Spline Implicitization of Planar Curves *

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Abstract. We present a new method for constructing a low degree implicit spline representation of a given parametric planar curve. To ensure the low degree condition, quadratic B-splines are used to approximate the given curve via orthogonal projection in Sobolev spaces. Adaptive knot removal, which based on spline wavelets, is used to reduce the number of segments. The B-spline segments are implicitized, and the resulting bivariate functions are joined along suitable transversal lines, yielding a globally continuous bivariate function. As shown by analyzing the asymptotic behavior of these transversal lines for stepsize $h \to 0$, the given curve can be implicitized with any desired accuracy.

§1. Introduction

Planar curves in Computer Aided Geometric Design can be defined in two different ways. In most applications, they are described by a *parametric representation*, $x = x(t)/w(t)$ and $y = y(t)/w(t)$ where $x(t), y(t)$, and $w(t)$ are often polynomials. Alternatively, the *implicit form* $f(x, y) = 0$ can be used.

Both the parametric and implicit representation have its advantages. The availability of both representations often results in simpler and more efficient computations. For example, if both implicit and parametric representations are available, the intersection of two curves is obtained easier than otherwise, as it can be found by solving a root finding problem in one variable.

From classical algebraic geometry, it is known that each rational parametric curve has an implicit representation, while the converse is not true. The process of converting the parametric equation into implicit form is called implicitization. A number of established methods for *exact* implicitization exists: resultants, Gröbner bases, and moving curve and surface, see [3] for further information.

However, *exact* implicitization has not found widespread use in CAGD. This is in part due to the following three facts:

- Exact implicitization often produces large data volumes, as the resulting implicit polynomials may have a huge number of coefficients.

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- The exact implicitization process is relatively complicated, especially, in the case of high polynomial degree. For instance, the resultant–based methods need the symbolic evaluation of large determinants.
- Even for regular parametric curves, an exactly implicitized parametric curve may have unwanted components (branches) or self intersections in the region of interest.

For these reasons, approximate implicitization has been proposed. A number of methods are available for approximate implicitization: Montaudouin and Tiller [9] employed a power series method to obtain local explicit approximation (about a regular point) to polynomial parametric curves and surfaces. Chuang and Hoffmann [2] extend this method using what they called "implicit approximation". Dokken [4] proposed a new way to approximate the parametric curve and surface, globally, in the sense that the approximation is valid within the whole domain of the curve segment or surface patch. Sederberg et al. [11] employed monoid curves and surfaces to find an approximate implicit equation and approximate inversion map of a planar rational parametric curve or a rational parametric surface .

This paper discusses the problem of constructing what we call a spline implicitization for planar curves: a partition of the plane into polygonal segments, and an bivariate polynomial for each segment, such that the collection of the zero contours approximately describes the given curve. On the boundaries, these polynomial pieces are joined to form a globally C^m spline function, for a suitable choice of m. In this paper we restrict ourselves to continuous functions $m = 0$. A methods for C^1 spline implicitization is currently under investigation.

The parametric and implicit representations of a planar curve have the same polynomial degree n. However, the number of the coefficients in the parametric case is $2(n + 1)$ while it is $(n + 1)(n + 2)/2$ in the implicit case. Consequently, in the implicit case, high polynomial degrees will lead to expensive computations.

Therefore, the main goal of this paper is to find a *low* degree spline implicit representation of a given parametric planar curve. To ensure the low degree condition, quadratic B-spline curves are used to approximate the given parametric curve (section 2). Then, adaptive knot removal, which is based on spline wavelets, is used to reduce the number of segments (section 3). The resulting quadratic B-spline segments are implicitized (section 4). Finally, these implicitized quadratic segments are joined together, forming a globally C^0 function. (section 5).

§2. B–spline Approximation of Planar Curves

Using the idea proposed in [10], we generate a quadratic B–spline approximation via orthogonal projection in Sobolev spaces. The uniform B–spline $B_{j,d,h}$ of order d with knot sequence $h\mathbb{Z}$, where h is the stepsize, form an orthonormal sequence in Sobolev space $H^{d-1,2}(\mathbb{R})$ endowed with the inner product:

$$(f, g)_{\mathbf{R},d,h} = \sum_{\mu=0}^{d-1} h^{2\mu-1}\ w_\mu(d)\ \ < \partial^\mu f, \partial^\mu g > \tag{1}$$

for certain positive weights $w_\mu(d)$. These weights are specified explicitly in [10]. For the case of finite intervals, the inner product (1) need to be modified near the boundary. Let $r = [r_1\ h, r_2\ h]$, $r_i \in \mathbb{Z}$, $r_2 - r_1 \geq d-1$. The sequence $B_{j,d,h}$, $r_1 - d < j < r_2 - 1$ is orthonormal with respect to the bilinear form,

$$(f,g)_{r,A_{1,2}(d,h)} = \sum_{i=1}^{2} f^{(d)}(r_i)\ A_i(d,h)\ g^{(d)}(r_i)^T \ + \ (f,g)_{\mathbb{R},d,h}\,, \qquad (2)$$

where $f^{(d)} := [f, \partial f, ..., \partial^{d-2} f]$ and the same for $g^{(d)}$. The matrices $A_i(d,h)$ are specified in [10].

