Graphical Generating of Minimal Surfaces Subject to the Plateau Problems *

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Abstract

This paper talks about computing the minimal surfaces subject to the well-known Plateau problem. The differential form of the Plateau problem is defined first and, the associated discrete schemes are generated by the finite element methods. The convergence properties of the discrete solutions are proved by steps and some 2-grid discretization algorithms have been implemented to speed up the computation. These approximation methods have been implemented for displaying such typical minimal surfaces on the Maple softwares.

Keywords: Plateau problem, variational form, convexity, Brouwer's fixed point theorem, maximum value principle, 2-grid discretization algorithm.

1 Introduction

The study of minimal surfaces is a branch of differential geometry, because the methods of differential calculus are applied to geometrical problems. One of the oldest questions is: "What is the surface of smallest area spanned by a given contour?" Such question seems nontrivial despite the fact that every physical soap film appears to know the answer. But to prove it in a theoretically way was very hard in the old years. People used to take the existence of the minimal surface within such a specified

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given contour, as the well-known Plateau problems (named after the Belgian physicist Joseph Plateau). However, we are known that it has already been proved (by Radó and Douglas) to be true in a general functional form during the 1930's [4].

From the point of view of local geometry, a minimal surface can also be defined as the surface which has the zero average mean curvature on each surface point, e.g., a saddle shape.

Previously unknown and certainly unexpected minimal surfaces were found by David Hoffman and his collaborators at GANG, the Center for Geometry, Analysis, Numerics, and Graphics at the University of Massachusetts in 1985 [7]. They first used their MESH computer graphics system to find these surfaces, and then later proved their existence with fully rigorous mathematics. This truly excited the minimal surface community and piqued their interest in computer graphics.

Solving such minimal surface problems in optimal geometry also requires specialized software systems because there is often no explicit parameterization of a desired minimal surface. Although in the past years, lots of softwares have been produced for computing and displaying minimal surfaces (e.g, the "Surface Evolver" [1] by applying the evolution techniques, which worked out by the geometric center of the University of Minnesota), the convergence theory of those special approaches are always left as open problems. However, there exist one finite element method for solving the Plateau problem based on the linear triangular domain partitions[8] with a complete convergence theory, but solving the discrete equations under partially defined boundary conditions are sometimes not cheap. In this paper, we discuss the finite element solutions to the discrete Plateau problem based on another representation form, which provide more possibilities and owns more general properties for getting the discrete solutions. One convenient property is for applying the symbolic computation to obtain the non-error discrete solutions, and the other is the new proof method fits more general type finite element spaces regarding to the mesh and interpolation type. The convergence properties and some speedup techniques (as the byproducts) are be proved and some typical experimental results have been demonstrated to show the error coincides the theoretical analysis.

This paper is also a revised version of the contributed paper [5] to the conference CST2002 in Prague, Czech Republic.

In the next section we will first define the differential equations to the Plateau problems, and extend the variational forms from the continuous Sobolev space to the finite element space. Some error resolutions for solving the linear equations by the finite element method will be previewed in Section 3, and we will focus on the proving of the convergence theory of the discrete solution to the Plateau problem mainly by Brouwer's fixed point theorem in Section 4. In section 5 we introduce two kinds of 2-grid discretization algorithms in order to speed up the normal computation and in Section 6 we show some numerical experiments of displaying the minimal surface on Maple software for testing these new approximating methods.

2 Preliminaries

Due to the fact that the minimal surfaces has the sufficient and necessary property of having zero average mean curvature at each surface point, we can define the minimal surfaces subject to the Plateau problem be the solution of a system of nonlinear partial differential equation, with restricted boundary conditions.

Let us also assume that such minimal surface can be represented explicitly by term $z = u(x, y)$ in a 3-dimensional space with fixed domain Ω and the boundary restriction

$$u = f(x, y), \ \ (x, y) \in \partial\Omega,$$

then $u(x, y)$ must solve the following system of partial differential equations:

$$\begin{cases} -div((1 + u_y^2)u_x, (1 + u_x^2)u_y) + 6u_{xy}u_xu_y = 0 \ \text{ in } \ \Omega, \\ \qquad u = f(x, y), \ \ \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is strictly convex. Here the strictly convexity can be sufficient to assume that the solution to the previous equation is unique if $f(x, y)$, which satisfies the bounded slope condition [10], is the restriction to $\partial\Omega$ of a function in the Sobolev space $W^{2,q}(\Omega)$, for some $q > 2$.

