A Note on the Approximation of $B - H$ Curves for Nonlinear Magnetic Field Computations *

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Abstract

In this paper, we deal with the approximation of discrete data sets from measurements of material characteristics, concentrating here on $B - H$ curves of magnetic materials. We propose an approach based on the regularizing method of smoothing splines, in combination with a discrepancy principle for the regularization parameter choice. This allows to guarantee monotone approximations of the $B - H$ curve, as it is essential both for physical reasons and for numerical purposes in nonlinear magnetic field computations.

Keywords smoothing splines, nonlinear field computations, Maxwell's equation, regularization

1 Introduction

Nonlinear numerical calculations are important in real life applications. Such nonlinearities are often due to material properties. For instance, the reluctivity

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$\nu(\cdot)$ in Maxwell's equations (see e.g. [6]), connects the magnetic induction $\mathbf{B}$ with the magnetic field intensity $\mathbf{H}$ by

$$\mathbf{H} = \nu(|\mathbf{B}|) \cdot \mathbf{B}\,.$$

If hysteresis effects and anisotropies are neglected, the reluctivity is a scalar function of the absolute value of the magnetic induction. In this paper we will consider only this example, but wish to mention the importance of nonlinearities as well as the applicability of the approximation approach considered here in other physical contexts such as piezoelectricity or heat conduction.

In practice, such material relations are given by a discrete sample of measured data that is naturally contaminated with noise. Therefore an appropriate approximation of the data points is necessary. The approximation of the reluctivity is an important task, since the final solution of Maxwell's equation depends on it.

The paper of B. Heise [5] is concerned with the interpolation of the sample and thus special assumptions on the reluctivity are made. Especially, monotonicity of the reluctivity is assumed, which does not always hold, though. Furthermore no noise is considered in the paper. We extend the work in the following directions:

1. approximation instead of interpolation of the data set
2. approximation of the $B-H$ curve itself instead of the reluctivity
3. incorporation of noise
4. approximation of the $B-H$ curve in a strictly monotone way

We base our work on the method of smoothing splines, see [9] and [1, 2] for an overview. While [9] is concerned with a twice continuously differentiable approximation of a given data set, we have to search our approximation in the space of only once continuously differentiable functions in order to be able to enforce a monotone spline approximation (see [4] for monotone spline interpolation). For a stochastic approach to spline approximation of noisy data, we refer to [10].

The method of smoothing splines was applied by the authors several times [8, 7]. However, the intention of this paper is to focus on the approximation of the $B-H$ curve and the resulting properties.

The paper is organized as follows: In Section 2 we formulate the problem and motivate it. Section 3 is concerned with the construction of the spline. Finally we do some numerical studies in Section 4 and in Section 5 conclusions are drawn and final remarks are given.

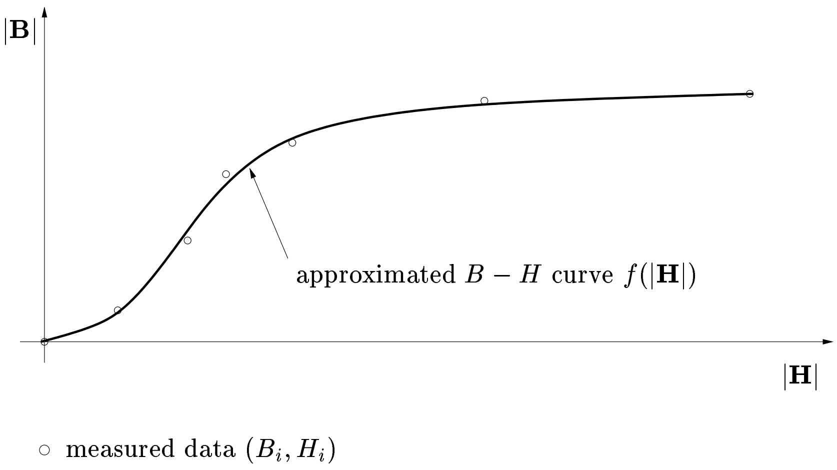

Figure 1: Principle of the $B - H$ curve approximation.

2 Problem Formulation

Maxwell's equations in the static case read as (see e.g. [6])

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathbf{J} \,, \\ \nabla \cdot \mathbf{B} &= 0 \,, \end{aligned} \tag{1}$$

with the impressed current density $\mathbf{J}$, the magnetic induction $\mathbf{B}$ and the magnetic field intensity $\mathbf{H}$. Considering the isotropic case and neglecting hysteresis effects, one can relate B and H via the reluctivity ν

$$\nu(|\mathbf{B}|) \cdot |\mathbf{B}| = |\mathbf{H}| \tag{2}$$

with $|\cdot|$ the Euclidian norm. The reluctivity can be seen as a scalar function

$$\nu(s) : \mathbb{R}_0^+ \to \mathbb{R}_0^+$$

and due to relation (2) we set

$$\nu(s) \cdot s = f^{-1}(s)$$

with f describing the $B - H$ curve (see Figure 1). Consequently the more general form of (2) reads as

$$|\mathbf{B}| = f(|\mathbf{H}|) \quad \text{or equiv.} \quad |\mathbf{H}| = f^{-1}(|\mathbf{B}|)\,.$$