The B-spline approximation of a given curve $\mathbf{g}$, with respect to the norm which is induced by the inner product (1), can be written as:

$$\mathbf{g}' = \sum_{j \in [r_1, r_2]} (Q_h \mathbf{g})_j\ B_{j,d,h} \qquad (3)$$

where $(Q_h\mathbf{g})_j$ is the j^{th} B-spline coefficient vector and $B_{j,d,h}$ is the j^{th} B-spline. According to [10], these B-spline coefficients are given by:

$$(Q_h \mathbf{g})_j := (\mathbf{g}, B_{j,d,h})_{r,A_{1,2}(d,h)} \qquad (4)$$

The quadratic B-spline coefficients $(Q_h\mathbf{g})_j$ are computed as the inner product of $\mathbf{g}$ and $B_{j,3,h}$ in Sobolev space $H^{2,2}(\mathbb{R})$, where $B_{j,3,h}$ are the quadratic B-splines. The following estimate [10] shows that the approximation order of Q_h is optimal.

$$\|(\mathbf{g} - Q_h\mathbf{g})\|_\infty \leq Ch^3 \|\partial^3 \mathbf{g}\|_\infty$$

where C is a certain constant.

The control points of the approximating B–spline curve can be generated by simple and efficient computations, as only (possibly numerical) integrations are needed. Also, no assumption about the given parametric representation have to be made, except that it should be at least C^1. By using sufficiently many segments, the initial approximation can be made as accurate as needed.

Remark 1. *In next section, we will use spline wavelets for data reduction process. For this reason, the stepsize h is restricted to be in the form $\frac{1}{2^i}$, $i \in \mathbb{Z}$ and the knot sequence is modified to be* $[0, 0, 0, h, 2h, ..., 1-2h, 1-h, 1, 1, 1]$. *Both B–spline basis and coefficients are modified according to the new knot sequence.*

Example

Throughout this paper, we will use the following example to illustrate the idea of this paper. We consider a polynomial parametric curve $\mathbf{g}$ of degree 20, as shown in Fig. 1.

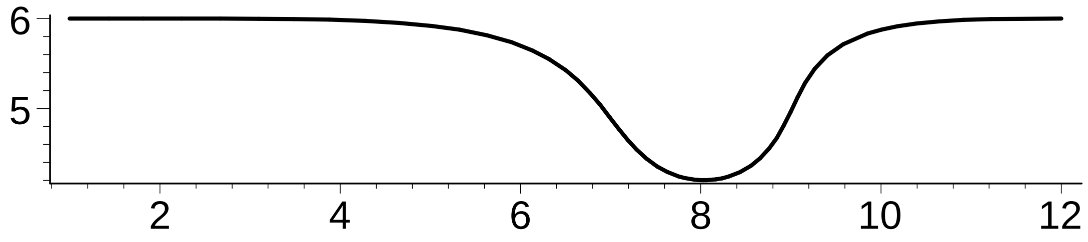

Fig. 1. A polynomial parametric curve of degree 20.

First, we approximate $\mathbf{g}$ using quadratic B–splines. Fig. 2 shows the error between $\mathbf{g}$ (black) and the quadratic B–spline approximation $\mathbf{g}'$ (gray) for stepsize $h = 1/128$. Note that the error had to be exaggerated by a factor $\delta = 25000$ to make it visible.

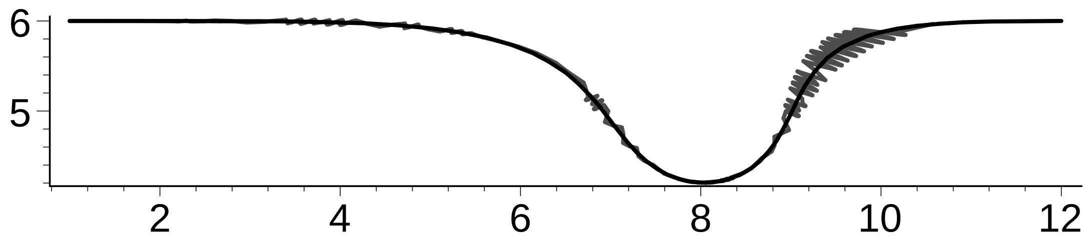

Fig. 2. The original curve (black) and the error introduced by approximating it with a quadratic B–spline curve (gray). The error is exaggerated (amplified) by a factor $\delta = 25000$ to make it visible.

