An Inequality for Lattice Polygons

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Abstract

We give an upper bound for the area of a lattice polygon in terms of the number of interior points. Together with Pick's formula, we obtain necessary and sufficient conditions for a triple of numbers to be the area, the interior, and the perimeter of a lattice polygon.

Introduction

Toric geometry is a powerful link connecting discrete and algebraic geometry. It was invented by Demazure [3] in order to study algebraic subgroups of the Cremona group in algebraic geometry. Stanley used it to prove the Dehn-Sommerville equations for convex polytopes [12]. Today, we have many applications in algebraic and discrete geometry (e.g. [7, 13, 10]). A quite surprising new application in computer-aided design has been found in [8].

From the algebraic geometry point of view, toric varieties are useful because most cohomological concepts have a concrete combinatorial interpretation. For instance, the Euler characteristic of a toric variety defined by a polyhedron is equal to the number of vertices of the polyhedron (see [5], p 59).

The main benefit for discrete geometers is that toric varieties provide a new way of proving theorems. Pick's theorem, giving the a formula for the area of a convex lattice polygon, has an elementary proof (for instance, see [4]); but the toric approach has lead to generalizations in higher dimension, for instance [1].

In this paper, we give necessary and sufficient conditions for three integers a, i, p to be the area, the number of interior points, and the number of boundary points (also: the perimeter) of a convex lattice polygon. Pick's theorem provides an equational constraint. A new inequality constraint bounds the area in terms of the number of interior points, if this number is not zero (there are lattice polygons with arbitrary large area and without interior points). This inequality is proven by toric geometry. The author tried to come up with an elementary proof, without success. The sufficiency of our conditions is proven by elementary methods.

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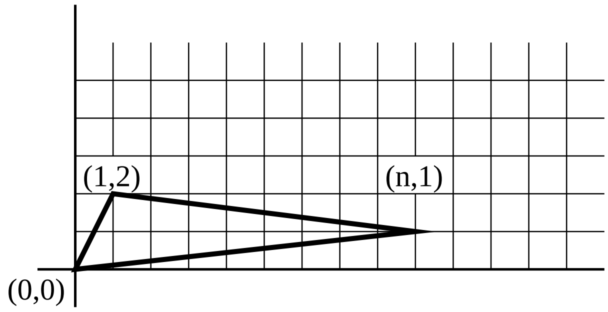

Figure 1: a polygon with perimeter 3.

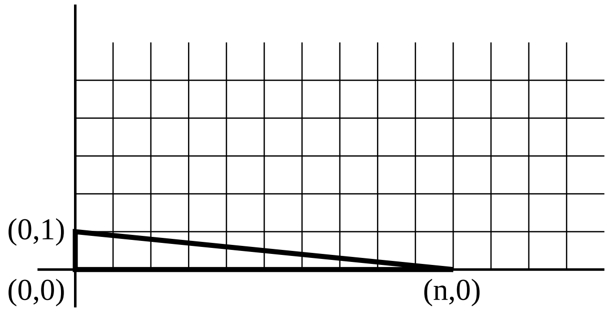

Figure 2: a polygon with zero interior.

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1 Lattice polygons

We consider convex polygons in $\mathbb{R}^2$ with vertices in $\mathbb{Z}^2$, also called *lattice polygons.* The *perimeter* of a lattice polytope is the number of lattice points on the boundary. The *area* is defined as usual. The *interior* is the number of lattice points in the interior. There is the following well-known relation between these numbers.

Theorem 1.1 (Pick). *For any lattice polygon Δ, we have*

$$\mathrm{Area}(\Delta) = \mathrm{Int}(\Delta) + \frac{1}{2}\,\mathrm{Per}(\Delta) - 1.$$

Our main goal is to try to find other relations between these three parameters, especially inequalities.

