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Solving Parameterized Linear Difference Equations in ΠΣ-Fields*

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Abstract

The described algorithms enable to find all solutions of parameterized linear difference equations of arbitrary order within a very general difference field setting, so called ΠΣ-fields. These algorithms not only allow to simplify indefinite nested multisums, but can be also used to prove and discover a huge class of definite multisums identities.

1. Introduction

M. Karr developed an algorithm for indefinite summation [Kar81, Kar85] based on the theory of difference fields [Coh65]. He introduced so called ΠΣ-fields, in which parameterized first order linear difference equations can be solved in full generality. This algorithm cannot only deal with series of (q-)hypergeometric terms [Gos78, PS95, PR97] or holonomic series [CS98] but with series of rational terms consisting of arbitrary nested indefinite sums and products. Karr's algorithm is, in a sense, the summation counterpart of Risch's algorithm [Ris70] for indefinite integration. Based on results from [Kar81, Sch02a, Sch02b] and Bronstein's denominator bound [Bro00] – a generalization of Abramov's denominator bound [Abr95] – in this work I streamline Karr's ideas and develop a simplified algorithm that allows to solve parameterized first order linear difference equations in ΠΣ-fields. Furthermore I generalize the reduction techniques presented in [Kar81] which enables to extend the above algorithm from solving first order linear difference equations in a given ΠΣ-field to searching for all solutions of a linear difference equation with arbitrary order. Although there are still open problems in this resulting algorithm, one finds all those solutions by increasing step by step the range in which the solutions may exist. All those algorithms

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are available in form of a package called `Sigma` [Sch00, Sch01] in the computer algebra system Mathematica.

In spite of exciting achievements [PWZ96] over the last years, symbolic summation became a well-recognized subbranch of computer algebra only recently. In particular by Zeilberger's idea of creative telescoping [Zei90] one obtains a recipe to compute recurrences that possess a given definite sum as solution. Hence one can prove definite sum identities which has for a long time been considered as algorithmically infeasible. I recognized in [Sch00] that creative telescoping is in the scope of our algorithm by solving a specific parameterized first order linear difference equation. By this observation one can compute recurrences for a huge class of definite multisums in the general ΠΣ-field setting that cannot be handled with the approaches [PS95, PR97, CS98] for (q-)hypergeometric or holonomic series. Moreover by solving linear difference equations with our proposed algorithms, one can find solutions of recurrences and thus not only prove various definite multisum identities, but even discover their closed form evaluations.

In [Bro00] M. Bronstein developed reduction techniques in an even more general setting, namely σ-derivations, by approaching the problem from the point of view of differential fields. As already sketched above, in my approach one comes directly from Karr's reduction techniques which are specialized for the ΠΣ-field situation. Contrary to [Bro00] I emphasize more algorithmic than theoretical aspects. In some sense the algorithms under discussion contain the algorithms introduced in [Pet92, Pet94, APP98, vH99] from the point of view of solving difference equations. But whereas in our approach one has to extend manually the underlying difference field by appropriate product extensions, for their case of (q-)hypergeometric series these extensions are found automatically. Combining these algorithms with the approach under discussion leads to a powerful tool to solve difference equations [Sch01, Chapter 1]. In particular in [AP94, HS99, Sch01] one considers further extensions like d'Alembertian extension, a subclass of Liouvillian extensions, in order to find additional solutions for a given difference equation. As it turns out in [Sch01, Chapter 1.3.4.2], indefinite summation for nested sums and therefore our summation algorithm play an essential role to simplify those d'Alembertian solutions further.

In the next section it is illustrated how closed form evaluations of nested indefinite and definite multisums can be found, by solving parameterized linear difference equations in ΠΣ-fields. Whereas in Section 3 this problem is specified in the general difference field setting, in Section 4 the domain is concretized to ΠΣ-fields. In Section 5 the basic reduction strategies are explained which enables to find all solutions of parameterized linear difference equations in a given ΠΣ-field. Finally in Section 6 the incremental reduction strategy, the inner core of the whole reduction process, is explored in more details. All these considerations will lead to algorithms that are carefully analyzed in Section 7.

2. Symbolic Summation in Difference Fields

`Sigma` [Sch00, Sch01] is a summation package, implemented in the computer algebra system Mathematica, that enables to discover and prove nested multisum identities. Based on results of this article the package allows to find all solutions of parameterized linear difference equations in a very general difference field setting, so called ΠΣ-fields. In the sequel we illustrate how one can discover closed form evaluations of nested multisums in the difference field setting by using the package `Sigma`. First some basic notions of difference fields are introduced.

Definition 2.1. A *difference field* (resp. ring) is a field (resp. ring) $\mathbb{F}$ together with a field (resp. ring) automorphism $\sigma : \mathbb{F} \to \mathbb{F}$. In the sequel a difference field (resp. ring) given by the field (resp. ring) $\mathbb{F}$ and automorphism σ is denoted by $(\mathbb{F}, \sigma)$. Moreover the subset $\mathbb{K} := \{k \in \mathbb{F} \,|\, \sigma(k) = k\}$ is called the *constant field* of the difference field $(\mathbb{F}, \sigma)$.

It is easy to see that the constant field $\mathbb{K}$ of a difference field $(\mathbb{F}, \sigma)$ is a subfield of $\mathbb{F}$. In the sequel we will assume that **all** fields are of characteristic 0. Then it is immediate that for any field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ we have $\sigma(q) = q$ for $q \in \mathbb{Q}$. Hence in any difference field, $\mathbb{Q}$ is a subfield of its constant field.

2.1. Indefinite Summation and First Order Linear Difference Equations

As M. Karr observed in [Kar81], a huge class of indefinite nested multisums can be simplified by solving first order linear difference equations in ΠΣ-fields. I will demonstrate this approach by the following elementary problem: find a closed form of $\sum_{k=0}^{n} k\,k!$. First one constructs a difference field for the given summation problem. Let $\mathbb{Q}(t_1, t_2)$ be the field of rational function, i.e. t_1, t_2 are indeterminates, and consider the field automorphism $\sigma : \mathbb{Q}(t_1, t_2) \to \mathbb{Q}(t_1, t_2)$ that is canonically defined by $\sigma(t_1) = t_1 + 1$ and $\sigma(t_2) = (t_1 + 1)\,t_2$. Note that the automorphism acts on t_1 and t_2 like the shift operator N on n and $n!$ via $Nn = n + 1$ and $Nn! = (n + 1)\,n!$. Hence the summation problem can be rephrased by a *first order linear difference equation* in terms of the difference field $(\mathbb{Q}(t_1, t_2), \sigma)$ as follows: find a solution $g \in \mathbb{Q}(t_1, t_2)$ of

$$\boxed{\sigma(g) - g = t_1\, t_2.}$$

Our package `Sigma` can compute the solution $g = t_2$ (Example 3.1) from which $(k + 1)! - k! = k\,k!$ immediately follows. Finally by telescoping one obtains the closed form evaluation $\sum_{k=0}^{n} k\,k! = (n + 1)! - 1$.

2.2. Definite Summation and Parameterized Linear Difference Equations

In [Sch00, Sch02a] I observed that one can find closed form evaluations for a huge class of definite nested multisums by solving parameterized linear difference equations in ΠΣ-fields. I will illustrate these ideas by finding a closed form of the two nested definite multisum $\mathrm{SUM}(n) := \sum_{k=0}^{n} \mathrm{H}_k \binom{n}{k}$ where $\mathrm{H}_k = \sum_{i=1}^{k} \frac{1}{i}$ denotes the k-th harmonic numbers.

Finding a recurrence: In a first step one can compute a recurrence

$$4\,(1+n)\,\mathrm{SUM}(n) - 2\,(3+2\,n)\,\mathrm{SUM}(1+n) + (2+n)\,\mathrm{SUM}(2+n) = 1 \quad (1)$$

for the definite sum $\mathrm{SUM}(n)$ by applying Zeilberger's creative telescoping trick [Zei90] in a difference field setting. First one constructs a difference field in which the creative telescoping problem can be formalized. For this let $Q(n)(t_1, t_2, t_3)$ be the field of rational functions over $\mathbb{Q}$ and consider the field automorphism $\sigma : \mathbb{Q}(n)(t_1, t_2, t_3) \to \mathbb{Q}(n)(t_1, t_2, t_3)$ canonically defined by

$$\sigma(n) = n, \quad \sigma(t_1) = t_1 + 1, \quad \sigma(t_2) = t_2 + \frac{1}{t_1+1}, \quad \sigma(t_3) = \frac{n - t_1}{t_1+1}\, t_3. \quad (2)$$

Note that the automorphism acts on t_1, t_2 and t_3 like the shift operator K on k, H_k and $\binom{n}{k}$ with $K\,k = k+1$, $K\,\mathrm{H}_k = \mathrm{H}_k + \frac{1}{k+1}$ and $K \binom{n}{k} = \frac{n-k}{k+1}\binom{n}{k}$. Therefore $f(n,k)$ can be rephrased in terms of the difference field $(\mathbb{Q}(n)(t_1, t_2, t_3), \sigma)$ by

$$\begin{aligned}
f(n,k) &= \mathrm{H}_k \binom{n}{k} \leftrightarrow t_2\,t_3 := f_1'\\
f(n+1,k) &= \frac{(n+1)\,\mathrm{H}_k \binom{n}{k}}{n+1-k} \leftrightarrow \frac{(n+1)\,t_2\,t_3}{n+1-t_1} := f_2'\\
f(n+2,k) &= \frac{(n+1)\,(n+2)\,\mathrm{H}_k \binom{n}{k}}{(n+1-k)\,(n+2-k)} \leftrightarrow \frac{(n+1)\,(n+2)\,t_2\,t_3}{(n+1-t_1)\,(n+2-t_1)} =: f_3'.
\end{aligned} \quad (3)$$

Then the *creative telescoping problem* is formulated in terms of the difference field $\mathbb{Q}(n)(t_1, t_2, t_3)$ as follows: find an element $g \in \mathbb{Q}(n)(t_1, t_2, t_3)$ and a vector $(0,0,0) \neq (c_1, c_2, c_3) \in \mathbb{Q}(n)^3$ such that

$$\boxed{\sigma(g) - g = c_1\,f_1' + c_2\,f_2' + c_3\,f_3'.} \quad (4)$$

Our package `Sigma` (Example 3.1) enables to handle exactly such kind of problems. In this example the solution

$$\begin{array}{l} c_1 := 4\,(1+n),\ c_2 := -2\,(3+2\,n),\\ c_3 := 2+n \end{array}\ g := \frac{(1+n)\,(-2+t_1-n+(2\,t_1-2\,t_1^2+t_1\,n)t_2)t_3}{(1-t_1+n)\,(2-t_1+n),} \quad (5)$$

is computed that can be rephrased in terms of k, H_k and $\binom{n}{k}$. Hence one obtains with $h(n,k) := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)\,\mathrm{H}_k)\binom{n}{k}}{(1-k+n)\,(2-k+n)}$ the *creative telescoping equation*

$$h(n,k+1) - h(n,k) = c_1\,f(n,k) + c_2\,f(n+1,k) + c_3\,f(n+2,k),$$

and summing the equation in k from 0 to n results in

$$h(n,n+1) - h(n,0) = c_1 \sum_{k=0}^{n} f(n,k) + c_2 \sum_{k=0}^{n} f(n+1,k) + c_2 \sum_{k=0}^{n} f(n+2,k).$$

By $\mathrm{SUM}(n+i) = \sum_{k=0}^{n} f(n,k) + \sum_{j=1}^{i} f(n+i, n+j)$ for $i \in \mathbb{N}_0$ recurrence (1) follows.

Solving linear recurrences: In order to find a closed form of the definite sum $\mathrm{SUM}(n)$, one solves recurrence (1) in terms of n and 2^n, H_n, $\sum_{i=1}^{n} \frac{1}{i\,2^i}$. As in the previous examples, one first constructs a difference field† for the given problem. Let $Q(t_1, t_2, t_3, t_4)$ be the field of rational functions over $\mathbb{Q}$ and consider the field automorphism $\sigma : \mathbb{Q}(t_1, t_2, t_3, t_4) \to \mathbb{Q}(t_1, t_2, t_3, t_4)$ canonically defined by

$$\sigma(t_1) = t_1 + 1 \quad \sigma(t_2) = 2\,t_2, \quad \sigma(t_3) = t_3 + \frac{1}{t_1+1}, \quad \sigma(t_4) = t_4 + \frac{1}{2\,(t_1+1)\,t_2}. \tag{6}$$

Note that the automorphism acts on t_1, t_2, t_3 and t_4 like the shift operator N on n, 2^n, H_n and $\sum_{i=1}^{n} \frac{1}{i\,2^i}$ with $N\,n = n+1$, $N\,2^n = 2\,2^n$, $N\,\mathrm{H}_n = \mathrm{H}_n + \frac{1}{n+1}$ and $N \sum_{i=0}^{n} \frac{1}{i\,2^i} = \sum_{i=0}^{n} \frac{1}{i\,2^i} + \frac{1}{2\,(n+1)2^{n+1}}$. Hence the problem of solving recurrence (1) in terms of n and 2^n, H_n, $\sum_{i=0}^{n} \frac{1}{i\,2^i}$, can be rephrased by a *linear difference equation* in terms of difference fields: find all $g \in \mathbb{Q}(t_1, \dots, t_4)$ such that

$$\boxed{a_1\,\sigma^2(g) + a_2\,\sigma(g) + a_3\,a_3\,g = 1} \tag{7}$$

where $a_1 := 2 + t_1$, $a_2 := -2(3 + 2\,t_1)$ and $a_3 := 4\,(1 + t_1)$. With our algorithms under discussion (Example 3.1) one computes two linearly independent solutions over $\mathbb{Q}$ of the homogeneous version of the difference equation, namely $g_1 := t_2$ and $g_2 := t_2\,t_3$, and one particular solution of the inhomogeneous difference equation itself, namely $g_3 := -t_2\,t_4$. Hence the set $\{k_1\,g_1 + k_2\,g_2 + g_3 \,|\, k_i \in \mathbb{Q}\}$ describes *all* solutions in $\mathbb{Q}(t_1, \dots, t_4)$ of the difference equation (7). Consequently in terms of the summation objects one obtains the complete solution in form of the set $\left\{k_1\,2^n + k_2\,2^n\,\mathrm{H}_n - 2^n \sum_{i=0}^{n} \frac{1}{i\,2^i} \,|\, k_i \in \mathbb{Q}\right\}$ for recurrence (1). Finally by comparing initial values of the original sum $\mathrm{SUM}(n)$ one finds the identity $\sum_{k=0}^{n} \mathrm{H}_k \binom{n}{k} = 2^n\big(\mathrm{H}_n - \sum_{i=1}^{n} \frac{1}{i\,2^i}\big)$.

3. The Solution Space for Difference Fields

The previous examples motivate us to solve *parameterized linear difference equations* in a difference field $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$:

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- Given a difference field $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, $a_1, \dots, a_m \in \mathbb{F}$ with $m \geq 1$ and $(a_1 \dots a_m) \neq (0, \dots, 0) =: \mathbf{0}$ and $f_1, \dots, f_n \in \mathbb{F}$ with $n \geq 1$.
- Find all $g \in \mathbb{F}$ and all $c_1, \dots, c_n \in \mathbb{K}$ with $a_1\,\sigma^{m-1}(g) + \dots + a_m\,g = c_1\,f_1 + \dots + c_n\,f_n$.

The solutions of the above problem are described by a set. For its definition note that in the difference field $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, $\mathbb{F}$ can be interpreted as a vector space over $\mathbb{K}$.

†Actually this difference field can be constructed automatically for the given recurrence (1). How this so called d'Alembertian extensions are computed, is explained in [AP94, HS99, Sch01].

Definition 3.1. Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$ and consider a subspace $\mathbb{V}$ of $\mathbb{F}$ as a vector space over $\mathbb{K}$. Let $\mathbf{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}^m$ and $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. We define the *solution space* for $\boldsymbol{a}, \boldsymbol{f}$ in $\mathbb{V}$ by

$$\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V}) = \{(c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{V} : a_1\, \sigma^{m-1}(g) + \dots + a_m\, g = c_1\, f_1 + \dots + c_n\, f_n\}.$$

It follows immediately that $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$ is a vector space over $\mathbb{K}$. The next proposition based on [Coh65, Theorem XII (page 272)] states that this vector space has even finite dimension.

Proposition 3.1. *Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$ and assume $\boldsymbol{f} \in \mathbb{F}^n$ and $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^m$. Let $\mathbb{V}$ be a subspace of $\mathbb{F}$ as a vector space over $\mathbb{K}$. Then $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$ is a vector space over $\mathbb{K}$ with maximal dimension $m+n-1$.*

Proof: By [Coh65, Theorem XII (page 272)] $\mathrm{V}(\boldsymbol{a}, (0), \mathbb{F})$ is a finite dimensional vector space with maximal dimension $m-1$. Since $\mathrm{V}(\boldsymbol{a}, (0), \mathbb{V})$ is a subspace of $\mathrm{V}(\boldsymbol{a}, (0), \mathbb{F})$ over $\mathbb{K}$, it follows that $\mathrm{V}(\boldsymbol{a}, (0), \mathbb{V})$ has maximal dimension $m-1$, say $d := \dim \mathrm{V}(\boldsymbol{a}, (0), \mathbb{V}) < m$. Now assume that $\dim \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V}) > n+d$, say there are $(c_{1i}, \dots, c_{ni}, g_i) \in \mathbb{K}^n \times \mathbb{V}$ for $1 \leq i \leq n+d+1$ which are linearly independent over $\mathbb{K}$ and solutions of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$. Then one can transform the matrix

$$\boldsymbol{M} := \begin{pmatrix} c_{11} & \dots & c_{n1} & g_1 \\ \vdots & \vdots & \vdots & \vdots \\ c_{1,n+d+1} & \dots & c_{n,n+d+1} & g_{n+d+1} \end{pmatrix}$$

by row operations over $\mathbb{K}$ to a matrix

$$\boldsymbol{M'} := \begin{pmatrix} c'_{11} & \dots & c'_{n1} & g'_1 \\ \vdots & \vdots & \vdots & \vdots \\ c'_{1,n+d+1} & \dots & c'_{n,n+d+1} & g'_{n+d+1} \end{pmatrix}$$

where the submatrix

$$\boldsymbol{C'} := \begin{pmatrix} c'_{11} & \dots & c'_{n1} \\ \vdots & \vdots & \vdots \\ c'_{1,n+d+1} & \dots & c'_{n,n+d+1} \end{pmatrix}$$

is in row reduced form and the rows in M' and the rows in M are a basis of the same vector space $\mathbb{W}$. Since we assumed that the $(c_{1i}, \dots, c_{ni}, g_i)$ are linearly independent over $\mathbb{K}$, it follows that all rows in M' have a nonzero entry and are linearly independent over $\mathbb{K}$. On the other side, only the first n rows in C' can have nonzero entries and therefore the last $d+1$ columns in M' must be of the form $(0, \dots, 0, g'_i)$ where $g'_i \neq 0$. Therefore we find $d+1$ linearly independent solutions over $\mathbb{K}$ with $\sigma_{\boldsymbol{a}} g_i = 0$ which contradicts to the assumption. □

In this article we develop algorithms that enable to find bases of solution spaces $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$ in ΠΣ-fields $(\mathbb{F}, \sigma)$ that will be specified later. In particular these algorithms under consideration (Remark 3.1) are available in form of a package `Sigma` in the computer algebra system Mathematica.

Example 3.1. With our package `Sigma` one can solve algorithmically all the difference equation problems in Section 2. After loading the package

In[1]:= << **Sigma`**

in the computer algebra system Mathematica one is able to compute a basis of the solution space $\mathrm{V}((1,-1),(t_1\,t_2),\mathbb{Q}(t_1)(t_2))$ where the difference field $(\mathbb{Q}(t_1)(t_2),\sigma)$ is canonically defined by $\sigma(t_1)=t_1+1$ and $\sigma(t_2)=(t_1+1)\,t_2$. Here $\{t_i,\alpha_i,\beta_i\}$ stands for $\sigma(t_i)=\alpha_i\,t_i+\beta_i$.

In[2]:= $\mathbf{SolveDifferenceVectorSpace}[\{\mathbf{1},-\mathbf{1}\},\{\mathbf{t_1\,t_2}\},\{\{\mathbf{t_1},\mathbf{1},\mathbf{1}\},\{\mathbf{t_2},\mathbf{t_1}+\mathbf{1},\mathbf{0}\}\}]$

Out[2]= $\{\{0,1\},\{1,\mathtt{t}_2\}\}$

This means that the elements in $\{(0,1),(0,t_2)\}$ form a basis of the solution space. Similarly one computes a basis of $\mathrm{V}((1,-1),(f_1',f_2',f_3'),\mathbb{Q}(t_1)(t_2)(t_3))$ where the parameters f_i' are defined as in (3) and the difference field as in (2).

In[3]:= $\mathbf{SolveDifferenceVectorSpace}\Big[\{\mathbf{1},-\mathbf{1}\},$
$$\{\mathbf{f}_1',\mathbf{f}_2',\mathbf{f}_3'\},\Big\{\{\mathbf{t_1},\mathbf{1},\mathbf{1}\},\Big\{\mathbf{t_2},\mathbf{1},\frac{\mathbf{1}}{\mathbf{1}+\mathbf{t_1}}\Big\},\Big\{\mathbf{t_3},\frac{\mathbf{n}-\mathbf{t_1}}{\mathbf{1}+\mathbf{t_1}},\mathbf{0}\Big\}\Big\}\Big]$$

Out[3]= $\Big\{\{0,0,0,1\},\Big\{4\ (1+\mathtt{n}),-6-4\ \mathtt{n},2+\mathtt{n},\big((1+\mathtt{n})\ \big(-2+\mathtt{t}_1+$
$$2\ \mathtt{t}_1\ \mathtt{t}_2-2\ \mathtt{t}_1{}^2\ \mathtt{t}_2+\mathtt{n}\ (-1+\mathtt{t}_1\ \mathtt{t}_2)\big)\ \mathtt{t}_3\big)/\big((1+\mathtt{n}-\mathtt{t}_1)\ (2+\mathtt{n}-\mathtt{t}_1)\big)\Big\}\Big\}$$

Hence $\{(0,0,0,1),(c_1,c_2,c_3,g)\}$ forms a basis of this solution space where the $c_i\in\mathbb{Q}$ and $g\in\mathbb{Q}(t_1,t_2,t_3)$ are defined as in (5). Moreover one is capable of computing a basis of the solution space $\mathrm{V}((a_1,a_2,a_3),(1),\mathbb{Q}(t_1)(t_2)(t_3)(t_4))$ with $a_1:=2+t_1$, $a_2:=-2(3+2\,t_1)$ and $a_3:=4\,(1+t_1)$ in the difference field $(\mathbb{Q}(t_1)(t_2)(t_3)(t_4),\sigma)$ as it is defined in (6)

In[4]:= $\mathbf{SolveDifferenceVectorSpace}\Big[\{\mathbf{a_1},\mathbf{a_2},\mathbf{a_3}\},\{\mathbf{1}\},$
$$\Big\{\{\mathbf{t_1},\mathbf{1},\mathbf{1}\},\{\mathbf{t_2},\mathbf{2},\mathbf{0}\},\Big\{\mathbf{t_3},\mathbf{1},\frac{\mathbf{1}}{\mathbf{1}+\mathbf{t_1}}\Big\},\Big\{\mathbf{t_4},\mathbf{1},\frac{\mathbf{1}}{\mathbf{2}\ (\mathbf{1}+\mathbf{t_1})\ \mathbf{t_2}}\Big\}\Big\}\Big]$$

Out[4]= $\{\{-1,\mathtt{t}_2\ \mathtt{t}_4\},\{0,\mathtt{t}_2\},\{0,\mathtt{t}_2\ \mathtt{t}_3\}\}$

and one obtains the basis $\{(-1,t_2\ t_4),(0,t_2),(0,t_2\ t_3)\}$ of the solution space.