§3. Data Reduction via Spline Wavelets

After computing the initial B–spline approximation, we apply a knot removal (or data reduction) procedure, in order to reduce the number of segments. Such a reduction means that we approximate the given B–spline in a space S by a B-spline in linear subspace of S. Knot removal has been discussed in a number of publications, see e.g. [5,8] and the references cited therein. In [5], the authors propose an optimal technique, by treating the knot removal procedure as a reverse approximative knot insertion process. It is based on a so–called "ranking list", is used to compare the evaluate the error introduced by removing a specific knot.

In the present work, a special method for our special situation (see Remark 1) is proposed. We propose a new technique, utilizing spline wavelets [1], for knot removal. The method is not optimal, but it is cheaper than all other methods since no sorting or ranking lists are required.

Consider a B–spline curve defined on a certain number of knots sufficient to guarantee the desired accuracy. First, the wavelets transform of the given B-spline curve is computed. Then, by setting all wavelets coefficients vectors with norm less than the threshold to be zero, we can remove blocks wavelets with zero coefficients vector. For each block, one of the two common knots can be removed from the knot sequence. The length of these blocks varies between 2 and 5 wavelets, depending on the location of the removed knot in the knot sequence (that is, if the knot is an inner knot or close to the boundary). Finally, the B–spline final representation $\mathbf{g}''$ of the given B–spline curve $\mathbf{g}'$ is computed over the reduced knot sequence K_{final}.

The error can be bounded simply by applying the wavelet synthesis to the set of removed wavelets. Due to the convex hull property of the B–splines, the error is bounded by the maximum absolute value of the control polygon.

Example Continued

We apply the procedure to the quadratic B–spline curve $\mathbf{g}'$. Applying the data reduction process, the number of knots is reduced from 133 to 17, where the threshold is equal to 10^{-4}. Fig. 3 shows the error between the original curve $\mathbf{g}$ (black) and the final B–spline representation curve $\mathbf{g}''$ (gray) over K_{final}. The knots are plotted as circles. The knots at the boundary have multiplicity 3. The error is exaggerated by a factor $\delta = 10$ to make it visible.

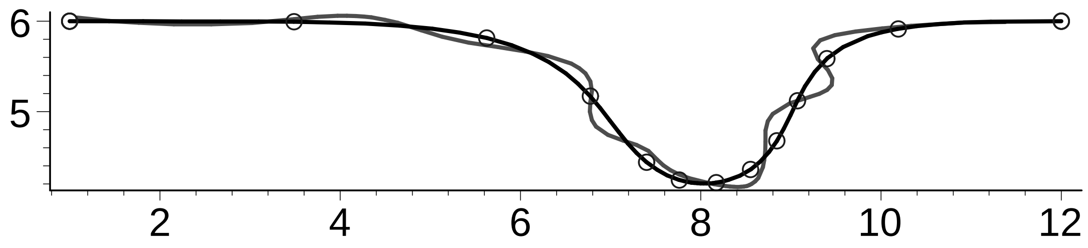

Fig. 3. The original curve (black) and the B–spline approximation curve after the data reduction (gray). The error is exaggerated (amplified) by a factor $\delta = 10$.

Fig. 4 shows – on a logarithmic scale – the values of wavelets coefficients of the curve $\mathbf{g}'$ before (left) and after (right) data reduction. The threshold is plotted as thick black line.