Due to the physical background, the functions $\nu(\cdot)$ and $f(\cdot)$ have to fulfill certain properties, namely

$$\begin{aligned} 0 < \nu(s) &\leq \nu_0 \quad \forall s \geq 0\,, \\ \lim_{s\to\infty} \nu(s) &= \nu_0\,, \\ f(0) &= 0\,, && (3) \\ f'(s) &> 0 \quad \forall s \geq 0\,, && (4) \end{aligned}$$

where ν_0 is the reluctivity in air.

Remark 2.1. *1. It is worth to mention that since f is strictly monotone, so the function $\nu(s) \cdot s$ is. This is an essential condition for the unique solvability of the nonlinear operator equation arising from Maxwell's equation (1) [5, 11].*

2. Furthermore the derivative

$$(f^{-1}(s))' = (\nu(s) \cdot s)' = \nu'(s) \cdot s + \nu(s) > 0$$

is strictly positive since the function $f(\cdot)$ is strictly monotone. This is important for the application of Newton's method to the nonlinear partial differential equation (1).

In practice, discrete measured pairs

$$(B_i, H_i) \quad i = 1, \ldots, N \tag{5}$$

are given (see Figure 1). The $B - H$ values are measured for several different impressed currents $\mathbf{J}_i$, $i = 1, \ldots, N$, such that

$$B_i = |\mathbf{B}(\mathbf{J}_i)|, \quad H_i = |\mathbf{H}(\mathbf{J}_i)| \quad \text{for} \quad i = 1, \ldots, N.$$

Thus, the reluctivity is given in discrete points, i.e.,

$$\nu_i = \frac{H_i}{B_i} \quad i = 1, \ldots, N.$$

3 Smoothing Splines

For the following section we denote by $\mathcal{C}^n([a,b])$ and $\mathcal{P}^n([a,b])$ the space of n times continuously differentiable functions and the set of polynomials of degree less or equal n on the interval $[a,b]$, respectively.

3.1 General

Let a set of discrete data points (5) be given and let the exact value corresponding to H_i (i.e., the one obtained without noise in the measurement) be denoted by $\bar{B}_i$, and the noise level by δ. Note that in the present application, the noise level is given by the (known) accuracy of the measurement equipment. In accordance with the given properties (3), (4) of f we assume:

A1. $\left.\begin{array}{l} 0 = H_1 \le H_i < H_{i+1} < \infty \text{ and} \\ 0 = B_1 = \bar{B}_1 \le \bar{B}_i < \bar{B}_{i+1} < \infty \end{array}\right\}$ for all $i = 2, \ldots, N-1$

A2. $|B_i - \bar{B}_i| \le \delta$ for all $i = 1, \ldots, N$

Now we apply the method of smoothing splines to a set (5) with the assumptions A1.,A2.. We have to minimize

$$\int_{H_1}^{H_N} (f''(s))^2 \, ds \to \min_{f \in \mathcal{V}} \tag{6}$$

under the constraint

$$\sum_{k=1}^{N} \left(\frac{f(H_k) - B_k}{w_k} \right)^2 \leq (c \cdot \delta)^2 \tag{7}$$

with $w_k \in \mathbb{R}$ some relative weights, $c \in \mathbb{R}^+$ and $\delta \in \mathbb{R}_0^+$ the given noise level. Using a Lagrange multiplier $\lambda \in \mathbb{R}$ (or equivalently, Tikhonov regularization with the regularization parameter $\theta = 1/\lambda$ chosen according to the discrepancy principle, see e.g. [3] pp. 121) we arrive at the objective function

$$J(f) = \lambda \left[\sum_{k=1}^{N} \left(\frac{f(H_k) - B_k}{w_k} \right)^2 + q - (c \cdot \delta)^2 \right] + \int_{H_1}^{H_N} (f''(s))^2 \, ds \tag{8}$$

that has to be minimized over the function space $\mathcal{V}$. The variable q is a slack variable, which is equal to zero, unless the data points can not be approximated by a straight line (see [9] for details). For further discussion we set $q = 0$. It is shown in [9] that the minimizer f^* fulfills

$$f^* \in \mathcal{V} \text{ and } \quad f^*|_{[H_k, H_{k+1}]} \in \mathcal{P}^3([H_k, H_{k+1}]), \quad k = 1, \ldots, N-1 \tag{9}$$

if $\mathcal{V} = \mathcal{C}^2([H_1, H_N])$ and for instance $f''(H_1) = f''(H_N) = 0$ is chosen. Especially this means that the minimizer can be represented by a polynomial on each subinterval, i.e.

$$f_k(s) = \sum_{j=1}^{4} f_{j,k} \cdot \zeta_{j,k}(s) \quad H_k \leq s \leq H_{k+1} \quad k = 1, \ldots, N-1 \tag{10}$$

where $\{\zeta_{j,k}\}_{j=1}^4$ is a basis of $\mathcal{P}^3([H_k, H_{k+1}])$. The coefficients $f_{j,k} \in \mathbb{R}$ are unknown and to be determined.

