation 5.1. If there exists a $d \in \mathbb{N}_0$ such that $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$, d is uniquely determined and* $\max(\|\boldsymbol{f}\| - p, d)$ *is a degree bound of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. *If there does not exist such a d then* $\max(\|\boldsymbol{f}\| - p, -1)$ *is a degree bound of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

Proof: We will proof the theorem by Corollary 4.1. Let $f \in \mathbb{F}[t]$ and $g \in \mathbb{W}$ be arbitrary but fixed such that $a_1\,\sigma(g) + a_2\,g = f$ and $\|f\| \leq \|\boldsymbol{f}\|$. We will show by case distinction that for an appropriate $b \in \mathbb{N}_0 \cup \{-1\}$ it follows that $\|g\| \leq b$ which will prove that b for the particular case is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

1. Assume there exists a $d \geq 0$ such that $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$. Then d is uniquely determined by Lemma 5.1. If $\|g\| + p > \|f\|$ and $\|g\| \geq 0$, it follows by Theorem 5.1 that $\|g\| = d$ and consequently $\|g\| = d = \max(\|f\| - p, d) \leq \max(\|\boldsymbol{f}\| - p, d)$. Otherwise, if $\|g\| + p \leq \|f\|$ or $\|g\| = -1$, we have $\|g\| \leq \max(\|f\| - p, d) \leq \max(\|\boldsymbol{f}\| - p, d)$. Thus for both cases we may apply Corollary 4.1 and hence $\max(\|\boldsymbol{f}\| - p, d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

2. Assume there does not exist such a d. Then by Theorem 5.1 it follows that $\|g\|+p = \|f\| \leq \|\boldsymbol{f}\|$ or $\|g\| = -1$ and thus by Corollary 4.1 $\max(\|\boldsymbol{f}\|-p, -1)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

□

Looking closer at the previous theorem, one immediately obtains a degree bound for the case $\boldsymbol{a} = (1, -1)$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$ which amounts to indefinite summation.

Corollary 5.1. *Let* $(\mathbb{F}(t), \sigma)$ *be a* Π*-extension of* $(\mathbb{F}, \sigma)$ *with constant field* $\mathbb{K}$*, let* $\mathbb{W}$ *be a subspace of* $\mathbb{F}(t)$ *over* $\mathbb{K}$ *and* $\boldsymbol{f} \in \mathbb{F}[t]^n$*. Then* $\max(\|\boldsymbol{f}\|, 0)$ *is a degree bound of* $\mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{W})$*.*

Proof: Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t$, $\alpha \in \mathbb{F}^*$. Since $\frac{1}{\alpha^0} = 1 \in \mathrm{H}_{(\mathbb{F},\sigma)}$, by Theorem 5.2 $\max(0, \|\boldsymbol{f}\|)$ is a degree bound of $\mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{W})$. □

5.2. A Generalization for Higher Order Linear Difference Equations

Finally we solve the degree bound problem of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ with $\boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$ for the more general Situation 5.2 that contains Situation 5.1.

Situatation 5.2. Assume $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, \dots, a_\lambda, \dots, a_\mu \;\dots, a_m) \in \mathbb{F}[t]^m$ with $\lambda < \mu$, $\|a_\lambda\| = \|a_\mu\| = p$ and

$$\|a_i\| < p \;\forall i \neq \lambda, \mu.$$

In particular suppose that $a_\lambda = t^p + r_1$ and $a_\mu = -c\, t^p + r_2$ for $c \in \mathbb{F}^*$, $p \geq 0$ and $r_1, r_2 \in \mathbb{F}[t]$ with $\|r_1\|, \|r_2\| < p$.

First we generalize Theorem 5.1.

Theorem 5.3. *Let* $(\mathbb{F}(t), \sigma)$ *be difference field extension of* $(\mathbb{F}, \sigma)$ *with* t *transcendental over* $\mathbb{F}$ *and* $\sigma(t) = \alpha\, t + \beta$ *where* $\alpha \in \mathbb{F}^*$ *and* $\beta \in \mathbb{F}$*. Assume* $\boldsymbol{a} \in \mathbb{F}[t]^m$ *as in Situation 5.2. If there exists a* $g \in \mathbb{F}(t)$ *with* $\|g\| \geq 0$ *such that* $\|\sigma_{\boldsymbol{a}} g\| < \|g\| + p$ *then* $\frac{\sigma^{\mu-m}(c)}{(\alpha)_{\mu-\lambda}^{\|g\|}} \in \mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$*.*

Proof: Let $d := \|g\| \geq 0$. It follows by Lemma 3.3 and Situation 5.2 that $\|a_\lambda\, \sigma^{m-\lambda}(g)\| = \|a_\mu\, \sigma^{m-\mu}(g)\| = p + d$ with $\lambda \neq \mu$ and $\|a_i\, \sigma^{m-i}(g)\| < p + d$ for all $i \neq \mu, \lambda$. Hence we have

$$0 = [\sigma_{\boldsymbol{a}} g]_{p+d} = \left[\sum_{i=1}^{m} a_i\, \sigma^{m-i}(g)\right]_{p+d} = \left[a_\lambda\, \sigma^{m-\lambda}(g) + a_\mu\, \sigma^{m-\mu}(g)\right]_{p+d}$$

and thus $\left[\sigma^{\mu-m}(a_\lambda)\, \sigma^{\mu-\lambda}(g) + \sigma^{\mu-m}(a_\mu)\, g\right]_{p+d} = 0$ by Lemma 3.4. By

$$\sigma^{\mu-m}(a_\lambda) = (\alpha)_{\mu-m}^{p}\, t^p + \sigma^{\mu-m}(r_1),\; \sigma^{\mu-m}(a_\mu) = -\sigma^{\mu-m}(c)\; (\alpha)_{\mu-m}^{p}\, t^p + \sigma^{\mu-m}(r_2)$$

it follows that $\left[b_1\,\sigma^{\mu-\lambda}(g)+b_2\,g\right]_{p+d}=0$ for

$$b_1:=t^p+\sigma^{\mu-m}(r_1)/\,(\alpha)^p_{\mu-m}\,,\qquad b_2:=-\sigma^{\mu-m}(c)\,t^p+\sigma^{\mu-m}(r_2)/\,(\alpha)^p_{\mu-m}\,.$$

Since $\|b_1\,\sigma^{\mu-\lambda}(g)+b_2\,g\|\leq p+d$, we have even $\|b_1\,\sigma^{\mu-\lambda}(g)+b_2\,g\|<p+d$. Hence we may apply Theorem 5.1 and thus we obtain $\frac{\sigma^{\mu-m}(c)}{(\alpha)^d_{\mu-\lambda}}\in\mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$. □

Finally we obtain a degree bound method for the Situation 5.2.