3.1. Some Conventions for Vectors and Matrices

In the following some notations and conventions will be introduced that are heavily used in the sequel. Let $\mathbb{F}$ be a vector space over $\mathbb{K}$ and, more generally, consider $\mathbb{F}^n$ as a vector space over $\mathbb{K}$. Then a vector $\boldsymbol{f}\in\mathbb{F}^n$ is considered either as a row **or** as a column vector. It will be convenient not to distinguish between these two types of presentations. This means that the vector $\boldsymbol{f}$ can be either interpreted as row vector $(f_1,\dots,f_n)$ or as column vector $\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}$. We will show that there cannot appear any ambiguous situations in the sequel. For the vector multiplication of the vectors $\boldsymbol{f}$ and $\boldsymbol{g}=(g_1,\dots,g_n)$ there cannot be confusion: $\sum_{i=1}^n f_i\,g_i=\boldsymbol{f}\,\boldsymbol{g}=\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}\begin{pmatrix}g_1\\ \vdots\\ g_n\end{pmatrix}=(f_1,\dots,f_n)\begin{pmatrix}g_1\\ \vdots\\ g_n\end{pmatrix}=\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}(g_1,\dots,g_n)$. Whereas a vector is always denoted by a small letter, matrices are denoted by capital letters, like $\boldsymbol{A} := \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \in \mathbb{F}^{m\times n}$ and $\boldsymbol{B} := \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \vdots & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix} \in \mathbb{F}^{n\times m}$. *Multiplying a matrix* $\boldsymbol{A}$ *with the vector* $\boldsymbol{f}$ *from the right* always means that the vector $\boldsymbol{f}$ is interpreted as a column vector, whereas *multiplying a matrix* $\boldsymbol{B}$ *with the vector* $\boldsymbol{f}$ *from the left* means always that the vector $\boldsymbol{f}$ is interpreted as a row vector, for instance $\boldsymbol{f}\cdot\boldsymbol{B} = (\sum_{i=0}^n b_{i1}\, f_i, \dots, \sum_{i=0}^n b_{im}\, f_i)$ and $\boldsymbol{A}\cdot\boldsymbol{f} = \begin{pmatrix} \sum_{i=0}^n a_{1i}\, f_{1i} \\ \vdots \\ \sum_{i=0}^n a_{mi}\, f_{mi} \end{pmatrix}$. Furthermore the multiplication of a matrix with a vector is denoted by the operation symbol $\cdot$. The usual matrix multiplication is denoted by $\boldsymbol{A}\,\boldsymbol{B}$. Moreover the construction $\boldsymbol{f}\wedge g = (f_1, \dots, f_n, g) \in \mathbb{F}^{n+1}$ stands for the *concatenation* of $\boldsymbol{f}$ with $g \in \mathbb{F}$. Similarly, one uses the construction $\boldsymbol{B}\wedge\boldsymbol{f} = \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \vdots & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix} \wedge \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} b_{11} & \dots & b_{1m} & f_1 \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & \dots & b_{nm} & f_n \end{pmatrix}$. For $h \in \mathbb{F}$ we write $h\,\boldsymbol{f} = (h\, f_1, \dots, h\, f_d) \in \mathbb{F}^d$. Furthermore if $\sigma : \mathbb{F} \to \mathbb{F}$ is a function, we write $\sigma(\boldsymbol{f}) = (\sigma(f_1), \dots, \sigma(f_n)) \in \mathbb{F}^n$. In the sequel we denote $\boldsymbol{0_n} := (0, \dots, 0) \in \mathbb{K}^n$ as the zero-vector of length n; if the length is clear from the context, we just write $\boldsymbol{0}$. Moreover we denote by $\boldsymbol{0_{m\times n}} \in \mathbb{K}^{m\times n}$ the $m\times n$-matrix with only zero-entries.

3.2. The Solution Space and Its Representation

Finally it is described how the solution space is represented in matrix notation. Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$, $\mathbb{V}$ be a subspace of $\mathbb{F}$ over $\mathbb{K}$, $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}^m$, and $\boldsymbol{f} \in \mathbb{F}^n$. For $g \in \mathbb{F}$ the notation $\sigma_{\boldsymbol{a}} g := a_1\, \sigma^{m-1}(g) + \dots + a_m\, g$. is introduced. Hence one obtains a compact description of the solution space, namely $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V}) = \{\boldsymbol{c}\wedge g \in \mathbb{K}^n \times \mathbb{V} \,|\, \sigma_{\boldsymbol{a}} g = \boldsymbol{c}\,\boldsymbol{f}\}$. Please note that the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$ is a finite dimensional vector space of $\mathbb{K}^n \times \mathbb{F}$ over $\mathbb{K}$. In the sequel it is convenient to describe a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$ by a matrix. Let $B = \{\boldsymbol{b_1}, \dots, \boldsymbol{b_d}\} \subseteq \mathbb{K}^n \times \mathbb{F}$ be a family of linearly independent vectors over $\mathbb{K}$ with $\boldsymbol{b_i} = (c_{i1}, \dots, c_{in}, g_i) \in \mathbb{K}^n \times \mathbb{F}$ such that

$$\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V}) = \{k_1\, \boldsymbol{b_1} + \dots + k_d\, \boldsymbol{b_d} \,|\, k_i \in \mathbb{K}\}.$$

Often the basis B of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$ will be represented by the *basis matrix*

$$\boldsymbol{M_B} := \begin{pmatrix} \boldsymbol{b_1} \\ \vdots \\ \boldsymbol{b_d} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1n} & g_1 \\ \vdots & \vdots & \vdots & \vdots \\ c_{d1} & \dots & c_{dn} & g_n \end{pmatrix},$$

i.e. one has $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V}) = \left\{\boldsymbol{k}\cdot\boldsymbol{M_B} \,|\, \boldsymbol{k} \in \mathbb{K}^d\right\}$. In particular for the special situation $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V}) = \{\boldsymbol{0_{n+1}}\}$ we define the basis matrix as $\boldsymbol{M_B} := \boldsymbol{0_{1\times(n+1)}}$. If the elements in B are not necessarily independent over $\mathbb{K}$, we say that B is a set of generators of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$. In this situation $\boldsymbol{M_B}$ is just called *generator matrix.*

Example 3.2. According Example 3.1 $\begin{pmatrix} 0 & 1 \\ 1 & t_2 \end{pmatrix}$ is a basis matrix of the solution space $\mathrm{V}((1,-1), (t_1\, t_2), \mathbb{Q}(t_1)(t_2))$, $\begin{pmatrix} 0 & 0 & 0 & 1 \\ c_1 & c_2 & c_3 & g \end{pmatrix}$ is a basis matrix of the solution space $\mathrm{V}((1,-1), (f_1', f_2', f_3'), \mathbb{Q}(t_1)(t_2)(t_3))$ and $\begin{pmatrix} -1 & t_2 & t_4 \\ 0 & t_2 & t_3 \\ 0 & & t_2 \end{pmatrix}$ is a basis matrix of the solution space $\mathrm{V}((a_1, a_2, a_3), (1), \mathbb{Q}(t_1)(t_2)(t_3)(t_4))$.

4. ΠΣ-Fields and Some Properties

As mentioned in previous sections, this work restricts to so called ΠΣ-fields that are introduced in [Kar81, Kar85] and further analyzed in [Bro00, Sch01, Sch02a]. In the following the basic definition and properties are introduced.

4.1. ΠΣ-Extensions

In order to define ΠΣ-fields, the notion of difference field extensions is needed.

Definition 4.1. Let $(\mathbb{E}, \sigma_{\mathbb{E}})$, $(\mathbb{F}, \sigma_{\mathbb{F}})$ be difference fields. $(\mathbb{E}, \sigma_{\mathbb{E}})$ is called a *difference field extension* of $(\mathbb{F}, \sigma_{\mathbb{F}})$, if $\mathbb{F} \subseteq \mathbb{E}$ and $\sigma_{\mathbb{F}}(f) = \sigma_{\mathbb{E}}(f)$ for all $f \in \mathbb{F}$.

If $(\mathbb{E}, \tilde{\sigma})$ is a difference field extension of $(\mathbb{F}, \sigma)$, we will not distinguish anymore that $\sigma : \mathbb{F} \to \mathbb{F}$ and $\tilde{\sigma} : \mathbb{E} \to \mathbb{E}$ are actually different automorphisms.
Later the following definitions are needed.

Definition 4.2. Let $\mathbb{F}[t]$ be a polynomial ring with coefficients in the field $\mathbb{F}$, i.e. t is transcendental over $\mathbb{F}$, and let $\mathbb{F}(t)$ be the field of *of rational functions* over $\mathbb{F}$, this means $\mathbb{F}(t)$ is the quotient field of $\mathbb{F}[t]$. $\frac{p}{q} \in \mathbb{F}(t)$ is in *reduced representation* if $p, q \in \mathbb{F}[t]$, $\gcd(p, q) = 1$ and q is monic.

In Section 2 all difference field extensions $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ are of the following type: $\mathbb{F}(t)$ is a field of rational functions and the automorphism $\sigma : \mathbb{F}(t) \to \mathbb{F}(t)$ is canonically defined by $\sigma(t) = \alpha\, t + \beta$ where $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. In this work all difference fields are constructed by exactly this type of difference field extensions.

Example 4.1. Let $\mathbb{Q}(t)$ be the field of rational functions and consider the automorphism $\sigma : \mathbb{Q}(t) \to \mathbb{Q}(t)$ canonically defined by $\sigma(t) = t + 1$. Now consider the field of rational functions $\mathbb{Q}(t)(k)$ and construct the difference field extension $(\mathbb{Q}(t)(k), \sigma)$ of $(\mathbb{Q}(t), \sigma)$ canonically defined by $\sigma(k) = k + t + 1$. One can easily check that for $g := \frac{(t+1)\,t}{2}$ one has $\sigma(g) = g + t + 1$. Since σ acts on g and t in the same way, the extension $(\mathbb{F}(t), \sigma)$ does not produce anything new. In particular one has $\sigma(g - k) = g - k$ and hence $g - t \in \mathrm{const}_{\sigma}\mathbb{Q}(t)(k)$. But since k is transcendental over $\mathbb{Q}(t)$, $g - t \notin \mathbb{Q}(t)$ and thus $\mathrm{const}_{\sigma}\mathbb{Q}(t)(k) \neq \mathrm{const}_{\sigma}\mathbb{Q}(t)$.

This example motivates to consider only those extensions in which the constant field remains the same. This restriction leads to ΠΣ-extensions and ΠΣ-fields.

Definition 4.3. $(\mathbb{F}(t), \sigma)$ is a Π-*extension* of $(\mathbb{F}, \sigma)$ if $\sigma(t) = \alpha\, t$ with $\alpha \in \mathbb{F}^*$, t is transcendental over $\mathbb{F}$ and $\mathrm{const}_{\sigma}\mathbb{F}(t) = \mathrm{const}_{\sigma}\mathbb{F}$.

According to [Kar81] we introduce the notion of the homogeneous group which plays an essential role in the theory of ΠΣ-fields.

Definition 4.4. The *homogeneous group* of $(\mathbb{F}, \sigma)$ is $\mathrm{H}_{(\mathbb{F},\sigma)} := \{\frac{\sigma(g)}{g} \mid g \in \mathbb{F}^*\}$.

One can easily check that $\mathrm{H}_{(\mathbb{F},\sigma)}$ forms a multiplicative group. With this notion one obtains an equivalent description of a Π-extension. This result can be found in [Kar85, Theorem 2.2] or [Sch01, Theorem 2.2.2].

Theorem 4.1. *Let $(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t$ where $\alpha\in\mathbb{F}^*$ and $t\neq 0$. Then $(\mathbb{F}(t),\sigma)$ is a* Π-extension *of $(\mathbb{F},\sigma)$ if and only if there does not exist an $n>0$ such that $\alpha^n\in\mathrm{H}_{(\mathbb{F},\sigma)}$.*

Next we define Σ-extensions according to Karr's notions.

Definition 4.5. $(\mathbb{F}(t),\sigma)$ is a Σ-*extension* of $(\mathbb{F},\sigma)$ if

1. $\sigma(t)=\alpha\,t+\beta$ with $\alpha,\beta\in\mathbb{F}^*$ and $t\notin\mathbb{F}$,
2. there does not exists a $g\in\mathbb{F}(t)\setminus\mathbb{F}$ with $\frac{\sigma(g)}{g}\in\mathbb{F}$, and
3. for all $n\in\mathbb{Z}^*$ we have that $\alpha^n\in\mathrm{H}_{(\mathbb{F},\sigma)}\Rightarrow\alpha\in\mathrm{H}_{(\mathbb{F},\sigma)}$.

Remark 4.1. Together with Remark 4.2 we explain and motivate the properties given in the definition of Σ-extensions. Actually we are interested in extensions, similarly to Π-extensions, where $\sigma(t)=\alpha\,t+\beta$ with $\alpha,\beta\notin\mathbb{F}^*$, t transcendental and $\mathrm{const}_\sigma\mathbb{F}(t)=\mathrm{const}_\sigma\mathbb{F}$. Under these aspects property (1.) fits to the desired goal. Unfortunately condition (3.) seems to be quite technical, and indeed is needed for computational aspects in [Sch02a, Sch02b] that are needed in Theorem 7.4. But since in most cases we are just interested in situation $\alpha=1$, property (3.) is obsolete by $1\in\mathrm{H}_{(\mathbb{F},\sigma)}$. Moreover the next result states that in a Σ-extension t is transcendental and $\mathrm{const}_\sigma\mathbb{F}(t)=\mathrm{const}_\sigma\mathbb{F}$.

The next theorem is a direct consequence of [Sch01, Theorem 2.2.3] which is a corrected version of [Kar81, Theorem 3] or [Kar85, Theorem 2.3].

Theorem 4.2. *Let $(\mathbb{F}(t),\sigma)$ be a Σ-extension of $(\mathbb{F},\sigma)$. Then $(\mathbb{F}(t),\sigma)$ is canonically defined by $\sigma(t)=\alpha\,t+\beta$ for some $\alpha,\beta\in\mathbb{F}^*$, t is transcendental over $\mathbb{F}$ and* $\mathrm{const}_\sigma\mathbb{F}(t)=\mathrm{const}_\sigma\mathbb{F}$.

Similarly to Π-extensions an alternative description of Σ-extensions is given. This result follows from [Kar81, Theorem 1] or [Kar85, Theorem 3] and is essentially the same as [Sch01, Corollary 2.2.3].

Theorem 4.3. *Let $(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$ where $\alpha,\beta\in\mathbb{F}^*$. Then $(\mathbb{F}(t),\sigma)$ is a* Σ-extension *of $(\mathbb{F},\sigma)$ if and only if there does not exist a $g\in\mathbb{F}$ with $\sigma(g)-\alpha\,g=\beta$, and property (3.) from Definition 4.5 holds.*

Remark 4.2. Finally I want remark that condition (2.) does not restrict to any subclass of extensions that possess the properties given in Theorem 4.2. In order to show this, assume that we have an extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ with properties (1.) and (3.), t transcendental over $\mathbb{F}$ and $\mathrm{const}_\sigma\mathbb{F}(t)=\mathrm{const}_\sigma\mathbb{F}$. In addition suppose that condition (2.) does not hold, i.e. there exists a $g\in\mathbb{F}(t)\setminus\mathbb{F}$

with $\frac{\sigma(g)}{g} \notin \mathbb{F}$. Then by Theorem 4.3 it follows that there exists a $g \in \mathbb{F}$ such that $\sigma(g) - \alpha\, g = \beta$. But then we have $\sigma(t - g) = \alpha\,(t - g)$. Furthermore $t - g$ is transcendental over $\mathbb{F}$ and $\mathrm{const}_\sigma\mathbb{F}(t - g) = \mathrm{const}_\sigma\mathbb{F}$. Hence by a change of basis we can work in the Π-extension $(\mathbb{F}(t - g), \sigma)$ of $(\mathbb{F}, \sigma)$. In some sense property (2.) just avoids that Σ- and Π-extensions have a common intersection. On the other side condition (2.) is an essential property which is needed to find degree and denominator bounds [Sch02a, Sch02b] of a given solution space that will be introduced in Sections 5.2 and 5.3.

Now we are ready to define ΠΣ-extensions.

Definition 4.6. $(\mathbb{F}(t), \sigma)$ is called a ΠΣ-extension of $(\mathbb{F}, \sigma)$, if $(\mathbb{F}(t), \sigma)$ is a Π- or a Σ-extension of $(\mathbb{F}, \sigma)$.

4.2. ΠΣ-Extensions and the Field of Rational Functions

The next lemma will be used over and over again; it gives the link between ΠΣ-extensions and its domain of rational functions. The proof is straightforward.

Lemma 4.1. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$. Then $\mathbb{F}(t)$ is a field of rational functions over $\mathbb{K}$. Furthermore, σ is an automorphism of the polynomial ring $\mathbb{F}[t]$, i.e. $(\mathbb{F}[t], \sigma)$ is a difference ring extension of $(\mathbb{F}, \sigma)$. Additionally, we have for all $f \in \mathbb{F}[t]$ that* $\deg(\sigma(f)) = \deg(f)$.

In this work we need the following notions for such a polynomial ring $\mathbb{F}[t]$ and its quotient field $\mathbb{F}(t)$. For $f = \sum_{i=0}^{n} f_i\, t_i \in \mathbb{F}[t]$ the i-th *coefficient* f_i of f will be denoted by $[f]_i$, i.e. $[f]_i = f_i$; if $i > n$, we have $[f]_i = 0$. Furthermore we define the rank function $\|\ \|$ of $\mathbb{F}[t]$ by

$$\|f\| := \begin{cases} -1 & \text{if } f = 0 \\ \deg(f) & \text{otherwise.} \end{cases}$$

Moreover for $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}[t]^n$ we introduce $\|\boldsymbol{f}\| := \max_i \|f_i\|$. With these notations a simple but important fact is formulated.

Lemma 4.2. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $f, g \in \mathbb{F}[t]$ such that $\sigma_{\boldsymbol{a}} g = f$. Then $\|f\| \leq \|\boldsymbol{a}\| + \|g\|$.*

Proof: If $g = 0$, we have $f = \sigma_{\boldsymbol{a}} g = 0$ and hence $-1 = \|f\| \leq \|\boldsymbol{a}\| + \|g\|$ holds by $\|g\| = -1$ and $\|\boldsymbol{a}\| \geq 0$. Otherwise assume that $g \neq 0$, i.e. $\|g\| \geq 0$. Then

$$\|f\| = \|\sigma_{\boldsymbol{a}} g\| = \|a_1\, \sigma^{m-1}(g) + \dots + a_m\, g\| \leq \max(\|a_1\, \sigma^{m-1}(g)\|, \dots, \|a_m\, g\|).$$

Please note that we have $\|a_i\, \sigma^{m-i}(g)\| \leq \|a_i\| + \|\sigma^{m-i}(g)\|$, if $a_i = 0$; otherwise, if $a_i \neq 0$, we even have equality. Moreover if $a_i = 0$ and $a_j \neq 0$ then $\|a_i\| + \|\sigma^{m-i}(g)\| < \|a_j\| + \|\sigma^{m-j}(g)\|$. Since there exists an j with $a_j \neq 0$, it follows that

$$\max(\|a_1\, \sigma^{m-1}(g)\|, \dots, \|a_m\, g\|) = \max(\|a_1\| + \|\sigma^{m-1}(g)\|, \dots, \|a_m\| + \|g\|).$$

By Lemma 4.1 we have $\|\sigma^i(g)\| = \|g\|$ for all $i \in \mathbb{Z}$ and thus

$$\max(\|a_1\| + \|\sigma^{m-1}(g)\|, \dots, \|a_m\| + \|g\|) = \max(\|a_1\|, \dots, \|a_m\|) + \|g\| = \|\boldsymbol{a}\| + \|g\|$$

which proves the lemma. □

4.3. ΠΣ-Fields and Some Properties

For the definition of ΠΣ-fields properties on the constant field are required.

Definition 4.7. A field $\mathbb{K}$ is called *computable*, if

- for any $k \in \mathbb{K}$ one is able to decide, if $k \in \mathbb{Z}$,
- polynomials in the polynomial ring $\mathbb{K}[t_1, \dots, t_n]$ can be factored over $\mathbb{K}$ and
- one knows how to compute for any $(c_1, \dots, c_n) \in \mathbb{K}^n$ a basis of the submodule $\left\{(n_1, \dots, n_k) \in \mathbb{Z}^k \,|\, c_1^{n_1} \cdots c_k^{n_k} = 1\right\}$ of $\mathbb{Z}^k$ over $\mathbb{Z}$.

Please note that by the following lemma the constant fields of the difference fields given in Section 2 are all computable.

Lemma 4.3. *Any field of rational functions $\mathbb{Q}(x_1, \dots, x_r)$ is computable.*

Finally ΠΣ-fields are essentially defined by ΠΣ-extensions. Unlike Karr's definition in this work we force *additionally* that the constant fields are computable.

Definition 4.8. Let $(\mathbb{F}, \sigma)$ be a difference field with constant field $\mathbb{K}$. $(\mathbb{F}, \sigma)$ is called a ΠΣ-*field* over $\mathbb{K}$, if $\mathbb{K}$ is computable, $\mathbb{F} := \mathbb{K}(t_1) \dots (t_n)$ for $n \geq 0$ and $(\mathbb{F}(t_1, \dots, t_{i-1})(t_i), \sigma)$ is a ΠΣ-extension‡ of $(\mathbb{F}(t_1, \dots, t_{i-1}), \sigma)$ for all $1 \leq i \leq n$.

Example 4.2. All difference fields in Section 2 are ΠΣ-fields.

In [Sch02c, Theorem 3.1] it is shown that for each basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$ one can define a canonical representation among all is basis matrices. This property will play an important role in Subsection 7.3, more precisely in Theorem 7.8.

Theorem 4.4. *Let $(\mathbb{F}, \sigma)$ be a ΠΣ-field over $\mathbb{K}$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^n$ and $\boldsymbol{f} \in \mathbb{F}^n$. Then there is an algorithm based on* gcd*-computations and Gaussian elimination that transforms a basis matrix of* $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$ *to a uniquely determined basis matrix.*

Definition 4.9. Let $(\mathbb{F}, \sigma)$ be a ΠΣ-field over $\mathbb{K}$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^n$ and $\boldsymbol{f} \in \mathbb{F}^n$. Then the uniquely defined basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$ obtained by the algorithm given in Theorem 4.4 is called *normalized.*

In [Kar81] algorithms are developed that find for a given $\alpha \in \mathbb{F}^*$ all $n \in \mathbb{Z}$ with $\alpha^n \in \mathrm{H}_{(\mathbb{F},\sigma)}$. Moreover by results from [Kar81] or Theorem 7.4, one can compute a basis of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$ for some $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}^2$ and $\boldsymbol{f} \in \mathbb{F}^n$. Hence by Theorems 4.1 and 4.3 one can decide algorithmically, if a difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ is a ΠΣ-extension. Starting from a computable field $\mathbb{K}$, this observation allows to construct algorithmically ΠΣ-fields for a given summation problem as it is carefully introduced in [Sch01, Chapter 2.2.5].

‡For the case $i = 0$ this means that $(\mathbb{F}(t_1), \sigma)$ is a ΠΣ-extension of $(\mathbb{F}, \sigma)$.

