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A Unique Representation of Solutions of Parameterized Linear Difference Equations in $\Pi\Sigma$-fields*

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Abstract

A very general class of multisum expressions can be formulated in $\Pi\Sigma$-fields, a certain subclass of difference fields. This allows to simplify and prove multisum expression and identities by solving parameterized linear difference equations in $\Pi\Sigma$-fields. In this article we explain how the solution space of such difference equations can be described in canonical form. In many cases this canonical form also leads to a compact representation of the solutions.

1. Introduction

In [Kar81, Kar85] an algorithm for indefinite summation is developed that is based on the theory of difference fields [Coh65]. In particular so called $\Pi\Sigma$-fields are introduced, in which parameterized first order linear difference equations can be solved in full generality. By results from [Bro00], I was able to streamline these ideas which results in a simpler algorithm in [Sch02c, Sch02a, Sch02b]. This algorithm is available in form of a summation package called `Sigma` [Sch00, Sch01] in the computer algebra system Mathematica. It cannot only deal with series of (q-)hypergeometric terms, like [Gos78, PS95, PR97], or holonomic series, like [CS98], but also with series of terms where for example the harmonic numbers can appear in the denominator.

Moreover `Sigma` can prove and discover a huge class of definite multisum identities. In [Sch00] I observed that one can apply Zeilberger's creative telescoping trick [Zei90] by solving a specific parameterized linear difference equation. This enables in many cases to compute a recurrence in the $\Pi\Sigma$-field setting which has a given definite multisum as a solution; therefore one can verify automatically a

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given multisum identity. Additionally, I have generalized Karr's ideas in [Sch02c] such that one can search for all solutions of linear difference equations with arbitrary order in ΠΣ-fields. Hence one can find solutions of recurrences and thus not only prove, but even discover definite multisums identities.

In [Sch02c] one of the important results is that these algorithms enable to search for all solutions of parameterized linear difference equations in ΠΣ-fields. In this proof the property is required that the solutions of such difference equations can be represented in a canonical form. This article delivers an algorithm that allows to transform these solutions of a linear difference equation into such a unique representation. Moreover it turns out that in most cases this canonical representation leads to a compacter description of the solutions. I want to emphasize that all these transformations of the solutions into a canonical representation are based on Gaussian elimination and gcd-computations. Moreover these constructions can be related to the theory of Gröbner basis.

2. Parameterized Linear Difference Equations in ΠΣ-Fields

`Sigma` [Sch00, Sch01] is a package, implemented in the computer algebra system Mathematica, that enables to discover and prove nested multisum identities.

Example 2.1. After loading the summation package

In[1]:= $<<$ **Sigma‘**

`Sigma - A summation package by Carsten Schneider`

we are able to insert the following definite summation problem:

In[2]:= $$\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{N}} \left(\frac{\binom{\mathbf{N}}{\mathbf{k}} (-\mathbf{1})^{\mathbf{k}}}{(\mathbf{1}+\mathbf{k})^{\mathbf{4}}} \right);$$

First we generate a recurrence by Zeilberger's creative telescoping trick [Zei90] that is satisfied by `mySum`.

In[3]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}, \mathbf{RecOrder}->\mathbf{3}]$

Out[3]= $$\{(1+N)\,(2+N)\,(3+N)\,(4+N)\,\mathrm{SUM}[N] - 3\,(2+N)\,(3+N)^2\,(4+N)\,\mathrm{SUM}[1+N] + (3+N)\,(4+N)\,(37+21\,N+3\,N^2)\,\mathrm{SUM}[2+N] - (4+N)^4\,\mathrm{SUM}[3+N] == -1\}$$

Next we solve the recurrence in terms of the Harmonic numbers $H_N = \sum_{k=1}^{N} \frac{1}{k}$ and generalized versions $H_N^{(o)} = \sum_{i=1}^{N} \frac{1}{k^o}$ with $o \in \mathbb{N}$.

In[4]:= $\mathbf{SolveRecurrence}[\mathbf{rec}[[\mathbf{1}]], \mathbf{SUM}[\mathbf{N}], \mathbf{Tower}-> \{\mathbf{H_N}, \mathbf{H_N^{(2)}}, \mathbf{H_N^{(3)}}\}]$

Out[4]= $$\Big\{\Big\{0, \frac{2+(1+N)\,H_N\,(2+(1+N)\,H_N) + H_N^{(2)} + N\,(2+N)\,H_N^{(2)}}{(1+N)^3}\Big\}, \Big\{0, \frac{-2\,N\,(2+N)+(1+N)\,H_N\,(2+(1+N)\,H_N) + H_N^{(2)} + N\,(2+N)\,H_N^{(2)}}{(1+N)^3}\Big\}, \Big\{0, \frac{(-1+N)\,N+(1+N)\,H_N\,(-1-3\,N+(1+N)\,H_N) + H_N^{(2)} + N\,(2+N)\,H_N^{(2)}}{(1+N)^3}\Big\},$$
$$\Big\{1, \frac{1}{6\,(1+N)^4}\big(-6\,N - 6\,N\,(1+N)\,H_N - 3\,N\,(1+N)^2\,H_N^2 + (1+N)^3\,H_N^3 + 3\,(1+N)^2\,(-N+(1+N)\,H_N)\,H_N^{(2)} + 2\,(1+N)^3\,H_N^{(3)}\big)\Big\}\Big\}$$

This has to be interpreted as follows: The first three expressions are linearly independent solutions of the homogeneous version of the recurrence, whereas the last expression is a specific solution of the recurrence itself.
Finally we obtain a closed form evaluation of `mySum` by finding a linear combination of those homogeneous solutions plus the specific inhomogeneous solution that have the same initial values as `mySum`.

In[5]:= **sol = FindLinearCombination[recSol, mySum, N, 3]**

Out[5]= $$\frac{1}{6\ (1+\mathrm{N})^4}\big(3\ (1+\mathrm{N})^2\ \mathrm{H}_\mathrm{N}^2+(1+\mathrm{N})^3\ \mathrm{H}_\mathrm{N}^3+ 3\ (1+\mathrm{N})^2\ \mathrm{H}_\mathrm{N}^{(2)}+3\ (1+\mathrm{N})\ \mathrm{H}_\mathrm{N}\ \big(2+(1+\mathrm{N})^2\ \mathrm{H}_\mathrm{N}^{(2)}\big)+ 2\ \big(3+\mathrm{H}_\mathrm{N}^{(3)}+3\ \mathrm{N}\ \mathrm{H}_\mathrm{N}^{(3)}+3\ \mathrm{N}^2\ \mathrm{H}_\mathrm{N}^{(3)}+\mathrm{N}^3\ \mathrm{H}_\mathrm{N}^{(3)}\big)\big)$$

The summation package `Sigma` provides algorithms that allow to solve parameterized linear difference equations in so called ΠΣ-fields. In particular the functions `GenerateRecurrence` and `SolveRecurrence` apply internally this difference field machinery which finally allows to discover a huge class of multisum identities. In the sequel we will illustrate how one can solve recurrence `rec` from the previous example in the difference field setting.