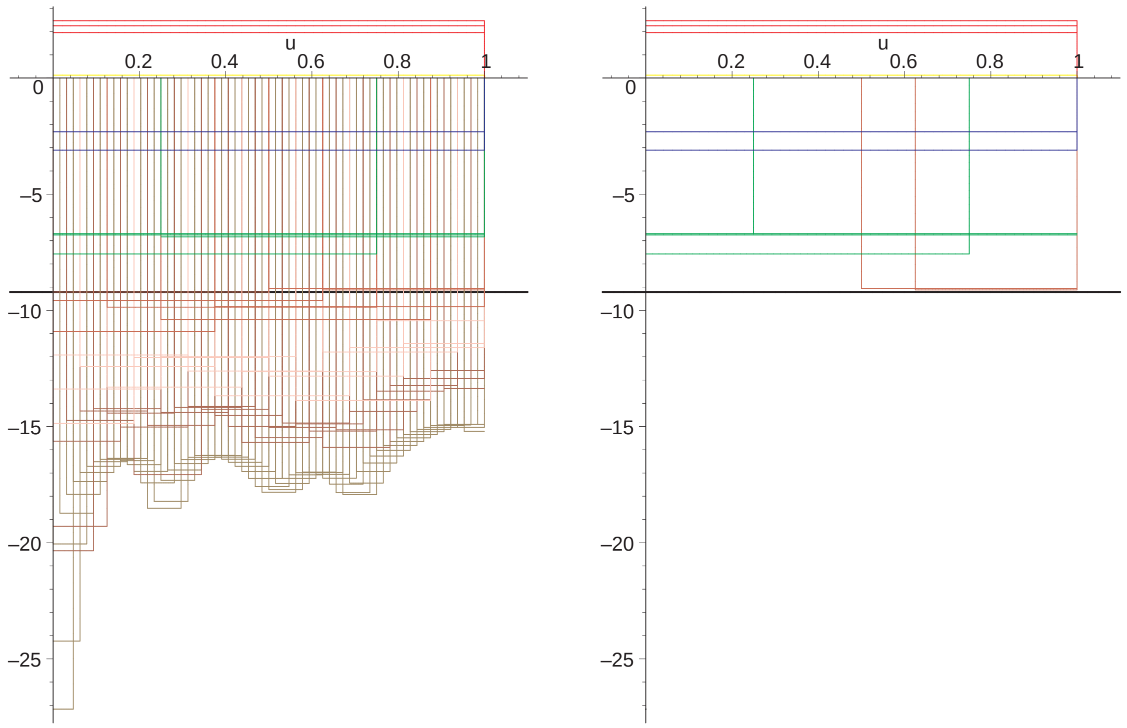

Fig. 4. The logarithmic value of wavelets coefficients before (left) and after (right) data reduction.

§4. Implicitization

After the data reduction process, we have a quadratic B–spline approximation $\mathbf{g}''$ defined over the reduced non–uniform knot sequence K_{final}. To implicitize this curve, we split the B–spline representation of this curve into the Bézier segments. Then, each quadratic Bézier segment is implicitized.

The conversion from B–spline representation of the curve to Bézier representation can easily be achieved via knot insertion. By increasing the knot multiplicity at each knot to be equal to the degree of the curve (in our case to be 2), the B–spline representation is converted to Bézier representation.

Each quadratic parametric Bézier segment has three control points. Let (p_0, q_0), (p_1, q_1) and (p_2, q_2) be the control points of one of these segments. Then the implicit form of this segment can be shown to be equal to

$$f(x,y) = \det \begin{pmatrix} Q_0(y)P_2(x) - P_0(x)Q_2(y) & Q_0(y)P_1(x) - P_0(x)Q_1(y) \\ Q_1(y)P_2(x) - P_1(x)Q_2(y) & Q_0(y)P_2(x) - P_0(x)Q_2(y) \end{pmatrix}$$

where

$$P_i = \binom{2}{i}(p_i - x), \qquad Q_i = \binom{2}{i}(q_i - y) \qquad \text{for} \qquad i = 0,1,2.$$

§5. Joining the Segments

We will now join the polynomial segments, which have been produced by the implicitization process. More precisely, we have to identify suitable transversal lines, such that these segments can be pieced together along them, giving a globally continuous function.

Any two consecutive segments are parabolas which meet with tangent continuity at their junction point p_0. Moreover, they intersect in two additional points p_1, p_1', see Fig. 5. Note that these two points can be conjugate complex! The transversal line has to be chosen as the line passing through the junction point p_0 and one of the other two intersection points p_1, p_1'. There are two possibilities to choose this line. We pick the line $L(p_0, p_1)$, which is closer to the normal vector of the curve. According to the following theorem, we can then always achieve a C^0 joint along the transversal line $L(p_0, p_1)$.

Theorem 2. *Given two quadratic functions $\mathbf{g}_1(x,y)$ and $\mathbf{g}_2(x,y)$ such that they have a common root and a parallel gradient at $p_0(x_0, y_0)$, and intersect at $p_1(x_1, y_1)$. Let $L(p_0, p_1)$ be the line joining $p_0(x_0, y_0)$ and $p_1(x_1, y_1)$. Then after multiplying $\mathbf{g}_2$ by a suitable constant, $\mathbf{g}_1(x,y)$ and $\mathbf{g}_2(x,y)$ are identical along the line $L(p_0, p_1)$.*