5. Reduction Strategies in ΠΣ-fields

In this section the main ideas are sketched that enable to search for parameterized linear difference equations in a ΠΣ-field $(\mathbb{F}(t), \sigma)$. Given $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}(t)^m$ and $\boldsymbol{f} \in \mathbb{F}(t)^n$ one can apply the following reduction techniques to find a basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$.

$$\begin{tikzcd}
\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t)) \arrow[d, shift right, "\text{normalization}"'] \\
\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}(t)) \arrow[u, shift right, "\text{by simplification}"'] \arrow[d, shift right, "\text{denominator elimination}"'] \\
\mathrm{V}(\boldsymbol{a''}, \boldsymbol{f''}, \mathbb{F}[t]) \arrow[u, shift right, "\text{by denominator bounding}"'] \arrow[d, shift right, "\text{degree elimination}"'] \\
\mathrm{V}(\boldsymbol{a'''}, \boldsymbol{f'''}, \{0\}) \arrow[u, shift right, "\text{by incremental reduction}"']
\end{tikzcd} \tag{8}$$

In the next subsections I explain in more details the methods for the different reduction steps.

5.1. Simplifications and Some Special Cases

Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$, $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}(t)^m$ and $\boldsymbol{f} \in \mathbb{F}(t)^n$. Here I explain the reduction

$$\begin{tikzcd}
\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t)) \arrow[d, shift right, "\text{normalization}"'] \\
\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}(t)), \arrow[u, shift right, "\text{by simplification}"']
\end{tikzcd} \tag{9}$$

i.e. how one reduces the problem $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ to $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}(t))$ for some normalized $\boldsymbol{a'} = (a'_1, \dots, a'_{m'}) \in \mathbb{F}[t]^m$ and $\boldsymbol{f'} \in \mathbb{F}[t]^n$ such that $m' \leq m$ and

$$a'_1 \neq 0 \neq a'_{m'}. \tag{10}$$

Then the subgoal is to find a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}(t))$ for such a normalized $\boldsymbol{a'}$ and $\boldsymbol{f'}$ and to reconstruct a basis of the original solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$.

If $a_1 \neq 0$, set $l := 1$, otherwise define l with $1 \leq l \leq m$ such that $0 = a_1 = \dots = a_{l-1} \neq a_l$. Similarly, if $a_m \neq 0$, set $k := m$, otherwise define k with $1 \leq k \leq m$ such that $a_k \neq a_{k+1} = \dots = a_m = 0$. Then we have

$$\begin{aligned} \boldsymbol{c}\,\boldsymbol{f} = \sigma_{\boldsymbol{a}} f = a_l\, \sigma^{m-l}(g) + \dots + a_k\, \sigma^{m-k}(g) = \boldsymbol{c}\,\boldsymbol{f} \\ \Leftrightarrow\ \ \sigma^{k-m}(a_l)\, \sigma^{k-l}(g) + \dots + \sigma^{k-m}(a_k)\, g = \boldsymbol{c}\, \sigma^{k-m}(\boldsymbol{f}) \end{aligned}$$

where $\sigma^{k-m}(a_l) \neq 0 \neq \sigma^{k-m}(a_k)$. Therefore define

$$\boldsymbol{a'} := \big(\sigma^{k-m}(a_l), \dots, \sigma^{k-m}(a_k)\big) \ \text{ and } \ \boldsymbol{f'} := \sigma^{k-m}(\boldsymbol{f}) \tag{11}$$

with $\boldsymbol{a'} \in \mathbb{F}(t)^{k-l+1}$ and $\boldsymbol{f'} \in \mathbb{F}(t)^n$, and find a basis of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}(t))$. Then one can compute a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ by the relation

$$\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t)) = \big\{\boldsymbol{c} \wedge \sigma^{m-k}(g) \,|\, \boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}(t))\big\}. \tag{12}$$

Here the previous considerations are summarized.

Theorem 5.1. *Let $(\mathbb{F},\sigma)$ be a difference field, $\boldsymbol{a}\in\mathbb{F}^m$ and $\boldsymbol{f}\in\mathbb{F}^n$, and define l and k as above. Define $\boldsymbol{a'}=(a'_1,\dots,a'_m)\in\mathbb{F}^m$ and $\boldsymbol{f'}\in\mathbb{F}^n$ as in (11). Then $a'_1\,a'_m\neq 0$. If $\boldsymbol{C}\wedge\boldsymbol{g}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F})$ then $\boldsymbol{C}_\wedge\sigma^{m-k}(\boldsymbol{g})$ is a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$.*

Therefore without loss of generality one may assume $\boldsymbol{a}$ with $a_1\,a_m\neq 0$. By Theorem 5.2 one finally achieves the reduction as it is stated in (9). Here the essential property (Lemma 4.1) is used that $\mathbb{F}[t]$ is a polynomial ring.

Theorem 5.2. *Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$, $\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}(t)^m$ and $\boldsymbol{f}=(f_1,\dots,f_n)\in\mathbb{F}(t)^n$ where $a_i=\frac{a_{i1}}{a_{i2}}$ and $f_i=\frac{f_{i1}}{f_{i2}}$ are in reduced representation. Let $d:=\mathrm{lcm}(a_{1,2},\dots,a_{1m},f_{2,1},\dots,f_{2n})\in\mathbb{F}[t]^*$ and define the vectors $\boldsymbol{a'}:=(a_1\,d,\dots,a_m\,d)\in\mathbb{F}[t]^m$ and $\boldsymbol{f'}:=(f_1\,d,\dots,f_n\,d)\in\mathbb{F}[t]^n$. Then we have $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(t))$.*

Hence by applying Theorems 5.1 and 5.2, one can compute a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ by computing a basis matrix of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(t))$ where $\boldsymbol{a'}\in\mathbb{F}[t]^{m'}$ and $\boldsymbol{f'}\in\mathbb{F}[t]^n$ have properties as stated in (10)

A Special Reduction: In particular if $a_i=0$ for all $1<i<m$ one is able to reduce the problem further to a first order linear difference equation problem.

Theorem 5.3. *Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-field, $\boldsymbol{f}\in\mathbb{F}(t)^n$ and $\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}^m$ with $m>1$, $a_1\,a_m\neq 0$ and $a_i=0$ for all $1<i<m$. Then $(\mathbb{F}(t),\sigma^{m-1})$ is a ΠΣ-field and we have $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},(\mathbb{F}(t),\sigma))=\mathrm{V}((a_1,a_m),\boldsymbol{f},(\mathbb{F}(t),\sigma^{m-1}))$.*

Proof: By Theorem 4 in [Kar85] it follows that $(\mathbb{F}(t),\sigma^{m-1})$ is a ΠΣ-field. The equality of the two solution spaces follows immediately. □

Two Shortcuts: Moreover if $(0)\neq\boldsymbol{a}\in\mathbb{F}^1$, one obtains a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$.

Theorem 5.4. *Let $(\mathbb{F}(t),\sigma)$ be a difference field with constant field $\mathbb{K}$, $a\in\mathbb{F}^*$ and $\boldsymbol{f}=(f_1,\dots,f_n)\in\mathbb{F}^n$. Then $\mathbf{Id}_{\boldsymbol{n}\wedge}\frac{\boldsymbol{f}}{a}$ is a basis matrix of $\mathrm{V}((a),\boldsymbol{f},\mathbb{F}(t))$ where $\mathbf{Id}_{\boldsymbol{n}}$ is the identity matrix of length n.*

Proof: Let $\boldsymbol{e_i}\in\mathbb{K}^n$ be the i-th unit vector, i.e. $\boldsymbol{e_i}\wedge f_i$ is the i-th row vector in $\mathbf{Id}_{\boldsymbol{n}}\wedge\boldsymbol{f}$. Clearly the elements in $B:=\{\boldsymbol{e_1}\wedge\frac{f_1}{a},\dots,\boldsymbol{e_n}\wedge\frac{f_n}{a}\}$ are linearly independent vectors over $\mathbb{K}$ with $a\,\frac{f_i}{a}=\boldsymbol{e_i}\,\boldsymbol{f}$. Hence B is a basis of a subspace $\mathbb{V}$ of $\mathrm{V}((a),\boldsymbol{f},\mathbb{F}(t))$ over $\mathbb{K}$. Now assume that there is a $\boldsymbol{c}\wedge g:=(c_1,\dots,c_n)\wedge g\in\mathrm{V}((a),\boldsymbol{f},\mathbb{F}(t))\setminus\mathbb{V}$. Then $a\,g=\boldsymbol{c}\,\boldsymbol{f}=(\sum_{i=1}^n c_i\,\boldsymbol{e_i})\boldsymbol{f}=\sum_{i=1}^n c_i(\boldsymbol{e_i}\,\boldsymbol{f})=\sum_{i=1}^n c_i\,g_i$ and hence $\boldsymbol{c}\wedge g\in\mathbb{V}$, a contradiction. □

Moreover by [Kar81, Proposition 10] there is a shortcut that can be heavily used.

Lemma 5.1. *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$ and $\mathbb{V}$ be a subspace of $\mathbb{F}$ over $\mathbb{K}$. If $\mathbb{V}\cap\mathbb{K}=\mathbb{K}$ then the identity matrix $\mathbf{Id}_{\boldsymbol{n+1}}$ of length $n+1$, otherwise $\mathbf{Id}_{\boldsymbol{n}}\wedge\mathbf{0}_{\boldsymbol{n}}$ is a basis matrix of $\mathrm{V}((1,-1),\mathbf{0}_{\boldsymbol{n}},\mathbb{V})$.*

Proof: We have

$$\mathrm{V}(\boldsymbol{a},\boldsymbol{0_n},\mathbb{V})=\{(c_1,\dots,c_n,g)\in\mathbb{K}^n\times\mathbb{V}\,|\,\sigma(g)-g=c_1\,0+\dots+c_n\,0\}.$$

If $\mathbb{V}\cap\mathbb{K}=\mathbb{K}$ then $\{g\in\mathbb{V}\,|\,\sigma(g)-g=0\}=\mathbb{K}$ and therefore it follows that $\mathrm{V}((1,-1),\boldsymbol{0_n},\mathbb{V})=\mathbb{K}^n\times\mathbb{K}$. Hence $\mathbf{Id}_{\boldsymbol{n}}\wedge\boldsymbol{0_n}$ is a basis matrix of our solution space. Otherwise, we must have $\mathbb{V}=\{0\}$ and therefore it follows that $\mathrm{V}((1,-1),\boldsymbol{0_n},\mathbb{V})=\mathbb{K}^n\times\{0\}$. Then clearly $\mathbf{Id}_{\boldsymbol{n}}\wedge\boldsymbol{0_n}$ is a basis matrix of the solution space. □

5.2. The Denominator Bound Method for Denominator Eliminations

The denominator bound method was introduced by S. Abramov in [Abr89b, Abr95] for one of the most simplest ΠΣ-fields $(\mathbb{K}(t),\sigma)$ over $\mathbb{K}$ with $\sigma(t)=t+1$. Based on a generalization by M. Bronstein in [Bro00] the following denominator elimination technique turns out to be essential to search for all solutions of linear difference equations in the general setting of ΠΣ-fields.
Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$, $\boldsymbol{0}\neq\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}(t)^m$ with $a_1\,a_m\neq0$ and $\boldsymbol{f}\in\mathbb{F}(t)^n$. Here I will give the main idea how one can achieve the reduction

$$\begin{array}{ccc} & \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t)) & \\ \text{denominator elimination} & \downarrow\quad\uparrow & \text{by denominator bounding} \\ & \mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}[t]) & \end{array}\qquad(13)$$

for some $\boldsymbol{a'}\in\mathbb{F}[t]^m$ and $\boldsymbol{f'}\in\mathbb{F}[t]^n$. With this strategy one has to compute only a basis of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}[t])$ in the polynomial ring $\mathbb{F}[t]$ which then gives the possibility to reconstruct a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ in its quotient field $\mathbb{F}(t)$. In this reduction the simple Lemma 5.2 gives the main idea.

Lemma 5.2. *Let $(\mathbb{F},\sigma)$ be a difference field, $\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}^m$ and $d\in\mathbb{F}^*$. Then for $\boldsymbol{a}':=\left(\frac{a_1}{\sigma^{m-1}(d)},\dots,\frac{a_{m-1}}{\sigma(d)},\frac{a_m}{d}\right)\in\mathbb{F}^m$ and $g\in\mathbb{F}$ we have $\sigma_{\boldsymbol{a}}g=\sigma_{\boldsymbol{a'}}(g\,d)$.*

Proof: We have

$$\begin{aligned}\sigma_{\boldsymbol{a}}g&=a_1\,\sigma^{m-1}(g)+\dots+a_{m-1}\,\sigma(g)+a_m\,g\\&=a_1\,\frac{\sigma^{m-1}(d)}{\sigma^{m-1}(d)}\,\sigma^{m-1}(g)+\dots+a_{m-1}\,\frac{\sigma(d)}{\sigma(d)}\,\sigma(g)+\frac{d}{d}\,a_m\,g\\&=\frac{a_1}{\sigma^{m-1}(d)}\,\sigma^{m-1}(g\,d)+\dots+\frac{a_{m-1}}{\sigma(d)}\,\sigma(g\,d)+\frac{a_m}{d}\,d\,g=\sigma_{\boldsymbol{a'}}(g\,d).\end{aligned}$$

□

The following proposition will lead to Theorem 5.5 which delivers the basic reduction of the denominator bound method. Furthermore this proposition is needed in Section 7.3, Theorem 7.7, to prove correctness of Algorithm 7.3.

Proposition 5.1. *Let* $(\mathbb{F},\sigma)$ *be a difference field with constant field* $\mathbb{K}$, $\mathbf{0}\neq\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}^m$ *and* $\boldsymbol{f}\in\mathbb{F}^n$. *Let* $d\in\mathbb{F}(t)^*$ *and set* $\boldsymbol{a}':=(\frac{a_1}{\sigma^{m-1}(d)},\dots,\frac{a_{m-1}}{\sigma(d)},\frac{a_m}{d})\in\mathbb{F}^m$. *If* $\boldsymbol{C}\wedge\boldsymbol{g}$ *is a basis matrix of a subspace of* $\mathrm{V}(\boldsymbol{a}',\boldsymbol{f},\mathbb{F})$ *over* $\mathbb{K}$ *then* $\boldsymbol{C}\wedge\frac{\boldsymbol{g}}{d}$ *is a basis matrix of a subspace of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$ *over* $\mathbb{K}$.

Proof: Let $\boldsymbol{C}\wedge\boldsymbol{g}$ be a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a}',\boldsymbol{f},\mathbb{F})$ over $\mathbb{K}$ for some $\boldsymbol{C}\in\mathbb{K}^{\lambda\times n}$ and $\boldsymbol{g}\in\mathbb{F}^\lambda$. Since the row vectors of $\boldsymbol{C}\wedge\boldsymbol{g}$ are linearly independent over $\mathbb{K}$, the row vectors of $\boldsymbol{C}\wedge\frac{\boldsymbol{g}}{d}$ are also linearly independent over $\mathbb{K}$. Moreover for any $\boldsymbol{k}\in\mathbb{K}^\lambda$ we have $\boldsymbol{k}\cdot(\boldsymbol{C}\wedge\frac{\boldsymbol{g}}{d})\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$ by Lemma 5.2. Hence $\boldsymbol{C}\wedge\frac{\boldsymbol{g}}{d}$ is a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$ over $\mathbb{K}$. $\square$

Next we introduce the subset $\mathbb{F}(t)^{(frac)}$ of $\mathbb{F}(t)$ as

$$\mathbb{F}(t)^{(frac)}:=\{\frac{p}{q}\in\mathbb{F}(t)\mid \tfrac{p}{q}\text{ is in reduced representation and}\deg(p)<deg(q)\}.$$

Clearly $\mathbb{F}[t]$ and $\mathbb{F}(t)^{(frac)}$ are subspaces of $\mathbb{F}(t)$ over $\mathbb{K}$. By polynomial division with remainder the following direct sum of vector spaces holds:

$$\mathbb{F}(t)=\mathbb{F}[t]\oplus\mathbb{F}(t)^{(frac)}.$$

In the reduction indicated by (13), the basic idea is to compute a particular $d\in\mathbb{F}[t]^*$ such that

$$\forall\boldsymbol{c}\wedge g\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]\oplus\mathbb{F}(t)^{(frac)}):\ d\,g\in\mathbb{F}[t]. \tag{14}$$

It is immediate that such a specific $d\in\mathbb{F}[t]^*$ bounds the denominator.

Definition 5.1. Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-extension of $(\mathbb{F},\sigma)$, $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{F}[t]^m$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$. Then $d\in\mathbb{F}[t]^*$ fulfilling condition (14) is called *denominator bound* of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$.

Theorem 5.5. *Let* $(\mathbb{F}(t),\sigma)$ *be a ΠΣ-extension of* $(\mathbb{F},\sigma)$ *with constant field* $\mathbb{K}$, $\mathbf{0}\neq\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}[t]^m$ *and* $\boldsymbol{f}\in\mathbb{F}[t]^n$. *Let* $d\in\mathbb{F}[t]^*$ *be a denominator bound of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ *and define* $\boldsymbol{a}':=(\frac{a_1}{\sigma^{m-1}(d)},\dots,\frac{a_{m-1}}{\sigma(d)},\frac{a_m}{d})\in\mathbb{F}(t)^m$. *If* $\boldsymbol{C}\wedge\boldsymbol{g}$ *is a basis matrix of* $\mathrm{V}(\boldsymbol{a}',\boldsymbol{f},\mathbb{F}[t])$ *then* $\boldsymbol{C}\wedge\frac{\boldsymbol{g}}{d}$ *is a basis matrix of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$.

Proof: By Lemma 5.2 it follows that

$$\boldsymbol{c}\wedge g\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))\Leftrightarrow\sigma_{\boldsymbol{a}}g=\boldsymbol{c}\,\boldsymbol{f}\Leftrightarrow\sigma_{\boldsymbol{a}'}(d\,g)=\boldsymbol{c}\,\boldsymbol{f}.$$

Since $d\,g\in\mathbb{F}[t]$ by property (14), we have

$$\boldsymbol{c}\wedge g\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))\Leftrightarrow\boldsymbol{c}\wedge(d\,g)\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]). \tag{15}$$

Let $\boldsymbol{C}\wedge\boldsymbol{g}$ be a basis matrix of $\mathrm{V}(\boldsymbol{a}',\boldsymbol{f},\mathbb{F}[t])$. Then by Proposition 5.1 $\boldsymbol{C}\wedge\frac{\boldsymbol{g}}{d}$ is a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$. Therefore by (15) $\boldsymbol{C}\wedge\frac{\boldsymbol{g}}{d}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$. $\square$

Hence by applying Theorem 5.5 one obtains $\boldsymbol{a}'\in\mathbb{F}(t)^m$ such that $\boldsymbol{C}\wedge\frac{\boldsymbol{g}}{d}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$, if $\boldsymbol{C}\wedge\boldsymbol{g}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{F}[t])$. So by clearing denominators in $\boldsymbol{a'}$ (Theorem 5.2) one succeeds in the reduction as stated in (13).

Remarks on the denominator bound problem in ΠΣ-fields: Using results from [Sch02a], which are based on [Kar81, Kar85, Bro00], there exists an algorithm with Specification 7.3 that solves the following problem in a ΠΣ-field $(\mathbb{F}(t), \sigma)$. If $(\mathbb{F}(t), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$, a denominator bound d of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$ can be computed. Otherwise, if $(\mathbb{F}(t), \sigma)$ is a Π-extension of $(\mathbb{F}, \sigma)$, one is able to compute a $u \in \mathbb{F}[t]^*$ such that $t^x\, u$ is a denominator bound for a big enough chosen $x \in \mathbb{N}_0$. If in addition $\boldsymbol{a} \in \mathbb{F}[t]^2$, such an $x \in \mathbb{N}_0$ can be also computed. Consequently there exists an algorithm with Specification 7.1 that solves the denominator bound problem for first order linear difference equations. This will result in Section 7.2 to an algorithm that solves parameterized first order linear difference equations in ΠΣ-fields in full generality. Moreover in [Sch02a] there are several investigations to solve the denominator bound problem in Π-extensions; here one is capable of determining an $x \in \mathbb{N}_0$ as described above for further subclasses of linear difference equations.

5.3. The Incremental Reduction for Polynomial Degree Eliminations

Whereas in this subsection an "oversimplified" sketch is given how the incremental reduction method for the polynomial degree elimination works, in Section 6 this incremental reduction method will be further analyzed and explained.
Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$, $\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ with $a_1\, a_m \neq 0$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. In the degree elimination strategy the goal is to reduce the problem from computing a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$ to computing a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f'}, \{0\})$

$$\begin{tikzcd} \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]) \arrow[d, shift right, "\text{degree elimination}"'] \\ \mathrm{V}(\boldsymbol{a}, \boldsymbol{f'}, \{0\}) \arrow[u, shift right, "\text{by incremental reduction}"'] \end{tikzcd}$$

for some $\boldsymbol{f'} \in \mathbb{F}[t]^\lambda$. Then in a second step one has to reconstruct the basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$ by a lifting process.

Determination of a degree bound: In a first step one tries to find a bound $b \in \mathbb{N}_0 \cup \{-1\}$ such that for all $\boldsymbol{c}\wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$ one has $\deg(g) \leq b$. Of course, for any $d \in \mathbb{N}_0 \cup \{-1\}$,

$$\mathbb{F}[t]_d := \{f \in \mathbb{F}[t] \mid \deg(f) \leq d\}$$

is a finite subspace of $\mathbb{F}[t]$ over $\mathbb{K}$. In particular we have $\mathbb{F}[t]_{-1} = \{0\}$. In other words, we try to find a $b \in \mathbb{N}_0 \cup \{-1\}$ such that

$$\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]) = \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_b). \tag{16}$$

Additionally we will assume that

$$b \geq \max(-1, \|\boldsymbol{f}\| - \|\boldsymbol{a}\|) \tag{17}$$

which guarantees that $\boldsymbol{f} \in \mathbb{F}[t]_{\|\boldsymbol{a}\|+b}$ by Lemma 4.2. As it will be shown later this is a necessary condition in order to proceed in the degree elimination technique.

Definition 5.2. Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. $b \in \mathbb{Z}$ is called *degree bound* of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$ if (16) and (17) hold.

In [Sch02b] one is focused to determine degree bounds for various subclasses of linear difference equations. In particular this work enables to determine degree bounds of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$, if $(\mathbb{F}(t), \sigma)$ is a $\Pi\Sigma$-field and $\boldsymbol{a} \in \mathbb{F}[t]^2$; more precisely there exists an algorithm that fulfills Specification 7.2. Together with the denominator elimination method introduced in the previous subsection and the incremental reduction technique that will be explained further this leads in Section 7.2 to algorithms that solve first order linear difference equations in $\Pi\Sigma$-fields.

Degree elimination: If one finds such a *degree bound* b of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$, one tries to eliminate the degrees by an incremental reduction technique.

$$\begin{tikzcd}
\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_b) \arrow[d, shift right] & & \\
\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_{b-1}}, \mathbb{F}[t]_{b-1}) \arrow[u, shift right] \arrow[r, shift right] & \dots \arrow[l, shift right] \arrow[r, shift right] & \mathrm{V}(\boldsymbol{a}, \boldsymbol{f_0}, \mathbb{F}[t]_0) \arrow[l, shift right] \arrow[d, shift right] \\
& & \mathrm{V}(\boldsymbol{a}, \boldsymbol{f_{-1}}, \mathbb{F}[t]_{-1}) \arrow[u, shift right]
\end{tikzcd} \tag{18}$$

where $\boldsymbol{f_d} \in \mathbb{F}[t]^{\lambda_d}_{\|\boldsymbol{a}\|+d}$ for $-1 \leq d \leq b$ with $\lambda_d \in \mathbb{N}$. This has to be read as follows: First has to compute a basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_{d-1}}, \mathbb{F}[t]_{d-1})$ for a specific $\boldsymbol{f_{d-1}} \in \mathbb{F}[t]^{\lambda_{d-1}}_{\|\boldsymbol{a}\|+d-1}$ which then allows to construct the basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f_d}, \mathbb{F}[t]_d)$. How this reduction works in details will be explained in Section 6.