Theorem 5.4. *Let $(\mathbb{F}(t),\sigma)$ be a Π- or a simple Σ-extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ and $\sigma(t)=\alpha\,t+\beta$ with $\alpha\in\mathbb{F}^*$ and $\beta\in\mathbb{F}$. Let $\mathbb{W}$ be subspace of $\mathbb{F}(t)$ over $\mathbb{K}$, $\boldsymbol{f}\in\mathbb{F}[t]^n$ and assume $\boldsymbol{a}\in\mathbb{F}[t]^m$ as in Situation 5.2. Furthermore suppose that $(\mathbb{F}(t),\sigma^{\mu-\lambda})$ is a Π- or simple Σ-extension of $(\mathbb{F},\sigma^{\mu-\lambda})$. If there exists a $d\in\mathbb{N}_0$ such that $\frac{\sigma^{\mu-m}(c)}{(\alpha)^d_{\mu-\lambda}}\in\mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$ then d is uniquely determined and* $\max(\|\boldsymbol{f}\|-p,d)$ *is a degree bound of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$*. If there does not exist such a d then* $\max(\|\boldsymbol{f}\|-p,-1)$ *is a degree bound of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$*.*

Proof: We will proof the theorem by Corollary 4.1. Let $f\in\mathbb{F}[t]$ and $g\in\mathbb{F}[t]$ be arbitrary but fixed such that $\sigma_{\boldsymbol{a}}g=f$ and $\|f\|\leq\|\boldsymbol{f}\|$. We will show by case distinction that for an appropriate $b\in\mathbb{N}_0\cup\{-1\}$ it follows that $\|g\|\leq b$ which will prove that b for the particular case is a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$.

1. Assume there exists a $d\geq 0$ such that $\frac{\sigma^{\mu-m}(c)}{(\alpha)^d_{\mu-\lambda}}\in\mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$. Then by Lemma 5.1 d is uniquely determined. If $\|g\|+p>\|f\|$ and $\|g\|\geq 0$, by Theorem 5.3 it follows that $\|g\|=d$ and therefore $\|g\|=d=\max(\|f\|,d)\leq\max(\|\boldsymbol{f}\|,d)$. Otherwise, if $\|g\|+p\leq\|f\|$ or $\|g\|=-1$, we have $\|g\|\leq\max(\|f\|,d)\leq\max(\|\boldsymbol{f}\|,d)$. Consequently in both cases we may apply Corollary 4.1 and $\max(\|\boldsymbol{f}\|,d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$.
2. Assume there does not exist such a d. Then by Theorem 5.3 it follows that $\|g\|+p=\|f\|\leq\|\boldsymbol{f}\|$ or $\|g\|=-1$ and thus by Corollary 4.1 $\max(\|\boldsymbol{f}\|-p,-1)$ is a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$.

□

By Theorem 3.5 and Corollary 3.1 $(\mathbb{F},\sigma^k)$ is a ΠΣ-field and $(\mathbb{F}(t),\sigma)$ is a Π- or a simple Σ-extension of $(\mathbb{F},\sigma)$ for any $k\in\mathbb{Z}^*$. Hence one can decide if $\frac{\sigma^{\mu-m}(c)}{(\alpha)^d_{\mu-\lambda}}\in\mathrm{H}_{(\mathbb{F},\sigma^{\mu-\lambda})}$ for some d and find such a d in case of existence by Theorem 3.4. Therefore we can apply Theorem 5.4 to compute a degree bound for the special case described in Situation 5.2.

6. Degree Bounds for Σ-Extensions

In this section we deliver degree bounds of parameterized linear difference equations for Σ-extensions. Similarly to the previous section we solve the degree bound problem for first order linear difference equations in Σ-extensions. By gathering together results from this and the previous section, we are capable of designing an algorithm in Section 7 that solves the degree bound problem for any first order linear difference equation in a given ΠΣ-field. Moreover we extend these degree bound techniques introduced in [Kar81] from the first order to the higher order case which allows to solve the degree bound problem for a special class of linear difference equations. Since for simple Σ-extensions this and the previous section deliver degree bounds for equivalent situations, one obtains further flexibility to choose the appropriate degree bound method.

In this section let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-field where $(\mathbb{F}(t),\sigma)$ is a Σ-extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ and let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$.

6.1. Degree Bounds of First Order Linear Difference Equations

We will solve degree bound problem for the solution space $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$ with $\mathbf{0}\neq\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}[t]^2$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$. If $\|a_1\|\neq\|a_2\|$, Theorem 4.2 provides a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$. What remains is the case $\|a_1\|=\|a_2\|\geq 0$.
Similarly to the Π-extension case the main idea is taken from Theorem 14 of [Kar81]. In the sequel we separate this result into theoretical and algorithmic aspects and give detailed proofs. First we will consider the following case.

Situatation 6.1. Assume $\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}^2$ with $a_1\neq 0\neq a_2$.

Lemma 6.1. *Let $(\mathbb{F}(t),\sigma)$ be a Σ-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$, $(\alpha,\beta\in\mathbb{F}^*)$ and $\boldsymbol{a}\in(\mathbb{F}[t]^*)^2$. If there exists a $g\in\mathbb{F}(t)$ such that $\|g\|>0$ and $\|a_1\,\sigma(g)-a_2\,g\|<\|g\|+\|\boldsymbol{a}\|-1$ then $\|a_1\|=\|a_2\|>0$.*

Proof: Let $g\in\mathbb{F}(t)$ with $\|g\|=d>0$ as stated above. Due to Theorem 4.2 it follows that $\|a_1\|=\|a_2\|$. Now assume $\|a_1\|=\|a_2\|=0$, i.e. $a_1,a_2\in\mathbb{F}$. Thus there is a $u\in\mathbb{F}$ with

$$\|\sigma(g)-u\,g\|<\|g\|-1. \tag{6}$$

Write $g=w\,t^d+r$ with $w\in\mathbb{F}^*$ and $\|r\|<d$. By Lemma 4.1 and (6) it follows that $\sigma(w\,t^d)-u\,w\,t^d=0$ and thus $\frac{\sigma(w\,t^d)}{w\,t^d}=u\in\mathbb{F}$. By Definition 3.4 $(\mathbb{F}(t),\sigma)$ is not a Σ-extension of $(\mathbb{F},\sigma)$, a contradiction. □

Corollary 6.1. *Let $(\mathbb{F}(t),\sigma)$ be a Σ-extension of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ and $\sigma(t)=\alpha\,t+\beta$ with $\alpha,\beta\in\mathbb{F}^*$. Let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$, $\boldsymbol{f}\in\mathbb{F}[t]^n$ and $\boldsymbol{a}\in(\mathbb{F}^*)^2$ as in Situation 6.1. Then $\|\boldsymbol{f}\|+1$ is a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{W})$.*

For the case $\|a_1\|=\|a_2\|=0$ Corollary 6.1 delivers a degree bound. What remains is the case $\|a_1\|=\|a_2\|>0$. More precisely we deal with Situation 6.2.