Proof: The restriction of $\mathbf{g}_1(x,y)$ and $\mathbf{g}_2(x,y)$ to the line $L(p_0, p_1)$ are two quadratic functions that have two common roots at p_0 and p_1. After multiplying $\mathbf{g}_2$ by a suitable constant, they have the same gradient at p_0. Therefore, $\mathbf{g}_1(x,y)$ and $\mathbf{g}_2(x,y)$ are identical along the line $L(p_0, p_1)$. □

After identifying suitable transversal lines, and multiplying the implicitized segments with suitable constants, we obtain a globally continuous function which is

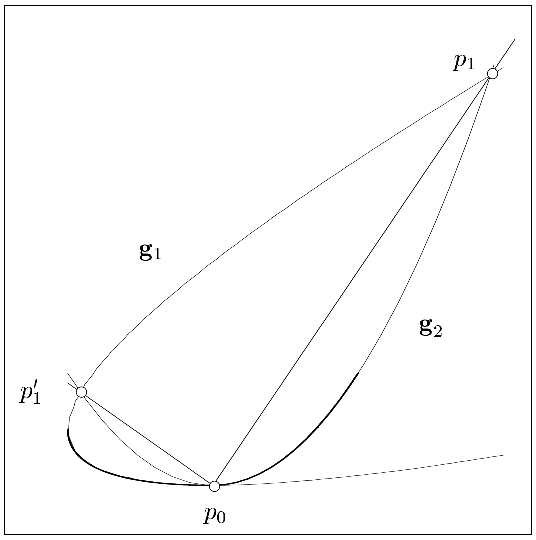

Fig. 5. Choosing the transversal line.

defined within a certain neighbourhood of the curve. Clearly, the segments cannot be extended continuously beyond the intersection points of the transversal lines. In the limit, for stepsize $h \to 0$, these intersection points form the the *evolute* of the given curve. This is due to the following result.

5.1 Asymptotic Behavior of the Transversal Lines

Next we analyze the asymptotic behavior of the transversal lines for decreasing stepsize. It will be shown, that the normal behaves nicely, provided that the curve has (in a certain neighbourhood) no inflection points. More precisely, the two additional intersections of the parabolas exist, and one of them tends to get closer and closer to the normal of the curve.

In order to get results which can be interpreted geometrically, we use the so–called canonical Taylor expansion [7] of the curve, which is derived in classical differential geometry.

Theorem 3. *Consider a curve which is parameterized by its arc length, and which has no inflection point, $\kappa \neq 0$. We apply the spline implicitization (orthogonal projection in Sobolev space and implicitization + joining of the segments) to the curve, where the knots of the quadratic B-Spline curve are uniformly spaced with stepsize h. If the stepsize h tends to zero, then the transversal line $L(p_0, p_1)$ tends to the normal vector of the curve at the point p_0.*

Proof: In the sequel we sketch the idea of the proof. Start from a curve which is parameterized by its arc length. The canonical Taylor expansion with respect to the arc length parameterization, $\mathbf{p}(s) = \{\, p_1(s), p_2(s) \,\}$, is given by

$$\begin{aligned} p_1(s) =& s - \frac{1}{6}\,\kappa_0^2\, s^3 - \frac{1}{8}\,\kappa_0\,\kappa_1\, s^4 + \mathcal{O}(s^5) \qquad \text{and} \\ p_2(s) =& \frac{1}{2}\,\kappa_0\, s^2 + \frac{1}{6}\,\kappa_1\, s^3 + \frac{1}{24}\,(\kappa_2 - \kappa_0^3)\, s^4 + \mathcal{O}(s^5) \end{aligned},$$

where $\kappa_0 = \kappa(0)$, $\kappa_1 = \kappa'(0)$, etc. This expansion is an immediate consequence of the Frenet–Serret formulas.

First, we approximate the curve $\mathbf{p}(s)$ with a quadratic B–spline curve defined over a knot sequence $[..., -2h, -h, 0, h, 2h, ...]$, as in section 2. The quadratic B–spline approximation of $\mathbf{p}(s)$ is given by piecewise function consists of m segments $(\mathbf{f}_i, i = 1..m)$. We consider two consecutive segments, for example the segments defined over the intervals $[-h, 0]$ and $[0, h]$ respectively, and call it $\mathbf{f}_{\text{left}}$ and $\mathbf{f}_{\text{right}}$. Based on the canonical Taylor expansion we obtain again Taylor expansions for these two segments.