Unlike in the continuous case, there is no isoperimetric inequality: there are lattice polygons with perimeter 3 with arbitrary large area. An example is the

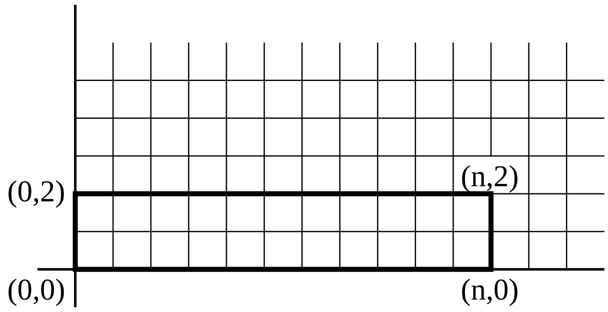

Figure 3: a polygon with $\mathrm{Per}(\Delta) = 2\ \mathrm{Int}(\Delta) + 6$.

family of polygons spanned by $\{(0,0),(1,2),(n,1)\}$, $n \geq 2$, with an area of $n - \frac{1}{2}$ and an interior of $n-2$ (see figure 1). In the other direction, we have

$$\mathrm{Per}(\Delta) \leq 2\ \mathrm{Area}(\Delta) + 2$$

as an obvious consequence of Pick's theorem. A bound of the perimeter in terms of the interior would be more interesting, but these seems impossible, too. The family of polygons spanned by $\{(0,0),(0,1),(n,0)\}$, $n \geq 1$, have zero interior and a perimeter of $n+2$ (see figure 2). Restricting to the case of positive interior, we can say something positive.

Theorem 1.2. *For any lattice polygon Δ with positive interior, we have the inequality*

$$\mathrm{Per}(\Delta) \leq 2\ \mathrm{Int}(\Delta) + 7.$$

If $\mathrm{Int}(\Delta) > 1$, *then the right hand side can be improved to* $2\ \mathrm{Int}(\Delta) + 6$.

The proof will be given in section2, because it requires some non-elementary facts about toric varieties.

The improved inequality is sharp for all values of interior greater than one. This follows from the family of polygons spanned by $\{(0,0),(0,2),(n,0),(n,2)\}$, $n \geq 3$ (see figure 3). An example with interior equal to 1 and perimeter equal to 9 is the polygon spanned by $\{(0,0),(0,3),(3,0)\}$ (see figure 4). In fact, this is the only lattice polygon with these parameters up to unimodular transformations. (This will be clear later from the proof of theorem 1.2.)

Let us now ask the question whether there are relations between area, interior, and perimeter, other than the theorems 1.1 and 1.2. The answer is no, as stated in the next theorem.

Theorem 1.3. *Let $a \in \frac{1}{2}\mathbb{N}$, $i, p \in \mathbb{N}$ be numbers fulfilling the following conditions:*

1. $p \geq 3$;

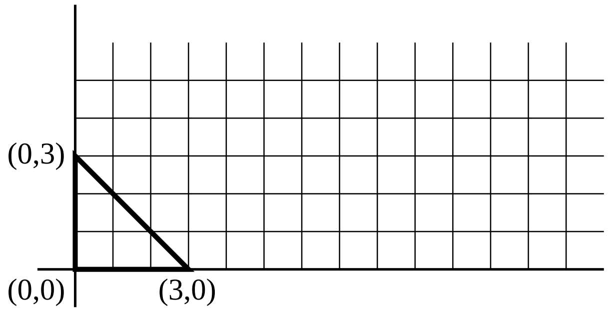

Figure 4: a polygon with $\mathrm{Per}(\Delta) = 9$ and $\mathrm{Int}(\Delta) = 1$.

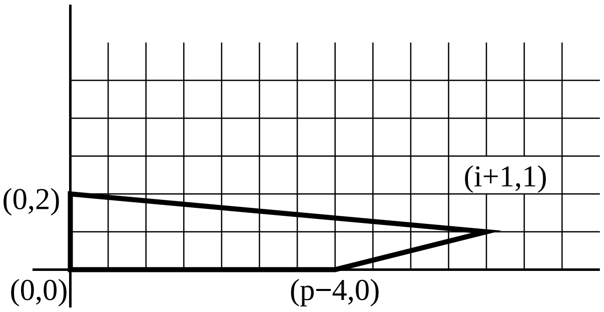

Figure 5: a polygon with $\mathrm{Per}(\Delta) = p$ and $\mathrm{Int}(\Delta) = i$.