Set the nonlinear bi-argument operator

$$A(u, v) = \int_\Omega [(1 + u_y^2)u_xv_x + (1 + u_x^2)u_yv_y + 6u_{xy}u_xu_yv]dxdy,$$

where u_x, u_{xy}... denote $\partial_x u, \partial_x(\partial_y u)$..., etc. Then the equation (2.1) is weakly equivalent to the following variational form:

Looking for $w \in H_0^1(\Omega) \cap W^{2,q}(\Omega)$, such that

$$A(w + \bar{f}, v) = 0, \quad \forall v \in H_0^1(\Omega) \cap W^{2,q}(\Omega), \tag{2.2}$$

where $\bar{f}$ is the "regular" extension of f from $\partial\Omega$ to Ω, especially, we could take $\bar{f} \in W^{2,q}(\Omega), q > 2$. (Obviously, $u = w + \bar{f}$ is the exact solution according to the previous uniqueness assumption.)

We now consider the discrete scheme for (2.2). Let $\Omega_h \subset \Omega$ be divided by mesh T_h with size h. $S_h(\Omega_h)$ be the finite element space constructed by the interpolation operator I_h [2, 3]. It is also assumed that the mesh partition is regular and all the grid nodes on $\partial\Omega_h$ also located on $\partial\Omega$.

Then the discrete solution $w_h \in S_h(\Omega_h)$ is defined by

$$A(w_h + \bar{f}, v) = 0, \qquad \forall v \in S_h(\Omega_h), \tag{2.3}$$

and $u_h = w_h + \bar{f}$ will be taken as the finite element solution to the Plateau problems.

The above discrete scheme appears to be a system of algebraic polynomial equations so that it can be either approximated by the Newton's method, or solved directly by the symbolic elimination methods [14] . Especially, the symbolic method is very promising for those Plateau problem with the parameter depending or incompletely defined boundary conditions since the complexity sometimes does not depend on the number of parameters [6].

For the continuous models, since w is assumed to be unique, one sees that $w^h = w|_{\Omega_h} \in H_0^1(\Omega_h) \cap W^{2,q}(\Omega_h)$ which satisfies

$$A(w^h + \bar{f}, v) = 0, \qquad \forall v \in H_0^1(\Omega_h) \cap W^{2,q}(\Omega_h)$$

also must be unique. Then we could in fact consider the original problem based on a certain convex polygon domain. For convenience, we still take w^h as w, Ω_h as Ω, etc., and will prove that w_h convergence to the real solution w when the mesh size h goes to zero in the following sections.

3 Some Error Resolutions for Solving Linear Finite Element Equations

Let $L(\cdot)$ be a linear operator defined by

$$L(u) = -div(a_{11}u_x + a_{21}u_y, a_{12}u_x + a_{22}u_y) + b_1u_x + b_2u_y + cu,$$

where $a_{ij}, b_i(i,j=1,2)$, and $c \geq 0$ are continuous functions satisfying

$$a_{11}v_x^2 + a_{22}v_y^2 + (a_{12} + a_{21})v_xv_y \geq c_0(\nabla v)^2 \quad \forall v \in S_h(\Omega)$$

for some $c_0 > 0$.

According to the maximum value principle [11], it can be proved that the following equation

$$A'(P_hu, v)(= (L(P_hu), v)) = (f, v) \quad \forall v \in S_h(\Omega) \tag{3.1}$$

where $(\cdot,\cdot)$ denote the inner product of Hilbert space $L_2(\Omega)$, will has a unique solution for the given condition f [12].

Namely, $P_h : H_0^1(\Omega) \to S_h(\Omega)$ is the Galerkin's projection operator which satisfies

$$A'(u - P_hu, v) = 0 \quad \forall v \in S_h(\Omega).$$

And we have the error estimation result [2, 3].

Theorem 3.1 *For $h << 1$, P_hu admits the following estimate*

$$\|u - P_hu\|_{1,\infty} \leq Ch^k\|u\|_{k+1,\infty}, \tag{3.2}$$

for $C > 0$ and $k \geq 1$.