In order to compute the intersection points p_1, p_1' of these two segments, we implicitize the parametric quadratic polynomial $\mathbf{f}_{\text{left}}$. Then we substitute the parametric form of $\mathbf{f}_{\text{right}}$ into the implicit form of $\mathbf{f}_{\text{left}}$. This gives a quartic equation in the curve local parameter S, where $s = hS$. But S^2 factors out, as both parabolic arcs are joined with tangent continuity. Solving the remaining quadratic equation we get two values S_1 and S_2 of the parameter S. By substituting these values into $\mathbf{f}_{\text{right}}$, we get the following two Laurent series for the coordinates $p_1' = (x_1, y_1)$ and $p_1 = (x_2, y_2)$ of the intersection points of the two segments $\mathbf{f}_{\text{left}}$ and $\mathbf{f}_{\text{right}}$,

$$\begin{aligned}
x_1 &= -\frac{\kappa_1}{\kappa_0^3} + \mathcal{O}(h^2) \\
y_1 &= \frac{1}{2}\frac{\kappa_1^2}{\kappa_0^5} + \mathcal{O}(h^2) \\
x_2 &= -\frac{\kappa_1}{\kappa_0^3} + \frac{2}{3}\frac{(3\kappa_1^2 - \kappa_0\kappa_2 + \kappa_0^4)}{\kappa_0^4}h + \mathcal{O}(h^2) \\
y_2 &= 8\frac{1}{\kappa_0^3}h^{-2} - \frac{1}{6}\frac{(12\kappa_0^4 + 39\kappa_1^2 - 20\kappa_0\kappa_2)}{\kappa_0^5} + \mathcal{O}(h)
\end{aligned} \tag{5}$$

Clearly, for sufficiently small stepsize $(h \to 0)$ and $\kappa_0 \neq 0$, the second intersection point (x_2, y_2) converges to normal of the curve. □

We illustrate this result by an example. Figure 6 shows the transversal lines for three different stepsizes. Obviously, one of them converges to the curve normal.

Thus, for sufficiently small stepsize h, we get a system of lines through the junction points of neighbouring segments, such that the implicit equations can be joined (at least) continuously along the transversal lines.

It is well known that no polynomial and rational curves – except straight lines – can be equipped with a closed–form arc length parameterization [6]. However, we can always reparameterize a general parametric curve by its arc length approximately, using numerical methods.

5.2 Singular Case: Inflection Points

If the curve has an inflection point $(\kappa_0 = 0)$, then we may consider the curve locally as a graph of a function. For a piecewise quadratic function, we can always join the segments of the implicit equation with C^0 continuity, as the parallels of the y-axis always form a system of suitable transversal lines. This is due to the fact that all these implicit quadratic curves share the infinite point of the y-axis.

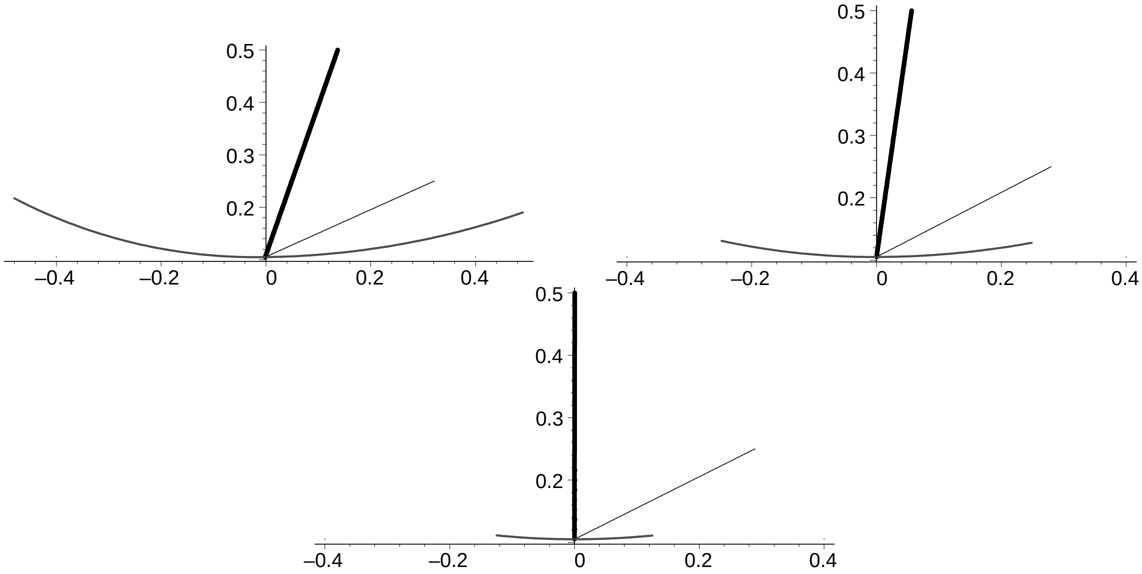

Fig. 6. The behavior of the transversal lines for different stepsize (h = $\frac{1}{8}$, $\frac{1}{32}$, and $\frac{1}{128}$ respectively).