2. *$a = i + \frac{1}{2}\, p - 1$;*
3. *if $i > 0$, then $p \leq 2\, i + 7$;*
4. *if $i > 1$, then $p \leq 2\, i + 6$.*

Then there is a lattice polygon Δ with $\mathrm{Area}(\Delta) = a$, $\mathrm{Int}(\Delta) = i$, *and* $\mathrm{Per}(\Delta) = p$.

Proof. Assume that $4 \leq p \leq 2i + 6$, in addition to the conditions above. Then the polygon Δ spanned by $\{(0,0), (0,2), (i+1,1), (p-4,0)\}$ has $\mathrm{Area}(\Delta) = a$, $\mathrm{Int}(\Delta) = i$, and $\mathrm{Per}(\Delta) = p$ (see figure5).

It remains to find polygons for the following cases:

1. $p = 3$;
2. $i = 0$;
3. $i = 1$ and $p = 9$.

But for these cases, we already have seen examples (see figure 1, figure 2, and figure 4). □

2 Toric Surfaces

In the following introduction of toric varieties, we follow [5] as a basic reference. Other introductions to toric varieties are [6, 9].

Let Δ be a lattice polygon contained in the positive quadrant. Let $v = (x, y)$ be a pair of variables. For any lattice point $p = (i, j)$ in the positive quadrant, we write t^p as a shorthand for $x^i * y^j$. Let $n := \mathrm{Per} + \mathrm{Int} - 1$. We define a map $f_\Delta : \mathbb{C}^2 \to \mathbb{P}^n_{\mathbb{C}}$ by

$$(t_1, t_2) \mapsto (t^{p_0} : \cdots : t^{p_n}),$$

where $p_0, \ldots, p_n$ are the lattice points of Δ. The projective closure of the image is a projective algebraic surface denoted by $X(\Delta)$. The surfaces constructed in this ways are called *toric*.

Example 2.1. ..

Obviously, any toric surface is rational. In some sense, the toric varieties are the simplest possible rational surfaces: they have a parametrization by monomials.

There is a close relation between the numerical characters of the lattice polygon – area, interior, perimeter – and some numerical characters of its associated toric surface. For instance, we have that the embedding dimension of the surface is equal to $\mathrm{Int} + \mathrm{Par} - 1$, as an immediate consequence of the construction. Here are some more well-known facts.

Theorem 2.2. *Let Δ be a lattice polygon, with associated toric surface X. Then the following hold.*

1. *The degree of X is equal to* $\deg(X) = 2\ \mathrm{Area}(\Delta)$.

2. *The interior of Δ is equal to the genus of the generic plane section, also called the sectional genus $p_1(X)$.*

Proof. See [5], p 111 and p 91. □

Proof of theorem 1.2. By theorem 2.2, the assertion is equivalent to

$$\deg(X) \leq 4\ p_1 + 4, \text{ if } p_1 \geq 2,$$

$$\deg(X) \leq 9, \text{ if } p_1 = 1,$$

where $X := X(\Delta)$. The first statement has been proven in [11] (theorem 6), for arbitrary rational surfaces. For the second statement, recall that the rational surfaces with sectional genus one are precisely the Del Pezzo surfaces, and that the degree of a Del Pezzo surface is less than or equal to 9 (see [2]). □

Remark 2.3. The only Del Pezzo surface of degree 9 is the 3-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$. This is a toric surface, and the corresponding lattice polygon is the one in figure 4. Hence our earlier statement on the uniqueness of a lattice polygon with interior equal to 1 and perimeter equal to 9 follows.

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