Moreover, we introduce the discrete Green functions as, given $z \in \Omega$, the Green functions $g^z_{h,x}, g^z_{h,y} \in S_h(\Omega)$ satisfying

$$A'(v, g^z_{h,x}) = v_x(z), \quad A'(v, g^z_{h,y}) = v_y(z) \quad \forall v \in S_h(\Omega). \tag{3.3}$$

It has been proved that [13]

$$\|g^z_{h,x}\|_{1,1} \leq C|\text{log}h|, \tag{3.4}$$

$$\|g^z_{h,y}\|_{1,1} \leq C|\text{log}h|, \tag{3.5}$$

for constant $C > 0$.

4 Approximating the Plateau Problems by the Finite Element Methods

Let us first consider the discrete problem (2.3) defined on a finite element space $S^1_h(\Omega)$ generated base on the the bilinear rectangular mesh partitions. The following lemma shows the sufficient and necessary conditions of being the finite element solution to the discrete equation (2.3).

Lemma 4.1 *For any w, w_h solves (2.2), (2,3) respectively, and for any $v \in S^1_h(\Omega)$, there holds*

$$A(w_h + \bar{f}, v) = A(w + \bar{f}, v) + A'(w + \bar{f}; w_h - w, v) + R(w + \bar{f}, w_h + \bar{f}, v) \tag{4.1}$$

where $A'(w + \bar{f}; \cdot, \cdot)$ is the bilinear operator defined from (3.1) with the coefficients $a_{ij}, b_i, c,\ i, j = 1, 2$, given by

$$a_{11} = 1 + u_y^2,\ a_{22} = 1 + u_x^2,\ \ a_{12} = a_{21} = -u_y u_x,$$

$$b_1 = 3u_y u_{xy} - 3u_{yy} u_x,\ \ b_2 = 3u_x u_{xy} - 3u_{xx} u_y,$$

and $c = 0$. Then $w_h \in S^1_h(\Omega)$ solves (2.3) if and only if

$$A'(w + \bar{f}; w - w_h, v) = R(w + \bar{f}, w_h + \bar{f}, v),\ \ \forall v \in S^1_h(\Omega). \tag{4.2}$$

Further more, if $\|\partial_{xy} w_h\|_{0,\infty} + \|w_h\|_{1,\infty} \leq K$, then the remainder R satisfying

$$R(w + \bar{f}, w_h + \bar{f}, v) \leq C(K)\|w - w_h\|^2_{1,\infty}\|v\|_{1,1}. \tag{4.3}$$

Proof: Set $G(t) = A(w + \bar{f} + t(w_h - w), v)$, then it follows from the identity

$$G(1) = G(0) + G_t(0) + \int_0^1 G_{tt}(t)(1 - t)dt.$$

It is easy to compute that

$$G_t(0) = A'(w + \bar{f}; w_h - w, v).$$

And, by taking

$$R(w + \bar{f}, w_h + \bar{f}, v) = \int_0^1 G_{tt}(t)(1-t)dt,$$

using the condition $\|w_h\|_{1,\infty} + \|\partial_{xy} w_h\|_{0,\infty} \leq K$ and a straightforward calculation, we get that

$$\begin{aligned} & |R(w + \bar{f}, w_h + \bar{f}, v)| \\ \leq\ & \max|G_{tt}(t)| \\ \leq\ & C(K)\|w - w_h\|^2_{1,\infty}\|v\|_{1,1}. \end{aligned}$$

Finally, if w_h solves (2.3), then $G(1) = 0$, thus complete the proof.

Let P_h be the Galerkin's projection with respect to the bilinear operator $A'(w + \bar{f}; \cdot, \cdot)$, we have the following theorem then:

Theorem 4.1 *For problems of the form (2.2), there exist a constant $C > 0$, such that for rectangular mesh T_h with its size $h << 1$, the corresponding finite element equation (2.3) defined on the finite element space $S^1_h(\Omega)$ contains a solution w_h satisfying*

$$\|w_h - P_h w\|_{1,\infty} \leq Ch^2|logh|,$$

and thus we have the following error estimation

$$\|w_h - w\|_{1,\infty} \leq Ch$$

by using Theorem 3.1.

