Boundary Element Tearing and Interconnecting Methods *

U. Langer[1], O. Steinbach[2]

[1]Institute for Computational Mathematics,
Johannes Kepler University of Linz,
Altenberger Strasse 69, A-4040 Linz, Austria

ulanger@numa.uni-linz.ac.at

[2]Institute for Applied Analysis and Numerical Simulation,
University of Stuttgart,
Pfaffenwaldring 57, D-70569 Stuttgart, Germany

steinbach@mathematik.uni-stuttgart.de

Abstract

In this paper we introduce the Boundary Element Tearing and Interconnecting (BETI) methods as boundary element counterparts of the well-established Finite Element Tearing and Interconnecting (FETI) methods. In some practical important applications such as far field computations, handling of singularities and moving parts etc., BETI methods have certainly some advantages over their finite element counterparts. This claim is especially true for the sparse versions of the BETI preconditioners resp. methods. Moreover, there is an unified framework for coupling, handling, and analyzing both methods. In particular, the FETI methods can benefit from preconditioning components constructed by boundary element techniques. The first numerical results confirm the efficiency and the robustness predicted by our analysis.

1 Introduction

The Finite Element Tearing and Interconnecting (FETI) methods were introduced by Farhat and Roux in 1991 [12], see also [13] for a more detailed description by the same authors. The classical FETI methods are non-overlapping Domain Decomposition (DD) methods and assume a conform triangulation of the

*This work has been supported by the Austrian Science Fund 'Fonds zur Förderung der wissenschaftlichen Forschung (FWF)' under grant SFB F013 'Numerical and Symbolic Scientific Computing' and the German Research Foundation 'Deutsche Forschungsgemeinschaft (DFG)' under the grant SFB 404 'Multifield Problems in Continuum Mechanics'. The paper was presented at the Oberwolfach Conference on 'New Trends in Boundary Elements' held in Oberwolfach, December 1 - 7, 2002.

total computational domain. In contrast to the iterative substructuring method, the Finite Element (FE) subspaces are given on each subdomain (substructure) including its boundary separately. The global continuity is then enforced by Lagrange multipliers, resulting in a saddle point problem that can be solved iteratively via its dual problem. Once the Lagrange multipliers are known, the primal variables can be easily calculated. The iteration process is nothing but a Preconditioned Conjugate Gradient (PCG) subspace iteration. The projection to the subspace ensures not only the solvability of the local FE Neumann problems involved in each iteration step and in each subdomain, but also the global information exchange. The FETI preconditioners typically involve the solution of a FE Dirichlet problem in each subdomain (multiplication by the usual DD Schur complement). Thus, in their standard exact versions, every iteration step of the FETI methods is quite expensive (see [24] for a first inexact version). Nevertheless, the FETI method is one of the most widely used DD methods in parallel codes including commercial codes. This fact is certainly related to the wide applicability of the FETI methods, the possibility of the use of standard Neumann and Dirichlet solvers in the solution process, the moderate dependence of the iteration number on the complexity of the problem [5, 25, 29], the scalability [35] and, last but not least, the robustness [5, 25]. More precisely, for elliptic problems, Mandel and Tezaur proved that the relative condition number of the dual system preconditioned by the so-called Dirichlet FETI preconditioner grows at most as $O((1+\log(H/h))^3)$ or $O((1+\log(H/h))^2)$ for the general case and for the case without cross points in the decomposition of the domain, respectively [29]. Here, H and h denote the average diameters of the subdomains and the finite elements, respectively. Thus, the number of unknowns belonging to one subdomain behaves like $O((H/h)^{-d})$, where d denotes the dimension of the computational domain. Klawonn and Widlund introduced new FETI preconditioners and showed the $O((1 + \log(H/h))^2)$ behavior of the relative condition number in the general case that permits cross points in the domain decomposition [25]. Moreover, they proved robustness with respect to jumps in the coefficients of the elliptic Partial Differential Equations (PDE). Similar results have recently been obtained by Brenner within the Schwarz framework [4, 5]. We refer the reader to [11] for some further development of the FETI methodology (FETI-2 and FETI-DP) and further references.

In this paper we generalize the tearing and interconnecting technique to symmetric Galerkin boundary element (BE) equations. During the last decade iterative substructuring solvers for symmetric BE element equations have been developed by Hsiao and Wendland [23], Langer [26], Steinbach [36, 38], Haase, Heise, Kuhn and Langer [15], Carstensen, Kuhn and Langer [7], Hsiao, Schnack and Wendland [20] Hsiao, Steinbach and Wendland [21] for elliptic boundary value problems in bounded and unbounded, two and three-dimensional domains, and have been successfully applied to real-life problems. Parallel implementations showed high performance on several platforms [15]. The key ingredients of an inexact DD BE preconditioner are preconditioners for the local discrete single layer potential operators and for the global discrete Steklov–Poincaré operator

(BE Schur complement). The problem of constructing effective discrete bounded extension operators as a key component in the inexact FE DD preconditioner (see, e.g., [16]) does not appear in inexact DD BE preconditioners. In contrast to iterative BE substructuring methods, the (exact) Boundary Element Tearing and Interconnecting (BETI) Methods needs the inversion of the local (subdomain) discrete Steklov–Poincaré operators and some preconditioner for the dual problem. In this sense the BETI methods are dual to the iterative BE substructuring methods. The inversion of the local discrete Steklov–Poincaré operators leads to the parallel solution of symmetric, but indefinite local problems. This corresponds to the solution of local FE Neumann problems in the FETI method. The BETI preconditioners for the dual problem are based on the parallel multiplications by the local discrete hypersingular operators which are available in the symmetric BETI scheme anyway. No solution of local Dirichlet problems is required. Moreover, using some sparse multiplication technique for the discrete hypersingular operators, we only need, up to some logarithmic factor, $O((H/h)^{d-1})$ arithmetical operations for one application of the sparse hypersingular BETI preconditioner. Thus, the cost for the preconditioning operation is almost proportional to the number of unknowns on the subdomain interfaces. As in the FETI method the BETI preconditioner can be scaled in such a way that the relative spectral condition number grows only like $O((1+\log(H/h))^2)$ and is independent of the jumps in the coefficients of the PDE. These properties are clearly reflected in our numerical experiments with the scaled hypersingular BETI preconditioner.

The rest of the paper is organized as follows. In Section 2, we introduce the BETI method for a symmetric DD BE discretization, propose some BETI preconditioners, and discuss the algorithmical aspects. Section 3 is devoted to the convergence analysis of the BETI method. Section 4 contains first numerical results for the BETI method. Finally, in Section 5, we draw some conclusions for using and developing the tearing and interconnecting technique in both the boundary and finite element worlds.

2 Formulation of BETI

As a model problem we consider the Dirichlet boundary value problem

$$-\mathrm{div}[\alpha(x)\nabla u(x)] = 0 \quad \text{for } x \in \Omega, \quad u(x) = g(x) \quad \text{for } x \in \Gamma = \partial\Omega \tag{2.1}$$

where the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is provided with some non–overlapping quasi regular domain decomposition

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i, \ \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j, \ \Gamma_i = \partial\Omega_i, \ \Gamma_{ij} = \Gamma_i \cap \Gamma_j, \ \Gamma_S = \bigcup_{i=1}^{p} \Gamma_i,$$

with the average subdomain size H, i.e. there exists some generic positive constant c such that (see, e.g., [5])

$$H_i := \operatorname{diam} \Omega_i, \quad H := \max_{i=1,\ldots,p} H_i, \quad c \cdot H \leq H_i \quad \text{for all } i = 1, \ldots, p.$$

We assume that, for $i = 1, \ldots, q < p$, the subdomains Ω_i does not touch the Dirichlet boundary $\Gamma = \partial\Omega$. Such subdomains are called floating subdomains. Moreover, we assume that the coefficient $\alpha(x)$ is piecewise constant,

$$\alpha(x) = \alpha_i \quad \text{for } x \in \Omega_i. \tag{2.2}$$

Instead of (2.1) we consider local boundary value problems

$$-\alpha_i \Delta u_i(x) = 0 \quad \text{for } x \in \Omega_i, \quad u_i(x) = g(x) \quad \text{for } x \in \Gamma_i \cap \Gamma \tag{2.3}$$

together with the transmission conditions

$$u_i(x) = u_j(x), \quad \alpha_i \frac{\partial}{\partial n_i} u_i(x) + \alpha_j \frac{\partial}{\partial n_j} u_j(x) = 0 \quad \text{for } x \in \Gamma_{ij}. \tag{2.4}$$

The fundamental solution of the Laplace operator is given by

$$U^*(x,y) = \begin{cases} -\dfrac{1}{2\pi} \log |x-y| & \text{for } d = 2, \\ \dfrac{1}{4\pi} \dfrac{1}{|x-y|} & \text{for } d = 3. \end{cases}$$