Remark 3.1. *In [4] it is shown that if assumptions A1., A2. are fulfilled with $\delta = 0$, then there exists a monotone (interpolating) cubic $\mathcal{C}^1$-spline. Thus a monotone function in $\mathcal{V} = \mathcal{C}^1([H_1, H_N])$ that is feasible in the sense of (7) is guaranteed. However this result is in general not valid for $\mathcal{V} = \mathcal{C}^2([H_1, H_N])$ and therefore f^* might not be monotone even if assumption A1. is fulfilled. Motivated by (9) on one hand and the monotonicity requirement on the other hand, we consequently concentrate on the approximation by $\mathcal{C}^1$-splines for further discussion.*

3.2 Realization

In order to construct a $\mathcal{C}^1$-spline we do not enforce continuity of the second derivative at the nodes H_k, $k = 2, \ldots, N-1$. Appropriate boundary conditions are imposed so that the solution of the minimization problem for (8) is unique, i.e., we set

$$f(H_1) = f_1, \quad f(H_N) = f_N, \quad f'(H_1) = f_1', \quad f'(H_N) = f_N', \tag{11}$$

with given f_1, f_N, f_1', f_N', e.g.,

$$f_1 = B_1, \quad f_N = B_N, \quad f_1' = \frac{B_2 - B_1}{H_2 - H_1}, \quad f_N' = \frac{B_N - B_{N-1}}{H_N - H_{N-1}}$$

where $B_2 > B_1$, $B_N > B_{N-1}$ is assumed.

The basis functions $\zeta_{i,k}$ are transformations to the interval $[H_k, H_{k+1}]$, $k = 1, \dots, N-1$ of the Hermite polynomials of degree 3 on the unit interval

$$\begin{aligned} \zeta_1(s) &= (1-s)^2(2s+1) \\ \zeta_2(s) &= s^2(3-2s) \\ \zeta_3(s) &= s(1-s)^2 \\ \zeta_4(s) &= -(1-s)s^2 \ , \end{aligned}$$

and read as

$$\begin{aligned} \zeta_{1,k}(s) &= \zeta_1((s-H_k)/(H_{k+1}-H_k)) \\ \zeta_{2,k}(s) &= \zeta_2((s-H_k)/(H_{k+1}-H_k)) \\ \zeta_{3,k}(s) &= \zeta_3((s-H_k)/(H_{k+1}-H_k)) \cdot (H_{k+1}-H_k) \\ \zeta_{4,k}(s) &= \zeta_4((s-H_k)/(H_{k+1}-H_k)) \cdot (H_{k+1}-H_k) \, . \end{aligned}$$

Hermite polynomials fulfill the conditions

$$\begin{aligned} \zeta_1(0) &= 1\,, \quad \zeta_i(0) = 0 \quad \text{for } i = 2,3,4 \\ \zeta_2(1) &= 1\,, \quad \zeta_i(1) = 0 \quad \text{for } i = 1,3,4 \\ \zeta_3'(0) &= 1\,, \quad \zeta_i'(0) = 0 \quad \text{for } i = 1,2,4 \\ \zeta_4'(1) &= 1\,, \quad \zeta_i'(1) = 0 \quad \text{for } i = 1,2,3\,. \end{aligned}$$

Consequently, by setting

$$f_{1,k} = f_{2,k-1} =: z_k \quad f_{3,k} = f_{4,k-1} =: \lambda_k \quad \forall k = 2, \dots, N-1$$

we enforce $f \in \mathcal{C}^1([H_1, H_N])$ and introduce two variables z_k, λ_k per subinterval $[H_k, H_{k+1}]$ instead of $f_{j,k}$, $j = 1, \dots, 4$.