Generally, however, the axes of the two parabolic segments are not parallel. Fig. 7 (left) shows a curve consists of two parabolic segments $\mathbf{g}_1$, $\mathbf{g}_2$ and the control polygon. The curve has an inflection point at F. The B-spline control points are showed as diamonds while the additional Bézier control points are showed as circles.

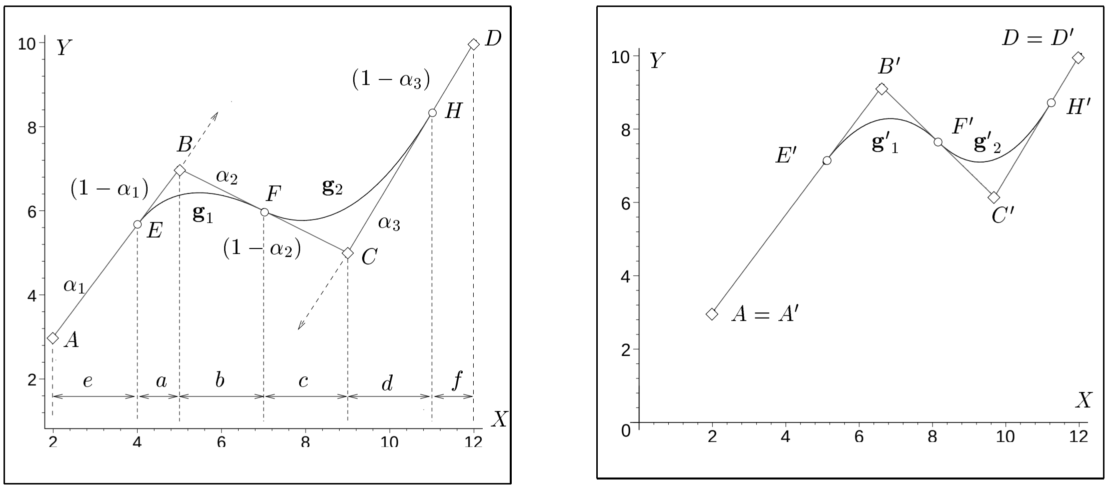

Fig. 7. Local modification of control points at inflection points.

From the above figure, it is clear that both axes of $\mathbf{g}_1$, $\mathbf{g}_2$ are generally not parallel. To force them to be parallel, we can modify the control points such that $a = b$ and $c = d$. This can be achieved by moving both points B and C along lines AB and CD respectively. Recall that the control points E, F and H come from the knot insertion process. Then, the control points E, F and H can be written as:

$$E = (1-\alpha_1)\ A + \alpha_1\ B, \qquad F = (1-\alpha_2)\ B + \alpha_2\ C, \qquad H = (1-\alpha_3)\ C + \alpha_3\ D$$

where α_i is a constant depend on the knot sequence. It is clear from Figure that:

$$e + a + b + c + d + f = x_D - x_A \tag{6}$$

and

$$e = \frac{\alpha_1}{(1-\alpha_1)}\ a, \qquad c = \frac{(1-\alpha_2)}{\alpha_2}\ b, \qquad f = \frac{(1-\alpha_3)}{\alpha_3}\ d$$

Substituting from above into (6) and choosing $a = b$, $c = d$ one finds the new control polygon. Fig. 7 (right) shows the new control polygon and the two parabolic segments $\mathbf{g}_1'$, $\mathbf{g}_2'$ after the modification.

§6. Example (finished)

We derive an implicit equation for the quadratic B-Spline curve from Figure 3, which was obtained after data reduction. In order to visualize the quality of the implicitization, Fig. 8 shows the *algebraic offsets* (thin lines) of $\mathbf{g}''$ (thick line) and the transversal lines through the junction points of the segments. By algebraic offsets (or parallel curves) of $\mathbf{g}''$ we mean the curves defined by the equation $g(x, y) = c$, where $g(x, y) = 0$ is the implicit equation of the original curve $\mathbf{g}''$ and the constant c is the "algebraic distance". To make the picture clearer, we enlarged a part of the curve and draw some additional algebraic offsets, see Fig. 9. From this Figure, it can clearly be seen that the algebraic offsets are C^0 but not C^1.