The solution of the local subproblems (2.3) can be represented in the form

$$u_i(x) = \int_{\Gamma_i} U^*(x,y) \frac{\partial}{\partial n_i} u_i(y) ds_y - \int_{\Gamma_i} \frac{\partial}{\partial n_i} U^*(x,y) u_i(y) ds_y \quad \text{for } x \in \Omega_i \tag{2.5}$$

called representation formula. Hence we have to find the complete Cauchy data $[u_i, t_i := \frac{\partial}{\partial n_i} u_i]$ on Γ_i. From (2.5) we derive a system of boundary integral equations,

$$\begin{pmatrix} u_i \\ t_i \end{pmatrix} = \begin{pmatrix} \frac{1}{2} I - K_i & V_i \\ D_i & \frac{1}{2} I + K_i' \end{pmatrix} \begin{pmatrix} u_i \\ t_i \end{pmatrix} \tag{2.6}$$

using the standard notations for the local single layer potential operator V_i, double layer potential operator K_i, adjoint double layer potential operator K_i', and the hypersingular integral operator D_i defined by

$$\begin{aligned}
(V_i t_i)(x) &:= \int_{\Gamma_i} U^*(x,y) t_i(y) ds_y, \ x \in \Gamma_i, \\
(K_i u_i)(x) &:= \int_{\Gamma_i} \frac{\partial}{\partial n_y} U^*(x,y) u_i(y) ds_y, \ x \in \Gamma_i, \\
(K_i' t_i)(x) &:= \int_{\Gamma_i} \frac{\partial}{\partial n_x} U^*(x,y) t_i(y) ds_y, \ x \in \Gamma_i, \text{ and} \\
(D_i u_i)(x) &:= -\frac{\partial}{\partial n_x} \int_{\Gamma_i} \frac{\partial}{\partial n_y} U^*(x,y) u_i(y) ds_y, \ x \in \Gamma_i,
\end{aligned}$$

respectively. The mapping properties of all of these boundary integral operators are well known [9], in particular, the local single layer potential $V_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$ is $H^{-1/2}(\Gamma_i)$–elliptic and therefore invertible [22]; for $d = 2$ we assume $\operatorname{diam} \Omega_i < 1$ that can be always obtained by scaling the computational domain. Using (2.6) the local Dirichlet–Neumann map can be written as

$$t_i(x) = \left[D_i + (\frac{1}{2}I + K_i')V_i^{-1}(\frac{1}{2}I + K_i) \right] u_i(x) \;=\; (S_i u_i)(x) \quad \text{for } x \in \Gamma_i \quad (2.7)$$

where $S_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ denotes the local Steklov–Poincaré operator. The transmission problem (2.3) and (2.4) then reads as follows: Find $u_i \in H^{1/2}(\Gamma_i)$ with $u_i(x) = g(x)$ for $x \in \Gamma_i \cap \Gamma$ and $u_i = u_j$ on Γ_{ij} such that

$$\alpha_i(S_i u_i)(x) + \alpha_j(S_j u_j)(x) = 0 \quad \text{for } x \in \Gamma_{ij}. \quad (2.8)$$

Let $H^{1/2}(\Gamma_S)$ be the trace space of $H^1(\Omega)$ restricted to the skeleton Γ_S. If we define $u_i = u_{|\Gamma_i}$ for $u \in H^{1/2}(\Gamma_S)$ the continuity condition $u_i = u_j$ on Γ_{ij} is satisfied automatically. Hence we have to find $\hat{u} \in H^{1/2}(\Gamma_S)$ with $\hat{u}(x) = 0$ for $x \in \Gamma$ such that

$$\sum_{i=1}^{p} \alpha_i \int_{\Gamma_i} [(S_i \hat{u}_{|\Gamma_i})(x) + (S_i \widetilde{g}_{|\Gamma_i})(x)] v_{|\Gamma_i}(x) ds_x \;=\; 0 \quad (2.9)$$

for all $v \in H^{1/2}(\Gamma_S)$ with $v(x) = 0$ for $x \in \Gamma$ where $\widetilde{g} \in H^{1/2}(\Gamma_S)$ is some arbitrary but fixed extension of the given Dirichlet data $g \in H^{1/2}(\Gamma)$.
Since the local Steklov–Poincaré operators S_i are defined via (2.7) in an implicit form only, we have to introduce computable approximations $\widetilde{S}_i$ first.
For $v_i \in H^{1/2}(\Gamma_i)$ the application of $S_i v_i$ is given by

$$(S_i v_i)(x) = (D_i v_i)(x) + (\frac{1}{2}I + K_i')w_i(x) \quad \text{for } x \in \Gamma_i,$$

where $w_i \in H^{-1/2}(\Gamma_i)$ is the unique solution of

$$\langle V_i w_i, \tau_i \rangle_{L_2(\Gamma_i)} \;=\; \langle (\frac{1}{2}I + K_i)v_i, \tau_i \rangle_{L_2(\Gamma_i)} \quad \text{for all } \tau_i \in H^{-1/2}(\Gamma_i). \quad (2.10)$$

Let $Z_{i,h} = \operatorname{span}\{\psi_k^i\}_{k=1}^{N_i} \subset H^{-1/2}(\Gamma_i)$ be some local boundary element space, e.g. of piecewise constant basis functions ψ_k^i, with respect to local quasi regular boundary meshes with average mesh size h_i. Instead of (2.10) we consider the Galerkin problem: Find $w_{i,h} \in Z_{i,h}$ such that

$$\langle V_i w_{i,h}, \tau_{i,h} \rangle_{L_2(\Gamma_i)} \;=\; \langle (\frac{1}{2}I + K_i)v_i, \tau_{i,h} \rangle_{L_2(\Gamma_i)} \quad \text{for all } \tau_{i,h} \in Z_{i,h}. \quad (2.11)$$

This variational problem possesses a unique solution satisfying the a priori error estimate [22]

$$||w_i - w_{i,h}||_{H^{-1/2}(\Gamma_i)} \;\leq\; c_i \cdot \inf_{\tau_{i,h} \in Z_{i,h}} ||w_i - \tau_{i,h}||_{H^{-1/2}(\Gamma_i)}. \quad (2.12)$$

Now we can define an approximate Steklov–Poincaré operator as

$$(\widetilde{S}_i v_i)(x) = (D_i v_i)(x) + (\frac{1}{2}I + K_i')w_{i,h}(x) \quad \text{for } x \in \Gamma_i. \tag{2.13}$$

Theorem 2.1 [38] *The approximate Steklov–Poincaré operator $\widetilde{S}_i$ as defined in (2.13) is bounded,*

$$||\widetilde{S}_i v_i||_{H^{-1/2}(\Gamma_i)} \le c_i \cdot ||v_i||_{H^{1/2}(\Gamma_i)} \quad \textit{for all}\, v_i \in H^{1/2}(\Gamma_i)$$

and satisfies the a priori error estimate

$$||(S_i - \widetilde{S}_i)v_i||_{H^{-1/2}(\Gamma_i)} \le c_i \cdot \inf_{\tau_{i,h} \in Z_{i,h}} ||S_i v_i - \tau_{i,h}||_{H^{-1/2}(\Gamma_i)}.$$

Moreover, there holds the estimate

$$\langle \widetilde{S}_i v_i, v_i \rangle_{L_2(\Gamma_i)} \ge \langle D_i v_i, v_i \rangle_{L_2(\Gamma_i)} \quad \textit{for all}\, v_i \in H^{1/2}(\Gamma_i).$$

Hence, the approximate Steklov–Poincaré operator $\widetilde{S}_i$ is elliptic whenever the hypersingular boundary integral operator D_i and therefore the exact Steklov–Poincaré operator S_i is elliptic.