By using the first order necessary conditions for a minimum, i.e.,

$$\frac{\partial J}{\partial z_k} = 0\,, \quad \frac{\partial J}{\partial \lambda_k} = 0 \quad \forall k = 2, \dots, N-1 \tag{12}$$

with the functional $J(f)$ defined in (10) we arrive at $2N-4$ necessary conditions for the coefficients. The resulting linear system

$$\tilde{T}\tilde{u} = \tilde{g}$$

has the following structure

$$\begin{aligned} \tilde{T} &= \begin{pmatrix} L(1) & D(2) & U(2) & 0 & \dots & 0 & 0 \\ 0 & L(2) & D(3) & U(3) & 0 & \dots & \dots \\ 0 & 0 & L(3) & D(4) & U(4) & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & L(N-2) & D(N-1) & U(N-1) \end{pmatrix} \in \mathbb{R}^{2N \times 2N-4} \,, \\ \tilde{g} &= (2 \cdot B_2, 0, 2 \cdot B_3, 0, \dots, 2 \cdot B_{N-1}, 0)^T \in \mathbb{R}^{2N-4} \,, \\ \tilde{u} &= (f_1, f_1', z_2, \lambda_2, \dots, z_{N-1}, \lambda_{N-1}, f_N, f_N')^T \in \mathbb{R}^{2N} \end{aligned}$$

with $D(\cdot)$, $L(\cdot)$, $U(\cdot) \in \mathbb{R}^{2\times 2}$. After the incorporation of the boundary conditions, i.e.,

$$
\begin{aligned}
g &= \tilde{g} + \tilde{T}u_l + \tilde{T}u_r \\
u_l &= (f_1, f_1', 0, 0, \ldots, 0)^T \\
u_r &= (0, 0, \ldots, 0, f_N, f_N')^T
\end{aligned}
$$

the reduced linear system $Tu = g$, $T \in \mathbb{R}^{2N-4\times 2N-4}$, with $g,\ u \in \mathbb{R}^{2N-4}$ is symmetric and positive definite (see [9]). The local matrices are given by:

$$
\begin{aligned}
D_{11}(k) &= 2\cdot\theta\cdot\int_{H_k}^{H_{k+1}} \zeta_{1,k}''(s)\zeta_{1,k}''(s) + \zeta_{2,k}''(s)\zeta_{2,k}''(s)\,ds + \frac{2}{w_k^2} \\
D_{12}(k) = D_{21}(k) &= 2\cdot\theta\cdot\int_{H_k}^{H_{k+1}} \zeta_{1,k}''(s)\zeta_{3,k}''(s) + \zeta_{2,k}''(s)\zeta_{4,k}''(s)\,ds \\
D_{22}(k) &= 2\cdot\theta\cdot\int_{H_k}^{H_{k+1}} \zeta_{3,k}''(s)\zeta_{3,k}''(s) + \zeta_{4,k}''(s)\zeta_{4,k}''(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
U_{11}(k) &= 2\cdot\theta\cdot\int_{H_k}^{H_{k+1}} \zeta_{2,k}''(s)\zeta_{1,k}''(s)\,ds \\
U_{12}(k) &= 2\cdot\theta\cdot\int_{H_k}^{H_{k+1}} \zeta_{4,k}''(s)\zeta_{1,k}''(s)\,ds \\
U_{21}(k) &= 2\cdot\theta\cdot\int_{H_k}^{H_{k+1}} \zeta_{2,k}''(s)\zeta_{3,k}''(s)\,ds \\
U_{22}(k) &= 2\cdot\theta\cdot\int_{H_k}^{H_{k+1}} \zeta_{4,k}''(s)\zeta_{3,k}''(s)\,ds
\end{aligned}
$$

and $L(k) = U(k)^T$.

For determining the regularization parameter θ a posteriori we use a discrepancy principle, see [3] pp. 121. Thus we are searching for the largest θ such that the residual

$$
\sum_{k=1}^{N} \left(\frac{f(H_k) - B_k}{w_k} \right)^2
$$

is of the order of magnitude of the measurement noise, i.e., inequality (7) holds with e.g. $c = 2$. In addition, the $B - H$ curve has to be strictly monotone. This property is easy to check, because f is monotone iff the spline is monotone on every subinterval. This leads to conditions on the coefficients $f_{j,k}$. Consequently the task is to find a monotone approximation which fulfills the discrepancy principle for a given data noise level.

Remark 3.2. *1. We could think of a $\mathcal{C}^2$-spline enforcing the monotonicity by enlarging the discrepancy parameter. But this would result in a loss of accuracy. Numerical studies have shown, that most of the examples yield either non-monotone $\mathcal{C}^2$-, or monotone $\mathcal{C}^1$-approximations. Achieving both goals, i.e. monotonicity and twice differentiability, seems to be hardly possible when using real life data. This corresponds also to the theoretical gap mentioned in Remark 3.1.*

2. *The noise level is given in terms of the points $(B_i, H_i)_{i=1}^N$. If we would deduce an estimate for the data noise in terms of the $(\nu_i, B_i)_{i=1}^N$, this would, due to the strong variation in scale of both function and derivative values of the curve under consideration, lead to a locally too pessimistic noise estimate and hence to a bad curve approximation.*

3. *The relative weights are chosen as $w_k = 1$, for $k = 1, \ldots, N$ in our applications.*

4. *In addition to Remark 2.1 it is easy to show that for the proposed approximation the estimates*

$$|\nu'(s)| \leq c_1 < \infty\,, \quad (\nu(s) \cdot s)' = \nu'(s) \cdot s + \nu(s) \leq c_2 < \infty$$

hold.