Example 5.1. Consider the $\Pi\Sigma$-field $(\mathbb{Q}(t_1, t_2), \sigma)$ over $\mathbb{Q}$ canonically defined by $\sigma(t_1) = t_1 + 1$ and $\sigma(t_2) = t_2 + \frac{1}{t_1+1}$ and set $\mathbb{F} := \mathbb{Q}(t_1)$. In order to find a basis matrix of $\mathrm{V}((1, -1), (t_2), \mathbb{F}(t_2))$, one first computes a denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$, in this case 1. Hence $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t)) = \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$. Then after computing a degree bound $b = 2$ of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t])$ by algorithms given in [Sch02b] one applies the incremental reduction technique.

$$\begin{tikzcd}
& \mathrm{V}((1,-1),(t_2),\mathbb{F}(t_2)) \arrow[d, equal] \\
\mathrm{V}((1,-1),(t_2),\mathbb{F}[t_2]_2) \arrow[d, shift right] \arrow[r, phantom, "\overset{\text{degree bound } b=2}{=}"] & \mathrm{V}((1,-1),(t_2),\mathbb{F}[t_2]) \\
\mathrm{V}\left((1,-1),\left(\frac{-1-2t_2-2t_1t_2}{(1+t_1)^2}, t_2\right),\mathbb{F}[t_2]_1\right) \arrow[u, shift right] \arrow[r, shift right] & \mathrm{V}\left((1,-1),\left(-\frac{1}{t_1+1},-1\right),\mathbb{F}[t_2]_0\right) \arrow[l, shift right] \arrow[d, shift right] \\
& \mathrm{V}((1,-1),(0,0),\mathbb{F}[t_2]_{-1}) \arrow[u, shift right]
\end{tikzcd}$$

In particular for the base case it follows that

$$\begin{aligned}
\mathrm{V}((1,-1),(0,0),\mathbb{F}[t_2]_{-1}) &= \left\{(c_1, c_2, g) \in \mathbb{Q}^2 \times \{0\} \mid \sigma(g) - g = c_1\, 0 + c_2\, 0\right\} \\
&= \{c_1\ (1,0,0) + c_2\ (0,1,0) \mid c_1, c_2 \in \mathbb{Q}\}
\end{aligned}$$

and one obtains the basis matrix $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$ of $\mathrm{V}((1,-1),(0,0),\mathbb{F}[t_2]_{-1})$. Later the reduction step, which reduces the problem from the solution range $\mathbb{F}[t_2]_1$ to $\mathbb{F}[t_2]_0$, will be considered in more details in Example 6.4. Finally with these reduction techniques one determines the basis matrix $\left(\begin{smallmatrix}0 & & 1\\ 1 & t_1 & (t_2-1)\end{smallmatrix}\right)$ of $\mathrm{V}((1,-1),(t_2),\mathbb{F}(t_2))$.

Example 5.2. Consider the ΠΣ-field $(\mathbb{Q}(t_1,t_2),\sigma)$ over $\mathbb{Q}$ canonically defined by $\sigma(t_1)=t_1+1$ and $\sigma(t_2)=(t_1+1)\,t_2$. In order to find the solution $g:=t_2$ of $\sigma(g)-g=t_1\,t_2$ in Section 2.1, the following incremental reduction process is involved.

$$\begin{tikzcd}
 & \mathrm{V}((1,-1),(t_1\,t_2),\mathbb{Q}(t_1)(t_2)) \arrow[d, equal] \\
\mathrm{V}((1,-1),(t_1\,t_2),\mathbb{Q}(t_1)[t_2]_1) \arrow[r, phantom, "\overset{\text{degree bound } b=1}{=}"] \arrow[d, shift right] & \mathrm{V}((1,-1),(t_1 t_2),\mathbb{Q}(t_1)[t_2]) \\
\mathrm{V}((1,-1),(0),\mathbb{Q}(t_1)[t_2]_0) \arrow[u, shift right] \arrow[r, shift left] & \mathrm{V}((1,-1),(0),\mathbb{Q}(t_1)[t_2]_{-1}) \arrow[l, shift left]
\end{tikzcd}$$

The complete reduction process for all subproblems is given in Example 7.1.

5.4. The First Base Case

As can be seen in Section 5.3, one has to compute a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f_{-1}},\{0\})$ in the end of the incremental reduction. Theorem 5.6 allows us to reduce this problem to a nullspace problem of $\mathbb{F}$ as a vector space over $\mathbb{K}$.

Definition 5.3. Let $\mathbb{F}$ be a vector space over $\mathbb{K}$ and consider $\mathbb{F}^n$ as a vector space over $\mathbb{K}$. Let $\boldsymbol{f}\in\mathbb{F}^n$. Then $\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f})=\{\boldsymbol{c}\in\mathbb{K}^n\,|\,\boldsymbol{c}\,\boldsymbol{f}=0\}$ is called the *nullspace* of $\boldsymbol{f}$ over $\mathbb{K}$.

If one considers $\mathbb{F}$ as a vector space over $\mathbb{K}$, $\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f})$ is clearly a subspace of $\mathbb{F}^n$ over $\mathbb{K}$. The next simple result relates $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\{0\})$ with $\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f})$.

Theorem 5.6. *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$ and assume $\boldsymbol{0}\neq\boldsymbol{a}\in\mathbb{F}^m$ and $\boldsymbol{f}\in\mathbb{F}^n$. Then $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\{0\})=\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f})\times\{0\}$.*

Proof: We have

$$\begin{aligned}
\boldsymbol{c}\wedge g\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\{0\}) &\Leftrightarrow \sigma_{\boldsymbol{a}}g=\boldsymbol{c}\,\boldsymbol{f}\ \&\ g=0\\
&\Leftrightarrow \boldsymbol{c}\,\boldsymbol{f}=\boldsymbol{0}\ \&\ g=0\\
&\Leftrightarrow \boldsymbol{c}\in\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f})\ \&\ g=0\\
&\Leftrightarrow \boldsymbol{c}\wedge g\in\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f})\times\{0\}.
\end{aligned}$$

□

Finally a basis matrix of $\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f})$ can be computed by linear algebra.

Lemma 5.3. *Let $(\mathbb{F},\sigma)$ be a ΠΣ-field over $\mathbb{K}$ and $\boldsymbol{f}\in\mathbb{F}^n$. Then $\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f})$ is a finite dimensional subspace of $\mathbb{K}^n$ whose basis can be computed by linear algebra.*

Proof: Let $\boldsymbol{f}=(f_1,\dots,f_n)\in\mathbb{F}^n$. Since $\mathbb{F}$ is a ΠΣ-field, it follows that $\mathbb{F}:=\mathbb{K}(t_1,\dots,t_e)$ can be written as the quotient field of a polynomial ring $\mathbb{K}[t_1,\dots,t_e]$. We can find a $d\in\mathbb{K}[t_1,\dots,t_e]^*$ such that

$$\boldsymbol{g}=(g_1,\dots,g_n):=(f_1\,d,\dots,f_n\,d)\in\mathbb{K}[t_1,\dots,t_e].$$

For $\boldsymbol{c} \in \mathbb{K}^n$ we have $\boldsymbol{c}\,\boldsymbol{f} = 0$ if and only if $\boldsymbol{c}\,\boldsymbol{g} = 0$ and therefore

$$\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f}) = \mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{g}).$$

Let $c_1, \dots, c_n$ be indeterminates and make the ansatz

$$c_1\, g_1 + \dots + c_n\, g_n = 0.$$

Then the coefficients of each monomial $t_1^{d_1} \dots t_e^{d_e}$ in $c_1\, g_1 + \dots + c_n\, g_n$ must vanish. Therefore we get a linear system of equations

$$\begin{array}{c} c_1\, p_{11} + \dots + c_n\, p_{1n} = 0 \\ \vdots \\ c_r\, p_{r1} + \dots + c_n\, p_{rn} = 0 \end{array} \tag{19}$$

where each equation corresponds to a coefficient of a monomial which must vanish. Since $p_{ij} \in \mathbb{K}$, finding all $(c_1, \dots, c_n) \in \mathbb{K}^n$ which are a solution of (19) is a simple linear algebra problem. In particular applying Gaussian elimination we get immediately a basis for the vector space

$$\{\boldsymbol{c} \in \mathbb{K}^n \,|\, \boldsymbol{c} \text{ is a solution of } (19)\},$$

thus for $\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{g})$ and consequently also for $\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f})$. □

6. The Incremental Reduction

In this section the incremental reduction method will be considered in details which enables to eliminate the polynomial degrees of the possible solutions as it was already illustrated in Section 5.3. More precisely one is concerned in computing a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$ for $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ with $l := \|\boldsymbol{a}\|$ and $\boldsymbol{f} \in \mathbb{F}[t]^n_{d+l}$ for some $d \in \mathbb{N}_0 \cup \{-1\}$. In particular if $d = -1$, one knows how to compute a basis matrix by linear algebra as it is described in Subsection 5.4. So in the sequel we assume that $d \in \mathbb{N}_0$.

6.1. A First Closer Look

In the sequel consider $\mathbb{F}[t]_{d-1}$ as a subspace of $\mathbb{F}[t]_d$ over $\mathbb{K}$ and

$$t^d\, \mathbb{F} := \left\{ f\, t^d \,|\, f \in \mathbb{F} \right\}$$

as a subspace of $\mathbb{F}[t]_d$ over $\mathbb{K}$. Then the following direct sum

$$\mathbb{F}[t]_d = \mathbb{F}[t]_{d-1} \oplus t^d\, \mathbb{F} \tag{20}$$

follows immediately. In diagram (18) of Section 5.3 it was already indicated to achieve the degree elimination

$$\begin{array}{c} \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d) \\ \downarrow \quad \uparrow \\ \mathrm{V}(\boldsymbol{a}, \tilde{\boldsymbol{f}}_{\boldsymbol{d-1}}, \mathbb{F}[t]_{d-1}) \end{array} \tag{21}$$

for some $\tilde{\boldsymbol{f}} \in \mathbb{F}[t]^{\lambda}_{d+l-1}$ with $\lambda \geq 1$. As will be explained in the sequel one first tries to solve some kind of difference equation problem in $t^d\,\mathbb{F}$, say $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$, which will be introduced in Definition 6.1. Having this solution in hands, one computes an $\tilde{\boldsymbol{f}} \in \mathbb{F}[t]^{\lambda}_{d+l-1}$ for some $\lambda \geq 1$ and tries to solve the problem $\mathrm{V}(\boldsymbol{a}, \tilde{\boldsymbol{f}}, \mathbb{F}[t]_{d-1})$. Then finally one can derive a solution for the original problem $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$ by using the solutions of $\mathrm{V}(\boldsymbol{a}, \tilde{\boldsymbol{f}}, \mathbb{F}[t]_{d-1})$ and $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$. In other words the solution in $\mathbb{F}[t]_d$ is obtained by combining solutions in $t^d\,\mathbb{F}$ and $\mathbb{F}[t]_{d-1}$ from specific subproblems. This is intuitively reflected by equation (20).
Finally the incremental solution space is introduced.

Definition 6.1. Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$. Let $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ with $l := \|\boldsymbol{a}\|$ and let $\boldsymbol{f} \in \mathbb{F}[t]_{d+l}$ for some $d \in \mathbb{N}_0$. We define the *incremental solution space* by

$$\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F}) := \{\boldsymbol{c} \wedge g \in \mathbb{K}^n \times t^d\,\mathbb{F} \mid \sigma_{\boldsymbol{a}} g - \boldsymbol{c}\boldsymbol{f} \in \mathbb{F}[t]_{d+l-1}\}.$$

Clearly the incremental solution space $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$ is a vector space over $\mathbb{K}$. In the next subsection it is shown that the incremental solution space is a finite dimensional vector space over $\mathbb{K}$, and it is explained how one can obtain a basis by solving parameterized linear difference equations in the difference field $(\mathbb{F}, \sigma)$. Having this in mind, the degree elimination (21) is done as follows:

$$\begin{tikzcd}
\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d) \arrow[dr, shift left, "1."'] & \\
& \mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F}) \arrow[ul, shift left, "3."'] \arrow[dl, "2."] \\
\mathrm{V}(\boldsymbol{a}, \tilde{\boldsymbol{f}}, \mathbb{F}[t]_{d-1}) \arrow[uu, "3."] &
\end{tikzcd} \tag{22}$$

1. First one attempts to compute a basis matrix of $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$.
2. With this basis matrix a specific $\tilde{\boldsymbol{f}} \in \mathbb{F}[t]^{\lambda}_{d+l-1}$, $\lambda \geq 1$, is computed which is explained later. Now one tries to compute a basis matrix of $\mathrm{V}(\boldsymbol{a}, \tilde{\boldsymbol{f}}, \mathbb{F}[t]_{d-1})$.
3. Given the basis matrices of $\mathrm{V}(\boldsymbol{a}, \tilde{\boldsymbol{f}}, \mathbb{F}[t]_{d-1})$ and $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$ one finally can compute a basis matrix of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$.

Finally we try to motivate how this specific $\tilde{\boldsymbol{f}}$ is computed. If $g \in t^d\,\mathbb{F}$, by Lemma 4.2 it follows that $\|\sigma_{\boldsymbol{a}} g\| \leq \|\boldsymbol{a}\| + \|g\| = l + d$. Furthermore for any $\boldsymbol{c} \in \mathbb{K}^n$ we have $\|\boldsymbol{c}\,\boldsymbol{f}\| \leq l + d$. In other words, the incremental solution space $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$ delivers us all $\boldsymbol{c} \in \mathbb{K}^n$ and all elements $g \in t^d\,\mathbb{F}$ such that the $l + d$-th coefficient, the coefficient of highest possible degree, of the polynomial $\sigma_{\boldsymbol{a}} g - \boldsymbol{c}\boldsymbol{f} \in \mathbb{F}[t]_{l+d-1}$ vanishes. This will be exactly the key-property for the reduction. Namely, if the set $\{\boldsymbol{c_1} \wedge g_1, \dots, \boldsymbol{c_\lambda} \wedge g_\lambda\}$ is a basis of $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$, define $\tilde{\boldsymbol{f}} := (h_1, \dots, h_\lambda)$ with

$$h_i := \sigma_{\boldsymbol{a}} g_i - \boldsymbol{c_i}\boldsymbol{f} \in \mathbb{F}[t]_{l+d-1}.$$

Then computing a basis of the solution space $\mathrm{V}(\boldsymbol{a}, \tilde{\boldsymbol{f}}, \mathbb{F}[t]_{d-1})$ will allow us to lift the problem to $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$ as it will be described further in Section 6.4.

6.2. An Algebraic Context: Filtrations and Graduations

As already described above, the problem $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$ is reduced to subproblems $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$ and $\mathrm{V}(\boldsymbol{a}, \tilde{\boldsymbol{f}}, \mathbb{F}[t]_{d-1})$. Then computing those basis matrices allows us to reconstruct a basis matrix $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$. Looking closer at (20) one obtains the direct sum $\mathbb{F}[t]_d = \bigoplus_{i=0}^d t^i\,\mathbb{F}$ of $\mathbb{F}[t]_d$. More generally there is the direct sum $\mathbb{F}[t] = \bigoplus_{i\in\mathbb{N}_0} t^i\,\mathbb{F}$ where $t^i\,\mathbb{F}$ is interpreted a subspace of $\mathbb{F}[t]$ over $\mathbb{K}$. Since

$$(t^i\,\mathbb{F})(t^j\,\mathbb{F}) = t^{i+j}\,\mathbb{F}.$$

for any $i, j \in \mathbb{N}_0$ the sequence $\langle t^i\,\mathbb{F}\rangle_{i\in\mathbb{N}_0}$ is a *graduation* of $\mathbb{F}[t]$. Furthermore we have that

$$\forall i, j \in \mathbb{N}_0\; \mathbb{F}[t]_i\,\mathbb{F}[t]_j = \mathbb{F}[t]_{i+j} \qquad \text{and} \qquad \bigcup_{i\in\mathbb{N}_0} \mathbb{F}[t]_i = \mathbb{F}[t],$$

and consequently $\langle\mathbb{F}[t]_i\rangle_{i\in\mathbb{N}_0}$ is a *filtration* of $\mathbb{F}[t]$. If one goes on in the reduction (22), see also Section 6.5, one actually computes the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$ by computing incremental solution spaces in the filtration $\langle t^i\,\mathbb{F}\rangle_{i\in\mathbb{N}_0}$ of $\mathbb{F}[t]$ and obtains step by step solution spaces in the graduation $\langle\mathbb{F}[t]_i\rangle_{i\in\mathbb{N}_0}$ of $\mathbb{F}[t]$.

6.3. The Incremental Solution Space

In the sequel we will explore some properties of the incremental solution space $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$, namely that it is a finite vector space over the constant field $\mathbb{K}$, and how one can find a basis matrix of $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$.

Example 6.1. Consider the ΠΣ-field $(\mathbb{Q}(t_1, t_2), \sigma)$ over $\mathbb{Q}$ canonically defined by $\sigma(t_1) = t_1 + 1$ and $\sigma(t_2) = t_2 + \frac{1}{t_1+1}$. For $c_1, c_2 \in \mathbb{Q}$ and $w \in \mathbb{Q}(t_1)$ we have

$$
\begin{aligned}
&(c_1, c_2, t_2\,w) \in \mathrm{I}\Big((1,-1), \big(\tfrac{-1-2\,t_2-2\,t_1\,t_2}{(1+t_1)^2}, t_2\big), t_2\,\mathbb{Q}(t_1)\Big)\\
\overset{d=1,l=0}{\Leftrightarrow}\quad & c_1\frac{-1-2\,t_2-2\,t_1\,t_2}{(1+t_1)^2} + c_2\,t_2 - \big(\boxed{\sigma(t_2\,w)} - t_2\,w\big) \in \mathbb{Q}(t_1)\\
\Leftrightarrow\quad & c_1\Big(-\frac{1}{(t_1+1)^2} - \frac{2}{t_1+1}\,t_2\Big) + c_2\,t_2 - \Big(\boxed{(t_2 + \frac{1}{t_1+1})\,\sigma(w)} - t_2\,w\Big) \in \mathbb{Q}(t_1)\\
\Leftrightarrow\quad & \frac{-2\,c_1}{t_1+1} + c_2 - (\sigma(w) - w) = 0 \quad\Leftrightarrow\quad (c_1, c_2, w) \in \mathrm{V}\Big((1,-1), \Big(\frac{-2}{t_1+1}, 1\Big), \mathbb{Q}(t_1)\Big).
\end{aligned}
$$

This observation enables to compute a basis matrix of $\mathrm{I}\big((1,-1), (\frac{1}{(t_1+1)^2}, 1), \mathbb{Q}(t_1)\big)$:

1. Computed a basis matrix $\left(\begin{smallmatrix} 0 & 0 & 1\\ 0 & 1 & t_1\end{smallmatrix}\right)$ of $\mathrm{V}\big((1,-1), (\frac{1}{(t_1+1)^2}, 1), \mathbb{Q}(t_1)\big)$.
2. Then $\left(\begin{smallmatrix} 0 & 0 & t_2\\ 0 & 1 & t_2\,t_1\end{smallmatrix}\right)$ is a basis matrix of $\mathrm{V}\big((1,-1), (\frac{1}{(t_1+1)^2}, 1), \mathbb{Q}(t_1)\big)$.

Example 6.2. Consider the ΠΣ-field $(\mathbb{Q}(t_1, t_2), \sigma)$ defined by $\sigma(t_1) = t_1 + 1$ and

$\sigma(t_2) = (t_1+1)\,t_2$; let $\boldsymbol{f} := (1 + (3 + 3\,t_1 + t_1^2)\,t_2 + (6 + 10\,t_1 + 6\,t_1^2 + t_1^3)\,t_2^2) = (f)$ and $\boldsymbol{a} := (t_2, 1, -t_2, t_2)$. Then for $c \in \mathbb{Q}$ and $w \in \mathbb{Q}(t_1)$ we have

$$\begin{aligned}
(c, t_2\,w) \in \mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t_2\,\mathbb{Q}(t_1)) &\overset{\text{d=1,l=1}}{\Leftrightarrow} c\,f - \sigma_{\boldsymbol{a}}(t_2\,w) \in \mathbb{Q}(t_1)[t_2]_1\\
\Leftrightarrow\quad & c\,f - (t_2\,\boxed{\sigma^3(w\,t_2)} + \sigma^2(t_2\,w) - t_2\,\sigma(t_2\,w) + t_2^2\,w) \in \mathbb{Q}(t_1)[t_2]_1\\
\Leftrightarrow\quad & c\,f - (t_2\,\boxed{(t_1+3)(t_1+2)(t_1+1)\,t_2\,\sigma^3(w)} +\\
& \qquad (t_1+2)(t_1+1)\,t_2\,\sigma^2(w) - t_2\,(t_1+1)\,t_2\sigma(w) + t_2^2\,w) \in \mathbb{Q}(t_1)[t_2]_1\\
\Leftrightarrow\quad & c\,(6 + 10\,t_1 + 6\,t_1^2 + t_1^3) - \big((t_1+3)(t_1+2)(t_1+1)\,\sigma^3(w) - (t_1+1)\,\sigma(w) + w\big) = 0\\
\Leftrightarrow\quad & (c, w) \in \mathrm{V}(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{f}}, \mathbb{Q}(t_1))
\end{aligned}$$

where $\tilde{\boldsymbol{f}} := (6 + 10\,t_1 + 6\,t_1^2 + t_1^3)$ and $\tilde{\boldsymbol{a}} := ((t_1+3)(t_1+2)(t_1+1), 0, -(t_1+1), 1)$.

The last example motivates us to define the so called σ-factorial, a generalization of the usual factorials.

Definition 6.2. Let $(\mathbb{F}, \sigma)$ be a difference field. Then we define the σ-factorial of $f \in \mathbb{F}$ shifted with $k \in \mathbb{N}_0$ by $(f)_k = \prod_{i=0}^{k-1} \sigma^i(f)$.

Example 6.3. Let $(\mathbb{F}(t), \sigma)$ be a Π-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\,t$. Then for $k \geq 0$ we have $\sigma^k(t) = (\alpha)_k\,t$.

Lemma 6.1 summarizes the observations from the previous Examples 6.1 and 6.2

Lemma 6.1. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ canonically defined by $\sigma(t) = \alpha\,t + \beta$ for some $\alpha \in \mathbb{F}^*$, $\beta \in \mathbb{F}$. Let $\mathbf{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ with $l := \|\boldsymbol{a}\|$ and $\boldsymbol{f} \in \mathbb{F}[t]^n_{d+l}$ for some $d \in \mathbb{N}_0$. Then*

$$\boldsymbol{c}\wedge(w\,t^d) \in \mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F}) \Leftrightarrow \boldsymbol{c}\wedge w \in \mathrm{V}(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{f}}, \mathbb{F})$$

where $\mathbf{0} \neq \tilde{\boldsymbol{a}} := \big([a_1]_l\;(\alpha)^d_{m-1}, \dots, [a_m]_l\,(\alpha)^d_0\big) \in \mathbb{F}^m$ and $\tilde{\boldsymbol{f}} := [\boldsymbol{f}]_{d+l} \in \mathbb{F}^n$.