Definition 2.1. A *difference field* (resp. ring) is a field (resp. ring) $\mathbb{F}$ together with a field (resp. ring) automorphism $\sigma:\mathbb{F}\to\mathbb{F}$. In the sequel a difference field (resp. ring) given by the field (resp. ring) $\mathbb{F}$ and automorphism σ is denoted by $(\mathbb{F},\sigma)$. Moreover the subset $\mathbb{K}:=\{k\in\mathbb{F}\,|\,\sigma(k)=k\}$ is called the *constant field* of the difference field $(\mathbb{F},\sigma)$.

It is easy to see that the constant field $\mathbb{K}$ of a difference field $(\mathbb{F},\sigma)$ is a subfield of $\mathbb{F}$. In the sequel we will assume that **all** fields are of characteristic 0. Then it is immediate that for any field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ we have $\sigma(q)=q$ for $q\in\mathbb{Q}$. Hence in any difference field, $\mathbb{Q}$ is a subfield of its constant field.

Example 2.2. Let $\mathbb{Q}(t_1,\dots,t_4)$ be the field of rational functions over $\mathbb{Q}$ with the field automorphism σ canonically defined by

$$\sigma(c)=c\ \ \forall c\in\mathbb{Q},\qquad \sigma(t_1)=t_1+1,\qquad \sigma(t_2)=t_2+\frac{1}{t_1+1},$$
$$\sigma(t_3)=t_3+\frac{1}{(t_1+1)^2},\qquad \sigma(t_4)=t_4+\frac{1}{(t_1+1)^3}.$$

Note that the automorphism σ acts on t_1, t_2, t_3 and t_4 like the shift operator S on N, H_N, $\mathrm{H}_N^{(2)}$ and $\mathrm{H}_N^{(3)}$ with $SN=N+1$ and

$$S\,\mathrm{H}_N=\mathrm{H}_N+\frac{1}{(N+1)},\quad S\,\mathrm{H}_N^{(2)}=\mathrm{H}_N^{(2)}+\frac{1}{(N+1)^2},\quad S\,\mathrm{H}_N^{(3)}=\mathrm{H}_N^{(3)}+\frac{1}{(N+1)^3}.$$

Furthermore let

$$a_1:=(1+t_1)\ (2+t_1)\ (3+t_1)\ (4+t_1),\qquad a_2:=3\ (2+t_1)\ (3+t_1)^2\ (4+t_1),$$
$$a_3:=(3+t_1)\ (4+t_1)\ (37+21\ t_1+3\ t_1^2),\quad a_4:=-(4+t_1)^4.$$

Then the problem of solving the recurrence `rec` in Example 2.1 in terms of N, H_N, $\mathrm{H}_N^{(2)}$ and $\mathrm{H}_N^{(3)}$ can be rephrased as the following problem in terms of the difference field $(\mathbb{Q}(t_1,\dots,t_4),\sigma)$: find all $g\in\mathbb{Q}(t_1,\dots,t_4)$ such that

$$a_1\,\sigma^3(g)+a_2\,\sigma^2(g)+a_3\,\sigma(g)+a_4\,g=-1. \tag{1}$$

The algorithms in `Sigma`, based on [Sch02c, Sch02a, Sch02b], deliver three linearly independent solutions s_1,s_2,s_3 over $\mathbb{Q}$ of the homogeneous version of the difference equation and one particular solution p of the recurrence itself, namely

$$\begin{aligned}
s_1 :=& \frac{2+2\,(1+t_1)\,t_2+(1+t_1)^2\,{t_2}^2+(1+t_1)^2\,t_3}{(1+t_1)^3},\\
s_2 :=& \frac{-2\,t_1\,(2+t_1)+2\,(1+t_1)\,t_2+(1+t_1)^2\,{t_2}^2+(1+t_1)^2\,t_3}{(1+t_1)^3},\\
s_3 :=& \frac{(-1+t_1)\,t_1-(1+t_1)\,(1+3\,t_1)\,t_2+(1+t_1)^2\,{t_2}^2+(1+t_1)^2\,t_3}{(1+t_1)^3}\ \text{and}\\
p :=& \frac{1}{6\,(1+t_1)^4}\big(-6\,t_1-6\,t_1\,(1+t_1)\,t_2-3\,t_1\,(1+t_1)^2\,{t_2}^2+(1+t_1)^3\,{t_2}^3+\\
&\big(-3\,t_1\,(1+t_1)^2+3\,(1+t_1)^3\,t_2\big)\,t_3+2\,(1+t_1)^3\,t_4\big).
\end{aligned}$$

Since the difference equation (1) has order 3, the set

$$\{k_1\,s_1+k_2\,s_2+k_3\,s_3+p\,|\,k_i\in\mathbb{Q}\}$$

describes *all* the solutions of (1) in $\mathbb{Q}(t_1,\dots,t_4)$. From this result the output of the function `SolveRecurrence` in Example 2.1 follows immediately.

As illustrated in [Sch01, Sch02c] one is able to discover and prove a huge class of indefinite and definite multisum identities by solving parameterized linear difference equations in ΠΣ-fields; in particular one can carry out indefinite summation, Zeilberger's creative telescoping idea and solving recurrences.

Solving Parameterized Linear Difference Equations

- GIVEN: A difference field $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$, $a_1,\dots,a_m\in\mathbb{F}$ with $m\geq 1$ and $(a_1\dots a_m)\neq(0,\dots,0)=:\mathbf{0}$ and $f_1,\dots,f_n\in\mathbb{F}$ with $n\geq 1$.
- FIND: All $g\in\mathbb{F}$ and all $c_1,\dots,c_n\in\mathbb{K}$ with $a_1\,\sigma^{m-1}(g)+\dots+a_m\,g=c_1\,f_1+\dots+c_n\,f_n$.

Note that in any difference field $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$, the field $\mathbb{F}$ can be interpreted as a vector space over $\mathbb{K}$. Hence the above problem can be described by the following set called solution space.

Definition 2.2. Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$ and consider a subspace $\mathbb{V}$ of $\mathbb{F}$ as a vector space over $\mathbb{K}$. Let $\mathbf{0}\neq\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}^m$ and $\boldsymbol{f}=(f_1,\dots,f_n)\in\mathbb{F}^n$. We define the *solution space* for $\boldsymbol{a},\boldsymbol{f}$ in $\mathbb{V}$ by

$$\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{V})=\{(c_1,\dots,c_n,g)\in\mathbb{K}^n\times\mathbb{V}:a_1\,\sigma^{m-1}(g)+\dots+a_m\,g=c_1\,f_1+\dots+c_n\,f_n\}.$$

It follows immediately that $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{V})$ is a vector space over $\mathbb{K}$. Moreover in [Sch02c] based on [Coh65] it is proven that this vector space has finite dimension.