3.3 Extrapolation

Finally a physical extrapolation of $f(s)$ for $s \to \infty$ is proposed. Since we know that $\lim_{s\to\infty} \nu(s) = \nu_0$ we extrapolate the function $\nu(s)$ rather than $f(s)$. Thus we make the ansatz

$$\nu(s) = \nu_0 + \beta \cdot e^{-\alpha s} \quad \forall s \geq B_N\,.$$

By using the conditions

$$\frac{H_N}{B_N} = \nu(B_N) \quad = \quad \nu_0 + \beta \cdot e^{-\alpha B_N} \tag{13}$$

$$\frac{B_N/B'(H_N) - H_N}{B_N^2} = \nu'(B_N) \quad = \quad -\alpha \cdot \beta \cdot e^{-\alpha B_N} \tag{14}$$

and in order to enforce $\mathcal{C}^1$-continuity, we set the coefficients α and β to

$$\alpha \quad = \quad \frac{H_N - B_N/B'(H_N)}{H_N B_N - \nu_0 B_N^2} \tag{15}$$

$$\beta \quad = \quad (H_N/B_N - \nu_0) \cdot e^{\alpha B_N}\,. \tag{16}$$

For possible other extrapolation approaches, see e.g. [5].

3.4 Convergence

In the situation of exact data ($\delta = 0$), results from spline interpolation (cf., e.g. [2]) yield convergence as the size $h = \max_{1\leq k\leq N-1} |H_{k+1} - H_k|$ tends to zero. We are here interested in the situation of noisy data, and of fixed nodes H_k, $k = 1, \ldots, N$, though, and in the question whether the result f_θ^δ of the smoothing spline technique with the discrepancy principle tends to the searched for exact curve as the noise level δ goes to zero. For this purpose we evoke the theory of regularization in Hilbert scales (cf., e.g., Section 8.5 in [3]). For using the smoothing spline reconstruction in Newton's method for computing the magnetic field from Maxwell's equations, we need to have closeness not only of the function values but also of the derivatives. Therefore, we choose the $W^{1,\infty}$ norm for describing convergence. To be able to work in the Hilbert space setting of Section 8.5 in [3] we use as the solution space $\mathcal{X} = H^a([B_1, B_N])$ with some $a > \frac{3}{2}$, which is contained in $W^{1,\infty}([B_1, B_N])$, and as the data space $\mathcal{Y} = L^2([B_1, B_N])$, since we have a bound on the data noise in the given values (cf. assumption A4.). Here $f^\dagger$ denotes the "exact curve", i.e., the curve through the exact values $\bar{B}_i$ with minimal H^a-norm. From Theorem 8.25 in [3] we can conclude that if this $f^\dagger$ is in $H^{a+u}([B_1, B_N])$ for some $0 < u \leq a + 4$, then

$$\|f_\theta^\delta - f^\dagger\|_{H^a} = O(\delta^{\frac{u}{a+u}}) \ .$$

4 Numerical Studies

In the following we show three typical data sets from real life measurements. The described technique is implemented in a C++ code **bhcurve**. The code is able to deal with different extrapolation techniques. Moreover different boundary conditions can be implemented. The parameter of the discrepancy principle is chosen automatically. Furthermore the monotonicity of $f(\cdot)$ is checked.

The CPU time for one $B - H$ curve approximation is in the range of a second on a standard PC, inclusively all checks.

First we consider a data sample (**Sample 1**) where the resulting reluctivity $\nu(\cdot)$ became monotone. The noise level is $\delta = 2 \cdot 10^{-2}$ and $N = 17$ sample points are given. In Figure 2 the approximation of the $B - H$ curve is shown up to $H = 1.3 \cdot 10^5$. In Figure 3 the extrapolation of the reluctivity is shown. By construction the global function is $\mathcal{C}^1$ and $\lim_{s\to\infty} = \nu_0$. Additionally the function $\nu(s) \cdot s$ is plotted; here the strict monotonicity can be seen. According to the data set **Sample 1** the first derivative of the approximation is shown in Figure 5. Obviously the reluctivity is only $\mathcal{C}^1$, but it is acceptable for our applications (e.g. if Newton's method is applied). Furthermore the $B - H$-curve of **Sample 1** with a too small regularization parameter θ is shown in Figure 7. It can be seen that this approximation is not monotone. By choosing θ correctly, we get a monotone approximation, see Figure 6 for comparison.