Situatation 6.2. Assume $(a_1,a_2)\in\mathbb{F}[t]^2$ with

$$a_1=(t^p+u_1\,t^{p-1}+r_1)\qquad\text{and}\qquad a_2=c\,(t^p+u_2\,t^{p-1}+r_2)$$

for some $c\in\mathbb{F}^*$, $u_1,u_2\in\mathbb{F}$, $p\geq 1$ and $r_1,r_2\in\mathbb{F}[t]$ with $\|r_1\|,\|r_2\|<p-1$.

The following considerations lead to an algorithm that allows to compute a degree bound of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$, if one can compute a basis of the solution space $\mathrm{V}(\boldsymbol{b}, \boldsymbol{v}, \mathbb{F})$ for any $\mathbf{0} \neq \boldsymbol{b} \in \mathbb{F}^2$ and $\boldsymbol{v} \in \mathbb{F}^2$. Together with the results from Subsection 5.1 and results from [Sch02b, Sch02a], in Section 7 we finally will be able to develop an algorithm that solves the degree bound problem for first order linear difference equation in a given ΠΣ-field.

Theorem 6.1. *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$, $\alpha, \beta \in \mathbb{F}^*$. Assume $a_1, a_2 \in \mathbb{F}[t]$ as in Situation 6.2. If there is a $g \in \mathbb{F}(t)$ with $\|g\| > 0$ and*

$$\|a_1 \sigma(g) - a_2\, g\| < \|g\| + p - 1 \tag{7}$$

then there exists a $w \in \mathbb{F}$ such that $\sigma(w) - \alpha\, w = \alpha\,(u_2 - u_1) - \|g\|\, \beta$ and $u_1 \neq u_2$.

Proof: Let $g \in \mathbb{F}(t)$ with $d = \|g\| > 0$ as stated above. By (7) it follows that

$$[a_1\, \sigma(g) + a_2\, g]_{p+d} = 0 \quad \text{and} \quad [a_1\, \sigma(g) + a_2\, g]_{p+d-1} = 0.$$

Now write $g = \sum_{i=0}^{d} g_i\, t^i + r$ where $g_i \in \mathbb{F}$, $g_d \neq 0$ and $r \in \mathbb{F}(t)$ with $\|r\| = -1$. Applying Lemma 4.1 it follows that $\sigma(g_d)\, \alpha^d + c\, g_d = 0$ and therefore

$$c = -\frac{\sigma(g_d)}{g_d}\, \alpha^d. \tag{8}$$

By $\sigma(g) = \sum_{i=0}^{d} \sigma(g_i)\,(\alpha\, t + \beta)^i + \sigma(r)$ and Lemma 3.4 we obtain

$$\begin{aligned}
[a_2\, g]_{p+d-1} &= [a_2]_p\, [g]_{d-1} + [a_2]_{p-1}\, [g]_d = c\, g_{d-1} + c\, u_2\, g_d = c\,(g_{d-1} + u_2\, g_d),\\
[a_1\, \sigma(g)]_{p+d-1} &= [a_1]_{p-1}\, [\sigma(g)]_d + [a_1]_p\, [\sigma(g)]_{d-1}\\
&= u_1\, \alpha^d\, \sigma(g_d) + \left[\sigma(g_d)\,(\alpha\, t + \beta)^d + \sigma(g_{d-1})\,(\alpha\, t + \beta)^{d-1}\right]_{d-1}\\
&= u_1\, \alpha^d\, \sigma(g_d) + d\, \alpha^{d-1}\, \beta\, \sigma(g_d) + \alpha^{d-1}\, \sigma(g_{d-1}).
\end{aligned}$$

Hence by Lemma 3.4 it follows that $0 = [a_1\, \sigma(g) + a_2\, g]_{p+d-1} = [a_1\, \sigma(g)]_{p+d-1} + [a_2\, g]_{p+d-1}$ and therefore

$$u_1\, \alpha^d\, \sigma(g_d) + d\, \alpha^{d-1}\, \beta\, \sigma(g_d) + \alpha^{d-1}\, \sigma(g_{d-1}) + c\,(g_{d-1} + u_2\, g_d) = 0.$$

Using (8) we may write

$$\begin{aligned}
&u_1\, \alpha^d\, \sigma(g_d) + d\, \alpha^{d-1}\, \beta\, \sigma(g_d) + \alpha^{d-1}\, \sigma(g_{d-1}) - \frac{\sigma(g_d)}{g_d}\, \alpha^d\,(g_{d-1} + u_2\, g_d) = 0\\
&\Leftrightarrow \quad \sigma(g_d)\,(u_1\, \alpha + d\, \beta - \alpha\, \frac{g_{d-1}}{g_d} - \alpha\, u_2) = -\sigma(g_{d-1})\\
&\Leftrightarrow \quad \sigma(\frac{g_{d-1}}{g_d}) - \alpha\, \frac{g_{d-1}}{g_d} = (u_2 - u_1)\, \alpha - d\, \beta
\end{aligned}$$

and thus for $w := \frac{g_{d-1}}{g_d}$ the first part is proven. Now assume that $u_1 = u_2$. Then $\sigma(w) - \alpha\, w = -\|g\|\, \beta$, thus $\sigma(\frac{w}{-\|g\|}) - \alpha\, \frac{w}{-\|g\|} = \beta$ and therefore by Theorem 3.3 $(\mathbb{F}(t), \sigma)$ is not a Σ-extension of $(\mathbb{F}, \sigma)$, a contradiction. □

Example 6.1. Consider the ΠΣ-field $(\mathbb{Q}(t)(z), \sigma)$ defined by $\sigma(t) = t + 1$ and $\sigma(z) = z + \frac{1}{z+1}$ as in Example 3.5 and take

$$\boldsymbol{a} = (a_1, a_2) = \Big(z \underbrace{-\frac{2}{1+t}}_{=:u_1}, \underbrace{-\frac{(1+t)^4}{t^4}}_{=:c} (z + \underbrace{\frac{1}{t+1}}_{=:u_2})\Big)$$

and $f = -(1 + (1 + t)\, z)\,(8 + 6\,t + 3\,(1 + t)\,(2 + t)\, z)$. There is the solution $g = t^4\, z^2\,(3 + z)$ for $\sigma_{\boldsymbol{a}} g = f$. Since inequality (7) is satisfied and $3 = \|g\| > 0$, it follows by Theorem 6.1 that $u_1 \neq u_2$. Moreover there must exist a $w \in \mathbb{Q}(t)$ such that $\sigma(w) - w = (u_2 - u_1) - \|g\|\, \frac{1}{t+1}$ holds. In deed $w = 1$ does the job.

The next lemma delivers the basic idea to find a degree bound of a first order linear difference equation described Situation 6.2.