Proposition 2.1. *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$ and assume $\boldsymbol{f}\in\mathbb{F}^n$ and $\boldsymbol{0}\neq\boldsymbol{a}\in\mathbb{F}^m$. Let $\mathbb{V}$ be a subspace of $\mathbb{F}$ as a vector space over $\mathbb{K}$. Then $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{V})$ is a vector space over $\mathbb{K}$ with maximal dimension $m+n-1$.*

Example 2.3. In Example 2.2 $\mathrm{V}((a_1,a_2,a_3,a_4),(-1),\mathbb{Q}(t_1,\dots,t_4))$ is a subspace of $\mathbb{Q}(t_1,\dots,t_e)$ over $\mathbb{Q}$ with the basis $\{(0,s_1),(0,s_2),(0,s_3),(1,p)\}$.

In [Sch02c, Sch02a, Sch02b] algorithms are developed that enable to search for a basis of the solution space $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$ in a huge class of difference fields, so called ΠΣ-fields. As indicated in Example 2.2, one can rephrase expressions involving nested sums and products in ΠΣ-fields. Since ΠΣ-fields can be constructed completely algorithmically, the user can be dispensed from working explicitly with difference fields. ΠΣ-fields and its important properties are introduced in [Kar81, Kar85] and further analyzed in [Bro00, Sch01, Sch02a]. In this work we focus only on the property that a ΠΣ-field $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ is represented by a field of rational functions $\mathbb{K}(t_1,\dots,t_e)$ over $\mathbb{K}$.

Example 2.4. The difference field $(\mathbb{Q}(t_1,\dots,t_4),\sigma)$ in Example 2.2 is a ΠΣ-field with constant field $\mathbb{Q}$.

In this work let $\mathbb{K}[t_1,\dots,t_n]$ be a polynomial ring with coefficients in the field $\mathbb{K}$ and let $\mathbb{K}(t_1,\dots,t_e)$ be its quotient field; i.e. $\mathbb{K}(t_1,\dots,t_e)$ is the field of rational functions over $\mathbb{K}$. Let $T:=[t_1,\dots,t_n]$ be the monoid (under multiplication) of *power products* $t_1^{i_1}\dots t_n^{i_n}$ with the unit element $1=t_1^0\dots t_e^0$. An *admissible ordering* $<$ on T is a total ordering that fulfills the following two properties:

- $1<t$ for all $t\in T\setminus\{1\}$, and
- for all $s,t,u\in T$ we have $s\,u<t\,u$, if $s<t$.

Let $<$ be such an admissible ordering and take $f\in\mathbb{K}[t_1,\dots,t_n]^*$ and $t\in T$. We denote by $[f]_t$ the *coefficient* of t in f. Moreover the *leading power product* of f is defined by

$$\mathrm{lpp}_<(f):=\max_<\{t\in T\mid [f]_t\neq 0\},$$

and the *leading coefficient* of f is defined by $\mathrm{lc}_<(f):=[f]_{\mathrm{lpp}_<(f)}$. In the sequel this admissible ordering will be always clear from the context, and hence we will suppress $<$ in lc and lpp. If $g\in\mathbb{K}(t_1,\dots,t_n)$ then there are uniquely determined $f_1,f_2\in\mathbb{K}[t_1,\dots,t_n]$ such that $f=\frac{f_1}{f_2}$ where $\gcd_{\mathbb{K}[t_1,\dots,t_n]}(f_1,f_2)=1$ and $\mathrm{lc}(f_2)=1$. In this case we write $\mathrm{den}(g)=f_2$ as the *denominator* of g.

3. A Unique Representation of the Solution Space

The main goal of this article is to provide an algorithm that transforms any basis of a solution space $\mathbb{V}:=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$ for a ΠΣ-field $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ to a canonical representation. This representation of the solution space $\mathbb{V}$ is uniquely defined up to a given admissible ordering $<$ on the monoid of power products $[t_1,\dots,t_e]$ in the field of rational functions $\mathbb{F}=\mathbb{K}(t_1,\dots,t_e)$. In particular the algorithm under discussion is based just on gcd-computations in $\mathbb{K}[t_1,\dots,t_e]$ and on Gaussian elimination.

Theorem 3.1. *Let $(\mathbb{F},\sigma)$ with $\mathbb{F}:=\mathbb{K}(t_1,\dots,t_e)$ be a ΠΣ-field over $\mathbb{K}$, $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{F}^m$ and $\boldsymbol{f}\in\mathbb{F}^n$. Then there exists an algorithm that transforms any basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$ by Gaussian elimination and* gcd*-computations to a uniquely determined basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$ for a given admissible ordering on $[t_1,\dots,t_e]$.*

This important aspect is needed in order to prove that the algorithms, developed in [Sch02c], enable to search for a basis of a given solution space in a ΠΣ-field. Moreover in many examples this specific basis representation of the solution space is very compact among the possible basis representations.

Example 3.1. By applying the algorithm, that will be explained later, to the given basis $\{(0,s_1),(0,s_2),(0,s_3),(1,p)\}$ from Example 2.3 and rephrasing this result (Example 3.4) in terms of N, H_N, $\mathrm{H}_N^{(2)}$ and $\mathrm{H}_N^{(3)}$ will lead to the following simpler description of the solutions for recurrence `rec` in Example 2.1.

$$\Big\{\Big\{0,\frac{2+(1+N)\,H_N\,(2+(1+N)\,H_N)+H_N^{(2)}+N\,(2+N)\,H_N^{(2)}}{(1+N)^3}\Big\},\Big\{0,\frac{1+(1+N)\,H_N}{(1+N)^2}\Big\},\Big\{0,\frac{1}{1+N}\Big\},$$

$$\Big\{1,\frac{1}{6\,(1+N)^4}\big(6+3\,(1+N)^2\,H_N^2+(1+N)^3\,H_N^3+3\,(1+N)^2\,H_N^{(2)}+$$

$$3\,(1+N)\,H_N\,\big(2+(1+N)^2\,H_N^{(2)}\big)+2\,(1+N)^3\,H_N^{(3)}\big)\Big\}\Big\}$$

Next we want to rephrase the above theorem in a precise problem specification. For this we introduce some further notations. Let $\mathbb{F}$ be a field and $\boldsymbol{f}=(f_1,\dots,f_n)\in\mathbb{F}^n$ with $n\geq1$. For $h\in\mathbb{F}$ we write $h\,\boldsymbol{f}=(h\,f_1,\dots,h\,f_n)\in\mathbb{F}^n$ and $\boldsymbol{f}\wedge h=(f_1,\dots,f_m,h)\in\mathbb{F}^{n+1}$. In addition we denote $\mathbf{0_n}:=(0,\dots,0)\in\mathbb{K}^n$ as the zero-vector of length n; if the length is clear from the context, we just write $\mathbf{0}$. Furthermore let $\mathbb{K}$ be a subfield of $\mathbb{F}$ and $\mathbb{V}\neq\{\mathbf{0}\}$ be a subspace of $\mathbb{K}^n\times\mathbb{F}$ over $\mathbb{K}$. In addition let $B=\{\boldsymbol{b_1},\dots,\boldsymbol{b_d}\}$, $d\geq1$, be a basis of $\mathbb{V}$ with $\boldsymbol{b_i}=(c_{i1},\dots,c_{in},g_i)\in\mathbb{K}^n\times\mathbb{F}$, i.e.