The next example (**Sample 2**) is related to a data set which results in a non-monotone $\nu(\cdot)$ approximation, see Figure 9. The noise level is $\delta = 10^{-2}$. Note, such data sets are typical for many magnetic materials but could not be handled by [5] since the set $\{\nu_i\}_{i=1}^N$ is not monotone. In Figure 8 the given data points and the approximation of the $B - H$-curve can be seen .

The last data set (**Sample 3**) with $\delta = 10^{-2}$ is another example of a (physically correct) non-monotone ν, see Figure 11. As displayed in Figure 10, the $B - H$ data set is approximated by a strictly monotone $f(s)$.

5 Conclusions and Further Remarks

In this paper we proposed a method for approximation of given discrete data points. For the special case of the magnetic reluctivity we presented a method of monotone approximations.

The method is very efficient and applicable to almost every given data set. Also the first derivative of $\nu(\cdot)$ is acceptable.

Acknowledgment

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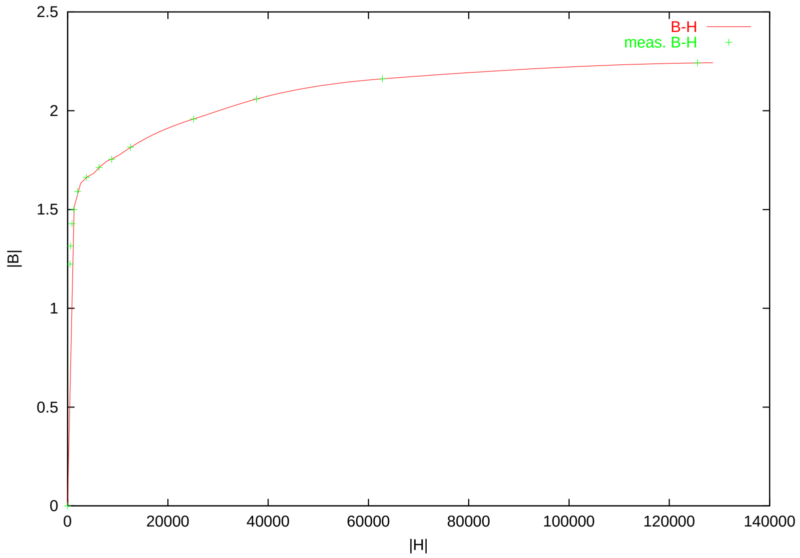

Figure 2: $B - H$-curve of **Sample 1**.

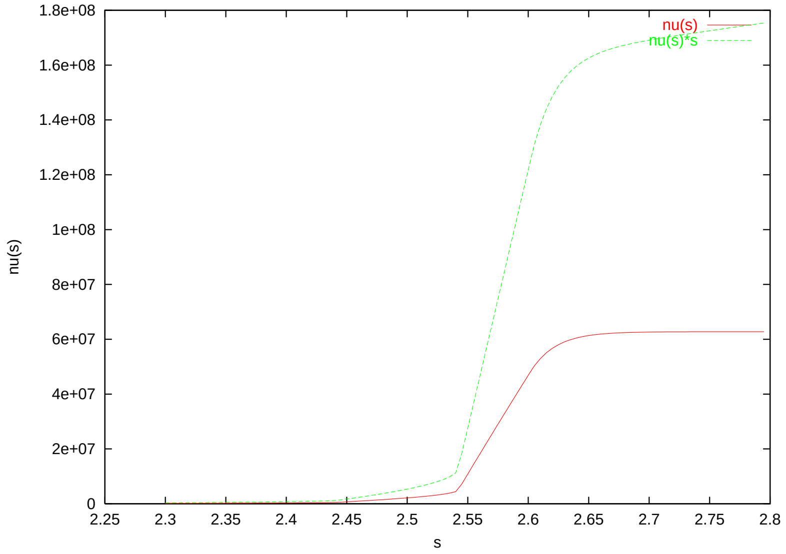

Figure 3: Reluctivity of **Sample 1**.

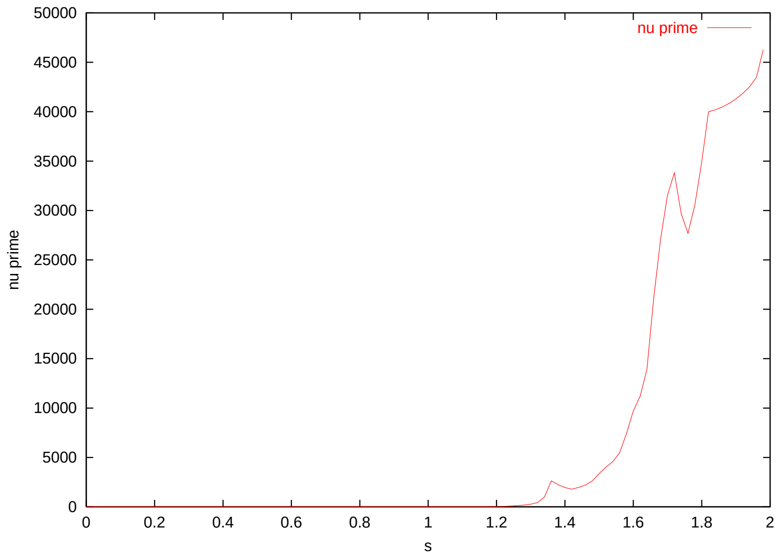

Figure 4: Reluctivity of **Sample 1**.