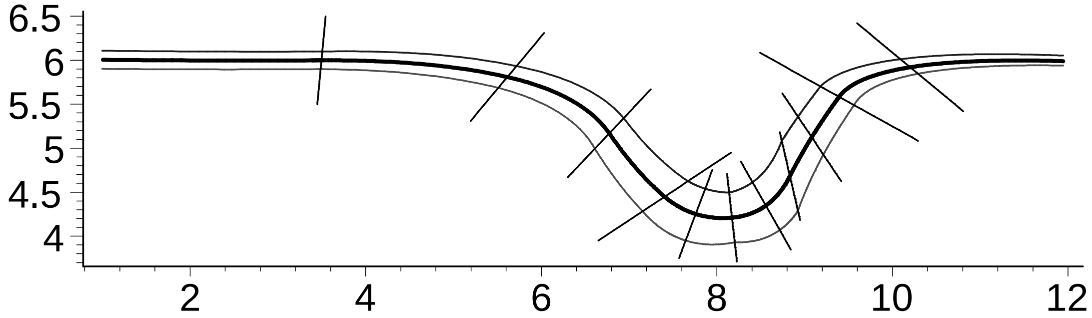

Fig. 8. Implicitized curve and algebraic offsets.

§7. Conclusion

We have derived a method for constructing a spline representation of a given parametric planar curve, which is valid within a certain neighbourhood of the given curve. The construction consists of four steps; B–spline curve approximation, knot removal, segment implicitization and segment joining. Compared to the existing methods for implicitization, our method has the following advantages.

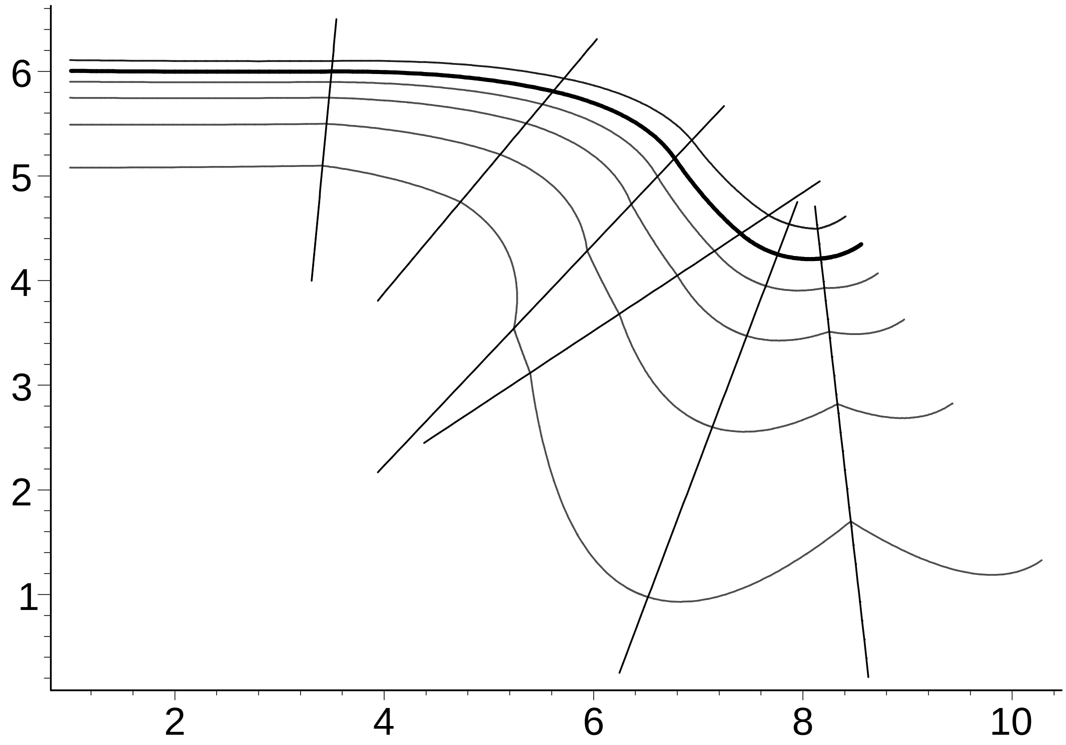

Fig. 9. Implicitized curve and algebraic offsets (enlarged).

- The method is computationally simple. In particular, no evaluations (symbolic or numerical) of large determinants are needed.
- It produces a low degree implicit representation. For instance, the intersection of a line with the implicitized curve can be found by computing the roots of a quadratic polynomial.
- The methods avoids unwanted branches or singularities, which otherwise could be present in the neighbourhood of the given curve.
- The implicit function is globally continuous.
- The method yields – for high degree curves – a smaller data volume as in the case of exact implicitization. In our example (degree 20), we have only 72 coefficients. Exact implicitization would produce 231 coefficients.

As a matter of future research, we plan to generalize this method to the C^1 case, and to surfaces.

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