Now, instead of (2.9) we consider a perturbed variational problem to find $\widetilde{u} \in H^{1/2}(\Gamma_S)$ with $\widetilde{u}(x) = 0$ for $x \in \Gamma$ such that

$$\sum_{i=1}^{p} \alpha_i \int_{\Gamma_i} [(\widetilde{S}_i \widetilde{u}_{|\Gamma_i})(x) + (\widetilde{S}_i \widetilde{g}_{|\Gamma_i})(x)] v_{|\Gamma_i}(x) ds_x \; = \; 0 \tag{2.14}$$

for all $v \in H^{1/2}(\Gamma_S)$ with $v(x) = 0$ for $x \in \Gamma$. Let

$$W_h = \text{span}\{\varphi_m\}_{m=1}^{M_0} \subset \Big\{ v \in H^{1/2}(\Gamma_S) : v(x) = 0 \text{ for } x \in \Gamma \Big\}$$

be a boundary element space on the skeleton Γ_S of, e.g., piecewise linear basis functions φ_m, with respect to a quasi regular boundary mesh with mesh size h_S. We define also local restrictions of W_h onto Γ_i, in particular, $W_{i,h} = \text{span}\{\varphi_m^i\}_{m=1}^{M_i}$. Obviously, for any $\varphi_m^i \in W_{i,h}$ there exists a unique basis function $\varphi_n \in W_h$ with $\varphi_m^i = \varphi_{n|\Gamma_i}$. By using the isomorphisms

$$\underline{v}_i \in \mathbb{R}^{M_i} \leftrightarrow v_{i,h} = \sum_{m=1}^{M_i} v_{i,m} \varphi_m^i \in W_{i,h}, \quad \underline{v} \in \mathbb{R}^{M_0} \leftrightarrow v_h = \sum_{n=1}^{M_0} v_n \varphi_n \in W_h$$

there exist connectivity matrices $A_i \in \mathbb{R}^{M_i \times M_0}$ such that $\underline{v}_i = A_i \underline{v}$.
The Galerkin variational formulation of (2.14) reads: Find $\widetilde{u}_h \in W_h$ such that

$$\sum_{i=1}^{p} \alpha_i \int_{\Gamma_i} [(\widetilde{S}_i \widetilde{u}_{h|\Gamma_i})(x) + (\widetilde{S}_i \widetilde{g}_{|\Gamma_i})(x)] v_{h|\Gamma_i}(x) ds_x \; = \; 0 \tag{2.15}$$

for all $v_h \in W_h$. Applying standard arguments we can state the following result, see [38]:

Theorem 2.2 *There exists a unique solution $\widetilde{u}_h \in W_h$ of the Galerkin variational problem (2.15) satisfying the a priori error estimate*

$$||\hat{u}-\widetilde{u}_h||_{H^{1/2}(\Gamma_S)} \le c_1 \inf_{v_h\in W_h} ||\hat{u}-v_h||_{H^{1/2}(\Gamma_S)} + c_2 \sum_{i=1}^{p} \inf_{\tau_{i,h}\in Z_{i,h}} ||S_i\hat{u}-\tau_{i,h}||_{H^{-1/2}(\Gamma_i)}.$$

Corollary 2.1 *When assuming the optimal regularity $\hat{u} \in H^2(\Gamma_S)$ as well as $S_i\hat{u} \in H^1_{pw}(\Gamma_i)$ for $i = 1, \dots, p$, we obtain the a priori error estimate*

$$||\hat{u} - \widetilde{u}_h||_{H^{1/2}(\Gamma_S)} \le \widetilde{c}_1 \cdot h_S^{3/2} \cdot ||\hat{u}||_{H^2(\Gamma_S)} + \widetilde{c}_2 \cdot \sum_{i=1}^{p} h_i^{3/2} \cdot ||S_i\hat{u}||_{H^1_{pw}(\Gamma_i)}.$$

The Galerkin variational problem (2.15) is equivalent to a system of linear equations,

$$\sum_{i=1}^{p} \alpha_i A_i^\top \widetilde{S}_{i,h} A_i \underline{\widetilde{u}} = \sum_{i=1}^{p} A_i^\top \underline{f}_i \tag{2.16}$$

with the discrete Steklov–Poincaré operator

$$\widetilde{S}_{i,h} = D_{i,h} + (\frac{1}{2}M_{i,h}^\top + K_{i,h}^\top)V_{i,h}^{-1}(\frac{1}{2}M_{i,h} + K_{i,h}) \tag{2.17}$$

and the boundary element matrices

$$\begin{aligned} V_{i,h}[\ell,k] &= \langle V_i\psi_k^i, \psi_\ell^i\rangle_{L_2(\Gamma_i)}, \\ D_{i,h}[m,n] &= \langle D_i\varphi_n^i, \varphi_m^i\rangle_{L_2(\Gamma_i)}, \\ K_{i,h}[\ell,n] &= \langle K\varphi_n^i, \psi_\ell^i\rangle_{L_2(\Gamma_i)}, \\ M_{i,h}[\ell,n] &= \langle \varphi_n^i, \psi_\ell^i\rangle_{L_2(\Gamma_i)} \end{aligned}$$

for $k, \ell = 1, \dots, N_i$; $m, n = 1, \dots, M_i$ and $i = 1, \dots, p$. When using the standard boundary element approach, the above stiffness matrices $V_{i,h}$, $K_{i,h}$ and $D_{i,h}$ are dense. Therefore, the memory demand and the effort for one matrix-by-vector multiplication is proportional to N_i^2. However, using sparse boundary element techniques such as the Fast Multipole Method (FMM) [14], we can reduce the complexity to $\mathcal{O}(N_i)$ up to some polylogarithmical factor that is $(\log N_i)^2$ in case of the FMM.

To derive the boundary element tearing and interconnecting algorithm we start with the solution of the linear system (2.16). This is equivalent to the solution of a minimization problem,

$$F(\underline{\widetilde{u}}) = \min_{\underline{v}\in\mathbb{R}^M} F(\underline{v}) \tag{2.18}$$

where the linear functional is given as

$$F(\underline{v}) := \sum_{i=1}^{p} \left[\frac{\alpha_i}{2}(\widetilde{S}_{i,h}A_i\underline{v}, A_i\underline{v}) - (\underline{f}_i, A_i\underline{v}) \right]. \tag{2.19}$$

By introducing local vectors $\underline{v}_i = A_i \underline{v}$ we obtain

$$F(\underline{v}) = \widetilde{F}(\underline{v}_1, \ldots, \underline{v}_p) := \sum_{i=1}^{p} \left[\frac{\alpha_i}{2} (\widetilde{S}_{i,h} \underline{v}_i, \underline{v}_i) - (\underline{f}_i, \underline{v}_i) \right]. \tag{2.20}$$

To describe the connection across the interfaces we introduce the constraint

$$\sum_{i=1}^{p} B_i \underline{v}_i = \underline{0} \tag{2.21}$$

where $B_i \in \mathbb{R}^{M \times M_i}$. Each row of the matrix $B = (B_1, \ldots, B_p)$ is connected with a pair of matching nodes across the interface. The entries of such a row are 1 and -1 for the indices corresponding to the matching nodes and 0 otherwise. Therefore, (2.21) implies that the corresponding boundary element functions $v_{i,h}$ are continuous across the interface (coupling boundaries) $\Gamma_C = \Gamma_S \setminus \Gamma$, i.e. $v_{i,h} = v_{j,h}$ on $\Gamma_i \cap \Gamma_j \neq \emptyset$. We assume here that the number of constraints at some matching node is equal to the number of matching subdomain minus one. This method of a minimal number of constraints respectively multipliers is called non-redundant, see, e.g., [25] for the use of redundant constraints.
Now the solution of (2.18) is equivalent to

$$F(\underline{\widetilde{u}}) = \widetilde{F}(\underline{\widetilde{u}}_1, \ldots, \underline{\widetilde{u}}_p) = \min_{\underline{v}_1, \ldots, \underline{v}_p : \sum_{i=1}^{p} B_i \underline{v}_i = \underline{0}} \widetilde{F}(\underline{v}_1, \ldots, \underline{v}_p). \tag{2.22}$$

By introducing the Lagrange multiplier $\underline{\lambda} \in \mathbb{R}^M$, the solution of the minimization problem (2.22) is given by solving the linear system

$$\begin{pmatrix} \alpha_1 \widetilde{S}_{1,h} & & & B_1^\top \\ & \ddots & & \vdots \\ & & \alpha_p \widetilde{S}_{p,h} & B_p^\top \\ B_1 & \ldots & B_p & 0 \end{pmatrix} \begin{pmatrix} \underline{\widetilde{u}}_1 \\ \vdots \\ \underline{\widetilde{u}}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_p \\ \underline{0} \end{pmatrix}. \tag{2.23}$$

For $i = 1, \ldots, q$ the discrete local (subdomain) Steklov–Poincaré operators $\widetilde{S}_{i,h}$ are singular due to the lack of a Dirichlet boundary condition. In this case, the local equations

$$\alpha_i \widetilde{S}_{i,h} \underline{\widetilde{u}}_i = \underline{f}_i - B_i^\top \underline{\lambda} \tag{2.24}$$

are solvable only when assuming the compatibility condition

$$(\underline{f}_i - B_i^\top \underline{\lambda}, \underline{e}_i) = 0, \tag{2.25}$$

where $\underline{e}_i = (1, \ldots, 1)^\top \in \mathbb{R}^{M_i}$. Instead of (2.24) we solve a modified system,

$$\alpha_i \left[\widetilde{S}_{i,h} + \beta_i \underline{e}_i \underline{e}_i^\top \right] \underline{\widetilde{u}}_i = \underline{f}_i - B_i^\top \underline{\lambda}, \tag{2.26}$$

where $\beta_i \in \mathbb{R}_+$ is some positive constant. Note that (2.26) is unique solvable for any right hand side, but yielding the normalization condition

$$\underline{e}_i^\top \underline{\widetilde{u}}_i = 0$$

when assuming the compatibility condition (2.25). Hence we can write the general solution of (2.24) as

$$\underline{\widetilde{u}}_i = \frac{1}{\alpha_i} \widetilde{S}^+_{i,h} (\underline{f}_i - B_i^\top \underline{\lambda}) + \gamma_i \underline{e}_i \quad \text{for } i = 1, \dots, q \tag{2.27}$$

where the application of

$$\widetilde{S}^+_{i,h} \; := \; \left[\widetilde{S}_{i,h} + \beta_i \underline{e}_i \underline{e}_i^\top \right]^{-1}$$

means the solution of (2.26) and where $\gamma_i \in \mathbb{R}$ has to be determined. In the remaining subdomains Ω_i, $i = q+1, \dots, p$, the discrete Steklov–Poincaré operator $\widetilde{S}_{i,h}$ is invertible,

$$\underline{\widetilde{u}}_i = \frac{1}{\alpha_i} \widetilde{S}^{-1}_{i,h} (\underline{f}_i - B_i^\top \underline{\lambda}) \quad \text{for } i = q+1, \dots, p. \tag{2.28}$$