Figure 5: First derivative of the approximation of **Sample 1**.

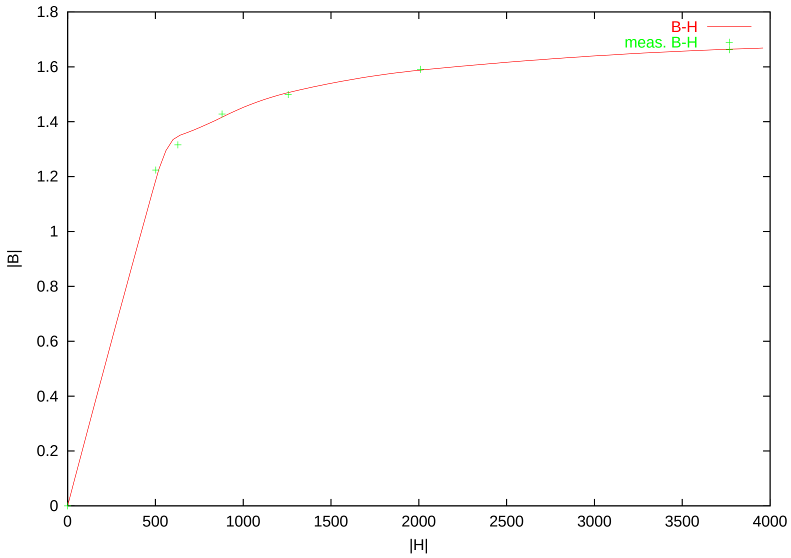

Figure 6: $B-H$-curve of **Sample 1** with correct θ.

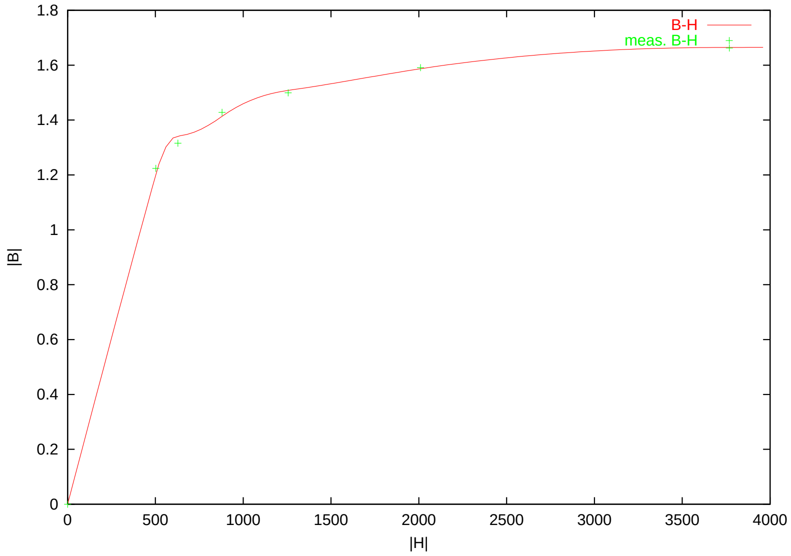

Figure 7: $B-H$-curve of **Sample 1** with too small θ.

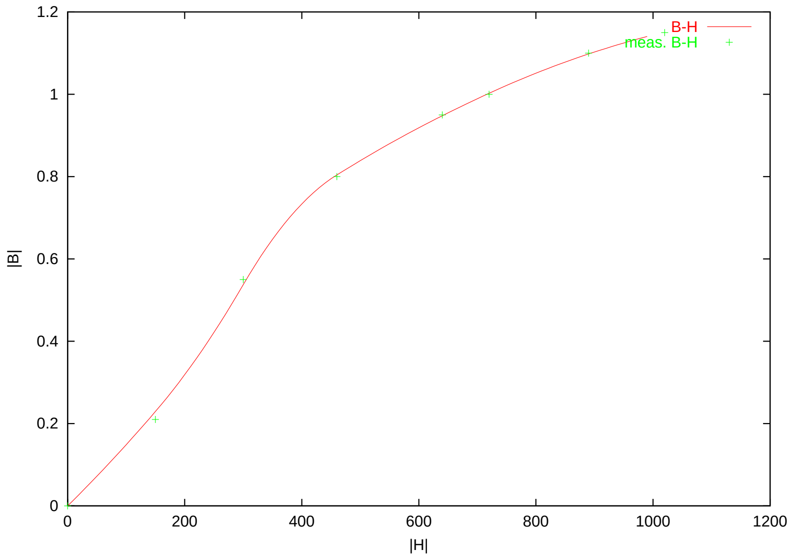

Figure 8: $B - H$-curve of **Sample 2**.