Proof: Define a nonlinear operator $\Phi : S^1_h(\Omega) \to S^1_h(\Omega)$ by

$$A'(w + \bar{f}, \Phi(v) - w, \phi) = R(w + \bar{f}, v + \bar{f}, \phi) \quad \forall \phi \in S^1_h(\Omega).$$

It can be proved by the maximum value principle and the fact that $S^1_h(\Omega)$ is of finite dimensional, Φ is well defined and a continuous operator. According to the result of Theorem 3.1 we can obtain the error estimates that

$$\|w - P_h w\|_{1,\infty} \leq C^* h, \;\; C^* > 0$$

Then define a set $B = \{v \in S_h(\Omega) : \|v - P_h w\|_{1,\infty} \leq C^* h\}$. Since $S^1_h(\Omega)$ is of linear, by the inverse estimation, there exist $C_1, C_2, C_3, C_4 > 0$, such that for all $v \in S^1_h(\Omega)$,

$$\begin{aligned} & \|\partial_{xy}(v - w)\|_{0,\infty} \\ \leq\ & \|\partial_{xy}(v - I_h w)\|_{0,\infty} + \|\partial_{xy}(I_h w - w)\|_{0,\infty} \\ \leq\ & C_1 h^{-1}(\|v - P_h w\|_{1,\infty} + \|I_h w - P_h w\|_{1,\infty}) + C_2 \\ \leq\ & C_3, \end{aligned}$$

Similarly we obtain

$$\|(v-w)\|_{1,\infty} \le C_4.$$

Thus we get $\|\partial_{xy} v\|_{0,\infty} + \|v\|_{1,\infty}$ is uniformly bounded.

Now we can prove that $\Phi(B) \subset B$ which means Φ is a contraction operator. In fact, when we substitute ϕ by the discrete Green functions $\phi = g^z_{h,x}$ and $\phi = g^z_{h,y}$ into

$$A'(w+\bar{f}; \Phi(v) - P_h w, \phi) = R(w+\bar{f}, v+\bar{f}, \phi),$$

according to Lemma 4.1, (3.4) and (3.5), for all $v \in B$, we get

$$\begin{aligned}
&\|\Phi(v) - P_h w\|_{1,\infty} \\
\le\ & C_0|\mathrm{log}h|\|w-v\|^2_{1,\infty} \\
\le\ & 2C_0|\mathrm{log}h|(\|P_h w - v\|^2_{1,\infty} + \|w - P_h w\|^2_{1,\infty}) \\
\le\ & 2C_0|\mathrm{log}h|(C^{*2}h^2 + C^{*2}h^2) \\
\le\ & Ch^2|\mathrm{log}h|(\le Ch^*),
\end{aligned}$$

where $C_0 > 0$ and $h << 1$.

By Brouwer's fixed point theorem, there exist a solution $w_h \in B$, such that

$$\Phi(w_h) = w_h.$$

And according to Lemma 4.1, w_h solves (2.3) and satisfies

$$\|w_h - P_h w\|_{1,\infty} \le Ch^2|\mathrm{log}h|.$$

Thus we complete the proof.

Based on the lower order finite element space like $S^1_h(\Omega)$, to generate the discrete system (2.3) is convenient but the uniqueness property of the solution u_h is at present not able to be proved. However we could prove that problem (2.3) will contains a unique solution if the finite element space belongs to a H^2 Sobolev space and we can also similarly prove

Theorem 4.2 *For problems of the form (2.2), there exist a constant $C > 0$, such that in the finite element subspace $S_h(\Omega)(\subset H^2(\Omega))$ with its mesh partition parameter $h << 1$, the finite element equation will has a unique solution u_h satisfying*

$$\|u_h\|_{2,\infty} \le C,$$

and

$$\|u_h - u\|_{m,\infty} \le Ch^{k-m+1}\|u\|_{k+1,\infty}$$

where $m = 0, 1$ and $k = 1, 2$.

Proof: For $m = 1$, the proof of the existence and convergence properties of u_h, those are basically the same as for proving Theorem 4.1, only need to change the holding condition for the inequality (4.3) into $\|w_h\|_{2,\infty} \leq K$ and later could prove $\|v\|_{2,\infty}$ is also uniformly bounded.

For uniqueness, we consider that u_h should still maintains the minimal area property over the domain Ω in the H^2 finite element space $S_h(\Omega)$ [6], e.g. u_h solves

$$\min_{v|_{\partial\Omega}=\bar{f}, v-\bar{f}\in S_h(\Omega)}\{A^*(v) = \int_\Omega (1+|\nabla v|^2)^{1/2}\}, \tag{4.4}$$

then u_h should be a unique solution since it is a strictly convex defined form [8].