$$\mathbb{V}=\{k_1\,\boldsymbol{b_1}+\dots+k_d\,\boldsymbol{b_d}\,|\,k_i\in\mathbb{K}\}.$$

The basis B is represented by the *basis matrix* $\boldsymbol{M_B}:=\begin{pmatrix}c_{11}&\dots&c_{1n}&g_1\\ \vdots&\vdots&\vdots&\vdots\\ c_{d1}&\dots&c_{dn}&g_d\end{pmatrix}$ in the sequel; in particular we have that

$$\mathbb{V}=\big\{\boldsymbol{k}\cdot\boldsymbol{M_B}\,|\,k\in\mathbb{K}^d\big\}.$$

Moreover we write $\boldsymbol{M_B}=\boldsymbol{C}\wedge\boldsymbol{g}$ for $\boldsymbol{C}:=\begin{pmatrix}c_{11}&\dots&c_{1n}\\ \vdots&\vdots&\vdots\\ c_{d1}&\dots&c_{dn}\end{pmatrix}$ and $\boldsymbol{g}:=\begin{pmatrix}g_1\\ \vdots\\ g_d\end{pmatrix}$. We call $\boldsymbol{C}$ the *parameter matrix* and $\boldsymbol{g}$ the *solution vector* of $\boldsymbol{M_B}$.

Example 3.2. In Example 2.2 we found the basis matrix $\begin{pmatrix} 1 & p \\ 0 & s_1 \\ 0 & s_2 \\ 0 & s_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} p \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}$ for the solution space $\mathrm{V}((a_1, a_2, a_3, a_4), (-1), \mathbb{Q}(t_1, \dots, t_4))$.

By the above remarks Theorem 3.1 can be rephrased to the following problem.

Find a unique basis matrix representation in the field of rational functions

- GIVEN: A field of rational function $\mathbb{F} := \mathbb{K}(t_1, \dots, t_e)$ over $\mathbb{K}$, a basis matrix $\boldsymbol{C} \wedge \boldsymbol{g}$ of a subspace $\mathbb{V} \neq \{\boldsymbol{0}\}$ of $\mathbb{K}^n \times \mathbb{F}$ over $\mathbb{K}$, and an admissible ordering $<$ on $[t_1, \dots, t_e]$.
- FIND: A unique basis matrix representation of $\mathbb{V}$ by gcd-computations and Gaussian elimination.

3.1. The Trivial Case $\mathbb{F} = \mathbb{K}$

If one changes a basis matrix of $\mathbb{V}$ by Gaussian-elimination, more precisely by row operations, one obtains again a basis matrix of $\mathbb{V}$. As it will turn out, all those matrices will be transformed to matrices in row reduced echelon form.

Definition 3.1. A matrix is in *row reduced echelon form* if the following holds: (1) The leftmost nonzero entry in any nonzero row is a 1. (2) If a row has the left most nonzero entry in the r-th column, all the other entries in the r-th column are 0 and the leftmost nonzero entries in subsequent rows are in columns to the right of the r-th column. (3) All zero rows come after all nonzero rows.

Example 3.3. The matrix $\boldsymbol{M}$ in Example 3.6 is transformed by row operations to $\boldsymbol{M'}$ which is in row reduced echelon form.

There is the following fact from linear algebra.

Lemma 3.1. *If a matrix $\boldsymbol{C}$ is transformed by row operations to a matrix $\boldsymbol{D}$ in row reduced echelon form, it is uniquely determined, i.e. for any other such matrix $\boldsymbol{D'}$, that one obtains by transforming $\boldsymbol{C}$ by row operations to a row reduced echelon form, we have $\boldsymbol{D} = \boldsymbol{D'}$.*

In case $\mathbb{K}(t_1, \dots, t_e) = \mathbb{K}$, i.e. $e = 0$, we can transform the basis matrix $\boldsymbol{C} \wedge \boldsymbol{g} \in \mathbb{K}^{(n+1)\times d}$ of $\mathbb{V}$ by row operations to a basis matrix $\boldsymbol{D} \wedge \boldsymbol{h}$ of $\mathbb{V}$ that is in row reduced echelon form. By the above lemma this $\boldsymbol{D} \wedge \boldsymbol{h}$ is uniquely determined which proves Theorem 3.1 for exactly the special case $e = 0$.

3.2. Elimination of Denominators

What remains to consider is the case $e > 0$. In a first step we reduce the problem from finding a unique basis matrix representation of a subspace of $\mathbb{K}^n \times \mathbb{F}$ to the problem of searching a unique basis matrix representation of a subspace of $\mathbb{K}^n \times \mathbb{K}[t_1, \dots, t_e]$.

Lemma 3.2. *Let $\mathbb{F} := \mathbb{K}(t_1, \dots, t_e)$ be a field of rational functions over $\mathbb{K}$, $\mathbb{V} \neq \{\mathbf{0}\}$ be a subspace of $\mathbb{K}^n \times \mathbb{F}$ over $\mathbb{K}$ and $u \in \mathbb{F}^*$. If $\boldsymbol{D}\wedge\boldsymbol{h}$ is a basis matrix of the subspace $\mathbb{W} := \{\boldsymbol{c}\wedge(g\,u) \,|\, \boldsymbol{c}\wedge g \in \mathbb{V}\}$ of $\mathbb{K}^n \times (d\,\mathbb{F})$ over $\mathbb{K}$, $\boldsymbol{D}\wedge\frac{\boldsymbol{h}}{u}$ is a basis matrix of $\mathbb{V}$ over $\mathbb{K}$.*

Proof: The d rows in $\boldsymbol{D}\wedge\boldsymbol{h}$ are linearly independent over $\mathbb{K}$ if and only if they are linearly independent in $\boldsymbol{D}\wedge\frac{\boldsymbol{h}}{u}$. Hence $\boldsymbol{D}\wedge\frac{\boldsymbol{h}}{u}$ is a basis matrix of a subspace $\mathbb{U}$ of $\mathbb{V}$ over $\mathbb{K}$. Take any $\boldsymbol{c}\wedge g \in \mathbb{V}$. Therefore $\boldsymbol{c}\wedge(g\,u) \in \mathbb{W}$. Consequently there is a $\boldsymbol{k} \in \mathbb{K}^d$ with $\boldsymbol{c}\wedge(g\,u) = \boldsymbol{k}\cdot(\boldsymbol{D}\wedge\boldsymbol{h})$, thus $\boldsymbol{c}\wedge g = \boldsymbol{k}\cdot(\boldsymbol{D}\wedge\frac{\boldsymbol{h}}{u})$, and hence $\boldsymbol{c}\wedge g \in \mathbb{U}$. Therefore $\mathbb{V} = \mathbb{U}$ which proves the lemma. □

Let $\boldsymbol{g} = (g_1, \dots, g_k)$ and compute by some gcd-computations

$$u := \operatorname{lcm}(\operatorname{den}(g_1), \dots, \operatorname{den}(g_k)) \in \mathbb{A}.$$

Then $\mathbb{W} := \{\boldsymbol{c}\wedge(g\,u) \,|\, \boldsymbol{c}\wedge g \in \mathbb{V}\}$ is a subspace of $\mathbb{K}^n \times \mathbb{K}[t_1, \dots, t_e]$. Now assume that we are able to determine a unique basis representation $\boldsymbol{D}\wedge\boldsymbol{h}$ for the vector space $\mathbb{W}$. Then by Lemma 3.2 we obtain a basis matrix $\boldsymbol{D}\wedge\frac{\boldsymbol{h}}{u}$ of $\mathbb{V}$ that is uniquely determined among the basis matrices of $\mathbb{V}$.