By defining $G = \left(B_1 \underline{e}_1 \dots B_q \underline{e}_q \right) \in \mathbb{R}^{M \times q}$ we obtain from (2.23)

$$\left[\sum_{i=1}^{q} \frac{1}{\alpha_i} B_i \widetilde{S}^+_{i,h} B_i^\top + \sum_{i=q+1}^{p} \frac{1}{\alpha_i} B_i \widetilde{S}^{-1}_{i,h} B_i^\top \right] \underline{\lambda} - G \underline{\gamma} = \tag{2.29}$$

$$= \sum_{i=1}^{q} \frac{1}{\alpha_i} B_i \widetilde{S}^+_{i,h} \underline{f}_i + \sum_{i=q+1}^{p} \frac{1}{\alpha_i} B_i \widetilde{S}^{-1}_{i,h} \underline{f}_i .$$

This can be written as

$$F \underline{\lambda} - G \underline{\gamma} \; = \; \underline{d} \tag{2.30}$$

subject to the constraint, see (2.25),

$$G^\top \underline{\lambda} = (\underline{e}_i^\top \underline{f}_i)_{i=1:q} \; =: \; \underline{e}. \tag{2.31}$$

Defining now the orthogonal projection $P = I - G(G^T G)^{-1} G^T$ from the space $\mathbf{\Lambda} := \mathbb{R}^M$ onto the subspace $\mathbf{\Lambda}_0 = \ker G^T = (\operatorname{range} G)^\perp$ with respect to the scalar product $(\cdot,\cdot) = (\cdot,\cdot)_{\mathbf{\Lambda}} = (\cdot,\cdot)_{\mathbb{R}^M}$, we can split the definition of $\underline{\lambda}$ from the definition of $\underline{\gamma}$. Indeed, applying P to (2.30) gives the equation

$$P F \underline{\lambda} \; = \; P \underline{d} \tag{2.32}$$

since $PG\underline{\gamma} = \underline{0}$. Together with the solvability condition (2.31), equation (2.32) is the final dual problem to find $\underline{\lambda} \in \mathbf{\Lambda}$. Once $\underline{\lambda}$ is defined, we obtain

$$\underline{\gamma} = (G^T G)^{-1} G^T (F \underline{\lambda} - \underline{d}) \tag{2.33}$$

from (2.30) and finally $\underline{\widetilde{u}}_i$ from (2.27) and (2.28), respectively. We mention that in the case of jumping coefficients the scalar product in $\mathbf{\Lambda}$ has to be changed according to the proposal made in [25], see also [4, 5]. Of course, the change of the scalar product changes the orthoprojection P too.

Algorithm 1 FETI subspace PCG iteration.

{initialization}

$\underline{\lambda}^0 = G(G^T G)^{-1}\underline{e}$ {forcing the constrains $G^T\underline{\lambda}^0 = \underline{e}$ for the initial guess}

$\underline{d}^0 = P(\underline{d} - F\underline{\lambda}^0)$ {compute the defect and project to the subspace $\mathbf{\Lambda}_0$}

$\underline{w}^0 = \underline{C}^{-1}\underline{d}^0$ {precondition step}

$\underline{s}^0 = \underline{z}^0 = P\underline{w}^0$ {project the correction to the subspace $\mathbf{\Lambda}_0$}

$\beta_0 = (\underline{w}^0, \underline{d}^0) = (\underline{z}^0, \underline{d}^0)$

{begin iteration loop}

for $n = 0$ **step** 1 **until** $\beta_n \le \varepsilon\beta_0$ **do**

$\underline{x}^n = PF\underline{s}^n$ {matrix-by-vector multiplication + projection}

$\alpha_n = (\underline{x}^n, \underline{s}^n)$

$\alpha = \beta_n / \alpha_n$

$\underline{\lambda}^{n+1} = \underline{\lambda}^n + \alpha\,\underline{s}^n$ {update of the iterate}

$\underline{d}^{n+1} = \underline{d}^n - \alpha\,\underline{x}^n$ {update of the defect}

$\underline{w}^{n+1} = C^{-1}\underline{d}^{n+1}$ {precondition step}

$\underline{z}^{n+1} = P\underline{w}^{n+1}$ {project the correction to the subspace $\mathbf{\Lambda}_0$}

$\beta_{n+1} = (\underline{w}^{n+1}, \underline{d}^{n+1}) = (\underline{z}^{n+1}, \underline{d}^{n+1})$

$\beta = \beta_{n+1}/\beta_n$

$\underline{s}^{n+1} = \underline{z}^n - \beta\,\underline{s}^n$ {update of the search direction in the subspace $\mathbf{\Lambda}_0$}

end for

{end iteration loop}

The dual problem (2.31)–(2.32) is now solved by a preconditioned conjugate gradient (PCG) iteration in the subspace $\mathbf{\Lambda}_0$ that is presented in Algorithm 1 as a projected PCG method.
The matrix by vector multiplication $F\underline{s}^n$ involves the application either of $\widetilde{S}_{i,h}^{+}$ $(i = 1, \ldots, q)$ or of $\widetilde{S}_{i,h}^{-1}$ $(i = q+1, \ldots, p)$. In the first case we have to solve the linear system

$$\alpha_i \left[\widetilde{S}_{i,h} + \beta_i \underline{e}_i \underline{e}_i^\top\right] \underline{v}_i = \underline{r}_i \tag{2.34}$$

where $\underline{e}_i^\top \underline{r}_i = 0$ ensures the normalization condition $\underline{e}_i^\top \underline{v}_i = 0$, while in the second case we have to solve

$$\alpha_i \widetilde{S}_{i,h} \underline{v}_i = \underline{r}_i. \tag{2.35}$$

Similar to the FE counterpart (cf. e.g. [27]), for $i = 1, \ldots, q$, the solution of (2.34) is equivalent to the solution of the extended system

$$\begin{pmatrix} V_{i,h} & -\frac{1}{2}M_{i,h} - K_{i,h} \\ \frac{1}{2}M_{i,h}^\top + K_{i,h}^\top & D_{i,h} + \beta_i \underline{e}_i \underline{e}_i^\top \end{pmatrix} \begin{pmatrix} \underline{w}_i \\ \underline{v}_i \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \frac{1}{\alpha_i}\underline{r}_i \end{pmatrix}. \tag{2.36}$$

In the same way, for $i = q+1, \ldots, p$, (2.35) is equivalent to

$$\begin{pmatrix} V_{i,h} & -\frac{1}{2}M_{i,h} - K_{i,h} \\ \frac{1}{2}M_{i,h}^\top + K_{i,h}^\top & D_{i,h} \end{pmatrix} \begin{pmatrix} \underline{w}_i \\ \underline{v}_i \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \frac{1}{\alpha_i}\underline{r}_i \end{pmatrix}. \tag{2.37}$$

Note that both matrices in (2.36)–(2.37) are positive definite but block skew–symmetric. Besides a direct solution procedure one may use preconditioned iterative schemes as discussed in [37] to solve (2.36) and (2.37).
When applying a Cholesky decomposition of the local discrete local single layer potentials $V_{i,h}$ only, the matrix-by-vector multiplication of the discrete Steklov–Poincaré operator $\widetilde{S}_{i,h}$ can be performed easily. Hence we can use a preconditioned conjugate gradient scheme as iterative solution procedure for (2.34) and (2.35). As preconditioner we may use a Galerkin matrix of the discrete single layer potential, see [30, 32, 39]. More precisely, the application of the preconditioner is given by

$$C_{S,i}^{-1} \;=\; \bar{M}_{i,h}^{-1}\bar{V}_{i,h}\bar{M}_{i,h}^{-1} \tag{2.38}$$

where

$$\bar{V}_{i,h}[m,n] \;=\; \langle V_i\varphi_n^i, \varphi_m^i\rangle_{L_2(\Gamma_i)}, \quad \bar{M}_{i,h}[m,n] \;=\; \langle \varphi_n^i, \varphi_m^i\rangle_{L_2(\Gamma_i)}$$

for $m,n = 1,\ldots,M_i$, $i = 1,\ldots,p$.
We mention that an alternative approach to realize the pseudoinverse $\widetilde{S}_{i,h}^{+}$ of the local Steklov–Poincaré operators by discrete counterparts of the local Poincaré–Steklov (Neumann-Dirichlet map) operator is explicitely available in the boundary element method. We will exploit this approach in connection with sparse approximation techniques in a forthcoming paper.