Proof: We have

$$\begin{gathered}
\boldsymbol{c}\wedge(w\,t^d) \in \mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})\\
\Updownarrow\\
\sigma_{\boldsymbol{a}}(w\,t^d) - \boldsymbol{c}\,\boldsymbol{f} \in \mathbb{F}[t]_{d+l-1}\\
\Updownarrow\\
a_1\,\sigma^{m-1}(w\,t^d) + \dots + a_m\,w\,t^d - \boldsymbol{c}\,\boldsymbol{f} \in \mathbb{F}[t]_{d+l-1}\\
\Updownarrow\\
a_1\,\sigma^{m-1}(w)\;(\alpha)^d_{m-1}\;t^d + \dots + a_{m-1}\,\sigma(w)\,\alpha^d\,t^d + a_m\,w\,t^d - \boldsymbol{c}\,\boldsymbol{f} \in \mathbb{F}[t]_{d+l-1}\\
\Updownarrow\\
\Big[a_1\,\sigma^{m-1}(w)\;(\alpha)^d_{m-1}\;t^d + \dots + a_{m-1}\,\sigma(w)\,\alpha^d\,t^d + a_m\,w\,t^d - \boldsymbol{c}\,\boldsymbol{f}\Big]_{d+l} = 0
\end{gathered}$$

$$\Updownarrow$$

$$[a_1]_l\ \sigma^{m-1}(w)\ (\alpha)^d_{m-1} + \cdots + [a_{m-1}]_l\ \sigma(w)\,\alpha^d + [a_m]_l\ w - \boldsymbol{c}\ \underbrace{[\boldsymbol{f}]_{d+l}}_{\tilde{\boldsymbol{f}}} = 0$$

$$\Updownarrow$$

$$\sigma_{\tilde{\boldsymbol{a}}} w = \boldsymbol{c}\,\tilde{\boldsymbol{f}}$$

$$\Updownarrow$$

$$\boldsymbol{c}\wedge w \in \mathrm{V}(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{f}}, \mathbb{F})$$

where $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{f}}$ as from above. □

The next theorem is a generalization of Theorem 16 in [Kar81] which is extended from the first order case to the higher order case of linear difference equations.

Theorem 6.1. *Let* $(\mathbb{F}(t), \sigma)$ *be a* ΠΣ*-extension of* $(\mathbb{F}, \sigma)$ *with constant field* $\mathbb{K}$ *canonically defined by* $\sigma(t) = \alpha\, t + \beta$ *for some* $\alpha \in \mathbb{F}^*$, $\beta \in \mathbb{F}$. *Let* $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ *with* $l := \|\boldsymbol{a}\|$, $\boldsymbol{f} \in \mathbb{F}[t]^n_{d+l}$ *for some* $d \in \mathbb{N}_0$ *and let* $\boldsymbol{0} \neq \tilde{\boldsymbol{a}} := \big([a_1]_l\ (\alpha)^d_{m-1}, \dots, [a_m]_l\,(\alpha)^d_0\big) \in \mathbb{F}^m$ *and* $\tilde{\boldsymbol{f}} := [\boldsymbol{f}]_{d+l} \in \mathbb{F}^n$. *Then* $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$ *is a finite dimensional vector space over* $\mathbb{K}$, *and* $\boldsymbol{C}\wedge\boldsymbol{w}$ *is a basis matrix of* $\mathrm{V}(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{f}}, \mathbb{F})$ *if and only if* $\boldsymbol{C}_\wedge(\boldsymbol{w}\,t^d)$ *is a basis matrix of* $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$.

Proof: By Proposition 3.1 $\mathrm{V}(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{f}}, \mathbb{F})$ is a finite dimensional vector space over $\mathbb{K}$. Hence there is a basis $\{\boldsymbol{c_i}\wedge w_i \,|\, 1 \leq i \leq r\} \subseteq \mathbb{K}^n \times \mathbb{F}$ for $\mathrm{V}(\boldsymbol{a}, \tilde{\boldsymbol{f}}, \mathbb{F})$. Then by Lemma 6.1 $\big\{\boldsymbol{c_i}\wedge w_i\, t^d \,|\, 1 \leq i \leq r\big\}$ spans the incremental vector space $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$ over $\mathbb{K}$. Hence $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$ is a finite dimensional vector space over $\mathbb{K}$. Contrary let $\big\{\boldsymbol{c_i}\wedge w_i\, t^d \,|\, 1 \leq i \leq r\big\} \subseteq \mathbb{K}^n \times (t^d\,\mathbb{F})$ be a basis for $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F})$. Then by Lemma 6.1 the set $\{\boldsymbol{c_i}\wedge w_i \,|\, 1 \leq i \leq r\}$ spans the vector space $\mathrm{V}(\boldsymbol{a}, \tilde{\boldsymbol{f}}, \mathbb{F})$ over $\mathbb{K}$. Moreover the set $\{\boldsymbol{c_i}\wedge w_i \in \mathbb{K}^n \times \mathbb{F} \,|\, 1 \leq i \leq r\}$ is linearly independent over $\mathbb{K}$ if and only if $\big\{\boldsymbol{c_i}\wedge w_i\, t^d \in \mathbb{K}^n \times (t^d\,\mathbb{F}) \,|\, 1 \leq i \leq r\big\}$ is linearly independent over $\mathbb{K}$. Hence the theorem is proven. □

The reduction motivated in Example 6.1 and formalized in Theorem 6.1 is represented by

$$\begin{array}{c} \mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F}) \\ \Big\downarrow 1. \quad 2. \Big\uparrow \\ \mathrm{V}(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{f}}, \mathbb{F}). \end{array}$$

6.4. The Incremental Reduction Theorem

The whole incremental reduction method (22) is based on Theorem 6.2 that will be considered in the following. As one can see in (22) or in the structure of Theorem 6.2, in a first step one is faced to compute a basis of $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$.

If it turns out that the incremental solution space consists only of the trivial solution

$$\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F}) = \{\boldsymbol{0_{n+1}}\}, \tag{23}$$

the following proposition tells us how to obtain a basis of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$ by computing a basis of $\mathrm{V}(\boldsymbol{a}, (0)\,, \mathbb{F}[t]_{d-1})$.

Proposition 6.1. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ with $l := \|\boldsymbol{a}\|$ and $\boldsymbol{f} \in \mathbb{F}[t]^n_{d+l}$ for some $d \in \mathbb{N}_0$. Then $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d) \supseteq \mathrm{V}(\boldsymbol{a}, \mathbf{0}_{\boldsymbol{n}}, \mathbb{F}[t]_{d-1}) \cap (\{\mathbf{0}_n\} \times \mathbb{F}[t]_{d-1})$. If additionally (23) holds then we even have equality.*

Proof: Define $\mathbb{W} := \mathrm{V}(\boldsymbol{a}, \mathbf{0}_{\boldsymbol{n}}, \mathbb{F}[t]_{d-1}) \cap (\mathbf{0}_n \times \mathbb{F}[t]_{d-1})$ and let $\boldsymbol{c} \wedge g \in \mathbb{W}$. Then $\boldsymbol{c} = \mathbf{0}_{\boldsymbol{n}}$ and $g \in \mathbb{F}[t]_{d-1}$ with $\sigma_{\boldsymbol{a}} g = \boldsymbol{c}\,\boldsymbol{f} = 0$, and hence $\boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$. Now assume that (23) but also $\mathbb{W} \subsetneq \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$ holds. We will prove that this leads to a contradiction. Take any $\boldsymbol{c} \wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d) \setminus \mathbb{W}$. Clearly $\boldsymbol{c} \wedge g \neq \mathbf{0}_{\boldsymbol{n+1}}$. First suppose $g \in \mathbb{F}[t]_{d-1}$. Hence $\|\sigma_{\boldsymbol{a}} g\| < d + l$ by Lemma 4.2 and therefore $0 = [\sigma_{\boldsymbol{a}} g]_{d+l} = [\boldsymbol{c}\,\boldsymbol{f}]_{d+l} = \boldsymbol{c}\,[\boldsymbol{f}]_{d+l}$. Consequently $\boldsymbol{c} \wedge 0 \in \mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$ and thus $\boldsymbol{c} = \mathbf{0}_{\boldsymbol{n}}$ by (23). Then $\boldsymbol{c} \wedge g = \mathbf{0}_{\boldsymbol{n}} \wedge g \in \mathbb{W}$, a contradiction. Otherwise assume $g \in \mathbb{F}[t]^d \setminus \mathbb{F}[t]_{d-1}$ and write $g = w\,t^d + r$ with $r \in \mathbb{F}[t]_{d-1}$ and $w \in \mathbb{F}^*$. Clearly $\sigma_{\boldsymbol{a}} g = \sigma_{\boldsymbol{a}}(w\,t^d) + \sigma_{\boldsymbol{a}} r$. By Lemma 4.2 it follows $\sigma_{\boldsymbol{a}} r \in \mathbb{F}[t]_{l+d-1}$ and thus $\boldsymbol{c}\,\boldsymbol{f} - \sigma_{\boldsymbol{a}}(w\,t^d) \in \mathbb{F}[t]_{l+d-1}$. But then $\boldsymbol{c} \wedge (w\,t^d) \in \mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$, a contradiction by (23). □

Now assume that (23) holds. Then note that we may write

$$\mathrm{V}(\boldsymbol{a}, \mathbf{0}_{\mathbf{1}}, \mathbb{F}[t]_{d-1}) = \mathbb{K} \times \mathbb{W}$$

for the subspace $\mathbb{W} = \{h \in \mathbb{F}[t]_{d-1} \mid \sigma_{\boldsymbol{a}} h = 0\}$ of $\mathbb{F}[t]_{d-1}$ over $\mathbb{K}$. Hence by Proposition 6.1 it follows that $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d) = \{\mathbf{0}_{\boldsymbol{n}}\} \times \mathbb{W}$. In other words, if $\mathbb{W} = \{0\}$, $\mathbf{0}_{\mathbf{1}\times(\boldsymbol{n}+\mathbf{1})}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$. Otherwise, if $\{h'_1, \dots, h'_l\}$ with $l \geq 1$ forms a basis of $\mathbb{W}$ then $\mathbf{0}_{\boldsymbol{l}\times\boldsymbol{n}} \wedge \boldsymbol{h'}$ with $\boldsymbol{h'} = (h'_1, \dots, h'_l) \in \mathbb{F}[t]^l_{d-1}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$.

Hence what remains is to extract a basis of the subspace $\{0\} \times \mathbb{W}$ of the solution space $\mathrm{V}(\boldsymbol{a}, \mathbf{0}_{\mathbf{1}}, \mathbb{F}[t]_{d-1})$. More generally let $\boldsymbol{D} \wedge \boldsymbol{h}$ with $\boldsymbol{D} \in \mathbb{K}_{\mu\times n}$ and $\boldsymbol{h} \in \mathbb{F}[t]^\mu_{d-1}$ be a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a}, \mathbf{0}_{\mathbf{1}}, \mathbb{F}[t]_{d-1})$. Then by linear algebra we obtain easily a basis of dimension at most $\mu - 1$ that generates a subspace of $\mathbb{W}$.

Determine a basis that generates a subspace of $\mathbb{W}$

1. Transform $\boldsymbol{D} \wedge \boldsymbol{h}$ by at most $\mu - 1$ row operations to a basis matrix of $\mathrm{V}(\boldsymbol{a}, \mathbf{0}_{\mathbf{1}}, \mathbb{F}[t]_{d-1})$ of the form
$$\begin{pmatrix} 1 & w \\ 0 & h'_1 \\ \cdots & \cdots \\ 0 & h'_{\mu-1} \end{pmatrix} \text{ with } w \in \mathbb{F} \text{ or } \begin{pmatrix} 0 & h'_1 \\ \cdots & \cdots \\ 0 & h'_\mu \end{pmatrix}. \tag{24}$$
2. If $(1, w)$ is the first row and $\mu = 1$ then set $\boldsymbol{h'} = (0)$. Otherwise set $\boldsymbol{h'} := (h'_1, \dots, h'_{\mu'})$ with $\mu - 1 \leq \mu' \leq \mu$ respectively.

Clearly $\mathbf{0}_{\boldsymbol{\mu'}\times\boldsymbol{n}} \wedge \boldsymbol{h'}$ is a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$ over $\mathbb{K}$. Moreover the entries in $\boldsymbol{h'}$ form a basis of a subspace of $\mathbb{W}$, if $\boldsymbol{h'} \neq (0)$. Furthermore, if $\boldsymbol{D} \wedge \boldsymbol{h}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_{d-1})$, the entries in $\boldsymbol{h'}$ constitute a basis of $\mathbb{W}$ itself. Hence by the above remarks $\mathbf{0}_{\boldsymbol{\mu'}\times\boldsymbol{n}} \wedge \boldsymbol{h'}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$, if additionally (23) holds. These aspects are summarized in the following corollary.

Corollary 6.1. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$,*

$\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ with $l := \|\boldsymbol{a}\|$ and $\boldsymbol{f} \in \mathbb{F}[t]^n_{d+l}$ for some $d \in \mathbb{N}_0$. Furthermore let $\boldsymbol{D}\wedge\boldsymbol{h}$ with $\boldsymbol{D} \in \mathbb{K}_{\mu\times n}$ and $\boldsymbol{h} \in \mathbb{F}[t]^{\mu}_{d-1}$ be a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a}, \mathbf{0_1}, \mathbb{F}[t]_{d-1})$ and take $\boldsymbol{h'} \in \mathbb{F}[t]^{\mu'}_{d-1}$ as described in (24). Then $\mathbf{0}_{\boldsymbol{\mu'\times n}}\wedge\boldsymbol{h'}$ is a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$. Moreover if (23) holds and $\boldsymbol{D}\wedge\boldsymbol{h}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_{d-1})$ then $\mathbf{0}_{\boldsymbol{\mu'\times n}}\wedge\boldsymbol{h'}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$.

Finally we state the incremental reduction theorem which is a generalization of [Kar81, Theorem 12] from the first to the higher order case of linear difference equations. In particular this result includes the special case (23) in step 3b.

Theorem 6.2 (Incremental Reduction Theorem). *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, canonically defined by $\sigma(t) = \alpha\, t + \beta$ for some $\alpha \in \mathbb{F}^*$, $\beta \in \mathbb{F}$. Let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ with $l := \|\boldsymbol{a}\|$ and $\boldsymbol{f} \in \mathbb{F}[t]^n_{d+l}$ for some $d \in \mathbb{N}_0$. Then one can carry out the following reduction:*

1. Let $\boldsymbol{C}\wedge\boldsymbol{g}$ be a basis matrix of a subspace of $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$ over $\mathbb{K}$ with $\boldsymbol{C} \in \mathbb{K}^{\lambda\times n}$ and $\boldsymbol{g} \in (t^d\,\mathbb{F})^{\lambda}$ for some $\lambda \geq 1$.

2. Take $\tilde{\boldsymbol{f}} := \boldsymbol{C}\cdot\boldsymbol{f} - \sigma_{\boldsymbol{a}}\boldsymbol{g} \in \mathbb{F}[t]^{\lambda}_{d+l-1}$ and let $\boldsymbol{D}\wedge\boldsymbol{h}$ be a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a}, \tilde{\boldsymbol{f}}, \mathbb{F}[t]_{d-1})$ over $\mathbb{K}$ with $\boldsymbol{D} \in \mathbb{K}^{\mu\times\lambda}$ and $\boldsymbol{h} \in \mathbb{F}[t]^{\mu}_{d-1}$ for some $\mu \geq 1$.

3a. If $\boldsymbol{C}\wedge\boldsymbol{g} \neq \mathbf{0}_{\mathbf{1}\times(\boldsymbol{n}+\mathbf{1})}$ then $\boldsymbol{(D\,C)\wedge(h + D\cdot g)}$ is a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$ over $\mathbb{K}$ with $\boldsymbol{D\,C} \in \mathbb{K}^{\mu\times n}$ and $\boldsymbol{h} + \boldsymbol{D}\cdot\boldsymbol{g} \in \mathbb{F}[t]^{\mu}_{d}$.

3b. Otherwise one obtains an $\boldsymbol{h'} \in \mathbb{F}[t]^{\mu'}_{d}$ with $\mu - 1 \leq \mu' \leq \mu$ as described in (24) such that $\mathbf{0}_{\boldsymbol{\mu'\times n}}\wedge\boldsymbol{h'}$ is basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$ over $\mathbb{K}$.

Moreover, if $\boldsymbol{C}\wedge\boldsymbol{g}$ and $\boldsymbol{D}\wedge\boldsymbol{h}$ are basis matrices of the vector spaces $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f}, t^d\,\mathbb{F})$ and $\mathrm{V}(\boldsymbol{a}, \tilde{\boldsymbol{f}}, \mathbb{F}[t]_{d-1})$ then $\boldsymbol{(D\,C)\wedge(h + D\cdot g)}$, or $\mathbf{0}_{\boldsymbol{\mu'\times n}}\wedge\boldsymbol{h'}$ respectively, is a basis matrix of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$.

Example 6.4. Let $(\mathbb{Q}(t_1, t_2), \sigma)$ be the ΠΣ-field over $\mathbb{Q}$ canonically defined by $\sigma(t_1) = t_1 + 1$ and $\sigma(t_2) = t_2 + \frac{1}{t_2+1}$. Then by Theorem 6.2 one can carry out the following reduction step which appears in the reduction process sketched in Example 5.1.

$$\begin{tikzcd}
\mathrm{V}\Big((1,-1), \overbrace{\big(\tfrac{-1-2\,t_2-2\,t_1\,t_2}{(1+t_1)^2}, t_2\big)}^{=:\boldsymbol{f}}, \mathbb{Q}(t_1)[t_2]_1\Big) \arrow[r, shift right, "1."'] & \mathrm{I}((1,-1), \boldsymbol{f}, t_2\,\mathbb{Q}(t_1)) \arrow[l, shift right, "3."'] \arrow[dl, "2."] \\
\mathrm{V}\Big(1,-1, \underbrace{\big(\tfrac{-1}{t_1+1}, -1\big)}_{=:\tilde{\boldsymbol{f}}}, \underbrace{\mathbb{Q}(t_1)[t_2]_0}_{\mathbb{Q}(t_1)}\Big) \arrow[u] &
\end{tikzcd}$$

1. First we compute a basis matrix $\boldsymbol{C}\wedge\boldsymbol{g} = \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)\wedge\left(\begin{smallmatrix}t_2\\ t_2\,t_1\end{smallmatrix}\right)$ of the incremental solution space $\mathrm{I}((1,-1), \boldsymbol{f}, t_2\,\mathbb{Q}(t_1))$ (see Example 6.1).
2. Let $\tilde{\boldsymbol{f}} := \boldsymbol{C}\cdot\boldsymbol{f} - (\sigma(\boldsymbol{g}) - \boldsymbol{g}) = \begin{pmatrix}\frac{-1}{t_1+1}\\ -1\end{pmatrix} \in \mathbb{Q}(t_1)[t_2]^2_0 = \mathbb{Q}(t_1)^2$ and compute a basis matrix, say $\boldsymbol{D}\wedge\boldsymbol{h} = \left(\begin{smallmatrix}0 & -1\\ 0 & 0\end{smallmatrix}\right)\wedge\left(\begin{smallmatrix}t_1\\ 1\end{smallmatrix}\right)$, of $\mathrm{V}\left((1,-1), \left(\frac{-1}{t_1+1}, -1\right), \mathbb{Q}(t_1)[t_2]_0\right)$.
3. Then $\boldsymbol{(D\,C)\wedge(h + D\cdot g)} = \left(\begin{smallmatrix}\mathbf{0} & \mathbf{-1}\\ \mathbf{0} & \mathbf{0}\end{smallmatrix}\right)\wedge\left(\begin{smallmatrix}t_1 - t_1\,t_2\\ 1\end{smallmatrix}\right)$ is a basis matrix the solution space of $\mathrm{V}\left((1,-1), \left(\frac{-1-2\,t_2-2\,t_1\,t_2}{(1+t_1)^2}, t_2\right), \mathbb{Q}(t_1)[t_2]_1\right)$.

To prove the fundamental reduction theorem, a simple lemma is introduced.

Lemma 6.2. *Let $\mathbb{F}$ be a field which is also a vector space over the field $\mathbb{K}$ and let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional subspaces of $\mathbb{K}^n\times\mathbb{F}$ over $\mathbb{K}$ with $\mathbb{V}\subseteq\mathbb{W}\subseteq\mathbb{K}^n\times\mathbb{F}$. Furthermore assume that $\boldsymbol{M}_{\mathbb{V}}\in\mathbb{F}^{d\times(n+1)}$ is a generator matrix of $\mathbb{V}$ and $\boldsymbol{M}_{\mathbb{W}}\in\mathbb{F}^{e\times(n+1)}$ of $\mathbb{W}$. Then there exists a matrix $\boldsymbol{K}\in\mathbb{K}^{d\times e}$ such that $\boldsymbol{M}_{\mathbb{V}}=\boldsymbol{K}\,\boldsymbol{M}_{\mathbb{W}}$.*

Proof: Let

$$\boldsymbol{M}_{\mathbb{V}}=\begin{pmatrix}\boldsymbol{v_1}\\ \vdots\\ \boldsymbol{v_d}\end{pmatrix}\text{ and }\boldsymbol{M}_{\mathbb{W}}=\begin{pmatrix}\boldsymbol{w_1}\\ \vdots\\ \boldsymbol{w_e}\end{pmatrix}$$

where the $\boldsymbol{v_i}$ and $\boldsymbol{w_i}$ vectors are interpreted as row-vectors. We have

$$\operatorname{span}_{\mathbb{K}}(\boldsymbol{v_1},\dots,\boldsymbol{v_d})=\mathbb{V}\subset\mathbb{W}=\operatorname{span}_{\mathbb{K}}(\boldsymbol{w_1},\dots,\boldsymbol{w_e}).$$

Thus there are vectors $\boldsymbol{k_i}=(k_{i1},\dots,k_{ie})\in\mathbb{K}^e$ for $1\leq i\leq d$ such that

$$\boldsymbol{v_i}=k_{i1}\boldsymbol{w_1}+\dots+k_{ie}\boldsymbol{w_e}=\boldsymbol{k_i}\cdot\boldsymbol{M}_{\mathbb{W}}$$

and therefore $\boldsymbol{M}_{\mathbb{V}}=\boldsymbol{K}\,\boldsymbol{M}_{\mathbb{W}}$ with $\boldsymbol{K}=\begin{pmatrix}\boldsymbol{k_1}\\ \vdots\\ \boldsymbol{k_d}\end{pmatrix}$ where the $\boldsymbol{k_i}$ are interpreted as row-vectors. □

Proof of Theorem 6.2

Let $\boldsymbol{C}\wedge\boldsymbol{g}$ be a basis matrix of a subspace of $\mathrm{I}(\boldsymbol{a},\boldsymbol{f},t^d\,\mathbb{F})$ as it is stated in the theorem. Then by the property of the incremental solution space it follows that $\tilde{\boldsymbol{f}}:=\boldsymbol{C}\cdot\boldsymbol{f}-\sigma_{\boldsymbol{a}}\boldsymbol{g}\in\mathbb{F}[t]^{\lambda}_{d+l-1}$. Now let $\boldsymbol{D}\wedge\boldsymbol{h}$ be a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a},\tilde{\boldsymbol{f}},\mathbb{F}[t]_{d-1})$ over $\mathbb{K}$ as stated in the theorem. Then we clearly have that $\boldsymbol{D}\,\boldsymbol{C}\in\mathbb{K}^{\mu\times n}$ and $\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g}\in\mathbb{F}[t]^{\mu}_{d}$.