Example 3.4. Consider the subspace $\mathbb{V} := \mathrm{V}((a_1, a_2, a_3, a_4), (-1), \mathbb{Q}(t_1, \dots, t_4))$ of $\mathbb{Q} \times \mathbb{Q}(t_1, \dots, t_4)$ over $\mathbb{Q}$ and its basis matrix from Example 3.2. Furthermore fix the lexicographic ordering $<$ on $[t_1, \dots, t_4]$ with $1 < t_1 < t_2 < t_3 < t_4$.Then we can determine $u := \operatorname{lcm}(\operatorname{den}(g_1), \operatorname{den}(g_2), \operatorname{den}(g_3), \operatorname{den}(p)) = (1+t_1)^4$ and we immediately obtain a basis matrix $\begin{pmatrix} 1 & g_1 \\ 0 & g_2 \\ 0 & g_3 \\ 0 & g_4 \end{pmatrix}$ with

$$\begin{aligned}
g_1 &:= (1+t_1)\,(2 + 2\,(1+t_1)\,t_2 + (1+t_1)^2\,t_2^2 + (1+t_1)^2\,t_3),\\
g_2 &:= (1+t_1)\,(-2\,t_1\,(2+t_1) + 2\,(1+t_1)\,t_2 + (1+t_1)^2\,t_2^2 + (1+t_1)^2\,t_3),\\
g_3 &:= (1+t_1)\,((-1+t_1)\,t_1 - (1+t_1)\,(1+3\,t_1)\,t_2 + (1+t_1)^2\,t_2^2 + (1+t_1)^2\,t_3),\\
g_4 &:= 6\,\big(-6\,t_1 - 6\,t_1\,(1+t_1)\,t_2 - 3\,t_1\,(1+t_1)^2\,t_2^2 + (1+t_1)^3\,t_2^3 +\\
&\qquad (-3\,t_1\,(1+t_1)^2 + 3\,(1+t_1)^3\,t_2)\,t_3 + 2\,(1+t_1)^3\,t_4\big)
\end{aligned}$$

for the subspace $\mathbb{W} := \{\boldsymbol{c}\wedge(g\,u) \,|\, \boldsymbol{c}\wedge g \in \mathbb{V}\}$ of $\mathbb{Q} \times \mathbb{Q}[t_1, \dots, t_4]$ over $\mathbb{Q}$. Later we will develop an algorithm based on Gaussian elimination that computes a unique basis matrix representation of $\mathbb{W}$, namely $\boldsymbol{B} = \begin{pmatrix} 1 & \bar{g} \\ 0 & q_1 \\ 0 & q_2 \\ 0 & q_3 \end{pmatrix}$ with

$$\begin{aligned}
\bar{g} :=& \frac{1}{6}\,\big(6 + 3\,(1+t_1)^2\,t_2^2 + (1+t_1)^3\,t_2^3 + 3\,(1+t_1)^2\,t_3 + 3\,(1+t_1)\,t_2\,\big(2 + (1+t_1)^2\,t_3\big)\\
&+ 2\,(1+t_1)^3\,t_4\big), \qquad q_1 := (1+t_1)\,(2 + (1+t_1)\,t_2\,(2 + (1+t_1)\,t_2) + t_3 + t_1\,(2+t_1)\,t_3),\\
q_2 :=& (1+t_1)^2\,(1 + (1+t_1)\,t_2), \qquad q_3 := (1+t_1)^3.
\end{aligned}$$

Hence by Lemma 3.2 we obtain the basis matrix $\begin{pmatrix} 1 & \bar{g}/u \\ 0 & q_1/u \\ 0 & q_2/u \\ 0 & q_3/u. \end{pmatrix}$ of $\mathbb{V}$ that is uniquely determined among the basis matrices of $\mathbb{V}$. Rephrasing this basis matrix in terms of N, H_N, $\mathrm{H}_N^{(2)}$ and $\mathrm{H}_N^{(3)}$ leads directly to the solution given in Example 3.1.

By the above lemma we are just concerned in solving the following problem.

Find a unique basis matrix representation in its polynomial ring

- GIVEN: A polynomial ring $\mathbb{A} := \mathbb{K}[t_1, \dots, t_e]$ over a field $\mathbb{K}$, a basis matrix $\boldsymbol{C}\wedge\boldsymbol{g}$ of a subspace $\mathbb{W} \neq \{\boldsymbol{0}\}$ of $\mathbb{K}^n \times \mathbb{A}$ over $\mathbb{K}$, and an admissible ordering $<$ on $[t_1, \dots, t_e]$.
- FIND: A unique basis matrix representation of $\mathbb{W}$ by Gaussian elimination.

3.3. A Unique Representation of the Parameter Matrix

Next we transform the basis matrix $\boldsymbol{C}\wedge\boldsymbol{g}$ of $\mathbb{W}$ by row operations to a basis matrix $\boldsymbol{D}\wedge\boldsymbol{h}$ of $\mathbb{W}$ where $\boldsymbol{D}$ is in row reduced echelon form. Then by Lemma 3.1 for any other such basis matrix $\boldsymbol{D'}\wedge\boldsymbol{h}$ of $\mathbb{W}$ where $\boldsymbol{D'}$ is in row reduced echelon form we must have $\boldsymbol{D} = \boldsymbol{D'}$. Hence one only has to deal with the following subproblem.

Find a unique solution vector with entries in $\mathbb{K}[t_1, \dots, t_e]$

- GIVEN: A polynomial ring $\mathbb{A} := \mathbb{K}[t_1, \dots, t_e]$ over a field $\mathbb{K}$ and a basis matrix $\boldsymbol{C}\wedge\boldsymbol{g}$ of a subspace $\mathbb{W} \neq \{\boldsymbol{0}\}$ of $\mathbb{K}^n \times \mathbb{A}$ over $\mathbb{K}$ where $\boldsymbol{C} \in \mathbb{K}^{d\times n}$ is in row reduced echelon form and $\boldsymbol{g} \in \mathbb{A}^d$. Furthermore an admissible ordering $<$ on $[t_1, \dots, t_e]$.
- FIND: A uniquely determined $\boldsymbol{h} \in \mathbb{A}^d$ by Gaussian elimination, such that $\boldsymbol{C}\wedge\boldsymbol{h}$ is a basis matrix of $\mathbb{W}$.