In case that $m = 0$, we use the Aubin-Nitsche duality argument [3] to get the L^2 error estimates.

5 The 2-grid Discretization Approaches

In this section, we introduce the 2-grid discretization techniques [16] to speed up the computation, based on the previous convergence theory.

Our basic assumption is, the finite element space should be a subspace belongs to $W^{2,q}(\Omega)$. We denote H be the maximum mesh size to the coarse grid partition and h the maximum mesh size to the refined grid partition (see Figure 1). And we also set $0 < H^2 < h << H$.

The 2-grid discretization algorithm, roughly speaking, is to use the coarse grid approximation on finite element space $S_H(\Omega)$ as an initial result for the recollection on the refined space $S_h(\Omega)$ later.

The 2-grid discretization algorithm 1.

1. Solve $w_H \in S_H(\Omega)$ such that

$$A(w_H + \bar{f}, v) = 0, \;\; \forall v \in S_H(\Omega).$$

2. Solve $e_h \in S_h(\Omega)$ from the linear equation

$$A'(w_H + \bar{f}; e_h, v) = -A(w_H + \bar{f}, v), \;\; \forall v \in S_h(\Omega).$$

3. Set $u^h = w_H + \bar{f} + e_h$.

Theorem 5.1 *Assume that u^h is the solution obtained by the 2-grid discretization algorithm 1, If $h << 1$, then by definition of the coarse solution $u_H = w_H + \bar{f}$, we could get*

$$\|u_h - u^h\|_1 \leq C\|u_h - u_H\|^2_{1,\infty}. \tag{5.1}$$

Consequently,

$$\|u - u^h\|_1 \le C(h + H^2) \tag{5.2}$$

for the constant $C > 0$*.*

Proof: From the algorithm, (4.1) and (4.3), we have

$$\begin{aligned}
& A'(w_H + \bar{f}; u_h - u^h, v) \\
&= A'(w_H + \bar{f}; u_h - u_H, v) - A'(w_H + \bar{f}; e_h, v) \\
&= A'(u_H; u_h - u_H, v) + A(w_H + \bar{f}, v) \\
&= -R(u_H, u_h, v) \\
&\le C\|u_H - u_h\|_{1,\infty}^2 \|v\|_{1,1}.
\end{aligned}$$

Since when h is sufficiently small, the bilinear operator $A'(u_H; \cdot, \cdot)$ will be well-posed, hence we can use the Inf-Sup condition [16]

$$\begin{aligned}
\|u^h - u_h\|_1 & \le C\text{sup}_{v \in S_h(\Omega)} \frac{A'(u_H; u_h - u^h, v)}{\|v\|_1} \\
& \le C\text{sup}_{v \in S_h(\Omega)} \frac{C\|u_H - u_h\|_{1,\infty}^2 \|v\|_{1,1}}{\|v\|_1} \\
& \le C\|u_H - u_h\|_{1,\infty}^2
\end{aligned}$$

By applying Theorem 4.2, for two different grid partition sizes, we could obtain

$$\|u - u_h\|_{1,\infty} \le Ch,$$

$$\|u - u_H\|_{1,\infty} \le CH.$$

Thus we can prove (5.2) by the triangular inequalities.

In fact, the 2-grid discretization algorithm 1 can be simplified by eliminating the lower differential order term from the operator A'.

If we define

$$A'_s(u; \cdot, \cdot) = A'(u; \cdot, \cdot) - N(u; \cdot, \cdot)$$

where

$$N(u; \cdot, v) = \int_\Omega (b_1(u)\partial_x(\cdot)v + b_2(u)\partial_y(\cdot)v)$$

and the coefficients $b_1(u), b_2(u)$ follows the definitions in Lemma 4.1, then we could have

The 2-grid discretization algorithm 2.