The orthoprojection P ensures the solvability of the local Neumann problems and the global information exchange. The application of $P = I - G(G^TG)^{-1}G^T$ to some vector $\underline{w} \in \mathbf{\Lambda}$ involves the direct solution of a small system with the $q \times q$ system matrix $\underline{G}^T\underline{G}$ that plays the role of some kind of a coarse grid problem that can be generated and factorized in advance.
The BETI preconditioner $\underline{C}$ should be spectrally equivalent to the BETI operator F on the subspace $\mathbf{\Lambda}_0 = \ker\underline{G}^T$, i.e.

$$\underline{\gamma}\,(C\underline{\lambda},\underline{\lambda}) \;\le\; (F\underline{\lambda},\underline{\lambda}) \;\le\; \overline{\gamma}\,(C\underline{\lambda},\underline{\lambda}) \quad \text{for all } \underline{\lambda} \in \mathbf{\Lambda}_0 \tag{2.39}$$

with positive spectral equivalence constants $\underline{\gamma}$ and $\overline{\gamma}$ such that the relative spectral condition number $\kappa(PC^{-1}P^TP^TFP)$ respectively its bound $\underline{\gamma}\,/\,\overline{\gamma}$ is as small as possible and the preconditioning operation $C^{-1}\underline{d}$ is as cheap as possible.
In contrast to the FETI method, we do not build C^{-1} from the local Schur complements $\widetilde{S}_{i,h}$, but from the local discrete hypersingular operators $D_{i,h}$. Thus, the BETI preconditioner will not require any solution of local Dirichlet problems. At first we propose the following three BETI preconditioners which are adapted to different situations:

- Hypersingular BETI preconditioner I (for moderate changing coefficients, no cross points):

$$C^{-1} = BD_hB^T = \sum_{i=1}^{p} \alpha_i B_i D_{i,h} B_i^T \tag{2.40}$$

with $D_h = \mathrm{diag}(\alpha_i D_{i,h})_{i=1:p}$;

- Hypersingular BETI preconditioner II (for moderate changing coefficients):

$$C^{-1} = (BB^\top)^{-1} B D_h B^\top (BB^\top)^{-1}; \tag{2.41}$$

- Scaled hypersingular BETI preconditioner:

$$C^{-1} = (BC_\alpha^{-1}B^\top)^{-1} BC_\alpha^{-1} D_h C_\alpha^{-1} B^\top (BC_\alpha^{-1}B^\top)^{-1} \tag{2.42}$$

  where $C_\alpha = \mathrm{diag}(C_{\alpha,i})_{i=1:p}$ and $C_{\alpha,i} = \mathrm{diag}(c_l^i)_{l=1:M_i}$ with appropriately chosen weights c_l^i, see [25] and [5].

The block diagonal components $D_{i,h}$ of D_h arise from the discretization of the hypersingular integral operator D_i on Γ_i. Therefore, this $M_i \times M_i$ matrix is fully populated. The matrix-by-vector multiplication $D_{i,h}\underline{d}_i$ requires M_i^2 operations that behaves like $O((H_i/h_i)^2$ and $O((H_i/h_i)^4$ for $d = 2$ and $d = 3$, respectively. Especially in 3D a reduction of the complexity of this multiplication operation is highly desirable. In addition to this we want to reduce the memory demand from M_i^2 to $\mathcal{O}(M_i)$ (up to some possible polylogarithmic factor). These goals can be achieved by using sparse approximations $\widetilde{D}_{i,h}$ of $D_{i,h}$ such as multipole representations [14, 8, 33], panel clustering [19], adaptive cross approximation [1, 2], $\mathcal{H}$– resp. $\mathcal{H}^2$–matrix techniques [18, 17], and wavelets [6, 34].
Using integration by parts [31], the bilinear form of the hypersingular boundary integral operator D_i can be rewritten as

$$\langle D_i u, v\rangle_{L_2(\Gamma_i)} = \frac{1}{4\pi} \int\limits_{\Gamma_i} \int\limits_{\Gamma_i} \frac{\mathrm{curl}_{\Gamma_i} u(y) \cdot \mathrm{curl}_{\Gamma_i} v(x)}{|x-y|} ds_y ds_x \tag{2.43}$$

for the case $d = 3$ (similarly, for $d = 2$), where

$$\mathrm{curl}_{\Gamma_i} := n_i(x) \times \nabla_x u^*(x) \quad \text{for } x \in \Gamma_i$$

and u^* is an extension of u into a neighborhood of Γ_i. When using a boundary element mesh of plane triangles and piecewise linear basis functions φ_n^i, $\mathrm{curl}_{\Gamma_i}\varphi_n^i \in \mathbb{R}^3$ is piecewise constant. Then the local Galerkin matrix $D_{i,h}$ can be represented in the form

$$D_{i,h} = C_{i,h}^\top \begin{pmatrix} V_{i,h} & & \\ & V_{i,h} & \\ & & V_{i,h} \end{pmatrix} C_{i,h}, \tag{2.44}$$

where $V_{i,h}$ is the Galerkin matrix of the related single layer potential with piecewise constant basis functions, and where $C_{i,h}$ is a matrix describing the piecewise constant surface curl. Hence a fast realization $\widetilde{D}_{i,h}$ of the discrete hypersingular integral operator is reduced to three fast applications $\widetilde{V}_{i,h}$ of the discrete single layer potential which can be done by a fast multipole method [14, 33]. Obviously, this approach is kernel–preserving with $\ker\widetilde{D}_{i,h} = \ker D_{i,h}$. Replacing

$D_{i,h}$ by $\widetilde{D}_{i,h}$ in (2.42), we arrive at the sparse version of the scaled hypersingular BETI preconditioner:

$$C^{-1} = (BC_\alpha^{-1}B^T)^{-1}BC_\alpha^{-1}\widetilde{D}_h C_\alpha^{-1}B^T(BC_\alpha^{-1}B^\top)^{-1} \tag{2.45}$$

called scaled sparse hypersingular BETI preconditioner, where C_α is the same scaling matrix as before, and $\widetilde{D}_h = \mathrm{diag}(\widetilde{D}_{i,h})_{i=1:p}$ is a sparse approximation of $D = \mathrm{diag}(D_{i,h})_{i=1:p}$ by means of the symmetric multipole technique described above. Now the preconditioning operation $C^{-1}\underline{d}^{n+1}$ in Algorithm 1 is almost optimal with respect to the operation count.

3 Convergence Analysis

The crucial point in the analysis of BETI methods is the observation that the BE (subdomain) Schur complement $S_{BEM,i} = \widetilde{S}_{i,h}$ can be related to some FE (subdomain) Schur complement $S_{FEM,i}$ via the spectral equivalence in $H^{1/2}(\Gamma_i)$. Before establishing this basic spectral equivalence, we prove the spectral equivalence of $D_{i,h}$ and $\widetilde{S}_{i,h}$. Here and in the following we use the notation $A_h \simeq B_h$ for the spectral equivalence of some symmetric and positive semidefinite (of course, with the same kernel) matrices A_h and B_h with independent of h, H, p and the coefficients jumps, positive spectral equivalence constants $\underline{c}$ and $\overline{c}$ (called universal constants), i.e. for all vectors $\underline{v}$ form the corresponding Euclidian vector space the following spectral equivalence inequalities are valid:

$$\underline{c}\,(B_h\underline{v},\underline{v}) \;\le\; (A_h\underline{v},\underline{v}) \;\le\; \overline{c}\,(B_h\underline{v},\underline{v}), \tag{3.1}$$

or, briefly, $\underline{c}\,B_h \;\le\; A_h \;\le\; \overline{c}\,B_h$.