Let $\boldsymbol{C}$ be in row reduced echelon form, more precisely assume that

$$\boldsymbol{C}\wedge\boldsymbol{g} := \begin{pmatrix} c_{11} & \dots & c_{n1} & g_1 \\ \dots & \dots & \dots & \dots \\ c_{1l} & \dots & c_{nl} & g_l \\ 0 & \dots & 0 & g_{l+1} \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & g_d \end{pmatrix} \tag{2}$$

where $(c_{1l}, \dots, c_{nl}) \neq \boldsymbol{0}$. Note that all rows except the last one are zero rows, if $l = 0$. Moreover note that $d \geq 1$, if $d = l$.

Example 3.5. In Example 3.4 the parameter matrix $\boldsymbol{C}$ of $\boldsymbol{M} = \boldsymbol{C}\wedge\boldsymbol{g}$ is already in row reduced echelon form.

If $\boldsymbol{C}\wedge\boldsymbol{g}$ stands for the basis matrix of a solution space $\mathbb{W}$ in the context of parameterized linear difference equations, $(g_1, \dots, g_l)$ stands for solutions of linear difference equations where the inhomogeneous part varies, whereas $(g_{l+1}, \dots, g_d)$ gives a basis of the solutions of the homogeneous version. In the sequel this two parts will be considered separately, first the homogeneous part $(g_{l+1}, \dots, g_d)$ and finally the inhomogeneous part $(g_1, \dots, g_l)$.

3.4. The Homogeneous Part of the Solutions

First we consider the special case $d = l$ in (2), i.e. $d \geq 1$.

Lemma 3.3. *Let $\mathbb{A} := \mathbb{K}[t_1, \dots, t_e]$ be a polynomial ring over $\mathbb{K}$ and $\boldsymbol{C} \wedge \boldsymbol{g}$ be a basis matrix of a subspace $\mathbb{W} \neq \{\mathbf{0}\}$ of $\mathbb{K}^n \times \mathbb{A}$ over $\mathbb{K}$ where $\boldsymbol{C} \in \mathbb{K}^{d \times n}$ is in row reduced echelon form, $\boldsymbol{g} \in \mathbb{A}^d$ and we have (2) with $(c_{1l}, \dots, c_{nl}) \neq \mathbf{0}$. If $d = l$, $\boldsymbol{C} \wedge \boldsymbol{g}$ is uniquely determined.*

Proof: Assume there are two such basis matrices $\boldsymbol{C_1} \wedge \boldsymbol{g_1} \neq \boldsymbol{C_2} \wedge \boldsymbol{g_2}$ of $\mathbb{W}$. We have $\boldsymbol{C_1} = \boldsymbol{C_2}$, since they are in row reduced echelon form. Hence there are two rows $\boldsymbol{c} \wedge u \in \mathbb{W}$ and $\boldsymbol{c} \wedge v \in \mathbb{W}$ with $u \neq v$ and therefore $\boldsymbol{0_n} \wedge (u - v) \in \mathbb{W}$. Since $\boldsymbol{C_1}$ is in row reduced echelon form and $d = l$, it follows $\mathbb{W} \cap (\{\boldsymbol{0_n}\} \times \mathbb{A}) = \{\boldsymbol{0_{n+1}}\}$, a contradiction. □

Hence for the case $d = l$, Theorem 3.1 holds. From now on we are only concerned in the case $\lambda := d - l > 0$ where we write

$$P := \{p_1, \dots, p_\lambda\} = \{g_{l+1} \dots, g_d\} \subseteq \mathbb{A}^*. \tag{3}$$

SUBPROBLEM (I) Find unique representatives of the homogeneous solutions

- GIVEN: A polynomial ring $\mathbb{A} := \mathbb{K}[t_1, \dots, t_e]$ over $\mathbb{K}$ and $(p_1, \dots, p_\lambda) \in \mathbb{A}^\lambda$ whose entries form a basis of a subspace $\mathbb{B}$ of $\mathbb{A}$ over $\mathbb{K}$. Furthermore an admissible ordering $<$ on $[t_1, \dots, t_e]$.
- FIND: A uniquely determined $\boldsymbol{q} \in \mathbb{A}^\lambda$ by Gaussian elimination such that its entries form a basis of the subspace $\mathbb{B}$ of $\mathbb{A}$ over $\mathbb{K}$.

In the sequel we define the set

$$X := \{x \in [t_1, \dots, t_e] \,|\, x \text{ is a power product that occurs in one of the } p_i\}. \tag{4}$$

By the admissible ordering $<$ on $[t_1, \dots, t_e]$ we obtain a unique ordering on the power products in X, say

$$X = \{x_1 > x_2 > \dots > x_r\}.$$

Moreover we can write

$$p_j = \sum_{i=1}^{r} k_{ji}\, x_i$$

with $k_{ij} \in \mathbb{K}$. Then we can set up the matrix

$$\boldsymbol{M} := \begin{pmatrix} k_{11} & \dots & k_{1r} \\ \dots & \dots & \dots \\ k_{\lambda 1} & \dots & k_{\lambda r} \end{pmatrix}$$

with $\boldsymbol{p} = \boldsymbol{M} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}$. Next transform this matrix by row operations to a matrix $\boldsymbol{M'}$ that is in row reduced echelon form. Clearly the entries in

$$\boldsymbol{q} = (q_1, \dots, q_\lambda) := \boldsymbol{M'} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \tag{5}$$

form a basis of $\mathbb{B}$. In particular by Lemma 3.1 it follows that this $\boldsymbol{q}$ is uniquely defined. Hence we can write

$$\boldsymbol{C}\wedge\boldsymbol{g} := \begin{pmatrix} c_{11} & \dots & c_{n1} & g_1 \\ \multicolumn{4}{c}{\dotfill} \\ c_{1l} & \dots & c_{nl} & g_l \\ 0 & \dots & 0 & q_1 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & \dots & 0 & q_\lambda \end{pmatrix} \tag{6}$$

where $\boldsymbol{C}$ and $\boldsymbol{q}$ are uniquely defined. Hence if $l = 0$, Theorem 3.1 holds.

Example 3.6. Fix the lexicographic ordering $<$ on $[t_1, \dots, t_4]$ with $1 < t_1 < t_2 < t_3 < t_4$. Then the polynomials $g_1, g_2, g_3 \in \mathbb{Q}[t_1, \dots, t_e]$ from Example 3.4 consist only of the power products

$$t_1^3\,t_3 > t_1^2\,t_3 > t_1\,t_3 > t_3 > t_1^3\,t_2^2 > t_2^2 > t_1\,t_2^2 > t_2^2 > t_1^3\,t_2 > t_1^2\,t_2 > t_1\,t_2 > t_2 > t_1^3 > t_1^2 > t_1 > 1.$$