Lemma 6.2. *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$, $\alpha, \beta \in \mathbb{F}^*$. Assume there exist a $d \in \mathbb{Z}$, a $w \in \mathbb{F}$ and a $u \in \mathbb{F}$ such that the difference equation $\sigma(w) - \alpha\, w = u\, \alpha + d\, \beta$ holds. Then d is uniquely determined.*

Proof: Assume there are $w_1, w_2 \in \mathbb{F}$ and $d_1, d_2 \in \mathbb{Z}$ with $d_1 < d_2$, $\sigma(w_1) - \alpha\, w_1 = u\, \alpha + d_1\, \beta$ and $\sigma(w_2) - \alpha\, w_2 = u\, \alpha + d_2\, \beta$. Then it follows that

$$\sigma(w_2 - w_1) - \alpha\,(w_2 - w_1) = (d_2 - d_1)\,\beta,$$

consequently $\sigma(\frac{w_2-w_1}{d_2-d_1}) - \alpha\, \frac{w_2-w_1}{d_2-d_1} = \beta$, and hence by Theorem 3.3 $(\mathbb{F}(t), \sigma)$ is not a Σ-extension of $(\mathbb{F}, \sigma)$, a contradiction. □

As for the case of Π-extensions in Theorem 5.2 one can derive a method that solves the degree bound problem for Σ-extensions.

Theorem 6.2. *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and $\sigma(t) = \alpha\, t + \beta$, $(\alpha, \beta \in \mathbb{F}^*)$. Let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$, let $\boldsymbol{f} \in \mathbb{F}[t]^n$ and assume $a_1, a_2 \in \mathbb{F}[t]$ as in Situation 6.2. If $u_1 = u_2$ then $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. Otherwise, if there exist a $d \in \mathbb{N}_0$ and a $w \in \mathbb{F}$ such that*

$$\sigma(w) - \alpha\, w = (u_2 - u_1)\,\alpha - d\,\beta, \tag{9}$$

d is uniquely determined and $\max(\|\boldsymbol{f}\| - p + 1, d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. If there is not such a d, $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

Proof: We will proof the theorem by Corollary 4.1. Let $f \in \mathbb{F}[t]$ and $g \in \mathbb{W}$ be arbitrary but fixed such that $a_1\, \sigma(g) + a_2\, g = f$ and $\|f\| \leq \|\boldsymbol{f}\|$. We will show by case distinction that for an appropriate $b \in \mathbb{N}_0 \cup \{-1\}$ it follows that $\|g\| \leq b$ which will prove that b for the particular case is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. If $u_1 = u_2$ then by Theorem 6.1 it follows either that $\|g\| \leq 0$ or that the inequality $\|g\| \leq \|f\| - p + 1 \leq \|\boldsymbol{f}\| - p + 1$ holds. Hence by Corollary 4.1 $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. Otherwise, assume $u_1 \neq u_2$.

1. Assume there exist a $d \geq 0$ and a $w \in \mathbb{F}$ such that (9) holds. Then by Lemma 6.2 d is uniquely determined.

If $\|g\| + p - 1 > \|f\|$ and $\|g\| > 0$, by Theorem 5.2 it follows that the inequality $\|g\| = d = \max(\|f\| - p + 1, d) \leq \max(\|\boldsymbol{f}\| - p + 1, d)$ holds. Otherwise, if $\|g\| + p - 1 \leq \|f\|$ or $\|g\| \leq 0$ then clearly we obtain the inequality $\|g\| \leq \max(\|f\| - p + 1, d) \leq \max(\|\boldsymbol{f}\| - p + 1, d)$. Thus in both cases, by Corollary 4.1, $\max(\|\boldsymbol{f}\| - p + 1, d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

2. Assume there do not exist such a d and a w. Then by Theorem 6.1 it follows that $\|g\| \leq \|f\| - p + 1 \leq \|\boldsymbol{f}\| - p + 1$ or $\|g\| \leq 0$ and thus $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ by Corollary 4.1.

□

6.2. A Generalization for Higher Order Linear Difference Equations

In the end we solve the degree bound problem of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ with $\boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$ for Situation 6.3 that contains Situation 6.2.

Situatation 6.3. Assume $\mathbf{0} \neq \boldsymbol{a} = (a_1, \dots, a_\lambda, \dots, a_\mu \; \dots, a_m) \in \mathbb{F}[t]^m$ with $\lambda < \mu$, $\|a_\lambda\| = \|a_\mu\| = p$ and

$$\|a_i\| < p - 1 \; \forall i \neq \lambda, \mu.$$

In particular suppose that

$$a_\lambda = (t^p + u_1\, t^{p-1} + r_1) \qquad \text{and} \qquad a_\mu = c\,(t^p + u_2\, t^{p-1} + r_2)$$

for some $c \in \mathbb{F}^*$, $u_1, u_2 \in \mathbb{F}$, $p > 0$ and $r_1, r_2 \in \mathbb{F}[t]$ with $\|r_1\|, \|r_2\| < p - 1$.

In order to achieve this, we first generalize Theorem 6.1 by Theorem 6.3. For this theorem we first show the following lemma.

Lemma 6.3. *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ and set $\beta_k := \sigma^k(t) - (\alpha)_k\, t$ for $k \in \mathbb{Z}^*$. Assume $\boldsymbol{a} \in \mathbb{F}[t]^m$ as in Situation 6.3 and suppose that $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a Σ-extension of $(\mathbb{F}, \sigma^{\mu-\lambda})$. If there exists a $g \in \mathbb{F}(t)$ with $\|g\| \geq 0$ and $\|\sigma_{\boldsymbol{a}} g\| < \|g\| + p - 1$ then we have*

$$\|b_1\, \sigma^{\mu-\lambda}(g) + b_2\, g\| < \|g\| + p - 1$$

where

$$\begin{aligned} b_1 &:= t^p + t^{p-1}\,(p\,\beta_{\mu-m} + \sigma^{\mu-m}(u_1))/\,(\alpha)_{\mu-m} \in \mathbb{F}[t]^*, \\ b_2 &:= \sigma^{\mu-m}(c)\, t^p + t^{p-1}\,(p\,\beta_{\mu-m}\,\sigma^{\mu-m}(c) + \sigma^{\mu-m}(u_2))/\,(\alpha)_{\mu-m} \in \mathbb{F}[t]^*. \end{aligned}$$

Proof: Let $d := \|g\| \geq 0$. It follows by Lemma 3.3 and Situation 6.3 that $\|a_\lambda\, \sigma^{m-\lambda}(g)\| = \|a_\mu\, \sigma^{m-\mu}(g)\| = p + d$ with $\lambda \neq \mu$ and $\|a_i\, \sigma^{m-i}(g)\| < p + d - 1$ for all $i \neq \mu, \lambda$. Hence for $i \in \{0, 1\}$ we have

$$0 = [\sigma_{\boldsymbol{a}} g]_{p+d-i} = \left[\sum_{i=1}^{m} a_i\, \sigma^{m-i}(g)\right]_{p+d-i} = \left[a_\lambda\, \sigma^{m-\lambda}(g) + a_\mu\, \sigma^{m-\mu}(g)\right]_{p+d-i}$$

and thus $0 = \left[\sigma^{\mu-m}(a_\lambda)\,\sigma^{\mu-\lambda}(g) + \sigma^{\mu-m}(a_\mu)\,g\right]_{p+d-i}$ by Lemma 3.4. By