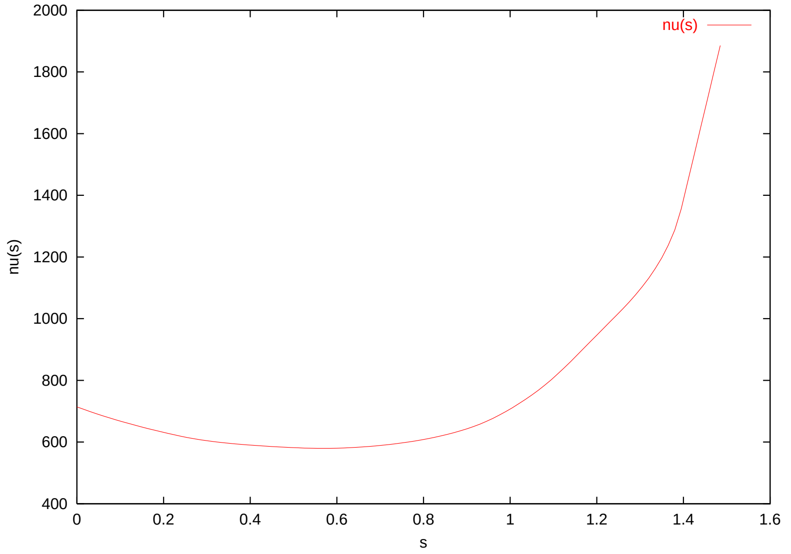

Figure 9: Reluctivity of **Sample 2**.

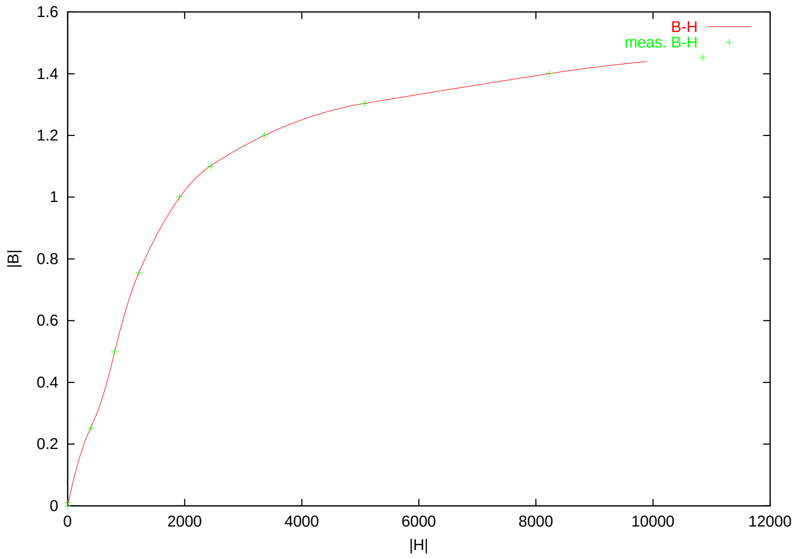

Figure 10: $B-H$-curve of **Sample 3**.

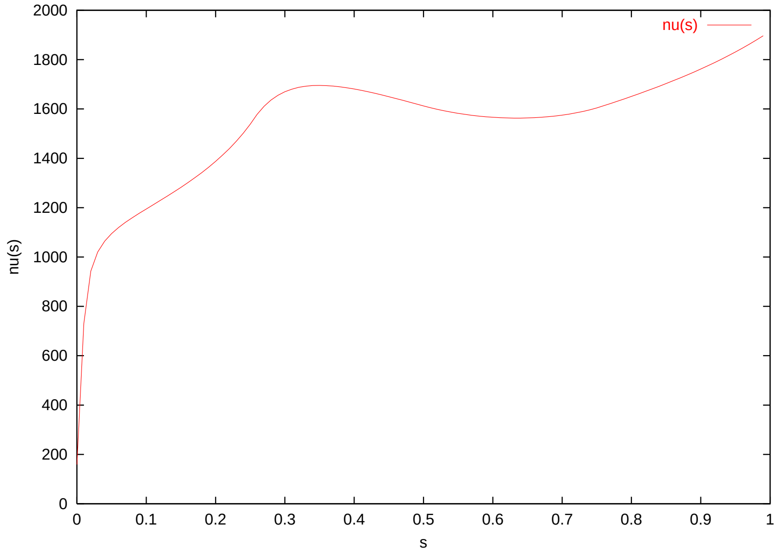

Figure 11: Reluctivity of **Sample 3**.