Hence we can write $(g_1, g_2, g_3) = \boldsymbol{M} \cdot \boldsymbol{x}$ where

$$\boldsymbol{M} := \begin{pmatrix} 1 & 3 & 3 & 1 & 1 & 3 & 3 & 1 & 0 & 2 & 4 & 2 & 0 & 0 & 2 \\ 1 & 3 & 3 & 1 & 1 & 3 & 3 & 1 & 0 & 2 & 4 & 2 & -2 & -6 & -4 \\ 1 & 3 & 3 & 1 & 1 & 3 & 3 & 1 & -3 & -7 & -5 & -1 & 1 & 0 & -1 \end{pmatrix}$$

and $\boldsymbol{x} := \left(t_1^3\,t_3, t_1^2\,t_3, t_1\,t_3, t_3, t_1^3\,t_2^2, t_2^2, t_1\,t_2^2, t_2^2, t_1^3\,t_2, t_1^2\,t_2, t_1\,t_2, t_2, t_1^3, t_1^2, t_1, 1 \right)$. Then by transforming $\boldsymbol{M}$ into row reduced echelon form we obtain

$$\boldsymbol{M'} = \begin{pmatrix} 1 & 3 & 3 & 1 & 1 & 3 & 3 & 1 & 0 & 2 & 4 & 2 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \end{pmatrix}$$

which yields to $(q_1, q_2, q_3) := \boldsymbol{M'} \cdot \boldsymbol{x}$ as given in $\boldsymbol{B}$ of Example 3.4.

3.5. The Inhomogeneous Part of the Solutions

Finally we consider the case $l \geq 1$ for (6) where we have to find a unique vector $(g_1, \dots, g_l)$. Then we determine a unique basis matrix $\mathbb{W}$ which proves Theorem 3.1. The following lemma[†] states how the possible $(g_1, \dots, g_l)$ look like.

Lemma 3.4. *Let $\mathbb{A} := \mathbb{K}[t_1, \dots, t_e]$ be a polynomial ring over $\mathbb{K}$, $\boldsymbol{C}$ be in row reduced echelon form and assume that*

$$\boldsymbol{M} = \boldsymbol{C}\wedge\boldsymbol{g} = \begin{pmatrix} c_{11} & \dots & c_{n1} & g_1 \\ \multicolumn{4}{c}{\dotfill} \\ c_{1d} & \dots & c_{nd} & g_l \\ 0 & \dots & 0 & q_1 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & \dots & 0 & q_\lambda \end{pmatrix} \quad \text{and} \quad \boldsymbol{M'} = \boldsymbol{C}\wedge\boldsymbol{g'} = \begin{pmatrix} c_{11} & \dots & c_{n1} & g'_1 \\ \multicolumn{4}{c}{\dotfill} \\ c_{1d} & \dots & c_{nd} & g'_l \\ 0 & \dots & 0 & q_1 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & \dots & 0 & q_\lambda \end{pmatrix}$$

are basis matrices of the subspace $\mathbb{W} \neq \{\boldsymbol{0}\}$ of $\mathbb{K}^n \times \mathbb{A}$ over $\mathbb{K}$ with $(c_{1d}, \dots, c_{nd}) \neq \boldsymbol{0}$ and $l \geq 1$. Then for any $1 \leq i \leq l$ we have $g'_i := g_i + \sum_{j=1}^{\lambda} \kappa_j\, q_j$ with $\kappa_j \in \mathbb{K}$.

Proof: Consider the rows $\boldsymbol{c}\wedge g_i$ and $\boldsymbol{c}\wedge g'_i$ with $\boldsymbol{c} = (c_{i1}, \dots, c_{in})$ for some $1 \leq i \leq l$. Since $\boldsymbol{c}\wedge g_i, \boldsymbol{c}\wedge g'_i \in \mathbb{W}$, we have $\boldsymbol{0}\wedge(g_i - g'_i) \in \mathbb{W}$. As $\boldsymbol{C}$ is in row reduced echelon form, it follows that $\{\boldsymbol{0}\wedge q_1, \dots, \boldsymbol{0}\wedge g_\lambda\}$ is a basis of $\mathbb{W} \cap (\{\boldsymbol{0_{n+1}}\} \times \mathbb{A})$. Hence $\boldsymbol{0}\wedge(g_i - g'_i) = \sum_{i=1}^{\lambda} k_i\, (\boldsymbol{0}\wedge q_i)$ for some $k_i \in \mathbb{K}$ which proves the lemma. □

[†]Note that Lemma 3.3 is a consequence of Lemma 3.4.

For $q_i\,\mathbb{K} := \{q_i\,k \mid k \in \mathbb{K}\}$ with $x \in [t_1, \dots, t_e]$ consider the direct sum

$$\mathbb{B} := \bigoplus_{i=1}^{\lambda} q_i\,\mathbb{K}$$

as a subspace of $\mathbb{A}$ over $\mathbb{K}$. Moreover we define $g + \mathbb{B} := \{g + b \mid b \in \mathbb{B}\}$ for any $g \in \mathbb{A}$. Then the previous Lemma 3.4 states that we have to deal with the following problem in order to find a unique vector $(g_1, \dots, g_l)$ in (6).

SUBPROBLEM (II) Find unique representatives for the inhomogeneous solutions

- GIVEN: A polynomial ring $\mathbb{A} := \mathbb{K}[t_1, \dots, t_e]$ over $\mathbb{K}$, $\boldsymbol{q} = (q_1, \dots, q_\lambda) \in \mathbb{A}^\lambda$ with (5) where $\boldsymbol{M'}$ is in row reduced echelon form and $g \in \mathbb{A}$.
- FIND: A uniquely determined $\bar{g} \in g + \mathbb{B}$.

Let

$$Q := \{q_1, \dots, q_\lambda\} \tag{7}$$

with $\lambda \geq 1$, $q \in Q$ and $f, f' \in \mathbb{A}$. We say that f is reduced by q to f', $f \xrightarrow{q} f'$, if the leading power product in q appears as $c\,\mathrm{lpp}(q)$ in f for some $c \in \mathbb{K}^*$ and we have

$$f' = f - c\,q. \tag{8}$$

In other words, if we have $f \xrightarrow{q} f'$, the term $c\,\mathrm{lpp}(g)$ is eliminated in f which leads to f'. Moreover we say that f is reduced by Q to f', $f \xrightarrow{Q} f'$, if there exists a $q \in Q$ with $f \xrightarrow{q} f'$.