Lemma 3.1 *The discrete hypersingular boundary integral operator $D_{i,h}$ is spectrally equivalent to the discrete Steklov–Poincaré operator $\widetilde{S}_{i,h}$, i.e.*

$$D_{i,h} \simeq \widetilde{S}_{i,h} := D_{i,h} + (\frac{1}{2}M_{i,h}^\top + K_{i,h}^\top)V_{i,h}^{-1}(\frac{1}{2}M_{i,h} + K_{i,h}). \tag{3.2}$$

Proof. For $\underline{v}_i \in \mathbb{R}^{M_i}$ we put $\underline{w}_i := V_{i,h}^{-1}(\frac{1}{2}M_{i,h} + K_{i,h})\underline{v}_i$. Then we have

$$\begin{aligned}(\widetilde{S}_{i,h}\underline{v}_i,\underline{v}_i) &= (D_{i,h}\underline{v}_i,\underline{v}_i) + ((\frac{1}{2}M_{i,h}^\top + K_{i,h}^\top)\underline{w}_i,\underline{v}_i) \\ &= (D_{i,h}\underline{v}_i,\underline{v}_i) + (V_{i,h}\underline{w}_i,\underline{w}_i) \;\ge\; (D_{i,h}\underline{v}_i,\underline{v}_i)\end{aligned}$$

since $V_{i,h}$ is positive definite. To prove the upper estimate we first consider the bilinear form of the continuous Steklov–Poincaré operator, for $v_i \in H^{1/2}(\Gamma_i)$ we define $w_i = V_i^{-1}(\frac{1}{2}I + K_i)v_i \in H^{-1/2}(\Gamma_i)$ and we have

$$\begin{aligned}\langle S_i v_i, v_i\rangle_{L_2(\Gamma_i)} &= \langle D_i v_i, v_i\rangle_{L_2(\Gamma_i)} + \langle(\frac{1}{2}I + K_i')V_i^{-1}(\frac{1}{2}I + K_i)v_i, v_i\rangle_{L_2(\Gamma_i)} \\ &= \langle D_i v_i, v_i\rangle_{L_2(\Gamma_i)} + \langle V_i w_i, w_i\rangle_{L_2(\Gamma_i)}.\end{aligned}$$

In the same manner we conclude

$$\langle \widetilde{S}_i v_i, v_i \rangle_{L_2(\Gamma_i)} \;=\; \langle D_i v_i, v_i \rangle_{L_2(\Gamma_i)} + \langle V_i w_{i,h}, w_{i,h} \rangle_{L_2(\Gamma_i)},$$

where $w_{i,h} \in Z_{i,h}$ is the unique solution of (2.12). From the Galerkin orthogonality we obtain

$$\langle V_i w_{i,h}, w_{i,h} \rangle_{L_2(\Gamma_i)} \;\le\; \langle V_i w_i, w_i \rangle_{L_2(\Gamma_i)}$$

and therefore

$$\langle \widetilde{S}_i v_i, v_i \rangle_{L_2(\Gamma_i)} \;\le\; \langle S_i v_i, v_i \rangle_{L_2(\Gamma_i)} \quad \text{for all } v_i \in H^{1/2}(\Gamma).$$

Since S_i and D_i are spectrally equivalent, we proved that the spectral equivalence inequalities

$$\underline{c}_{DS}\, S_{BEM,i} \;\le\; D_{i,h} \;\le\; \overline{c}_{DS}\, S_{BEM,i} \tag{3.3}$$

hold with $\overline{c}_{DS} = 1$ and some universal positive constant $\underline{c}_{DS}$. This completes the proof. ∎

This observation now allows us to carry over the FETI analysis to BETI methods. More precisely, only for this purpose, we generate a quasi-regular finite element mesh in every subdomain Ω_i starting from the subdomain boundary mesh. This is always possible since the subdomain boundary mesh was supposed to be quasi-regular as well. The same discretization parameter h_i can be used to describe the quasi-regularity of the mesh. We assume a triangular mesh for $d = 2$ and a tetrahedral mesh for $d = 3$. Let us now denote the subdomain finite element stiffness matrix derived from the Laplace operator on the basis of linear elements by $K_{FEM,i}$. Numbering the unknowns on the subdomain boundary Γ_i first, then $K_{FEM,i}$ has the following block structure:

$$K_{FEM,i} = \begin{pmatrix} K_{C,i} & K_{CI,i} \\ K_{IC,i} & K_{I,i} \end{pmatrix}, \tag{3.4}$$

where the indices C and I indicate the correspondence to the subdomain boundary and the interior unknowns, respectively. The finite element Schur complement matrix arising from the elimination of the interior unknowns can be represented in the form (see, e.g. [16])

$$S_{FEM,i} = K_{C,i} - K_{CI,i} K_{I,i}^{-1} K_{IC,i}. \tag{3.5}$$

Lemma 3.2 *The local boundary element Schur complement matrix $S_{BEM,i} = \widetilde{S}_{i,h}$ is spectrally equivalent to the local finite element Schur complement matrix $S_{FEM,i}$, i.e.*

$$S_{BEM,i} \;\simeq\; S_{FEM,i} \tag{3.6}$$

for all $i = 1, \ldots, p$.

Proof. It is well-known [10] that the energy of the local hypersingular operator is equivalent to the $H^{1/2}(\Gamma_i)$-semi-norm squared, i.e. there exist universal positive constants $\underline{c}_D$ and $\overline{c}_D$ such that for all $v_{i,h} \in W_{i,h}$ and the corresponding coefficient vectors $\underline{v}_i \in \mathbb{R}^{M_i}$ the equivalence inequalities

$$\underline{c}_D \, |v_{i,h}|^2_{H^{1/2}(\Gamma_i)} \;\leq\; (D_{i,h}\underline{v}_i, \underline{v}_i) = \langle D_i v_{i,h}, v_{i,h}\rangle_{L_2(\Gamma_i)} \;\leq\; \overline{c}_D \, |v_{i,h}|^2_{H^{1/2}(\Gamma_i)} \quad (3.7)$$

hold. Now, from Lemma 3.1 and the equivalence inequalities (3.7), we immediately get the inequalities

$$\underline{c}_B \, |v_{i,h}|^2_{H^{1/2}(\Gamma_i)} \;\leq\; (S_{BEM,i}\underline{v}_i, \underline{v}_i) \;\leq\; \overline{c}_B \, |v_{i,h}|^2_{H^{1/2}(\Gamma_i)} \quad (3.8)$$

with universal positive constants $\underline{c}_B$ and $\overline{c}_B$. Similar equivalence inequalities are true for the finite element Schur complement $S_{FEM,i}$. Indeed, there are universal positive constants $\underline{c}_F$ and $\overline{c}_F$ such that for all $v_{i,h} \in W_{i,h}$ and the corresponding coefficient vectors $\underline{v}_i \in \mathbb{R}^{M_i}$ the equivalence inequalities

$$\underline{c}_F \, |v_{i,h}|^2_{H^{1/2}(\Gamma_i)} \;\leq\; (S_{FEM,i}\underline{v}_i, \underline{v}_i) \;\leq\; \overline{c}_F \, |v_{i,h}|^2_{H^{1/2}(\Gamma_i)} \quad (3.9)$$

are valid (see, e.g., [3], lemma 3.8, p. 8 in [28], or theorem 3.5, p.64 in [38] for the proof). Combining (3.8) and (3.9) gives the spectral equivalence inequalities

$$\overline{c}_F^{-1} \underline{c}_B \, S_{FEM,i} \;\leq\; S_{BEM,i} \;\leq\; \overline{c}_B \, \underline{c}_F^{-1} \, S_{FEM,i} \quad (3.10)$$

that completes the proof. ∎

The following lemma shows that the same spectral equivalence is true for the corresponding Moore-Penrose pseudoinverses on the subspace $B_i^T \mathbf{\Lambda}_0$.

Lemma 3.3 *There are universal positive constants $\underline{c}_{BF}^+$ and $\overline{c}_{BF}^+$ such that*

$$\underline{c}_{BF}^+ (S_{FEM,i}^{-1} B_i^T \underline{\lambda}, B_i^T \underline{\lambda}) \;\leq\; (S_{BEM,i}^{-1} B_i^T \underline{\lambda}, B_i^T \underline{\lambda}) \;\leq\; \overline{c}_{BF}^+ (S_{FEM,i}^{-1} B_i^T \underline{\lambda}, B_i^T \underline{\lambda}) \quad (3.11)$$

for all $\underline{\lambda} \in \mathbf{\Lambda}_0$ and for all $i = q+1, \ldots, p$ and

$$\underline{c}_{BF}^+ (S_{FEM,i}^{+} B_i^T \underline{\lambda}, B_i^T \underline{\lambda}) \;\leq\; (S_{BEM,i}^{+} B_i^T \underline{\lambda}, B_i^T \underline{\lambda}) \;\leq\; \overline{c}_{BF}^+ (S_{FEM,i}^{+} B_i^T \underline{\lambda}, B_i^T \underline{\lambda}) \quad (3.12)$$

for all $\underline{\lambda} \in \mathbf{\Lambda}_0$ and for all $i = 1, \ldots, q$.