The special case in 3b: If $\boldsymbol{C}\wedge\boldsymbol{g}=\boldsymbol{0_{1\times(n+1)}}$ holds, it follows that $\tilde{\boldsymbol{f}}=(0)$, in particular $\lambda=1$. Then by Corollary 6.1 the statement in 3b holds. Moreover, if $\boldsymbol{C}\wedge\boldsymbol{g}$ is a basis matrix of $\mathrm{I}(\boldsymbol{a},\boldsymbol{f},t^d\,\mathbb{F})$, condition (23) holds. If additionally $\boldsymbol{D}\wedge\boldsymbol{h}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a},\tilde{\boldsymbol{f}},\mathbb{F}[t]_{d-1})$ then by Corollary 6.1 $\boldsymbol{0_{\mu'\times n}}\wedge\boldsymbol{h'}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$ which proves the theorem for the special case 3b.
What remains to consider is the case 3a, in particular we may assume that

$$\boldsymbol{C}\wedge\boldsymbol{g}\neq\boldsymbol{0_{1\times(n+1)}}.\tag{25}$$

Step 1: We show that $(\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\,\boldsymbol{g})$ generates a subspace of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$ over $\mathbb{K}$. We have

$$\begin{aligned}\sigma_{\boldsymbol{a}}\boldsymbol{h}=\boldsymbol{D}\cdot\tilde{\boldsymbol{f}}=\boldsymbol{D}\cdot(\boldsymbol{C}\cdot\boldsymbol{f}-\sigma_{\boldsymbol{a}}\boldsymbol{g})&\Leftrightarrow\sigma_{\boldsymbol{a}}\boldsymbol{h}=\boldsymbol{D}\cdot(\boldsymbol{C}\cdot\boldsymbol{f})-\boldsymbol{D}\cdot\sigma_{\boldsymbol{a}}\boldsymbol{g}\\&\Leftrightarrow\sigma_{\boldsymbol{a}}\boldsymbol{h}+\boldsymbol{D}\cdot\sigma_{\boldsymbol{a}}\boldsymbol{g}=(\boldsymbol{D}\,\boldsymbol{C})\cdot\boldsymbol{f}\Leftrightarrow\sigma_{\boldsymbol{a}}(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})=(\boldsymbol{D}\,\boldsymbol{C})\cdot\boldsymbol{f}\end{aligned}$$

and by $\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g} \in \mathbb{F}[t]_d^{\mu}$ it follows that $(\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})$ generates a subspace of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$ over $\mathbb{K}$.

Step 2: Next we show that $(\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})$ is a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$ over $\mathbb{K}$. If

$$(\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g}) = \boldsymbol{0}_{\boldsymbol{1}\times(\boldsymbol{n}+\boldsymbol{1})} \tag{26}$$

then by convention it is a basis matrix and represents the vector space $\{\boldsymbol{0}\} \subseteq \mathbb{K}^n \times \mathbb{F}$. Otherwise, assume that the basis matrix is not of the form (26). We will show that the rows in the matrix $(\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})$ are linearly independent over $\mathbb{K}$ which proves that it is a basis matrix. Assume the rows are linearly dependent. Then there is a $\boldsymbol{0} \neq \boldsymbol{k} \in \mathbb{K}^{\mu}$ such that

$$\boldsymbol{k}\cdot((\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})) = \boldsymbol{0}. \tag{27}$$

- Now assume that
$$\boldsymbol{k}\cdot\boldsymbol{D} = \boldsymbol{0}. \tag{28}$$
$\boldsymbol{D}\wedge\boldsymbol{h}$ is a basis matrix by assumption. If $\boldsymbol{D}\wedge\boldsymbol{h}$ consists of exactly one zero-row, we are in the case (26), a contradiction. Therefore we may assume that the rows are nonzero and linearly independent over $\mathbb{K}$, i.e. we have $\boldsymbol{k}\cdot(\boldsymbol{D}\wedge\boldsymbol{h}) \neq 0$. Hence by (28) it follows that
$$0 \neq \boldsymbol{k}\,\boldsymbol{h} \in \mathbb{F}[t]_{d-1}. \tag{29}$$
Since $\boldsymbol{g} \in (t^d\,\mathbb{F})^{\lambda}$, we conclude that $\boldsymbol{k}\,(\boldsymbol{D}\cdot\boldsymbol{g}) \in t^d\,\mathbb{F}$. Therefore by (29) we have $\boldsymbol{0} \neq \boldsymbol{k}\,\boldsymbol{h} + \boldsymbol{k}\,(\boldsymbol{D}\cdot\boldsymbol{g}) = \boldsymbol{k}\,(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})$ and thus $\boldsymbol{k}\cdot((\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})) \neq \boldsymbol{0}$, a contradiction to (27).
- Otherwise, assume that $\boldsymbol{v} := \boldsymbol{k}\cdot\boldsymbol{D} \neq \boldsymbol{0}$. Then by (27) we have
$$\begin{aligned}\boldsymbol{0} &= \boldsymbol{k}\cdot((\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})) = (\boldsymbol{k}\cdot(\boldsymbol{D}\,\boldsymbol{C}))\wedge(\boldsymbol{k}\,(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g}))\\ &= (\boldsymbol{k}\cdot\boldsymbol{D})\cdot\boldsymbol{C})\wedge(\boldsymbol{k}\,\boldsymbol{h}+(\boldsymbol{k}\cdot\boldsymbol{D})\cdot\boldsymbol{g})) = (\boldsymbol{v}\cdot\boldsymbol{C})\wedge(\boldsymbol{k}\,\boldsymbol{h}+\boldsymbol{v}\,\boldsymbol{g})\end{aligned}$$
and thus
$$\boldsymbol{v}\cdot\boldsymbol{C} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{k}\,\boldsymbol{h}+\boldsymbol{v}\,\boldsymbol{g} = 0. \tag{30}$$
But $\boldsymbol{C}\wedge\boldsymbol{g}$ is a basis matrix with (25). Therefore the rows must be linearly independent over $\mathbb{K}$, i.e. $\boldsymbol{v}\cdot(\boldsymbol{C}\wedge\boldsymbol{g}) \neq 0$. Hence by (30) we have $0 \neq \boldsymbol{v}\,\boldsymbol{g} \in t^d\,\mathbb{F}$. As $\boldsymbol{k}\,\boldsymbol{h} \in \mathbb{F}[t]_{d-1}$, we finally get $\boldsymbol{k}\,\boldsymbol{h}+\boldsymbol{v}\,\boldsymbol{g} \neq 0$, a contradiction to (30).

Altogether it follows that $(\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})$ is a basis matrix of a subspace, say $\mathbb{W}$, of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$ over $\mathbb{K}$ which proves the first part of the theorem.

Step 3: Now assume that $\boldsymbol{C}\wedge\boldsymbol{g}$ is a basis matrix of $\mathrm{I}(\boldsymbol{a},\boldsymbol{f},t^d\,\mathbb{F})$ and $\boldsymbol{D}\wedge\boldsymbol{h}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a},\tilde{\boldsymbol{f}},\mathbb{F}[t]_{d-1})$ respectively. What remains to show is that $\mathbb{W} = \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$. Clearly $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$ is a finite dimension vector space over $\mathbb{K}$ by Proposition 3.1. Hence we can take a basis matrix $\tilde{\boldsymbol{E}}\wedge\tilde{\boldsymbol{h}}$ of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$, say $\tilde{\boldsymbol{E}} \in \mathbb{K}^{\nu\times n}$, $\tilde{\boldsymbol{h}} \in \mathbb{F}[t]_d^{\nu}$, and write $\tilde{\boldsymbol{h}} = \boldsymbol{h_1} + \boldsymbol{h_2} \in (t^d\,\mathbb{F})^{\nu} \oplus \mathbb{F}[t]_{d-1}^{\nu}$. Let $\mathbb{V}$ be the vector space that is generated by $\tilde{\boldsymbol{E}}\wedge\boldsymbol{h_1}$. Since

$$\boldsymbol{0} = \sigma_{\boldsymbol{a}}\tilde{\boldsymbol{h}} - \tilde{\boldsymbol{E}}\cdot\boldsymbol{f} = \sigma_{\boldsymbol{a}}\boldsymbol{h_1} + \sigma_{\boldsymbol{a}}\boldsymbol{h_2} - \tilde{\boldsymbol{E}}\cdot\boldsymbol{f} \tag{31}$$

by assumption and $\sigma_{\boldsymbol{a}}\boldsymbol{h_2} \in \mathbb{F}[t]_{d+l-1}^{\nu}$ by Lemma 4.2, it follows that $\sigma_{\boldsymbol{a}}\boldsymbol{h_1} - \tilde{\boldsymbol{E}}\boldsymbol{f} \in \mathbb{F}[t]_{d+l-1}^{s}$. Therefore $\mathbb{V} \subset \mathrm{I}(\boldsymbol{a},\boldsymbol{f},t^d\,\mathbb{F})$ and thus by Lemma 6.2 we find a matrix $\tilde{\boldsymbol{D}} \in \mathbb{K}^{\lambda\times\nu}$ such that $\tilde{\boldsymbol{E}}\wedge\boldsymbol{h_1} = \tilde{\boldsymbol{D}}(\boldsymbol{C}\wedge\boldsymbol{g}) = (\tilde{\boldsymbol{D}}\,\boldsymbol{C})\wedge(\tilde{\boldsymbol{D}}\cdot\boldsymbol{g})$, this means

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{D}}\,\boldsymbol{C} \text{ and } \boldsymbol{h_1} = \tilde{\boldsymbol{D}}\cdot\boldsymbol{g}. \tag{32}$$

By (31) we have

$$\sigma_{\boldsymbol{a}}\boldsymbol{h_2} = \tilde{\boldsymbol{E}}\cdot\boldsymbol{f} - \sigma_{\boldsymbol{a}}\boldsymbol{h_1} \overset{(32)}{=} (\tilde{\boldsymbol{D}}\,\boldsymbol{C})\cdot\boldsymbol{f} - \sigma_{\boldsymbol{a}}(\tilde{\boldsymbol{D}}\cdot\boldsymbol{g}) = \tilde{\boldsymbol{D}}\cdot(\boldsymbol{C}\cdot\boldsymbol{f} - \sigma_{\boldsymbol{a}}\boldsymbol{g})$$

and hence

$$\sigma_{\boldsymbol{a}}\boldsymbol{h_2} = \tilde{\boldsymbol{D}}\cdot\tilde{\boldsymbol{f}}. \tag{33}$$

Let $\mathbb{U}$ be the vector space over $\mathbb{K}$ that is generated by $\tilde{\boldsymbol{D}}\wedge\boldsymbol{h_2}$. Then by (33) it follows that $\mathbb{U} \subseteq \mathrm{V}(\boldsymbol{a},\tilde{\boldsymbol{f}},\mathbb{F}[t]_{d-1})$ and thus by Lemma 6.2 we find a matrix $\boldsymbol{K} \in \mathbb{K}^{\nu\times\mu}$ such that $\tilde{\boldsymbol{D}}\wedge\boldsymbol{h_2} = \boldsymbol{K}\,(\boldsymbol{D}\wedge\boldsymbol{h}) = (\boldsymbol{K}\,\boldsymbol{D})\wedge(\boldsymbol{K}\cdot\boldsymbol{h})$, this means

$$\tilde{\boldsymbol{D}} = \boldsymbol{K}\,\boldsymbol{D} \text{ and } \boldsymbol{h_2} = \boldsymbol{K}\cdot\boldsymbol{h}. \tag{34}$$

Then

$$\begin{aligned}\tilde{\boldsymbol{E}}\wedge\tilde{\boldsymbol{h}} = \tilde{\boldsymbol{E}}\wedge(\boldsymbol{h_1}+\boldsymbol{h_2}) &\overset{(32)}{=} (\tilde{\boldsymbol{D}}\,\boldsymbol{C})\wedge(\tilde{\boldsymbol{D}}\cdot\boldsymbol{g}+\boldsymbol{h_2})\\ &\overset{(34)}{=} (\boldsymbol{K}\,\boldsymbol{D}\,\boldsymbol{C})\wedge((\boldsymbol{K}\,\boldsymbol{D})\cdot\boldsymbol{g}+\boldsymbol{K}\cdot\boldsymbol{h}) = \boldsymbol{K}\,((\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{D}\cdot\boldsymbol{g}+\boldsymbol{h}))\end{aligned}$$

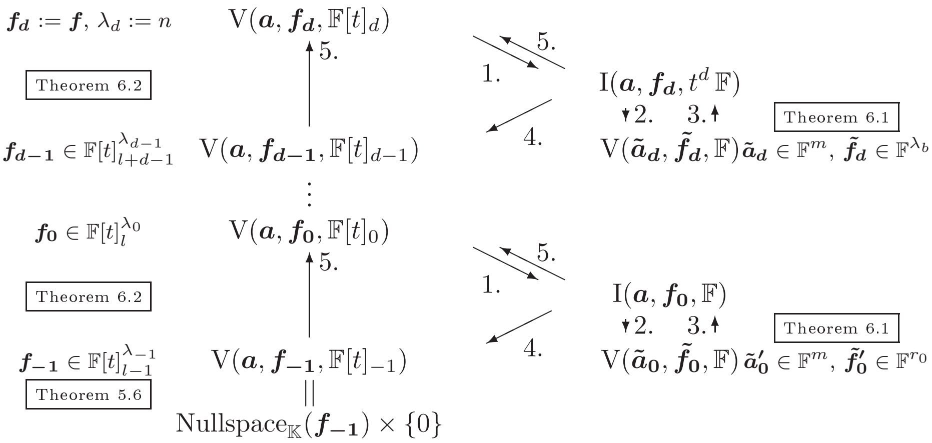
and it follows that $\mathbb{W} \supseteq \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$ which proves the theorem. (In particular, $\boldsymbol{K}$ is a basis transformation, i.e. $\nu = \mu$ and $\boldsymbol{K}$ is invertible.) □

Remark 6.1. If $\boldsymbol{C}\wedge\boldsymbol{g} = \boldsymbol{0}_{\boldsymbol{1}\times(\boldsymbol{n}+\boldsymbol{1})}$ and $\boldsymbol{D}\wedge\boldsymbol{h}$ are basis matrices of $\mathrm{I}(\boldsymbol{a},\boldsymbol{f},t^d\,\mathbb{F})$ and $\mathrm{V}(\boldsymbol{a},\tilde{\boldsymbol{f}},\mathbb{F}[t]_{d-1})$, $(\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})$ generates $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$, since in any case the proof-steps 1 and 3 hold; but the rows in $(\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})$ are linearly dependent over $\mathbb{K}$. A quite expensive transformation to a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$ can be avoided, by applying situation 3b. This subcase delivers the desired basis matrix by some cheap row operations in the matrix $\boldsymbol{D}\wedge\boldsymbol{h}$.

Remark 6.2. Finally I want to indicate that Theorem 6.2 can be generalized from a ΠΣ-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ to a difference ring extension $(\mathbb{A}[t],\sigma)$ of $(\mathbb{A},\sigma)$ where t must be transcendental over a commutative ring $\mathbb{A}$ but the ring $\mathbb{A}$ even might have zero-divisors. In [Sch01] reduction strategies are developed to find at least partially the solutions of parameterized linear difference equations where for instance elements $x \in \mathbb{A}$ can appear with $\sigma(x) = -x$ and $x^2 = 1$. These extensions enable to work with summation objects like $(-1)^n$.

6.5. The Incremental Reduction Process

By exploiting the incremental reduction theorem recursively one can carry out a reduction process as it is already indicated in diagram (18). More precisely by applying Theorems 6.1 and 6.2 with the given matrix operations one obtains an *incremental reduction process* of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$.



Definition 6.3. Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$, $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ with $l := \|\boldsymbol{a}\|$ and $\boldsymbol{f} \in \mathbb{F}[t]_{d+l}^n$ for some $d \in \mathbb{N}_0 \cup \{-1\}$. Then by an *incremental reduction process* of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$ we understand a diagram as above. We call $\{(\boldsymbol{\tilde{a}_d}, \boldsymbol{\tilde{f}_d}), \dots, (\boldsymbol{\tilde{a}_0}, \boldsymbol{\tilde{f}_0})\}$ the *subproblems* of the reduction process.

Proposition 6.2 states that if the basis matrices of the subproblems within an incremental reduction process are normalized (Definition 4.9), the subproblems in this incremental reduction process are uniquely determined. But this means that the whole incremental reduction process is uniquely defined.

Proposition 6.2. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$, $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ with $l := \|\boldsymbol{a}\|$ and $\boldsymbol{f} \in \mathbb{F}[t]_{d+l}^n$ for some $d \in \mathbb{N}_0 \cup \{-1\}$. Consider a reduction process of the solution space $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_d)$ where the basis matrices of the $d+1$ subproblems are normalized. Then the subproblems are uniquely defined.*

Proof: By Theorem 6.2 the first subproblem $(\boldsymbol{\tilde{a}_d}, \boldsymbol{\tilde{f}_d})$ is uniquely defined. Now assume that the first r subproblems are uniquely defined for some $1 \leq r \leq d$. By assumption the basis matrix of $\mathrm{V}(\boldsymbol{\tilde{a}_r}, \boldsymbol{\tilde{f}_r}, \mathbb{F})$ is normalized and hence uniquely defined. Hence by Theorem 6.1 the basis matrix of the solution space $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f_r}, t^r\,\mathbb{F})$ is uniquely defined. But then by Theorem 6.2 $\boldsymbol{f_{r-1}}$ is uniquely defined. By Theorem 6.1 we have to find a basis matrix of $\mathrm{I}(\boldsymbol{a}, \boldsymbol{f_{r-1}}, t^{r-1}\,\mathbb{F})$. In order to achieve this, we have to find a basis matrix of $\mathrm{V}(\boldsymbol{\tilde{a}_{r-1}}, \boldsymbol{\tilde{f}_{r-1}}, \mathbb{F})$ where $(\boldsymbol{\tilde{a}_{r-1}}, \boldsymbol{\tilde{f}_{r-1}})$ are uniquely defined. But this is the $d-r+1$-th subproblem in our incremental reduction process. Hence by induction on r all $d+1$ subproblems are uniquely defined. $\square$

7. Algorithms to Solve Linear Difference Equations

In the following I want to emphasize that one is able to develop algorithms to solve parameterized linear difference equations in ΠΣ-fields with the reduction techniques introduced in the last two sections. For this let $(\mathbb{F}(t),\sigma)$ be a ΠΣ-field, $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{F}[t]^m$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$.

The incremental reduction process: First we look closer at the incremental reduction process introduced in Subsection 6.5. For this let $l:=\|\boldsymbol{a}\|$ and take $d\in\mathbb{N}_0\cup\{-1\}$ such that $\|\boldsymbol{f}\|\in\mathbb{F}[t]_{l+d}$. Then the main observation is the following: If one is capable of solving parameterized linear difference equations of order $m-1$ in the difference field $(\mathbb{F},\sigma)$, in particular if one can compute a basis matrix of all the subproblems in an incremental reduction process then one is able to compute a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$.

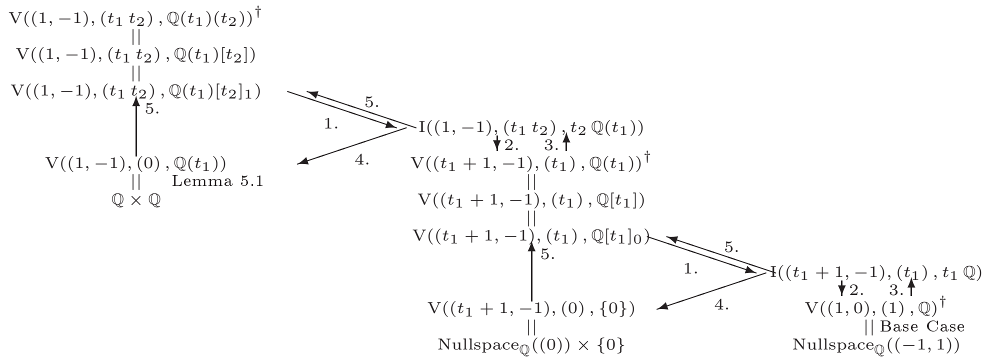
Combining all reduction techniques: Moreover, an algorithm can be designed which computes a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$, if the full reduction strategy as in (5) can be applied:

1. Clearly the simplifications in Subsection 5.1 work in any ΠΣ-field, and hence one can reduce the problem to find a basis matrix of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(t))$ with (10).
2. Furthermore, if one can compute a denominator bound of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(t))$, one is able to reduce the problem to the problem of computing a basis matrix of $\mathrm{V}(\boldsymbol{a''},\boldsymbol{f''},\mathbb{F}[t])$ (Subsection 5.2).
3. Moreover, if one is capable of determining a degree bound of $\mathrm{V}(\boldsymbol{a''},\boldsymbol{f''},\mathbb{F}[t])$, one can apply the incremental reduction technique on $\mathrm{V}(\boldsymbol{a''},\boldsymbol{f''},\mathbb{F}[t]_b)$ and obtains its basis matrix (Subsection 5.3).

Finally one reconstructs a basis matrix of the original problem $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ as it is described in Subsections 5.1 and 5.2.

A recursive reduction process - the first order case: As already indicated in Section 5 these reduction strategies deliver an algorithm to compute all solutions of parameterized first order linear difference equations: First by results from [Sch02a, Sch02b] there exist algorithms to compute a denominator bound of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(t))$ and a degree bound of $\mathrm{V}(\boldsymbol{a''},\boldsymbol{f''},\mathbb{F}[t])$ for the cases $\mathbf{0}\neq\boldsymbol{a'}\in(\mathbb{F}[t]^*)^2$ and $\mathbf{0}\neq\boldsymbol{a''}\in(\mathbb{F}[t]^*)^2$. Second the subproblems in an incremental reduction process are again parameterized first order linear difference equations in the ΠΣ-field $(\mathbb{F},\sigma)$. But recursively these problems can be solved again by our reductions strategies.
In order to compute the solution space in Example 5.2, the reduction techniques are applied recursively which results in a *recursive reduction process.*

Example 7.1. Let $(\mathbb{Q}(t_1)(t_2),\sigma)$ be the ΠΣ-field over $\mathbb{Q}$ canonically defined by $t_1=t_1+1$ and $t_2=(t_1+1)\,t_2$. In order to find a $g\in\mathbb{Q}(t_1,t_2)$ such that $\sigma(g)-g=t_1\,t_2$, we compute a basis of the solution space $\mathrm{V}((1,-1),(t_1\,t_2)\,,\mathbb{Q}(t_1)(t_2))$ by applying our reduction techniques recursively.



7.1. The Second Base Case

Looking closer at the recursive reduction process (the labels † in Example 7.1), one can follow a path with a new base case, namely

$$\mathrm{V}((1,-1),(t_1\,t_2),\mathbb{Q}(t_1,t_2)) \longrightarrow \mathrm{V}((t_1+1,-1),(t_1),\mathbb{Q}(t_1)) \longrightarrow \mathrm{V}((1,0),(1),\mathbb{Q}).$$

In the general case, for a ΠΣ-field $(\mathbb{F},\sigma)$ over $\mathbb{K}$ with $\mathbb{F} := \mathbb{K}(t_1,\dots,t_e)$, $\mathbf{0} \neq \boldsymbol{a_e} \in \mathbb{F}^{m_e}$ and $\boldsymbol{f_e} \in \mathbb{F}^{n_e}$ the following reduction path pops up:

$$\begin{array}{ccc} \mathrm{V}(\boldsymbol{a_e},\boldsymbol{f_e},\mathbb{K}(t_1,\dots,t_e)) & & \\ \downarrow\uparrow & & \\ \mathrm{V}(\boldsymbol{a_{e-1}},\boldsymbol{f_{e-1}},\mathbb{K}(t_1,\dots,t_{e-1})) & \rightleftarrows \dots \rightleftarrows & \mathrm{V}(\boldsymbol{a_1},\boldsymbol{f_1},\mathbb{K}(t_1)) \\ & & \downarrow\uparrow \\ & & \mathrm{V}(\boldsymbol{a_0},\boldsymbol{f_0},\mathbb{K}). \end{array}$$

Finally one has to determine a basis of $\mathrm{V}(\boldsymbol{a_0},\boldsymbol{f_0},\mathbb{K})$ for some $\mathbf{0} \neq \boldsymbol{a_0} \in \mathbb{F}^{n_0}$ and $\boldsymbol{f_0} \in \mathbb{F}^{m_0}$. Theorem 7.1 allows us to handle this second base case.

Theorem 7.1. *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$, $\boldsymbol{f} \in \mathbb{F}^n$ and $\mathbf{0} \neq \boldsymbol{a} = (a_1,\dots,a_m) \in \mathbb{F}^m$. Then $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{K}) = \mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f}\wedge(-\sum_{i=1}^m a_i))$.*

Proof: Let $\boldsymbol{c} \in \mathbb{K}^n$ and $g \in \mathbb{K}$. It follows that

$$\begin{aligned} \boldsymbol{c}\wedge g \in \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{K}) \Leftrightarrow \boldsymbol{c}\,\boldsymbol{f} - \sigma_{\boldsymbol{a}} g = 0 \Leftrightarrow \boldsymbol{c}\,\boldsymbol{f} - g\left(\sum_{i=0}^{m} a_i\right) = 0 \\ \Leftrightarrow \boldsymbol{c}\wedge g \in \mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f}\wedge u). \end{aligned}$$

□

Remark 7.1. Given $\boldsymbol{f} \in \mathbb{K}^n$ in a field $\mathbb{K}$ a basis of $\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f})$ can be immediately computed by linear algebra.