$$\begin{aligned}\sigma^{\mu-m}(a_\lambda) =&((\alpha)_{\mu-m}\,t + \beta_{\mu-m})^p + \sigma^{\mu-m}(u_1)\,((\alpha)_{\mu-m}\,t + \beta_{\mu-m})^{p-1} + \sigma^{\mu-m}(r_1)\\ =&\,(\alpha)^p_{\mu-m}\,t^p + t^{p-1}\,(p\,(\alpha)^{p-1}_{\mu-m}\,\beta_{\mu-m} + \sigma^{\mu-m}(u_1)\,(\alpha)^{p-1}_{\mu-m}) + \tilde{r}_1,\\ \sigma^{\mu-m}(a_\mu) =&\sigma^{\mu-m}(c)\,((\alpha)_{\mu-m}\,t + \beta_{\mu-m})^p +\\ &\sigma^{\mu-m}(u_2)\,((\alpha)_{\mu-m}\,t + \beta_{\mu-m})^{p-1} + \sigma^{\mu-m}(r_1)\\ =&\sigma^{\mu-m}(c)\,(\alpha)^p_{\mu-m}\,t^p +\\ &t^{p-1}\,(p\,(\alpha)^{p-1}_{\mu-m}\,\beta_{\mu-m}\,\sigma^{\mu-m}(c) + \sigma^{\mu-m}(u_2)\,(\alpha)^{p-1}_{\mu-m}) + \tilde{r}_2\end{aligned}$$

for some $\tilde{r}_1, \tilde{r}_2 \in \mathbb{F}[t]$ with $\|\tilde{r}_1\|, \|\tilde{r}_2\| < p-2$ it follows that

$$\left[b_1\,\sigma^{\mu-\lambda}(g) + b_2\,g\right]_{p+d-i} = 0$$

for $i \in \{0,1\}$ with $b_1, b_2 \in \mathbb{F}[t]^*$ as above. Hence by $\|b_1\,\sigma^{\mu-\lambda}(g) + b_2\,g\| \leq \|g\| + p$, it follows that $\|b_1\,\sigma^{\mu-\lambda}(g) + b_2\,g\| < \|g\| + p - 1$. □

Theorem 6.3. *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\,t + \beta$ and set $\beta_k := \sigma^k(t) - (\alpha)_k\,t$ for $k \in \mathbb{Z}^*$. Assume $\boldsymbol{a} \in \mathbb{F}[t]^m$ as in Situation 6.3 and suppose that $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a Σ-extension of $(\mathbb{F}, \sigma^{\mu-\lambda})$. Define*

$$\begin{aligned}v_1 :&= (p\,\beta_{\mu-m} + \sigma^{\mu-m}(u_1))/\,(\alpha)_{m-\mu} \in \mathbb{F},\\ v_2 :&= (p\,\beta_{\mu-m}\,\sigma^{\mu-m}(c) + \sigma^{\mu-m}(u_2))/\,(\alpha)_{\mu-m} \in \mathbb{F}.\end{aligned} \tag{10}$$

If there exists a $g \in \mathbb{F}(t)$ with $\|g\| > 0$ and $\|\sigma_{\boldsymbol{a}} g\| < \|g\| + p - 1$, there exists a $w \in \mathbb{F}$ with $\sigma^{\mu-\lambda}(w) - (\alpha)_{\mu-\lambda}\,w = (\alpha)_{\mu-\lambda}\,(v_2 - v_1) - \|g\|\,\beta_{\mu-\lambda}$. Moreover $v_1 \neq v_2$.

Proof: Assume there exists a $g \in \mathbb{F}(t)$ with $\|\sigma_{\boldsymbol{a}} g\| < \|g\| + p - 1$. Then by Lemma 6.3 there are $b_1 := t^p + v_1\,t^{p-1}$ and $b_2 := \sigma^{\mu-m}(c)\,t^p + v_2\,t^{p-1}$ such that

$$\|b_1\,\sigma^{\mu-\lambda}(g) + b_2\,g\| < \|g\| + p - 1.$$

As $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is a Σ-extension, in particular $(\alpha)_{\mu-\lambda}, \beta_{\mu-\lambda} \in \mathbb{F}^*$, we may apply Theorem 6.1 and obtain

$$\sigma^{\mu-\lambda}(w) - (\alpha)_{\mu-\lambda}\,w = (\alpha)_{\mu-\lambda}\,(v_2 - v_1) - \|g\|\,\beta_{\mu-\lambda}$$

for some $w \in \mathbb{F}$. Now assume that $v_1 = v_2$. Then $\sigma^{\mu-\lambda}(w) - (\alpha)_{\mu-\lambda}\,w = -\|g\|\,\beta_{\mu-\lambda}$ and therefore $\sigma^{\mu-\lambda}(\frac{w}{-\|g\|}) - (\alpha)_{\mu-\lambda}\,\frac{w}{-\|g\|} = \beta_{\mu-\lambda}$. By Theorem 3.3 $(\mathbb{F}(t), \sigma^{\mu-\lambda})$ is not a Σ-extension of $(\mathbb{F}, \sigma^{\mu-\lambda})$, a contradiction. □

Finally one obtains a degree bound method for Situation 6.3.

Theorem 6.4. *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with constant field, $\sigma(t) = \alpha\,t + \beta$ and set $\beta_k := \sigma^k(t) - (\alpha)_k\,t$ for $k \in \mathbb{Z}^*$. Let $\mathbb{W}$ be a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. Let $\boldsymbol{f} \in \mathbb{F}[t]^n$, assume $\boldsymbol{a} \in \mathbb{F}[t]^m$ as in Situation 6.3 and suppose that $(\mathbb{F}(t), \sigma^{\mu-\lambda})$*

is a Σ*-extension of* $(\mathbb{F}, \sigma^{\mu-\lambda})$*, in particular* $\beta_{\mu-\lambda} \in \mathbb{F}^*$*. Define* $v_1, v_2 \in \mathbb{F}$ *as in (10). If* $v_1 = v_2$ *then* $\max(\|\boldsymbol{f}\| - p + 1, 0)$ *is a degree bound of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$*. Otherwise, if there exist a* $d \in \mathbb{N}_0$ *and a* $w \in \mathbb{F}$ *such that*

$$\sigma^{\mu-\lambda}(w) - (\alpha)_{\mu-\lambda}\, w = (v_2 - v_1)\, (\alpha)_{\mu-\lambda} - d\,\beta_{\mu-\lambda}, \tag{11}$$

d *is uniquely determined and* $\max(\|\boldsymbol{f}\| - p + 1, d)$ *is a degree bound of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$*. If there is not such a* d *then* $\max(\|\boldsymbol{f}\| - p + 1, 0)$ *is a degree bound of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$*.*

Proof: We will proof the theorem by Corollary 4.1. Let $f \in \mathbb{F}[t]$ and $g \in \mathbb{W}$ be arbitrary but fixed such that $a_1\, \sigma(g) + a_2\, g = f$ and $\|f\| \leq \|\boldsymbol{f}\|$. We will show by case distinction that for an appropriate $b \in \mathbb{N}_0 \cup \{-1\}$ it follows that $\|g\| \leq b$ which will prove that b for the particular case is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. If $v_1 = v_2$ then by Theorem 6.3 it follows that $\|g\| \leq \|f\| - p + 1 \leq \|\boldsymbol{f}\| - p + 1$ or $\|g\| \leq 0$ and thus by Corollary 4.1 $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. Otherwise, assume $v_1 \neq v_2$.