Proof. For all $i = q+1, \ldots, p$, the local boundary and finite element Schur complements $S_{BEM,i}$ and $S_{FEM,i}$ are symmetric and positive definite (spd). It is well-known that, for spd matrices A and B, the spectral equivalence inequalities $\underline{c}\, B_h \;\leq\; A_h \;\leq\; \overline{c}\, B_h$ imply the spectral equivalence inequalities $\underline{c}\, A_h^{-1} \;\leq\; B_h^{-1} \;\leq\; \overline{c}\, A_h^{-1}$ and vice versa. Therefore, for the non-floating subdomains, the spectral inequalities (3.11) follow from inequalities (3.6). The same is true for the floating subdomains, i.e. for all $i = 1, \ldots, q$. Indeed, for $\underline{v}_i$ satisfying $\underline{e}_i^\top \underline{v}_i = 0$ we have

$$\left[S_{BEM,i} + \beta_i \underline{e}_i \underline{e}_i^\top\right] \underline{v}_i \;=\; S_{BEM,i} \underline{v}_i \,.$$

On the other hand, for $\underline{\lambda} \in \mathbf{\Lambda}_0$ we have

$$\underline{v}_i \;=\; S^{+}_{BEM,i} \;=\; \left[S_{BEM,i} + \beta \underline{e}_i \underline{e}_i^{\top}\right] B_i^{\top} \underline{\lambda} \quad \text{with } \underline{e}_i^{\top} \underline{v}_i = 0.$$

Using analogues considerations for the Moore-Penrose pseudoinverse of the finite element Schur complement, $S^{+}_{FEM,i} = [S_{FEM,i} + \beta_i \underline{e}_i \underline{e}_i^{\top}]^{-1}$, the assertion follows from the spd matrices $S_{BEM,i} + \beta_i \underline{e}_i \underline{e}_i^{\top}$ and $S_{FEM,i} + \beta_i \underline{e}_i \underline{e}_i$. ∎

Using these lemmas and the results obtained for the FETI method, we can prove our first main result for the scaled hypersingular BETI preconditioner (2.42).

Theorem 3.1 *For the scaled hypersingular BETI preconditioner (2.42), the condition estimate*

$$\kappa(PC^{-1}P^T P^T F P) \leq c \left(1 + \log \frac{H}{h}\right)^2 \tag{3.13}$$

holds, where the positive constant c is independent of h, H, p and the α_i's (coefficient jumps). The matrix-by-vector operation $D_{i,h}\underline{v}_i$ that is the most expensive operation in the preconditioning step costs $ops(D_{i,h}\underline{v}_i) = O((H/h)^2$ and $ops(D_{i,h}\underline{v}_i) = O((H/h)^4$ arithmetical operation for $d = 2$ and for $d = 3$, respectively.

Proof. In a first step, we show that the inverse of the scaled hypersingular BETI preconditioner (2.42)

$$C^{-1}_{BETI} = (BC_\alpha^{-1}B^{\top})^{-1} BC_\alpha^{-1} D_h C_\alpha^{-1} B^{\top} (BC_\alpha^{-1}B^{\top})^{-1} \tag{3.14}$$

is spectrally equivalent to the FETI preconditioner

$$C^{-1}_{FETI} = (BC_\alpha^{-1}B^{\top})^{-1} BC_\alpha^{-1} S_{FEM} C_\alpha^{-1} B^{\top} (BC_\alpha^{-1}B^{\top})^{-1} \tag{3.15}$$

that was proposed by Klawonn and Widlund in [25], where

$$S_{FEM} = \operatorname{diag}(\alpha_i S_{FEM,i})_{i=1:p}. \tag{3.16}$$

Indeed, using Lemma 3.1 and Lemma 3.2, we get

$$\begin{aligned}
(C^{-1}_{BETI}\underline{\lambda}, \underline{\lambda}) &= \sum_{i=1}^{p} \alpha_i (D_{h,i} C^{-1}_{\alpha,i} B_i^{\top} (BC_\alpha^{-1}B^{\top})^{-1} \underline{\lambda}, C^{-1}_{\alpha,i} B_i^{\top} (BC_\alpha^{-1}B^{\top})^{-1} \underline{\lambda}) \\
&\leq \overline{c}_{DS} \sum_{i=1}^{p} \alpha_i (S_{BEM,i} \underline{w}_i, \underline{w}_i) \\
&\leq \overline{c}_{DS} \overline{c}_B \underline{c}_F^{-1} \sum_{i=1}^{p} \alpha_i (S_{FEM,i} \underline{w}_i, \underline{w}_i) \\
&= \overline{c}_{DS} \overline{c}_B \underline{c}_F^{-1} (C^{-1}_{FETI} \underline{\lambda}, \underline{\lambda}) \quad \text{for all } \underline{\lambda} \in \mathbf{\Lambda}_0,
\end{aligned} \tag{3.17}$$

where $\underline{w}_i$ substitutes $C_{\alpha,i}^{-1} B_i^\top (B C_\alpha^{-1} B^\top)^{-1} \underline{\lambda}$. Therefore, we proved that

$$(C_{FETI}\underline{\lambda}, \underline{\lambda}) \le \overline{c}_{CC} \, (C_{BETI}\underline{\lambda}, \underline{\lambda}) \quad \text{for all } \underline{\lambda} \in \mathbf{\Lambda}_0 \tag{3.18}$$

with $\overline{c}_{CC} = \overline{c}_{DS} \overline{c}_B \underline{c}_F^{-1}$. Similarly, we get

$$\underline{c}_{CC} \, (C_{BETI}\underline{\lambda}, \underline{\lambda}) \le (C_{FETI}\underline{\lambda}, \underline{\lambda}) \quad \text{for all } \underline{\lambda} \in \mathbf{\Lambda}_0 \tag{3.19}$$

with $\underline{c}_{CC} = \underline{c}_{DS} \underline{c}_B \overline{c}_F^{-1}$. Therefore, $C_{BETI} \simeq C_{FETI}$.
Now, using Lemma 3.2, inequalities (3.18)-(3.19), and the FETI spectral equivalence inequalities

$$(C_{FETI}\underline{\lambda}, \underline{\lambda}) \le (F_{FETI}\underline{\lambda}, \underline{\lambda}) \le c_{FETI} \left(1 + \log \frac{H}{h}\right)^2 (C_{FETI}\underline{\lambda}, \underline{\lambda}) \tag{3.20}$$

for all $\underline{\lambda} \in \mathbf{\Lambda}_0$, which were proved by Klawonn and Widlund in [25], theorem 1 (see also Brenner [4, 5]), where c_{FETI} is a positive universal constant and

$$F_{FETI} = \sum_{i=1}^p \frac{1}{\alpha_i} B_i S_{FEM,i}^+ B_i^\top,$$

we finally obtain

$$\begin{aligned}
(F\underline{\lambda}, \underline{\lambda}) &= \sum_{i=1}^p \frac{1}{\alpha_i} (S_{BEM,i}^+ B_i^\top \underline{\lambda}, B_i^\top \underline{\lambda}) \\
&\le \overline{c}_{BF}^+ \sum_{i=1}^p \frac{1}{\alpha_i} (S_{FEM,i}^+ B_i^\top \underline{\lambda}, B_i^\top \underline{\lambda}) \\
&\le \overline{c}_{BF}^+ c_{FETI} \left(1 + \log \frac{H}{h}\right)^2 (C_{FETI}\underline{\lambda}, \underline{\lambda}) \\
&\le \overline{c}_{CC} \overline{c}_{BF}^+ c_{FETI} \left(1 + \log \frac{H}{h}\right)^2 (C_{BETI}\underline{\lambda}, \underline{\lambda}) \quad \text{for all } \underline{\lambda} \in \mathbf{\Lambda}_0.
\end{aligned}$$

Similarly, we get

$$(F\underline{\lambda}, \underline{\lambda}) \ge \underline{c}_{CC} \underline{c}_{BF}^+ (C_{BETI}\underline{\lambda}, \underline{\lambda}) \quad \text{for all } \underline{\lambda} \in \mathbf{\Lambda}_0.$$

Therefore, we verified the spectral equivalence inequalities (2.39) with the constants $\underline{\gamma} = \underline{c}_{CC} \underline{c}_{BF}^+$ and $\overline{\gamma} = \overline{c}_{CC} \overline{c}_{BF}^+ c_{FETI} \left(1 + \log(H/h))^2\right)$, i.e. the condition estimate (3.13) is proved. The complexity estimates given in Theorem 3.1 immediately follow from the fact that $D_{i,h}$ is fully-populated. ∎

In order to prove the same condition number estimate for the sparse version, we need an additional result formulated in the following lemma.

Lemma 3.4 *The sparse representations of the discrete local hypersingular operators $\widetilde{D}_{i,h}$ are symmetric, positive semidefinite, and spectrally equivalent to their full counterparts $D_{i,h}$, i.e.*

$$\widetilde{D}_{i,h} \simeq D_{i,h} \quad \textit{for all } i = 1, \ldots, p \tag{3.21}$$

In particular, $ker \widetilde{D}_{i,h} = ker D_{i,h}$ for all $i = 1, \ldots, p$.