Proposition 3.1. *Let $\mathbb{A} := \mathbb{K}[t_1, \dots, t_e]$ be a polynomial ring over $\mathbb{K}$ and consider $Q := \{q_1, \dots, q_\lambda\} \subseteq \mathbb{B} \subseteq \mathbb{A}$ with (5) where $\boldsymbol{M'}$ is in row reduced echelon form and $g \in \mathbb{A}$. Then after at most λ reductions of g by Q one obtains a uniquely defined $\bar{g} \in g + \mathbb{B}$ that cannot be reduced further, i.e. for some $\kappa_i \in \mathbb{K}$ we have*

$$\bar{g} = g + \sum_{i=1}^{\lambda} \kappa_i\,q_i. \tag{9}$$

Proof: Consider a chain of reductions

$$g = g_1 \xrightarrow{Q} g_2 \xrightarrow{Q} g_3 \cdots g_{j-1} \xrightarrow{Q} g_\nu$$

with $2 \leq \nu \leq \lambda$. Now choose one of these reductions, say $g_{l-1} \xrightarrow{Q} g_l$, where a $c\,x$ with $c \in \mathbb{K}^*$ and $x \in [t_1, \dots, t_e]$ is eliminated. Then by construction of Q, where (5) holds and $\boldsymbol{M}$ is in row reduced echelon form, there cannot occur such a term $\kappa\,x$ for some $\kappa \in \mathbb{K}^*$ in any of the g_i with $l \leq i \leq j$. But this means that after at most λ reduction steps, we cannot reduce further which leads to an $\bar{g}$ with (9). Moreover for any g_i for $1 \leq i < l$ exactly the power product x occurs in form of $c\,x$ where c is fixed. Hence any chain of reductions is -up to a reordering of the reduction steps- exactly the same. In particular the resulting polynomial, that cannot be reduced further, must be always $\bar{g}$. □

Example 3.7. Let us going back to Example 3.6. Here we found already a unique vector $\boldsymbol{q} = (q_1, q_2, q_3)$, i.e. a unique basis of the homogeneous version of the linear difference equation (1). Looking at the leading power products we have $\mathrm{lpp}(q_1) = t_1^3 t_3$, $\mathrm{lpp}(q_2) = t_1^3 t_2$ and $\mathrm{lpp}(h_3) = t_1^3$. Then one can see immediately that $g := g_1$ in Example 3.6 is free of any term with power products $\mathrm{lpp}(h_2)$ or $\mathrm{lpp}(h_3)$, but the term $-\frac{1}{2} t_1^3 t_3$ occurs. Hence we can apply the reduction $g \xrightarrow{h_1} g'$ with $g' := g + \frac{1}{2} h_1$. Clearly the resulting g' is free of any term with power products $\mathrm{lpp}(h_1)$, $\mathrm{lpp}(h_2)$ and $\mathrm{lpp}(h_3)$, and consequently g' is not further reducible. Hence we found $\bar{g} := g'$ as our uniquely determined part that is needed to define the desired basis matrix $\boldsymbol{B}$ of $\mathbb{W}$ in Example 3.4.

Remark 3.1. In the end I indicate that behind all these constructions one can find the more general concept of Gröbner basis. I will use all the notations and definitions as they are given in [Win96, Chapter 8].

First let us consider the simpler case that $1 \notin X$ in (4). Moreover forget all algebraic relations in the power products of X, i.e. interpret all the elements in

$$X = \{x_1, \ldots, x_r\} \tag{10}$$

as new variables. Then each polynomial $p_j \in P$ from (3) can be written in the form

$$p_j = \sum_{i=1}^{r} c_i\, x_i \in \mathbb{K}[x_1, \ldots, x_r] \tag{11}$$

with $c_i \in \mathbb{K}$. Then it follows for instance by Buchberger's Theorem [Win96, Theorem 8.3.1] and the product criterion [Win96, Theorem 8.5.1] that for any admissible ordering on $[x_1, \ldots, x_r]$ the set P forms a Gröbner basis in $\mathbb{K}[x_1, \ldots, x_r]$. Moreover the set Q from (7) forms a normed reduced minimal Gröbner basis according to an admissible ordering $<$ with $x_1 > \cdots > x_r$. This follows immediately by (5) and the fact that $\boldsymbol{M'}$ is in row reduced echelon form. In particular by [Win96, Theorem 8.3.6] Q is uniquely defined for the admissible ordering $<$. This is exactly what we needed in order to choose a uniquely defined vector $\boldsymbol{q} = (q_1, \ldots, q_\lambda)$ for a given P.

Note that there might be power products that occur in $(g_1, \ldots, g_l)$ but not in X. In this case interpret that power products as new variables, say $y_1, \ldots, y_s$, and define an admissible ordering on $[x_1, \ldots, x_r, y_1, \ldots, y_s]$ with

$$x_1 > \cdots > x_r > y_1 > \cdots > y_s.$$

By this construction we may write

$$g_i = \sum_{i=1}^{r} c_i\, x_i + \sum_{i=1}^{s} k_i\, y_i \in \mathbb{K}[x_1, \ldots, x_r, y_1, \ldots, y_s]$$

for $c_i, k_i \in \mathbb{K}$. By definition of the admissible ordering $<$, Q forms a normed reduced minimal Gröbner basis in $\mathbb{K}[x_1, \ldots, x_r, y_1, \ldots, y_r]$ w.r.t. $<$. Moreover the reduction defined in (8) is included in the more general concept of polynomial reduction given in [Win96, Definition 8.2.4]. In our situation only this specialized reduction is needed due to the simple Gröbner basis structure of Q. Here the elements $g_i \in \mathbb{K}[x_1, \dots, x_r, y_1, \dots, y_s]$ are reduced to $\bar{g}_i$ modulo the ideal generated by Q until they cannot be reduced further. Again by Gröbner basis theory, [Win96, Theorem 8.3.4], these $\bar{g}_i$ are uniquely defined which is exactly the required property in order to obtain a uniquely defined $(\bar{g}_1, \dots, \bar{g}_l)$.

Finally we have to consider the case $1 \in X$, in particular we assume that $x_r = 1$ for (10). The problem is that 1 might be in the ideal generated by the set P. But then $\{1\}$ is the normed reduced minimal Gröbner basis of P, and not Q. In order to avoid this situation, one can introduce, besides $x_1, \dots, x_{r-1}$ an additional indeterminate z and writes

$$p'_j := k_r\, z + \sum_{i=1}^{r-1} k_i x_i \in \mathbb{K}[x_1, \dots, x_{r-1}, y_1, \dots, y_s, z]$$

for p_i given in (11). Then it follows again that

$$P' := \{p'_1, \dots, p'_\lambda\}$$

forms a Gröbner basis for any admissible ordering. Moreover the set $\{q'_1, \dots, q'_\lambda\}$ where the q'_j are defined by

$$q'_j := \boldsymbol{M'} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \\ z \end{pmatrix}$$

forms a normed reduced minimal Gröbner basis in $\mathbb{K}[x_1, \dots, x_{r-1}, y_1, \dots, y_s, z]$, which is uniquely defined, w.r.t. an admissible ordering $<$ with

$$x_1 > \dots > x_{r-1} > \dots > y_1 > \dots > y_s > z.$$

Similarly one has to introduce $g'_i \in \mathbb{K}[x_1, \dots, x_{r-1}, y_1, \dots, y_s, z]$ by replacing the constant term $c \in \mathbb{K}$ in g_i by $c\,z$. Then one can reduce g_i by the Gröbner basis Q' in $\mathbb{K}[x_1, \dots, x_{r-1}, y_1, \dots, y_s, z]$ modulo the ideal generated by Q' to $\bar{g}'_i$ that cannot be reduced further. Then again this $\bar{g}'_i$ is uniquely defined. Finally by substituting $z \to 1$ one just obtains $(q_1, \dots, q_\lambda)$ and $(\bar{g}_1, \dots, \bar{g}_l)$ as it is needed to obtain a uniquely determined basis matrix of a given vector space $\mathbb{W}$.

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