1. Assume there exist a $d \geq 0$ and a $w \in \mathbb{F}$ such that (11) holds. Then by Lemma 6.2 d is uniquely determined. If $\|g\| + p - 1 > \|f\|$ and $\|g\| > 0$, it follows that $\|g\| = d = \max(\|f\| - p + 1, d) \leq \max(\|\boldsymbol{f}\| - p + 1, d)$ by Theorem 6.3. Otherwise, if $\|g\| + p - 1 \leq \|f\|$ or $\|g\| \leq 0$ then clearly we have $\|g\| \leq \max(\|f\| - p + 1, d) \leq \max(\|\boldsymbol{f}\| - p + 1, d)$. Thus by Corollary 4.1 $\max(\|\boldsymbol{f}\| - p + 1, d)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.
2. Assume there do not exist such a d and a w. Then by Theorem 6.3 it follows that $\|g\| \leq 0$ or $\|g\| \leq \|f\| - p + 1 \leq \|\boldsymbol{f}\| - p + 1$ and therefore by Corollary 4.1 $\max(\|\boldsymbol{f}\| - p + 1, 0)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$.

□

By Theorem 3.5 and Corollary 3.1 $(\mathbb{F}, \sigma^k)$ is a ΠΣ-field and $(\mathbb{F}(t), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$ for any $k \in \mathbb{Z}^*$. Using results from Section 7 especially from Theorem 7.2 it follows that one can compute a basis of parameterized first order linear difference equations in ΠΣ-fields. Hence we can decide if there exists a $d \in \mathbb{N}_0$ and a $w \in \mathbb{F}$ such that (11) holds. And in case of existence we can determine them. Therefore we can apply Theorem 5.4 to compute a degree bound for the special case described in Situation 6.3.

7. A Degree Bound Algorithm for the First Order Case

Combining results from Subsections 4.2, 5.1 and 6.1 one can design an algorithm that solves the degree bound problem for first order linear difference equations in a ΠΣ-field $(\mathbb{F}(t), \sigma)$, if one is able to solve parameterized first order linear difference equations in the ΠΣ-field $(\mathbb{F}, \sigma)$.

Algorithm 7.1. Compute a degree bound.

b=`DegreeBound(`$(\mathbb{F}(t), \sigma), \boldsymbol{a}, \boldsymbol{f}$`)`

Input: A ΠΣ-field $(\mathbb{F}(t), \sigma)$ over $\mathbb{K}$ with $\sigma(t) = \alpha\, t + \beta$, $\mathbf{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}^2$ and $\boldsymbol{f} \in \mathbb{F}^n$.

Output: A degree bound of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ for any subspace $\mathbb{W}$ of $\mathbb{F}(t)$ over $\mathbb{K}$.

(1) IF $\|a_1\| \neq \|a_2\|$ THEN RETURN $\max(\|\boldsymbol{f}\| - \|\boldsymbol{a}\|, -1)$.
(2) Set $p := \|\boldsymbol{a}\|$; If $\beta = 0$ {
(3) Set $c := -\frac{[a_2]_p}{[a_1]_p}$.
(4) IF there exists a $d \in \mathbb{N}_0$ with $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$ take it and RETURN $\max(\|\boldsymbol{f}\| - p, d)$.
(5) OTHERWISE RETURN $\max(\|\boldsymbol{f}\| - p, -1)$.
(6) } ELSE {
(7) IF $p = 0$ THEN RETURN $\|\boldsymbol{f}\| + 1$.
(8) Set $u_1 := \frac{[a_1]_{p-1}}{[a_1]_p}$, $u_2 := \frac{[a_2]_{p-1}}{[a_2]_p}$.
(9) IF $u_1 = u_2$ THEN RETURN $\max(\|\boldsymbol{f}\| - p + 1, 0)$.
(10) IF there are $d \in \mathbb{N}_0$ and $w \in \mathbb{F}$ with (9) take d and RETURN $\max(\|\boldsymbol{f}\| - p + 1, d)$.
(11) OTHERWISE RETURN $\max(\|\boldsymbol{f}\| - p + 1, 0)$.
(12) }

Theorem 7.1. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-field over $\mathbb{K}$, $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}^2$ and $\boldsymbol{f} \in \mathbb{F}^n$. Assume that one can solve parameterized first order linear difference equations in the ΠΣ-field $(\mathbb{F}, \sigma)$. Then there exists an algorithm that computes a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ for any subspace $\mathbb{W}$ of $\mathbb{F}(t)$ over $\mathbb{K}$.*

Proof: We will collect all the results of the previous sections and show that Algorithm 7.1 allows to compute the degree bound as claimed above. Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-field over $\mathbb{K}$ with $\sigma(t) = \alpha\, t + \beta$, $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}^2$ and $\boldsymbol{f} \in \mathbb{F}^n$. Furthermore suppose that $\mathbb{W}$ is a subspace of $\mathbb{F}(t)$ over $\mathbb{K}$. If $\|a_1\| \neq \|a_2\|$, we may apply Theorem 4.2 and $\max(\|\boldsymbol{f}\| - \|\boldsymbol{a}\|, -1)$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ in line (1). Otherwise we make a case distinction for Π- and Σ-extensions. If $\beta = 0$, we are dealing with a Π-extension. Then we compute a $c \in \mathbb{F}^*$ and obtain $\frac{\boldsymbol{a}}{[a_1]_p} = (t^p + r_1, -c\, t^p + r_2)$ as it is assumed in Situation 5.1. By Theorem 3.4 there exists an algorithm that decides, if there exists a $d \in \mathbb{N}_0$ with $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$, and that computes such a d if it exists. Then Theorem 5.2 guarantees that any result in the Π-extension case is a degree bound of $\mathrm{V}(\frac{\boldsymbol{a}}{[a_1]_p}, \frac{\boldsymbol{f}}{[a_1]_p}, \mathbb{W})$. But then it follows immediately that the result is also a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. Finally we consider the Σ-extension case. If $p = \|a_1\| = \|a_2\| = 0$ then $\boldsymbol{a} \in (\mathbb{F}^*)^2$ and hence $\|\boldsymbol{f}\| + 1$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$ by Corollary 6.1. Otherwise we compute $u_1, u_2 \in \mathbb{F}$ and obtain $\frac{\boldsymbol{a}}{[a_1]_p} = \left(t^p + u_1\, t^{p-1} + r_1, \frac{[a_2]_p}{[a_1]_p}(t^p + u_2\, t^{p-1} + r_2)\right)$ as it is assumed in Situation 6.2. By assumption one can solve parameterized first order linear difference equations in the ΠΣ-field $(\mathbb{F}, \sigma)$. Hence one can decide, if there exist a $w \in \mathbb{F}$ and $d \in \mathbb{N}_0$ such that $\sigma(w) - \alpha\, w = (u_2 - u_1)\, \alpha - d\, \beta$. Moreover one can compute them in case of existence. Then Theorem 6.2 guarantees that any result for the Σ-extension case is a degree bound of $\mathrm{V}(\frac{\boldsymbol{a}}{[a_1]_p}, \frac{\boldsymbol{f}}{[a_1]_p}, \mathbb{W})$. But then it follows immediately that the result is also a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$. □