1. Solve $w_H \in S_H(\Omega)$ such that

$$A(w_H + \bar{f}, v) = 0, \ \ \forall v \in S_H(\Omega).$$

2. Solve $e_h \in S_h(\Omega)$ from the linear equation

$$A'_s(w_H + \bar{f}; e_h, v) = -A(w_H + \bar{f}, v), \ \ \forall v \in S_h(\Omega).$$

3. Set $u^h_s = w_H + \bar{f} + e_h$.

Theorem 5.2 *Assume that u^h is the solution obtained by 2-grid discretization algorithm 2, and $u \in H^3(\Omega)$, then we could still get*

$$\|u_h - u^h_s\|_1 \leq C(H^2 + h). \tag{5.3}$$

Proof: By definitions and the facts from Lemma 4.1 we could obtain that

$$\begin{aligned} & A'_s(w_H + \bar{f}; u_h - u^h_s, v) \\ = & A'_s(w_H + \bar{f}; u_h - u_H, v) - A'_s(w_H + \bar{f}; e_h, v) \\ = & A'(u_H; u_h - u_H, v) + A(w_H + \bar{f}, v) - N(u_H; u_h - u_H, v) \\ = & -R(u_H, u_h, v) - N(u_H; u_h - u_H, v). \end{aligned}$$

Since $u \in H^3(\Omega)$,

$$\begin{aligned} & N(u; u_h - u_H, v) \\ = & \int_\Omega (b_1 \partial_x (u_h - u_H) v + b_2 \partial_y (u_h - u_H) v) \\ = & -\int_\Omega (b_1 (u_h - u_H) v_x + b_2 (u_h - u_H) v_y + 2(u^2_{xy} - u_{xx} u_{yy})(u_h - u_H) v), \end{aligned}$$

and $\|u_H\|_{2,\infty}$ is uniformly bounded according to Theorem 4.2, we have

$$\begin{aligned} & |N(u_H; u_h - u_H, v)| \\ \leq & \|b_1(u_H)\|_{0,\infty} \|u_h - u_H\|_{0,\infty} \|v\|_{1,1} + \|b_2(u_H)\|_{0,\infty} \|u_h - u_H\|_{0,\infty} \|v\|_{1,1} \\ & + 2\|u^2_{H,xy} - u_{H,xx} u_{H,yy}\|_{0,\infty} \|u_h - u_H\|_{0,\infty} \|v\|_{0,1} \\ \leq & 6\|u_H\|^2_{2,\infty} \|u_h - u_H\|_{0,\infty} \|v\|_{1,1} \\ \leq & C\|u_h - u_H\|_{0,\infty} \|v\|_{1,1}. \end{aligned}$$

By the fact that the bilinear operator $A'_s(u_H; \cdot, \cdot)$ is positive definite, we could still get that

$$\begin{aligned} \|u^h_s - u_h\|_1 & \leq C \sup_{v \in S_h(\Omega)} \frac{A'_s(u_H; u_h - u^h, v)}{\|v\|_1} \\ & \leq C \sup_{v \in S_h(\Omega)} \frac{\|u_H - u_h\|^2_{1,\infty} \|v\|_{1,1} + \|u_h - u^H\|_{0,\infty} \|v\|_{1,1}}{\|v\|_1} \\ & \leq C(\|u_H - u_h\|^2_{1,\infty} + \|u_h - u^H\|_{0,\infty}) \\ & \leq C(H^2 + h). \end{aligned}$$

Thus complete the proof.

However, to generate the discrete equations based on the $W^{2,q}$ finite element space is not convenient in practical. But we might notice that the 2-grid discretization algorithm 2 is also well-defined if the finite element space is generated based on the bilinear rectangular mesh interpolations (since we only use the first order derivate function of u_H in the discrete schemes). Although we could not prove whether using 2-grid discretization algorithm 2 still admits the error estimates of Theorem 5.2 under the linear approach (even if we know priorly that the discrete system has a unique solution), we could find it works resonablely on some typical cases from the experiments result (see Example 2 in the next section).

6 Examples of Generating Minimal Surfaces

To illustrate the features of the approximation methods proposed in this paper, we now show some model examples.

Example 1.

For $\Omega = [0,1]\times[0,1]$, we use the bilinear rectangular mesh interpolation to generate the finite element interpolation with mesh size $h = 1/16$. The border curves are the restriction of smooth function $\bar{f} = 0.25 - (x - 0.5)^2$. Then the finite element solution after sufficient iteration steps is displayed in Figure 2, a saddle shape.

Example 2.