7.2. The Reduction Algorithm for ΠΣ-fields

By the remarks in the beginning of Section 7 one obtains an algorithm to solve parameterized linear difference equations, if one is able to apply recursively our reduction techniques. This will be possible, if one can compute all denominator and degree bounds within a recursive reduction process.

In order to formalize this in precise terms, we define two input-output specifications of algorithms which deliver exactly the desired denominator and degree bounds that are needed in a recursive reduction process.

Specification 7.1. for a denominator bound algorithm

d=`DenBound`$((\mathbb{F}(t),\sigma),\boldsymbol{a},\boldsymbol{f})$

Input: A ΠΣ-field $(\mathbb{F}(t),\sigma)$ over $\mathbb{K}$, $\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}[t]^m$ with $a_1\,a_m\neq 0$, and $\boldsymbol{f}\in\mathbb{F}[t]^n$.

Output: A denominator bound $d\in\mathbb{F}[t]^*$ of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$.

Specification 7.2. for a degree bound algorithm

d=`DegreeBound`$((\mathbb{F}(t),\sigma),\boldsymbol{a},\boldsymbol{f})$

Input: A ΠΣ-field $(\mathbb{F}(t),\sigma)$ over $\mathbb{K}$, $\boldsymbol{0}\neq\boldsymbol{a}\in\mathbb{F}[t]^m$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$.

Output: A degree bound $b\in\mathbb{N}_0\cup\{-1\}$ of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$

Now we are ready to define if a ΠΣ-field is m-solvable with $m\geq 1$. In this case parameterized linear difference equations of order less than m can be solved.

Definition 7.1. Let $m\geq 1$. In the following we define inductively if a ΠΣ-field $(\mathbb{F},\sigma)$ is *m-solvable.* If $(\mathbb{F},\sigma)$ is the constant field or $m=1$, $(\mathbb{F},\sigma)$ is called m-solvable. Furthermore a ΠΣ-field $(\mathbb{F}(t),\sigma)$ is called m-solvable for $m\geq 2$, if $(\mathbb{F},\sigma)$ is m-solvable and there exist algorithms `DenBound` and `DegreeBound` that fulfill Specifications 7.1 and 7.2 with input `DenBound`$((\mathbb{F}(t),\sigma),\boldsymbol{a},\boldsymbol{f})$ and `DegreeBound`$((\mathbb{F}(t),\sigma),\boldsymbol{a},\boldsymbol{f})$ for any $\boldsymbol{a}=(a_1,\dots,a_{m'})\in\mathbb{F}[t]^{m'}$ for some $2\leq m'\leq m$ with $a_1\,a_{m'}\neq 0$ and any $\boldsymbol{f}\in\mathbb{F}[t]^n$ for some $n\geq 1$.

If a ΠΣ-field is m-solvable and algorithms `DegreeBound` or `DenBound` are applied, they will always fulfill Specifications 7.1 or 7.2.
In particular in [Abr89b, Abr95, vH98] and [Abr89a, Pet92, SAA95, PWZ96] algorithms are developed that fulfill Specifications 7.1 and 7.2 for any $m\geq 2$ in a ΠΣ-field $(\mathbb{K}(t),\sigma)$ over $\mathbb{K}$ with $\sigma(t)=t+1$. All these results immediately lead to the following theorem.

Theorem 7.2. *A ΠΣ-field $(\mathbb{K}(t),\sigma)$ over $\mathbb{K}$ with $\sigma(t)=t+1$ is m-solvable for any $m\geq 2$.*

Furthermore in Theorem 7.4 we will show by results from [Sch02a, Sch02b], based on [Kar81, Bro00], that any ΠΣ-field is 2-solvable. In combination with the following considerations this results in algorithms to solve parameterized first order linear difference equations in full generality.
Now we are ready to write down the algorithms that follow exactly the reduction process as it is illustrated in Example 7.1.

Algorithm 7.1. Solving parameterized linear difference equations in m-solvable ΠΣ-fields.

$\boldsymbol{B}$=`SolveSolutionSpace(`$(\mathbb{E},\sigma),\boldsymbol{a},\boldsymbol{f}$`)`

Input: An m-solvable ΠΣ-field $(\mathbb{E},\sigma)$ over $\mathbb{K}$ with $\mathbb{E}=\mathbb{K}(t_1,\dots,t_e)$ and $e\geq 0$; $\boldsymbol{0}\neq\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{E}^m$ and $\boldsymbol{f}\in\mathbb{E}^n$.

Output: A basis matrix $\boldsymbol{B}$ of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$.

(***Base case II** - Section 7.1*)

(1) IF $e=0$ compute a basis matrix $\boldsymbol{B}$ of $\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f}\wedge(-\sum_{i=1}^m a_i))$; RETURN $\boldsymbol{B}$.

(***Reduction step I:** Simplifications - Subsection 5.1*)

Let $\mathbb{F}:=\mathbb{K}(t_1,\dots,t_{e-1})$, i.e. $(\mathbb{F}(t_e),\sigma)$ is a ΠΣ-extension of $(\mathbb{F},\sigma)$.

(2) If $a_m\neq 0$, set $k:=m$, otherwise define k such that $a_k\neq a_{k+1}=\dots=a_m=0$. Transform $\boldsymbol{a},\boldsymbol{f}$ by (11) to $\boldsymbol{a'}=(a'_1,\dots,a'_{m'})\in\mathbb{F}(t_e)^{m'}$ and $\boldsymbol{f'}\in\mathbb{F}(t_e)^n$ with $a'_1\,a'_{m'}\neq 0$ and $m'\leq m$; clear denominators in $\boldsymbol{a'}$, $\boldsymbol{f'}$ as in Theorem 5.2 which results in $\boldsymbol{a'}\in\mathbb{F}[t_e]^{m'}$, $\boldsymbol{f'}\in\mathbb{F}[t_e]^n$.

(3) IF $\boldsymbol{a'}\in\mathbb{F}[t_e]^1$ RETURN $\mathbf{Id}_{\boldsymbol{n}}\wedge\sigma^{m-k}(\frac{\boldsymbol{f}}{a'_1})$.

(***Reduction step II:** Denominator elimination - Subsection 5.2*)

(4) Compute by $d:=$ `DenBound`$((\mathbb{F}(t_e),\sigma),\boldsymbol{a'},\boldsymbol{f'})$ a denominator bound of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(t_e))$.

(5) Set $\boldsymbol{a''}:=(\frac{a'_1}{\sigma^{m'-1}(d)},\dots,\frac{a_{m'}}{d})\in\mathbb{F}(t_e)^{m'}$ as in Theorem 5.5, and clear denominators in $\boldsymbol{a''}$ and $\boldsymbol{f''}:=\boldsymbol{f'}$ by Theorem 5.2 which results in $\boldsymbol{a''}\in\mathbb{F}[t_e]^{m'}$ and $\boldsymbol{f''}\in\mathbb{F}[t_e]^n$.

(***Reduction step III:** Polynomial degree elimination - Subsection 5.3*)

(6) Compute by $b:=$ `DegreeBound`$((\mathbb{F}(t_e),\sigma),\boldsymbol{a''},\boldsymbol{f''})$ a degree bound of $\mathrm{V}(\boldsymbol{a''},\boldsymbol{f''},\mathbb{F}[t_e])$.

(7) Set $\boldsymbol{C}\wedge\boldsymbol{w}$:=`IncrementalReduction`$((\mathbb{F}(t_e),\sigma),b,\boldsymbol{a},\boldsymbol{f})$ by using Algorithm 7.2.

(8) RETURN $\boldsymbol{C}_\wedge\sigma^{m-k}(\frac{\boldsymbol{w}}{d})$.

Algorithm 7.2. The incremental reduction process in m-solvable ΠΣ-fields.

$\boldsymbol{B}$=`InrementalReduction(`$(\mathbb{F}(t),\sigma),d,\boldsymbol{a},\boldsymbol{f}$`)`

Input: An m-solvable ΠΣ-field $(\mathbb{F}(t),\sigma)$ over $\mathbb{K}$ and $d\in\mathbb{N}_0\cup\{-1\}$; $\boldsymbol{0}\neq\boldsymbol{a}=(a_1,\dots,a_m)\in\mathbb{F}[t]^m$ with $l:=\|\boldsymbol{a}\|$ and $\boldsymbol{f}\in\mathbb{F}[t]^n_{l+d}$.

Output: A basis matrix $\boldsymbol{B}$ of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$.

(***Base case I** - Subsection 5.4*)

(1) IF $d=-1$, compute a basis matrix $\boldsymbol{B}$ of $\mathrm{Nullspace}_{\mathbb{K}}(\boldsymbol{f})\times\{0\}$; RETURN $\boldsymbol{B}$.

(***Degree Elimination** by incremental reduction - Subsection 6.4*)

(2) Set $\boldsymbol{0}\neq\tilde{\boldsymbol{a}}:=\Big([a_1]_l\;(\alpha)^d_{m-1},\dots,[a_m]_l\;(\alpha)^d_0\Big)\in\mathbb{F}^m$ and $\tilde{\boldsymbol{f}}:=[\boldsymbol{f}]_{d+l}\in\mathbb{F}^n$.

(*Computation of the subproblems in an incremental reduction process*)

(3) Set $\boldsymbol{C}\wedge\boldsymbol{w}:=$ `SolveSolutionSpace`$((\mathbb{F},\sigma),\tilde{\boldsymbol{a}},\tilde{\boldsymbol{f}})$ with $\boldsymbol{C}\in\mathbb{K}^{\lambda\times n},\boldsymbol{g}\in(t^d\,\mathbb{F})^\lambda$ by Alg. 7.1.

(4) Set $\boldsymbol{g}:=\boldsymbol{w}\,t^d$ and $\tilde{\boldsymbol{f}}':=\boldsymbol{C}\cdot\boldsymbol{f}-\sigma_{\boldsymbol{a}}\boldsymbol{g}\in\mathbb{F}[t]^\lambda_{d-1}$.

(5) Set $\boldsymbol{D}\wedge\boldsymbol{h}:=$ `IncrementalReduction`$((\mathbb{F}(t),\sigma),d-1,\boldsymbol{a},\tilde{\boldsymbol{f}}')$ with $\boldsymbol{D}\in\mathbb{K}^{\mu\times\lambda},\boldsymbol{h}\in\mathbb{F}[t]^\mu_{d-1}$.

(6) IF $\boldsymbol{D}\wedge\boldsymbol{h}\neq\boldsymbol{0}_{\boldsymbol{1}\times(\boldsymbol{n}+\boldsymbol{1})}$ THEN RETURN $(\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})$.

(7) Compute $\boldsymbol{h'}\in\mathbb{F}[t]^{\mu'}_{d-1}$ as in (24); RETURN $\boldsymbol{0}_{\boldsymbol{\mu'}\times\boldsymbol{n}}\wedge\boldsymbol{h'}$.

First the correctness of Algorithm 7.2 is shown in an m-solvable ΠΣ-field $(\mathbb{F}(t),\sigma)$ under the assumption that Algorithm 7.1 works correct in the ΠΣ-field $(\mathbb{F},\sigma)$.

Lemma 7.1. *Let $(\mathbb{F}(t),\sigma)$ be an m-solvable ΠΣ-field over $\mathbb{K}$ and $d\in\mathbb{N}_0\cup\{-1\}$; let $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{F}[t]^m$ with $l:=\|\boldsymbol{a}\|$ and $\boldsymbol{f}\in\mathbb{F}[t]^n_{l+d}$. Assume that Algorithm 7.1 terminates and works correct for any valid input with the ΠΣ-field $(\mathbb{F},\sigma)$. Then Algorithm 7.2 with input* `InrementalReduction`$((\mathbb{F}(t),\sigma),d,\boldsymbol{a},\boldsymbol{f})$ *terminates and computes a basis matrix of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$.

Proof: If $d=-1$, we obtain in line (1) by Theorem 5.6 a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$ and we are done. Now assume as induction assumption that Algorithm 7.2 with `IncrementalReduction`$((\mathbb{F}(t),\sigma),d-1,\boldsymbol{a},\tilde{\boldsymbol{f}}')$ works correct for any $\tilde{\boldsymbol{f}}'\in\mathbb{F}[t]^\lambda_{d-1}$ for some $\lambda\geq 1$. Hence in line (5) we obtain a basis matrix of $\mathrm{V}(\boldsymbol{a},\tilde{\boldsymbol{f}}',\mathbb{F}[t]_{d-1})$. By definition $(\mathbb{F},\sigma)$ is an m-solvable ΠΣ-field. Thus we obtain a basis matrix $\boldsymbol{D}\wedge\boldsymbol{h}$ of $\mathrm{V}(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{f}},\mathbb{F})$ in line (3) by assumption. Hence by Theorem 6.2 $(\boldsymbol{D}\,\boldsymbol{C})\wedge(\boldsymbol{h}+\boldsymbol{D}\cdot\boldsymbol{g})$ is a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$ if $\boldsymbol{D}\wedge\boldsymbol{h}\neq\mathbf{0}_{\mathbf{1}\times\boldsymbol{n}+\mathbf{1}}$; otherwise $\mathbf{0}_{(\boldsymbol{\mu}-\mathbf{1})\times\boldsymbol{n}}\wedge\boldsymbol{h}'$ is a basis matrix. Thus by induction on d Algorithm 7.2 works correctly for any $d\geq -1$. Clearly the algorithm terminates. □

Remark 7.2. Assume that Algorithm 7.1 terminates and works correct for any valid input with the ΠΣ-field $(\mathbb{F},\sigma)$ and consider Algorithm 7.2 with input as in Lemma 7.1, i.e. `InrementalReduction`$((\mathbb{F}(t),\sigma),d,\boldsymbol{a},\boldsymbol{f})$. Then the algorithm calls itself exactly d times where in line (3) exactly $d+1$ subproblems (Definition 6.3) for an incremental reduction process are computed.

Finally the correctness of Algorithm 7.2 is shown.

Theorem 7.3. *Algorithm 7.1 terminates and is correct.*

Proof: Let $(\mathbb{E},\sigma)$ with $\mathbb{E}=\mathbb{K}(t_1,\dots,t_e)$ be an m-solvable ΠΣ-field over $\mathbb{K}$ with $e\geq 0$, $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{E}^m$ and $\boldsymbol{f}\in\mathbb{E}^n$. If $e=0$, by Theorem 7.1 we compute a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{K})$ in line (1). Otherwise let $\mathbb{F}:=\mathbb{K}(t_1,\dots,t_{e-1})$ and assume as induction assumption that Algorithm 7.1 terminates and works correct for any valid input in the ΠΣ-field $(\mathbb{F},\sigma)$. Now perform step (3). Then we obtain $\boldsymbol{a}'=(a'_1,\dots,a'_{m'})\in\mathbb{F}[t_e]^{m'}$ with $m'\leq m$, $a'_1\,a'_2\neq 0$ and $\boldsymbol{f}'\in\mathbb{F}[t_e]^n$ as described in line (2). If $\boldsymbol{a}'\in\mathbb{F}[t_e]^1$ in line (3), by Theorems 5.1 and 5.4 the result is correct. Now assume that $\boldsymbol{a}'\notin\mathbb{F}[t_e]^{m'}$ with $m'\geq 2$ where $(\mathbb{F},\sigma)$ is m'-solvable by definition. Since the input of `DenBound` in line (4) fulfills Specification 7.2, we compute a denominator bound $d\in\mathbb{F}[t_e]^*$ of $\mathrm{V}(\boldsymbol{a}',\boldsymbol{f}',\mathbb{F}(t_e))$. Now take $\boldsymbol{a}''\in\mathbb{F}[t_e]^{m'}$ and $\boldsymbol{f}''\in\mathbb{F}[t_e]^n$ as described in line (5). Clearly, in line (6) we compute a degree bound of $\mathrm{V}(\boldsymbol{a}',\boldsymbol{f}',\mathbb{F}[t_e])$ due to the correct input for `DegreeBound`. Then by Lemma 7.1 and our induction assumption it follows that we obtain a basis matrix $\boldsymbol{C}\wedge\boldsymbol{w}$ of $\mathrm{V}(\boldsymbol{a}'',\boldsymbol{f}'',\mathbb{F}[t_e]_b)$ and hence of $\mathrm{V}(\boldsymbol{a}'',\boldsymbol{f}'',\mathbb{F}[t_e])$ in line (7). Since d is a denominator bound of $\mathrm{V}(\boldsymbol{a}',\boldsymbol{f}',\mathbb{F}(t_e))$, by Theorems 5.2 and 5.5 $\boldsymbol{C}\wedge\frac{\boldsymbol{w}}{\boldsymbol{d}}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a}',\boldsymbol{f}',\mathbb{F}(t_e))$. But then by Theorems 5.1 and 5.2 $\boldsymbol{C}_\wedge\sigma^{m-k}(\frac{w}{d})$ is a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_e))$. □

By results from [Sch02a, Sch02b] we show that all ΠΣ-fields are 2-solvable.

Theorem 7.4. *Any ΠΣ-field is 2-solvable. In particular there exists an algorithm that solves any parameterized first order linear difference equation in a ΠΣ-field.*

Proof: The proof will be done by induction on the number $e \geq 0$ of extensions in the ΠΣ-field $(\mathbb{K}(t_1, \dots, t_e), \sigma)$ over $\mathbb{K}$. For $e = 0$, the theorem clearly holds by Theorem 7.1. Now assume that the theorem holds for the ΠΣ-field $(\mathbb{F}, \sigma)$ with $\mathbb{F} := \mathbb{K}(t_1, \dots, t_e)$ and consider the ΠΣ-extension $(\mathbb{F}(t_{e+1}), \sigma)$ of $(\mathbb{F}, \sigma)$. Then by [Sch02a, Theorem 8.1] and [Sch02b, Corollary 7.1] there exist algorithms with input `DenBound`$((\mathbb{F}(t_{e+1}), \sigma), \boldsymbol{a}, \boldsymbol{f})$ and `DegreeBound`$((\mathbb{F}(t_{e+1}), \sigma), \boldsymbol{a}, \boldsymbol{f})$ that fulfill Specification 7.1 and 7.2 for any $\boldsymbol{a} \in (\mathbb{F}[t_{e+1}]^*)^2$ and $\boldsymbol{f} \in \mathbb{F}[t_{e+1}]^n$. Hence the ΠΣ-field $(\mathbb{F}(t_{e+1}), \sigma)$ is 2-solvable. But then by Theorem 7.3 one can solve parameterized first order linear difference equations in the ΠΣ-field $(\mathbb{F}(t), \sigma)$. Therefore the induction step holds. □

7.3. Solving Linear Difference Equations by Increasing the Solution Range

In many cases algorithms `DenBound` and `DegreeBound` with Specifications 7.1 and 7.2 are not known for the general case $m \geq 3$ of ΠΣ-fields, contrary only the rational case, Theorem 7.2, and some special cases in [Sch02a, Sch02b] are well studied so far. But by [Sch02a, Theorem 6.4] based on the work of [Bro00] there exists at least an algorithm that fulfills Specification 7.3.

Specification 7.3. for a restricted denominator bound algorithm

d=`DenBoundH`$((\mathbb{F}, \sigma), \boldsymbol{a}, \boldsymbol{f})$

Input: A ΠΣ-field $(\mathbb{F}(t), \sigma)$, $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ with $a_1\, a_m \neq 0$, and $\boldsymbol{f} \in \mathbb{F}[t]^n$.

Output: A $d \in \mathbb{F}[t]^*$ with the following property: If $(\mathbb{F}(t), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$, d is a denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$. Otherwise there exists an $x \in \mathbb{N}_0$ such that $d\, t^x$ is a denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$.

Theorem 7.5. *There exists an algorithm that fulfills Specification 7.3.*

By this result one only needs an $x \in \mathbb{N}_0$ to complete the denominator bound and an $y \in \mathbb{N}_0$ to approximate the degree bound, in order to simulate Algorithm 7.1. This idea leads to Algorithm 7.3 that will be motivated further in the sequel.
Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-field, $\boldsymbol{a'} = (a'_1, \dots, a'_{m'}) \in \mathbb{F}[t]^{m'}$ with $a'_1\, a'_m \neq 0$ and $\boldsymbol{f'} \in \mathbb{F}[t]^n$. Suppose we computed a $d \in \mathbb{F}[t]^*$ by `DenBoundH`$((\mathbb{F}(t_e), \sigma), \boldsymbol{a'}, \boldsymbol{f'})$ that fulfills Specification 7.3. Then one can choose as in line (5) of Algorithm 7.3 an $x \in \mathbb{N}_0$ such that $d\, t^x$ is a denominator bound of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f'}, \mathbb{F}(t))$. Then after computing $\boldsymbol{a''}$ and $\boldsymbol{f''}$ as in line (6), one is faced with the problem to choose a b that approximates a degree bound of $\mathrm{V}(\boldsymbol{a''}, \boldsymbol{f''}, \mathbb{F}[t])$. By definition we must have (17). Hence we might choose any $y \in \mathbb{N}_0$ and take $b := \max(\|\boldsymbol{f''}\| - \|\boldsymbol{a''}\|, y)$ as the degree bound approximation. The following result, a refinement of Theorem 5.5, motivates us to choose a variation of that approximation.

Theorem 7.6. *Let $(\mathbb{F}(t), \sigma)$ be a ΠΣ-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$, $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{F}[t]^m$ and $\boldsymbol{f} \in \mathbb{F}[t]^n$. Let $d \in \mathbb{F}[t]^*$ be a denominator bound of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$, define $\boldsymbol{a'} := (\frac{a_1}{\sigma^{m-1}(d)}, \dots, \frac{a_{m-1}}{\sigma(d)}, \frac{a_m}{d}) \in \mathbb{F}(t)^m$ and let $y \in \mathbb{N}_0$. If $\boldsymbol{C} \wedge \boldsymbol{g}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{F}[t]_{y+\|d\|})$ then $\boldsymbol{C}_{\wedge} \frac{\boldsymbol{g}}{d}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_y \oplus \mathbb{F}(t)^{(frac)})$.*

Proof: As in the proof of Theorem 5.5 we obtain equivalence (15). Now let $\boldsymbol{c}\wedge(d\,g) \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_{y+\|d\|})$. We will show that

$$g \in \mathbb{F}[t]_y \oplus \mathbb{F}(t)^{(frac)}. \tag{35}$$

Write $g = g_0 + g_1 \in \mathbb{F}[t] \oplus \mathbb{F}(t)^{(frac)}$ where $g_1 = \frac{a}{b}$ is in reduced representation. Since $d\,g \in \mathbb{F}[t]$ and $d\,g_0$ we have $d\,g_1 \in \mathbb{F}[t]$. In order to show (35), we show first that

$$\|d\,g_1\| < y + \|d\|. \tag{36}$$

If $g_1 = 0$ then (36) holds. Otherwise assume $g_1 \neq 0$. Then $0 \leq \|a\| < \|b\| \leq \|d\|$ and $b\,u = d$ for some $u \in \mathbb{F}[t]^*$ with $\|b\| + \|u\| = \|d\|$. Hence $\|d\,g_1\| = \|a\,u\| = \|a\| + \|u\| < \|b\| + \|u\| = \|d\|$. Therefore (36) holds in any case. Since $\|d\,g\| = \max(\|d\,g_0\|, \|d\,g_1\|) \leq y + \|d\|$, by (36) it follows that $\|d\,g_0\| < y + \|d\|$ which proves (35). Hence by (15) we obtain

$$\boldsymbol{c}\wedge(d\,g) \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_{y+\|d\|}) \Rightarrow \boldsymbol{c}\wedge g \in \mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_y \oplus \mathbb{F}(t)^{(frac)}). \tag{37}$$

Let $\boldsymbol{C}\wedge\boldsymbol{g}$ be a basis matrix of $\mathrm{V}(\boldsymbol{a'}, \boldsymbol{f}, \mathbb{F}[t])$. Then by Proposition 5.1 $\boldsymbol{C}_\wedge \frac{\boldsymbol{g}}{d}$ is a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}(t))$. Therefore by (37) $\boldsymbol{C}_\wedge \frac{\boldsymbol{g}}{d}$ is a basis matrix of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{F}[t]_y \oplus \mathbb{F}(t)^{(frac)})$. □

Actually we want that the polynomial part in the solution $\mathbb{F}[t] \oplus \mathbb{F}(t)^{(frac)}$ has degree bound y, i.e. the solution should be in $\mathbb{F}[t]_y \oplus \mathbb{F}(t)^{(frac)}$. Then the previous theorem explains why in line (7) of Algorithm 7.3 we choose $b := y + \max(\boldsymbol{f''} - \boldsymbol{a''}, \|d\| + x)$ as the approximated degree bound of $\mathrm{V}(\boldsymbol{a''}, \boldsymbol{f''}, \mathbb{F}[t])$.
Hence one only needs an $x \in \mathbb{N}_0$ to complete the denominator bound and an $y \in \mathbb{N}_0$ to approximate the degree bound. Loosely spoken, the main idea is to insert manually this missing tuples (x, y) in the above algorithm. In order to formalize this, a bounding-matrix is introduced that allows to specify these tuples (x, y) for each extension t_i in a ΠΣ-field $(\mathbb{F}(t_1, \dots, t_e), \sigma)$.