In order to prove Theorem 7.1 one uses the fact that one can decide in a ΠΣ-field $(\mathbb{F}, \sigma)$, if there exists a $d \in \mathbb{N}_0$ with $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$ for some $c, \alpha \in \mathbb{F}^*$ and that one can compute such a d in case of existence. In [Kar81] M. Karr developed an algorithm that solves this problem which is reflected in Theorem 3.4. Note that this algorithm is completely independent of solving parameterized linear difference equation. Hence only in line (10) of Algorithm 7.1 one needs the ability to solve parameterized first order liner difference equations in the ΠΣ-field $(\mathbb{F}, \sigma)$.

As already indicated in Section 4, solving the degree bound problem –besides other reduction techniques described in [Sch02b, Sch02a]– is an essential step to deal with parameterized linear difference equations. In particular Theorem 7.1 plays a key role to obtain the following result from [Sch02b, Theorem 7.4].

Theorem 7.2. *There exists an algorithm that solves parameterized first order linear difference equations in* ΠΣ*-fields. More precisely, one can compute a basis of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$ *for a* ΠΣ*-field* $(\mathbb{F}, \sigma)$ *with* $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^2$ *and* $\boldsymbol{f} \in \mathbb{F}^n$*.*

This result is proven in [Sch02b] by induction on the number of extensions e in the ΠΣ-field $(\mathbb{F}, \sigma)$ over $\mathbb{K}$ with $\mathbb{F} := \mathbb{K}(t_1, \ldots, t_e)$. Inside of the induction step, Theorem 7.1 is needed that enables to apply further reduction techniques as it is indicated in the beginning of Section 4. Now it is obvious that this result of [Sch02b] combined with Theorem 7.1 gives the argument that one can solve the degree bound problem for the first order case in any ΠΣ-field.

Corollary 7.1. *Let* $(\mathbb{F}(t), \sigma)$ *be a* ΠΣ*-field over* $\mathbb{K}$*,* $\mathbb{W}$ *be a subspace of* $\mathbb{F}(t)$ *over* $\mathbb{K}$*,* $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^2$ *and* $\boldsymbol{f} \in \mathbb{F}^n$*. Then Algorithm 7.1 enables to compute a degree bound of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{W})$*.*

As already motivated in the introduction, one can describe nested sum and products in a natural way in the ΠΣ-setting. By transforming nested multisums into ΠΣ-fields, in many cases one can reduce sum-quantifiers, this means one is capable of reducing the nested level of the recursively defined sums and products. In [Sch01, Section 1.2.4] it turns out that one has to construct a ΠΣ-field in a very subtle way such that the nested level of a given multisum can be really reduced. In work under development these aspects are carefully analyzed and algorithms are developed that enable to reduce the nested level of a given multisum. The following properties of Algorithm 7.1 are essential for these developments.

Proposition 7.1. *Let* $(\mathbb{F}(t), \sigma)$ *and* $(\mathbb{G}(t), \sigma)$ *be* ΠΣ*-fields which are isomorph by a permutation. Then for any* $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^2$ *and* $\boldsymbol{f} \in \mathbb{F}[t]^n$ *we have*

$$\texttt{DegreeBound}((\mathbb{F}(\texttt{t}), \sigma), \boldsymbol{a}, \boldsymbol{f}) = \texttt{DegreeBound}((\mathbb{G}(\texttt{t}), \sigma), \boldsymbol{a}, \boldsymbol{f}).$$

Proof: Looking closer at Algorithm 7.1, for $\texttt{DegreeBound}((\mathbb{F}(\texttt{t}), \sigma), \boldsymbol{a}, \boldsymbol{f})$ and $\texttt{DegreeBound}((\mathbb{G}(\texttt{t}), \sigma), \boldsymbol{a}, \boldsymbol{f})$ there can be only two situations where the computation steps differ, namely in lines (4) and (10). First let us look at line (4). By Lemma 3.7 we have that $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$ if and only if $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{G},\sigma)}$. Hence there does not exist a $d \in \mathbb{N}_0$ with $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$ if and only if $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{G},\sigma)}$. Furthermore, if there exists such a d, it is unique by Lemma 5.1 and hence we obtain the same d in both cases. Thus in line (4) the computations are exactly the same. Similarly one can proof that in line (10) one finds the same d in both situations, or one fails to find such a d in both circumstances by using Lemmas 3.7 and 6.2. □

Moreover Theorem 7.3 is of important interest; the proof needs following lemmas.

Lemma 7.1. *Let $(\mathbb{F}(t), \sigma)$ be a Σ-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$, let $\boldsymbol{a} \in (\mathbb{F}[t]^*)^2$ be as in Situation 6.2 and suppose that there exists an $h \in \mathbb{F}[t]^*$ with $\sigma_{\boldsymbol{a}} h = 0$. If $u_1 \neq u_2$ then there exist a $g \in \mathbb{F}^*$ and a $d \in \mathbb{N}_0$ with (9).*

Proof: Let $\boldsymbol{a}$ as in Situation 6.2, in particular $p := \|\boldsymbol{a}\| > 0$, and let $h \in \mathbb{F}[t]^*$ with $\sigma_{\boldsymbol{a}} h = 0$. Furthermore assume that $u_1 \neq u_2$ and suppose that there do not exist a $g \in \mathbb{F}^*$ and a $d \in \mathbb{N}_0$ with (9). Hence by Theorem 6.2 $\max(\|0\| - p + 1, 0) = 0$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, (0), \mathbb{F}[t])$ and thus $h \in \mathbb{F}^*$. Consequently

$$c\,(t^p + u_2\, t^{p-1} + r_2) = a_2 = -\frac{\sigma(h)}{h}\, a_1 = -\frac{\sigma(h)}{h}\, (t^p + u_1\, t^{p-1} + r_1)$$

and therefore $c = -\frac{\sigma(h)}{h}$ and $u_1 = u_2$, a contradiction. □

Lemma 7.2. *Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t$ and $\boldsymbol{a} \in \mathbb{F}[t]$ as in Situation 5.1. If there exists an $h \in \mathbb{F}[t]^*$ with $\sigma_{\boldsymbol{a}} h = 0$ then there exists a $d \in \mathbb{N}_0$ such that $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$.*