This trivial example shows the convergence property for the approximated solutions. By given the initial border function the restriction of $xy(1-x)(1-y)$ on the same domain, we know the exact solution to this Plateau problem should be uniquely a plane $z = 0$.

Figure 3 shows the maximum error between the discrete solution and the exact minimal surface which we expected will not be greater than $2*10^{-4}$, if we use the same grid partition as the last example. And based on the relatively coarser grid partition by $h = 1/8$ or $h = 1/4$, we get the maximum errors will be less than $8*10^{-4}$ or $4*10^{-3}$, respectively, that coincides the error estimations of Theorem 4.1.

Figure 4 indicates the result solved by the 2-grid dsicretization algorithm 2, whose initial iteration is based on the coarse grid partition $h = 1/8$ on the finite element space $S_H^1(\Omega)$. We can obviously find the solution u_s^h has the same efficiency as that solved directly from the refined grid partition (Figure 3), but computing u_s^h is much more cheaper than computing u_h on the dense grid partition $h = 1/16$.

Those experimental data for the computations are obtained from the Solaris operating system computer galaxy.risc.uni-linz.ac.at with 1152M memory.

7 Conclusion and Remarks

In this paper, we discussed the finite element methods for solving the Plateau problems on convex domains, and proved the error analysis to all the approaches. Solving these discrete forms to the Plateau problems will be more flexible especially under the general boundary conditions. We have also implemented some 2-grid discretization algorithms which could speed up the computations. There also exists possibilities of applying the parallel algorithms based on the locally mesh refinement [6, 17] on the large scale domains. For purpose of the final generating of those minimal surfaces, we carried out all the numerical examples on the Maple software.

References

[1] K.Brakke, *"The surface evolver. Experiment"*, Math. 1 (1992), no. 2, 141–165.

[2] S.Brenner and R. Scott, *"The Mathematical Theory of Finite Element Methods"*, Springer-Verlag, 1994.

[3] P.Ciarlet, *"The Finite Element Method for Elliptic Problems"*, North-Holland, 1978.

[4] J.Gray, *"The Hilbert Challenge"*, Oxford University Press, 2000.

[5] H.Gu, *"The Finite Element Approximation to the Minimal Surfaces Subject to the Plateau Problems"*, Proceedings of The Sixth International Conference on Computational Structure Technology, (Civil-comp Press) Prague, Czech Rep., Sept. 4-6, 2002.

[6] H.Gu, *"The Software Generating of Minamal Surface Subject to the Plateau Problems"*, Ph.D. Thesis, RISC-Linz, 2002. (to appear)

[7] D.Hoffman, *"Natural minimal surfaces. Via theory and computation"*, Science Television, NewYork; American Mathematical Society, Providence, RI, 1990.

[8] C.Johnson and V.Thomeé, *"Error estimates for a finite element approximation of a minimal surface"*, Math. of Comp. Vol. 29, Num. 130, 1975 343-349.

[9] M.Marion and J.Xu, *"Error estimates on a new nonlinear Galerkin method based on two-grid finite elements"*, SIAM J. Numer. Anal. 32 (1995), no.4, 1170–1184.

[10] C.Morry, *"Multiple Intergrals in the Calculus of Variations, Die Grundlehren der math"*, Wissenschften, Band 130, Springer-Verlag, New York, 1966.

[11] M.Protter, H.Weinerger, *"Maximum Principles in Differential Equations"*, Corrected reprint of the 1967 original. Springer-Verlag, New York, 1984.

[12] A.Schatz, *"An observation concerning Ritz-Galerkin Methods with indefinite bilinear forms"*, Math. Comp. 28(1974), 959-962.

[13] V.Thomeé, J.Xu and N.Zhang, *"Supperconvergence of the gradient in piecewise linear finite element approximation to a parabolic problem"*, SIAM J. Numer. Anal., 26(1989), 553-573.

[14] D.Wang, *"Elimination Methods"*, Springer-Verlag, Vienna, 2001.
[15] J.Xu, *"Two-Grid Discretization Techniques for Linear and Nonlinear Partial Differential Equation"*, SIAM J. Numer. Anal., 1996.
[16] J.Xu and A.Zhou, *"Local and parallel finite element algorithms based on two-grid discretizations"*, Math. Comp., 69(2000), 881-909.
[17] J.Xu and A.Zhou, *"Local and parallel finite element algorithms based on two-grid discretizations for nonlinear problems"*, Adv. Comput. Math., 14(2001), 393-327.