Definition 7.2. Let $(\mathbb{F}(t_1, \dots, t_e), \sigma)$ be a ΠΣ-field. For $e > 0$ we call a matrix $\left(\begin{smallmatrix} x_1 & \dots & x_e \\ y_1 & \dots & y_e \end{smallmatrix}\right) \in \mathbb{N}_0^{2\times e}$ *bounding-matrix with length e for $\mathbb{F}(t_1, \dots, t_e)$*, if for all $1 \leq i \leq e$ we have $x_i = 0$ or $(\mathbb{F}(t_1, \dots, t_i), \sigma)$ is a Π-extension of $(\mathbb{F}(t_1, \dots, t_{i-1}), \sigma)$. In case $e = 0$ the bounding-matrix is defined as the empty list ().

With the concept of bounding matrices one can search for all solutions of linear difference equations in ΠΣ-fields by the following modified algorithm.

Algorithm 7.3. Finding solutions of parameterized linear difference equations in ΠΣ-fields.
$\boldsymbol{B}$=`SolveSolutionSpaceH`$((\mathbb{E}, \sigma), \boldsymbol{M}, \boldsymbol{a}, \boldsymbol{f})$

Input: A ΠΣ-field $(\mathbb{E}, \sigma)$ over $\mathbb{K}$ with $\mathbb{E} = \mathbb{H}(t_1, \dots, t_e)$ and $e \geq 0$ where $(\mathbb{H}, \sigma)$ is m-solvable; a bounding-matrix $\boldsymbol{M}$ with length e for $\mathbb{E}$, $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{E}^m$ and $\boldsymbol{f} \in \mathbb{E}^n$.

Output: A normalized basis matrix $\boldsymbol{B}$ of a subspace of $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{E})$ over $\mathbb{K}$.

(1) IF $e = 0$ RETURN `SolveSolutionSpace`$((\mathbb{E}, \sigma), \boldsymbol{a}, \boldsymbol{f})$

Let $\mathbb{F} := \mathbb{H}(t_1,\dots,t_{e-1})$, i.e. $(\mathbb{F}(t_e),\sigma)$ is a ΠΣ-extension of $(\mathbb{F},\sigma)$.

(2) Normalize $\boldsymbol{a}$, $\boldsymbol{f}$ as in line (2) of Algorithm 7.1 which results to an k with $1 \leq k \leq m$ and to $\boldsymbol{a'} = (a'_1,\dots,a'_{m'}) \in \mathbb{F}[t_e]^{m'}$ and $\boldsymbol{f'} \in \mathbb{F}[t_e]^n$ with $a'_1\, a'_{m'} \neq 0$ and $m' \leq m$.

(3) IF $\boldsymbol{a'} \in \mathbb{F}[t_e]^1$ normalize $\mathbf{Id}_{\boldsymbol{n}\wedge}\sigma^{m-k}(\frac{\boldsymbol{f}}{a'_1})$ to $\boldsymbol{B}$ (Definition 4.9); RETURN $\boldsymbol{B}$.

(4) Let $\boldsymbol{M} = \boldsymbol{M_0}{}_\wedge(\begin{smallmatrix}x\\y\end{smallmatrix})$; if $e = 1$, $\boldsymbol{M_0}$ is the empty list ().

(5) $\boxed{\text{Approximate a denominator bound by setting } d := \texttt{DenBoundH}((\mathbb{F}(t_e),\sigma),\boldsymbol{a'},\boldsymbol{f'})\, t_e^x.}$

(6) Set $\boldsymbol{a''} := (\frac{a'_1}{\sigma^{m'-1}(d)},\dots,\frac{a_{m'}}{d}) \in \mathbb{F}(t_e)^{m'}$ and clear denominators in $\boldsymbol{a''}$ which results in $\boldsymbol{a''} \in \mathbb{F}[t_e]^{m'}$ and $\boldsymbol{f''} \in \mathbb{F}[t_e]^n$ (like in line (5) of Algorithm 7.1).

(7) $\boxed{\text{Approximate a degree bound by setting } b := y + \max(\|\boldsymbol{f''}\| - \|\boldsymbol{a''}\|, x + \|d\|).}$

(8) Set $\boldsymbol{C}\wedge\boldsymbol{w}$:=$\texttt{IncrementalReductionH}((\mathbb{F}(t_e),\sigma),\boldsymbol{M_0},b,\boldsymbol{a},\boldsymbol{f})$ by using Algorithm 7.4.

(9) Normalize $\boldsymbol{C}_\wedge\sigma^{m-k}(\frac{\boldsymbol{w}}{d})$ to $\boldsymbol{B}$ (Definition 4.9); RETURN $\boldsymbol{B}$.

Algorithm 7.4. The incremental reduction process.

$\boldsymbol{B}$=$\texttt{InrementalReductionH}((\mathbb{F}(t),\sigma),\boldsymbol{M},d,\boldsymbol{a},\boldsymbol{f})$

Input: A ΠΣ-field $(\mathbb{F}(t),\sigma)$ over $\mathbb{K}$ with $\mathbb{F} = \mathbb{H}(t_1,\dots,t_e)$ and $e \geq 0$ where $(\mathbb{H},\sigma)$ is m-solvable; a bounding-matrix $\boldsymbol{M}$ with length $e+1$ for $\mathbb{F}(t)$ and $d \in \mathbb{N}_0 \cup \{-1\}$; $\boldsymbol{0} \neq \boldsymbol{a} = (a_1,\dots,a_m) \in \mathbb{F}[t]^m$ with $l := \|\boldsymbol{a}\|$ and $\boldsymbol{f} \in \mathbb{F}[t]^n_{l+d}$.

Output: A basis matrix $\boldsymbol{B}$ of a subspace of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_d)$ over $\mathbb{K}$.

Exactly the same lines as in Algorithm 7.2 up to the replacing of line (5) by:

(5) Set $\boldsymbol{D}\wedge\boldsymbol{h} := \texttt{IncrementalReductionH}((\mathbb{F}(t),\sigma),d-1,\boldsymbol{a},\tilde{\boldsymbol{f}}')$ with $\boldsymbol{C} \in \mathbb{K}^{\lambda\times n}, \boldsymbol{g} \in (t^d\,\mathbb{F})^\lambda$.

Remark 7.3. The normalization steps in lines (3) and (9) are not necessary to prove correctness of Algorithm 7.3 in Theorem 7.7. Nevertheless this property is essential for Theorem 7.8 that states that we can find all solutions of a given solution space by adapting appropriately the bounding-matrix. Although the normalization is based on linear algebra, i.e. on Gaussian elimination (Theorem 4.4), this transformation of the basis matrix might be very expensive. In particular if one deals with the creative telescoping problem or with highly nested indefinite sums this transformation seems to be quite infeasible. But fortunately exactly those problems are formulated in parameterized first order linear difference equations, hence Algorithm 7.1 might be applied (Theorem 7.4) without any normalization steps. Moreover for recurrences of higher order, that come from typical summation problems, those normalization steps are quite cheap.

Similarly as above, one shows that Algorithm 7.2 works correct in a ΠΣ-field $(\mathbb{F}(t),\sigma)$ under the assumption that Algorithm 7.1 works correct in $(\mathbb{F},\sigma)$.

Lemma 7.2. *Let $(\mathbb{F}(t),\sigma)$ with $\mathbb{F} := \mathbb{H}(t_1,\dots,t_e)$ be a ΠΣ-field over $\mathbb{K}$ where $(\mathbb{H},\sigma)$ is m-solvable, $\boldsymbol{M} = \boldsymbol{M_0}{}_\wedge(\begin{smallmatrix}x\\y\end{smallmatrix})$ be a bounding-matrix with length $e+1$ for $\mathbb{F}(t)$ and $d \in \mathbb{N}_0 \cup \{-1\}$; let $\boldsymbol{0} \neq \boldsymbol{a} \in \mathbb{F}[t]^m$ with $l := \|\boldsymbol{a}\|$ and $\boldsymbol{f} \in \mathbb{F}[t]^n_{l+d}$. Assume that Algorithm 7.3 terminates and works correct for any valid input in the ΠΣ-field $(\mathbb{F},\sigma)$. Then Algorithm 7.4 terminates and computes a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ over $\mathbb{K}$ for $\texttt{InrementalReductionH}((\mathbb{F}(t),\sigma),d,\boldsymbol{M_0},\boldsymbol{a},\boldsymbol{f})$.*

Proof: The proof is essentially the same as for Lemma 7.1 where one just does not use the last statement in Theorem 6.2. □

First we analyze the subproblems in the incremental reduction process of the solution space $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}[t]_d)$ under the assumption that Algorithm 7.3 computes for any valid input a basis matrix of $\mathrm{V}(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{f}},\mathbb{E})$ for some $\tilde{\boldsymbol{a}}\in\mathbb{E}^{\mu}$ and $\tilde{\boldsymbol{f}}\in\mathbb{E}^{\nu}$.

Lemma 7.3. *Let $(\mathbb{F}(t),\sigma)$ with $\mathbb{F}:=\mathbb{H}(t_1,\dots,t_e)$ be a ΠΣ-field over $\mathbb{K}$ where $(\mathbb{H},\sigma)$ is m-solvable, $\boldsymbol{M}=\boldsymbol{M_0}_{\wedge}\binom{x}{y}$ be a bounding-matrix with length $e+1$ for $\mathbb{F}(t)$ and $d\in\mathbb{N}_0\cup\{-1\}$; furthermore let $\boldsymbol{0}\neq\boldsymbol{a}\in\mathbb{F}[t]^m$ with $l:=\|\boldsymbol{a}\|$ and $\boldsymbol{f}\in\mathbb{F}[t]^n_{l+d}$. Assume that Algorithm 7.3 terminates and computes for any valid input* `SolveSolutionSpaceH`$((\mathbb{F}(t),\sigma),\boldsymbol{M_0},\tilde{\boldsymbol{a}},\tilde{\boldsymbol{f}})$ *a basis matrix of* $\mathrm{V}(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{f}},\mathbb{F}(t))$.

1. *Then Algorithm 7.4 terminates and computes a basis matrix of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ *for* `InrementalReductionH`$((\mathbb{F}(t),\sigma),d,\boldsymbol{M},\boldsymbol{a},\boldsymbol{f})$.
2. *The algorithm calls itself d times where in line (5) the $d+1$ uniquely defined subproblems in the incremental reduction (Definition 6.3) are computed.*

Proof: The proof of the first part is essentially the same as for Lemma 7.1. Also one sees immediately that in line (5) $d+1$ subproblems of the incremental reduction process are computed (Remark 7.2). Since in line (9) the basis matrices are normalized, the uniqueness of the subproblems in the incremental reduction follows by Proposition 6.2. □

Now we prove the two main results. First correctness of Algorithm 7.3 is shown.

Theorem 7.7. *Let $(\mathbb{E},\sigma)$ with $\mathbb{E}:=\mathbb{H}(t_1,\dots,t_e)$ be a ΠΣ-field over $\mathbb{K}$ where $(\mathbb{H},\sigma)$ is m-solvable. Let $\boldsymbol{0}\neq\boldsymbol{a}\in\mathbb{E}^m$, $\boldsymbol{f}\in\mathbb{E}^n$ and $\boldsymbol{B}$ be a bounding-matrix with length e for $\mathbb{E}$. Then for* `SolveSolutionSpaceH`$((\mathbb{E},\sigma),\boldsymbol{a},\boldsymbol{f},\boldsymbol{B})$ *Algorithm 7.3 computes a basis-matrix of a subspace of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$ *over* $\mathbb{K}$.

Proof: If $e=0$, by Theorem 7.3 we compute a basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$ in line (1). Otherwise let $\mathbb{F}:=\mathbb{H}(t_1,\dots,t_{e-1})$ and assume as induction assumption that Algorithm 7.1 terminates and works correct for any valid input with the ΠΣ-field $(\mathbb{F},\sigma)$. Now transform $\boldsymbol{a}$ and $\boldsymbol{f}$ to $\boldsymbol{a'}$ and $\boldsymbol{f'}$ like in line (2). If one exits in line (3), the result is a normalized basis matrix of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_e))$ by Theorems 5.1 and 5.4. Clearly b is chosen such that $\boldsymbol{f}\in\mathbb{F}[t_e]^n_{b+l}$. Hence by Lemma 7.2 we obtain in line (8) a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t_e]_b)$ over $\mathbb{K}$ and hence also of a subspace of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t_e])$ over $\mathbb{K}$. But then by Proposition 5.1 and Theorem 5.1 $\boldsymbol{C}_{\wedge}\sigma^{m-k}(\frac{\boldsymbol{w}}{d})$ is a basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_e))$ over $\mathbb{K}$. Finally one returns a normalized basis matrix of a subspace of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_e))$ over $\mathbb{K}$. □

Finally we show that by choosing an appropriate bounding-matrix, we are able to find all solutions of a parameterized linear difference equation in ΠΣ-fields.

Theorem 7.8. *Let $(\mathbb{E},\sigma)$ with $\mathbb{E}:=\mathbb{H}(t_1,\dots,t_e)$ be a ΠΣ-field where $(\mathbb{H},\sigma)$ is m-solvable. Let $\boldsymbol{0}\neq\boldsymbol{a}\in\mathbb{E}^m$ and $\boldsymbol{f}\in\mathbb{E}^n$. Then there exists a bounding-matrix $\boldsymbol{B}$ with length e for $\mathbb{E}$ such that for* `SolveSolutionSpaceH`$((\mathbb{E},\sigma),\boldsymbol{a},\boldsymbol{f},\boldsymbol{B})$ *Algorithm 7.3 computes a basis-matrix of* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$.

Proof: If $e = 0$, take the empty list () as bounding-matrix, and the theorem holds. Now assume $e \geq 1$ and set $\mathbb{F} := \mathbb{H}(t_1, \dots, t_{e-1})$. In order to prove the theorem, we prove the following stronger result. Let

$$S := \{(\boldsymbol{a_1}, \boldsymbol{f_1}), \dots, (\boldsymbol{a_k}, \boldsymbol{f_k})\}$$

with $\boldsymbol{0} \neq \boldsymbol{a_i} \in \mathbb{F}(t_e)^{m_i}$ and $\boldsymbol{f_i} \in \mathbb{F}(t_e)^{n_i}$ for some $m_i, n_i \geq 1$. Then there exists a bounding-matrix $\boldsymbol{B}$ with length e for $\mathbb{F}(t_e) = \mathbb{H}(t_1, \dots, t_e)$ such that one computes with $\texttt{SolveSolutionSpaceH}((\mathbb{F}(t_e), \sigma), \boldsymbol{a_i}, \boldsymbol{f_i}, \boldsymbol{B})$ a basis-matrix of $\mathrm{V}(\boldsymbol{a_i}, \boldsymbol{f_i}, \mathbb{F}(t_e))$ for all $1 \leq i \leq k$. Having this result in hands, the theorem follows immediately by considering the special case $k = 1$.
Now assume that the more general assumption holds for the ΠΣ-field $(\mathbb{F}, \sigma)$ and let S be as above. Now adapt $(\boldsymbol{a_i}, \boldsymbol{f_i})$, as it is performed in line (3) to $(\boldsymbol{a'_i}, \boldsymbol{f'_i})$. For any $1 \leq i \leq k$ with $\boldsymbol{a'_i} \in \mathbb{F}(t_e)^1$ we obtain a basis matrix of $\mathrm{V}(\boldsymbol{a'_i}, \boldsymbol{f'_i}, \mathbb{F}(t_e))$ in line (3). Therefore we can restrict S to those $\boldsymbol{a'_i}$ with $\boldsymbol{a'_i} \notin \mathbb{F}(t_e)^1$ and write

$$S := \{(\boldsymbol{a'_1}, \boldsymbol{f'_1}), \dots, (\boldsymbol{a'_{k'}}, \boldsymbol{f'_{k'}})\}$$

for some $k' \leq k$. If $k' = 0$ we are done. Otherwise suppose $k' > 0$. Let $d_i \in \mathbb{F}[t_e]^*$ for $1 \leq i \leq k'$ be the polynomial obtained by $\texttt{DenBoundH}((\mathbb{F}(t_e), \sigma), \boldsymbol{a'_i}, \boldsymbol{f'_i})$. Furthermore let $x_i \in \mathbb{N}_0$ be minimal such that $d_i\, t_e^{x_i}$ is a denominator bound of $\mathrm{V}(\boldsymbol{a'_i}, \boldsymbol{f'_i}, \mathbb{F}(t_e))$. Now we set $x := \max(x_1, \dots, x_{k'})$. Note that $x_i = 0$ for all $1 \leq i \leq k'$ and hence $x = 0$, if $(\mathbb{F}(t_e), \sigma)$ is a Σ-extension of $(\mathbb{F}, \sigma)$. Furthermore $d_i\, t_e^x$ is a denominator bound of $\mathrm{V}(\boldsymbol{a'_i}, \boldsymbol{f'_i}, \mathbb{F}(t_e))$ for all $1 \leq i \leq k'$. Next adapt $(\boldsymbol{a'_i}, \boldsymbol{f'_i})$ for the denominator bound $d_i\, t^x$ to $(\boldsymbol{a''_i}, \boldsymbol{f''_i})$ as it is performed in line (6). Now let y be minimal such that $b_i := y + \max(\|\boldsymbol{f''}\| - \|\boldsymbol{a''}\|, \|d_i\| + x)$ is a degree bound of $\mathrm{V}(\boldsymbol{a''_i}, \boldsymbol{f''_i}, \mathbb{F}[t_e])$ for all i with $1 \leq i \leq k'$. With those degree bounds b_i we consider the uniquely determined incremental reduction process of $\mathrm{V}(\boldsymbol{a''_i}, \boldsymbol{f''_i}, \mathbb{F}[t_e]_{b_i})$ for all $1 \leq i \leq k'$ where the basis matrices of the subproblems are normalized. In this incremental reduction processes of $\mathrm{V}(\boldsymbol{a''_i}, \boldsymbol{f''_i}, \mathbb{F}[t_e]_{b_i})$ for $1 \leq i \leq k'$ let

$$S_i := \{(\boldsymbol{a''_{ib}}, \boldsymbol{f''_{ib}}), \dots, (\boldsymbol{a''_{i0}}, \boldsymbol{f''_{i0}})\}$$

be the uniquely determined subproblems. Then by induction assumption there exists a bounding-matrix $\boldsymbol{B_0} \in \mathbb{N}_0^{2\times(e-1)}$ of length $e - 1$ for $\mathbb{F}$ such that for all $(\boldsymbol{b}, \boldsymbol{g}) \in \bigcup_{i=1}^{k'} S_i$ Algorithm 7.3 with $\texttt{SolveSolutionSpaceH}((\mathbb{F}, \sigma), \boldsymbol{B_0}, \boldsymbol{b}, \boldsymbol{g})$ computes a basis-matrix of $\mathrm{V}(\boldsymbol{b}, \boldsymbol{g}, \mathbb{F})$. Hence by applying Algorithm 7.4 with input $\texttt{InrementalReduction}((\mathbb{F}, \sigma), b_i, \boldsymbol{a''_i}, \boldsymbol{f''_i}, \boldsymbol{B_0})$ one computes a basis matrix $\boldsymbol{C_i} \wedge \boldsymbol{w_i}$ of $\mathrm{V}(\boldsymbol{a''_i}, \boldsymbol{f''_i}, \mathbb{F}[t_e]_b)$ for all $1 \leq i \leq k'$ by Lemma 7.3. Clearly $\boldsymbol{B} := \boldsymbol{B_0}{\wedge}\binom{x}{y}$ is a bounding-matrix of length e for $\mathbb{F}(t_e)$. Since $d_i\, t^x$ is a denominator bound of $\mathrm{V}(\boldsymbol{a'_i}, \boldsymbol{f'_i}, \mathbb{F}(t_e))$, by Theorems 5.2 and 5.5 $\boldsymbol{C_i}{\wedge}\frac{\boldsymbol{w_i}}{d_i\, t^x}$ is a basis-matrix of $\mathrm{V}(\boldsymbol{a'_i}, \boldsymbol{f'_i}, \mathbb{F}(t_e))$ for all $1 \leq i \leq k'$. But then by Theorems 5.1 and 5.2 $\boldsymbol{C_i}{\wedge}\sigma^{m-k}(\frac{\boldsymbol{w_i}}{d_i\, t^x})$ is a basis matrix of $\mathrm{V}(\boldsymbol{a_i}, \boldsymbol{f_i}, \mathbb{F}(t_e))$ for all $1 \leq i \leq k'$. Hence the induction step holds and the theorem is proven. □

Remark 7.4. As illustrated in Example 3.1, Algorithms 7.1 and 7.3 are available in the package `Sigma` in form of the function call `SolveDifferenceVectorSpace`. Some further remarks are given about the implementation of Algorithm 7.3.

- Let $(\mathbb{E},\sigma)$ with $\mathbb{E} := \mathbb{H}(t_1,\dots,t_e)$ be a ΠΣ-field where $(\mathbb{H},\sigma)$ is m-solvable, $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{E}^m$ and $\boldsymbol{f} \in \mathbb{E}^n$. Then by calling `SolveDifferenceVectorSpace` without choosing any bounding-matrix as input, Algorithm 7.3 will be applied with `SolveSolutionSpaceH`$(\boldsymbol{a},\boldsymbol{f},\boldsymbol{M},(\mathbb{E},\sigma))$ by using automatically a bounding-matrix $\boldsymbol{M}$ for $\mathbb{E}$ of length e. More precisely the bounding-matrix $\boldsymbol{M}$ is of the form $\left(\begin{smallmatrix} c & \cdots & c \\ 1 & \cdots & 1 \end{smallmatrix}\right) \in \mathbb{N}_0^{2\times e}$ where $c=1$, if $(\mathbb{F}(t_1,\dots,t_i),\sigma)$ is a Π-extension of $(\mathbb{F}(t_1,\dots,t_{i-1}),\sigma)$, otherwise $c=0$. It turned out that with this simple choice one computes a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$ in many cases.
- In some specific instances there are denominator and degree bound algorithms developed in [Sch02a, Sch02b]. If one runs into such special cases, these bounds are used in lines (5) or (7) instead of using the bounding-matrix mechanism.

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