Proof: Let $\boldsymbol{a} \in \mathbb{F}[t]$ as in Situation 5.1, in particular $p := \|\boldsymbol{a}\| \geq 0$, and assume that there does not exist a $d \in \mathbb{N}_0$ such that $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F},\sigma)}$. Then by Theorem 5.2 it follows that $\max(\|0\| - p, -1) = -1$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, (0), \mathbb{F}[t])$. Hence there does not exist an $h \in \mathbb{F}[t]^*$ such that $\sigma_{\boldsymbol{a}} h = 0$. □

Theorem 7.3. *Let $(\mathbb{F}(x_1, \ldots, x_e)(t)(s), \sigma)$ and $(\mathbb{F}(s)(x_1, \ldots, x_e)(t), \sigma)$ be ΠΣ-fields which are isomorph by a permutation and let $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}(x_1, \ldots, x_e)[t]^2$ such that there exists an $h \in \mathbb{F}(x_1, \ldots, x_e)^*$ with $\sigma_{\boldsymbol{a}} h = 0$; let $\boldsymbol{f} \in \mathbb{F}(x_1, \ldots, x_e)[t]^n$. Then we have*

$$\texttt{DegreeBound}((\mathbb{F}(x_1, \ldots, x_e)(t), \sigma), \boldsymbol{a}, \boldsymbol{f}) = \texttt{DegreeBound}((\mathbb{F}(s)(x_1, \ldots, x_e)(t), \sigma), \boldsymbol{a}, \boldsymbol{f}).$$

Proof: In the following we will consider the computation steps for both ΠΣ-fields $(\mathbb{F}(x_1, \ldots, x_e)(t)(s), \sigma)$ and $(\mathbb{F}(s)(x_1, \ldots, x_e)(t), \sigma)$ and will prove that the output will be always the same. Assume that $\sigma(t) = \alpha\, t + \beta$. If we have $\|a_1\| \neq \|a_2\|$, in both cases the output will be the same in line (1). Now assume that $\beta = 0$. Then by Lemma 7.2 we find a $d \in \mathbb{N}_0$ such that $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F}(x_1,\ldots,x_e),\sigma)}$. For this d we also have $\frac{c}{\alpha^d} \in \mathrm{H}_{(\mathbb{F}(s)(x_1,\ldots,x_e),\sigma)}$. Since d is unique by Lemma 5.1, it follows that in both cases we find the same d and consequently the output in line (4) is in both cases the same. Now assume that $\beta \neq 0$. If $u_1 = u_2$, in both cases we compute the same output in line (9). Now assume that $u_1 \neq u_2$. Then by Lemma 7.1 it follows that one can find a $g \in \mathbb{F}(x_1, \ldots, x_e)^*$ and a $d \in \mathbb{N}_0$ with (9). But then we find also a $g \in \mathbb{F}(s)(x_1, \ldots, x_e)^*$ for the same d with (9). Since d is unique by Lemma 6.2, it follows that in both cases we find the same d and consequently the output in line (10) is in both cases the same. Hence in each situation the output is the same which proves the theorem. □

8. Some Further Results

In [Sch01] there are various investigations to find degree bounds. In particular in [Sch01, Corollary 3.4.12] the following result pops up.

Theorem 8.1. *Let* $(\mathbb{F}, \sigma)$ *be a* ΠΣ*-field over* $\mathbb{K}$ *and let* $(\mathbb{F}(t), \sigma)$ *be a proper sum extension of* $(\mathbb{F}, \sigma)$*; let* $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^m$ *and* $f \in \mathbb{F}[t]$ *with* $l := \|f\|$*. If there is a* $g \in \mathbb{F}[t]^*$ *with* $\sigma_{\boldsymbol{a}} g = f$ *and* $n := \deg(g)$ *then there are* $g_i \in \mathbb{F}[t]$ *with* $0 \leq i \leq n - l - 1$ *such that* $\sigma_{\boldsymbol{a}} g_i = 0$ *and* $\deg(g_i) = i$*.*

For a ΠΣ-field $(\mathbb{F}(t), \sigma)$ where$(\mathbb{F}(t), \sigma)$ is a proper sum extension of $(\mathbb{F}, \sigma)$ this result delivers a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$, if $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$.

Corollary 8.1. *Let* $(\mathbb{F}(t), \sigma)$ *be a* ΠΣ*-field over* $\mathbb{K}$ *and where* $(\mathbb{F}(t), \sigma)$ *is a proper sum extension of* $(\mathbb{F}, \sigma)$*. Let* $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^m$*,* $\boldsymbol{f} \in \mathbb{F}[t]^n$ *and assume there are* $k \geq 0$ *linearly independent* $g \in \mathbb{F}$ *over* $\mathbb{K}$ *with* $\sigma_{\boldsymbol{a}} g = 0$*. Then* $m + \|\boldsymbol{f}\| - \max(k, 1)$ *is a degree bound of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$*.*

Proof: Assume $b := m + \|\boldsymbol{f}\| - \max(k, 1)$ is not a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$, i.e. there exists a $\boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]) \setminus \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_b)$. This means that $d := \deg(g) > b$ and $\sigma_{\boldsymbol{a}} g = \boldsymbol{c}\, \boldsymbol{f} =: f$. Clearly we have $l := \|f\| \leq \|\boldsymbol{f}\|$. By Theorem 8.1 there are $d - l + \max(k, 1) - 1$ linearly independent solutions g over $\mathbb{K}$ with $\sigma_{\boldsymbol{a}} g = 0$. Thus by $d - l + \max(k, 1) - 1 \geq m + \|\boldsymbol{f}\| - \max(k, 1) + 1 + \max(k, 1) - 1 \geq m$ it follows that there is a subspace of $\mathrm{V}(\boldsymbol{a}, (0), \mathbb{F}[t])$ which is generated by a basis of the form $B := \{(0, g_1), \ldots, (0, g_m)\}$. Since $(1, 0) \in \mathrm{V}(\boldsymbol{a}, (0), \mathbb{F}[t])$, $B \cup \{(1, 0)\}$ forms a basis of a subspace of $\mathrm{V}(\boldsymbol{a}, (0), \mathbb{F}[t])$ over $\mathbb{K}$. Hence $\mathrm{V}(\boldsymbol{a}, (0), \mathbb{F}(t))$ has at least dimension $m + 1$, a contradiction to Proposition 2.1. □

Note that this degree bound contains Corollary 6.1, if one restricts to proper sum extensions and sets $k = 0$, i.e. $m + \|\boldsymbol{f}\|$ is a degree bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$. In many cases, the dimension k of the subspace $\mathbb{V} := \{g \in \mathbb{F} \mid \sigma_{\boldsymbol{a}} g = 0\}$ of $\mathbb{F}$ over $\mathbb{K}$ is not zero, but contrary respectably high. As already pointed out in [Sch01, Section 3.4.9], one anyway has to compute that vector space $\mathbb{V}$, if one wants to find a basis of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$ according to the reduction techniques given in [Sch02b]. Hence one obtains the number k for free which allows to reduce tremendously the degree bound in many situations.

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