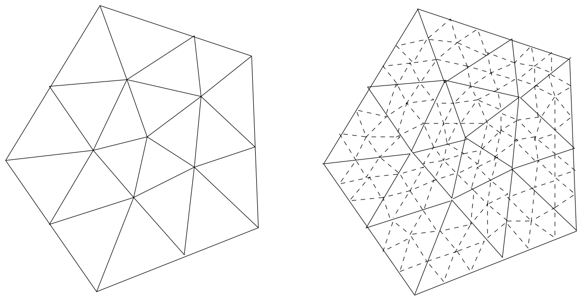

Figure 1: The coarse and refined grid partition

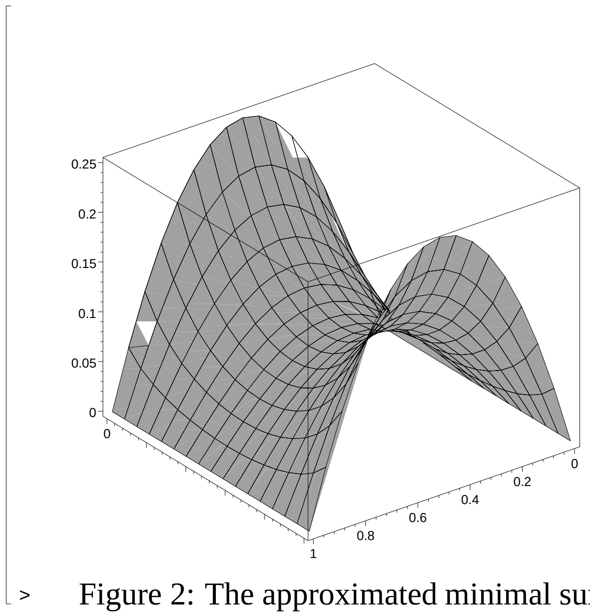

Figure 2: The approximated minimal surface

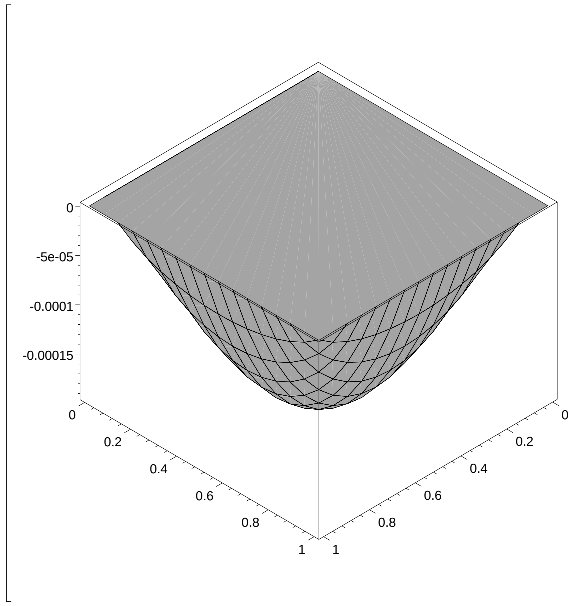

Figure 3: The shape of the discrete solution which also indicate the error

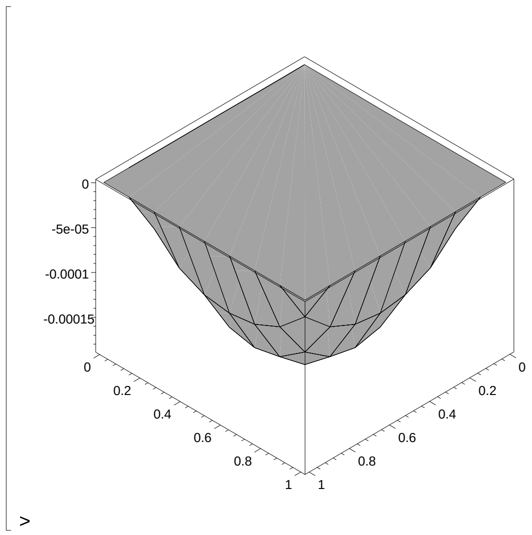

Figure 4: The solution solved by the 2-grid discretization algorithm 2