Proof. The sparse representations of the discrete local hypersingular operators are given as

$$\widetilde{D}_{i,h} = C_{i,h}^{\top} \begin{pmatrix} \widetilde{V}_{i,h} & & \\ & \widetilde{V}_{i,h} & \\ & & \widetilde{V}_{i,h} \end{pmatrix} C_{i,h}$$

where the $\widetilde{V}_{i,h}$ are sparse representations of the discrete local single layer potentials. Since the transformation matrix $C_{i,h}$ describing the surface curl coincides with the exact representation in (2.44), we conclude $\ker \widetilde{D}_{i,h} = \ker D_{i,h}$. Now, the spectral equivalence inequalities (3.21) follow from the spectral equivalence inequalities $\widetilde{V}_{i,h} \simeq V_{i,h}$. This can be obtained when choosing the multipole parameters in an appropriate way, see [33] for details. ∎

Theorem 3.2 *For the scaled sparse hypersingular BETI preconditioner (2.45), the condition estimate*

$$\kappa(PC^{-1}P^TP^TFP) \leq c\left(1 + \log \frac{H}{h}\right)^2 \tag{3.22}$$

holds, where the positive constant c is independent of h, H, p and the α_i's (coefficient jumps). Now the matrix-by-vector operation $\widetilde{D}_{i,h}\underline{v}_i$ only costs ops$(\widetilde{D}_{i,h}\underline{v}_i) = O((H/h)^{d-1}(1 + \log \frac{H}{h})^2)$ arithmetical operation.

Proof. Taking into account Lemma 3.4, we can easily prove estimates of the type (3.17) - (3.19) with $D_{i,h}$ replaced by $\widetilde{D}_{i,h}$. The rest of the proof is identical to the proof of Theorem 3.1. ∎

4 Numerical Results

As a first test example we consider the boundary value problem

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \quad u(x) = g(x) \quad \text{for } x \in \Gamma = \partial\Omega$$

where $g(x) = 4\sum_{i=1}^{d} x_i$ and $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \subset \mathbb{R}^d$ is given as depicted in Figure 1 for $d = 2$ and $d = 3$.

The trial space $W_h = \text{span}\{\varphi_m\}_{m=1}^{M_0}$ is formed by piecewise linear basis functions φ_m with respect to a uniform mesh on the coupling boundary Γ_{12}. The local boundary element spaces $Z_{i,h} = \text{span}\{\psi_k^i\}_{k=1}^{N_i}$ are spanned by piecewise constant basis functions ψ_k^i with respect to a uniform mesh of the local subdomain boundaries Γ_i. The local single layer potentials $V_{i,h}$ are inverted by applying a Cholesky decomposition ($d = 2$) or a sufficiently accurate (up to the discretization error) inner conjugate gradient iteration ($d = 3$). The resulting linear system (2.29) is solved by a preconditioned conjugate gradient scheme

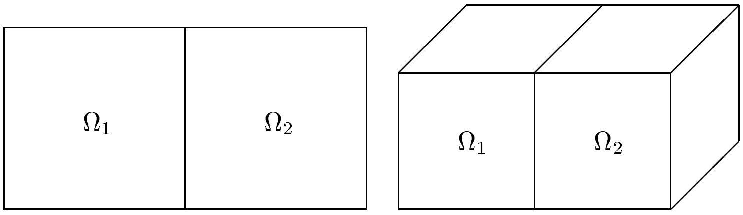

Figure 1: Domain decomposition with $p = 2$ subdomains ($d = 2, 3$).

either by using no preconditioner ($C = I$) or the hypersingular BETI preconditioner (2.40). As stopping criteria a relative error reduction of $\varepsilon = 10^{-8}$ was used. The number of iterations needed are shown in Table 1 ($d = 2$) and in Table 2 ($d = 3$), where L denotes the refinement level. In order to have a complexity comparison with the finite element discretization, we also indicate the number N_i^{FEM} of local finite element unknowns that would correspond to the boundary unkowns. It can clearly be seen from both tables that the grow in the number of iterations in the unpreconditioned case ($C = I$) is stopped by our hypersingular BETI preconditioner (2.40) as expected by our analysis.

L	M	N_i	N_i^{FEM}	$C^{-1} = I$	C^{-1} =(2.40)
3	7	32	49	4	4
4	15	64	225	8	7
5	31	128	961	11	10
6	63	256	3969	13	10
7	127	512	16129	17	10
8	255	1024	65025	25	10

Table 1: Numerical results for $p = 2$ subdomains ($d = 2$).

L	M	N_i	N_i^{FEM}	$C^{-1} = I$	C^{-1} = (2.40)
2	25	384	343	6	6
3	113	1536	3357	11	9
4	481	6144	29791	13	11

Table 2: Numerical results for $p = 2$ subdomains ($d = 3$).

As a second test example we consider a potential equation with piecewise constant coefficients,

$$-\text{div}[\alpha(x)\nabla u(x)] = 0 \quad \text{for } x \in \Omega, \quad u(x) = g(x) \quad \text{for } x \in \Gamma = \partial\Omega$$

where Ω and $\alpha(x)$ are as depicted in Figure 2, and $g(x) = 4(x_1 + x_2)$.

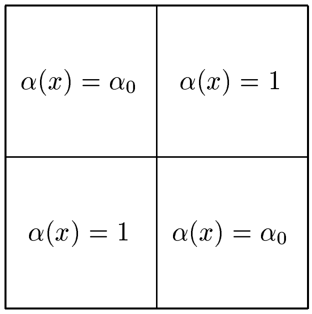

Figure 2: Domain decomposition with piecewise constant coefficients.

Table 3 presents the number of iterations for different preconditioners, different refinement levels L, and different jumps in the coefficients. Again we have a very good agreement with our analysis. Without preconditioner the numbers of iterations grow with growing refinement levels and the iteration is sensitive against coefficient jumps. The hypersingular BETI preconditioner II is better than the hypersingular BETI preconditioner I (we have a cross point !), but both preconditioners are sensitive against coefficient jumps. Only the scaled hypersingular BETI preconditioner (2.45) is really stable against coefficient jumps and shows only a $\log(H/h)$ grow (H=constant !) in the number of PCG iterations as predicted by our analysis.

5 Concluding Remarks

In this paper we presented the boundary element counterpart BETI of the FETI method that is nowadays well established in the finite element community. Our BETI preconditioners were constructed from the discrete hypersingular operator that is available in the symmetric domain decomposition boundary element method anyway. Our analysis showed and our numerical experiments confirmed that the BETI methods exhibit the same nice numerical and practical properties as the FETI methods.

The standard boundary element discretization leads to fully populated matrices. This drawback of the boundary element method can be avoided by means of sparse approximation techniques. We exploited this technique for the construction of the symmetric and positive definite scaled sparse hypersingular BETI preconditioner (2.45) that appears to be almost optimal with respect to the operation count as well as the memory demand. Up to some polylogarithmic factor the complexity of these resources is proportional to the number of the subdomain boundary unknowns. In a forthcoming paper we will develop sparse inexact BETI versions the total complexity of which is basically proportional to the number of the subdomain boundary unknowns. It follows from our analysis

L	N_i	M	$C^{-1} = I$	C^{-1} =(2.40)	C^{-1} =(2.41)	C^{-1} =(2.42)
			$\alpha_0 = 1$			
4	64	61	17	16	11	11
5	128	125	21	17	12	12
6	256	253	30	17	13	13
7	512	509	40	18	14	14
8	1024	1021	54	19	14	14
			$\alpha_0 = 2$			
4	64	61	19	27	13	16
5	128	125	24	29	13	18
6	256	253	33	31	14	20
7	512	509	44	30	14	21
8	1024	1021	60	31	15	22
			$\alpha_0 = 100$			
4	64	61	21	34	20	20
5	128	125	27	39	21	22
6	256	253	35	41	22	23
7	512	509	50	44	22	25
8	1024	1021	69	46	24	26
			$\alpha_0 = 10000$			
4	64	61	13	15	24	13
5	128	125	17	17	29	15
6	256	253	22	19	30	16
7	512	509	30	19	31	17
8	1024	1021	39	19	31	19

Table 3: Numerical results for $p = 4$ subdomains ($d = 2$) and again $\varepsilon = 10^{-8}$.

that the scaled sparse hypersingular BETI preconditioner (2.45) can be used as optimal preconditioner in the FETI methods as well.

There is another, very useful opportunity to make a marriage between BETI and FETI methods. Since the FEM and the BEM have certain complementary properties, it is sometimes very useful to couple these discretization techniques and to benefit from both worlds (see e.g. [15]). This concerns not only the treatment of unbounded domains (BEM), but also the right handling of singularities (BEM), moving parts (BEM), volume forces (FEM), non-linearities (FEM) etc. Thus, combining our BETI techniques with the FETI methods gives new, quite attractive tearing and interconnecting parallel solvers for large scale coupled FE-BE-DD equations.

References

[1] M. Bebendorf. Approximation of boundary element matrices. *Numer. Math.*, 86:565–589, 2000.

[2] M. Bebendorf and S. Rjasanow. Adaptive low-rank approximation of collocation matrices. *Bericht Nr. 51, Universität des Saarlandes*, 2001.

[3] J.H. Bramble, J.E. Pasciak, and A.H. Schatz. The construction of preconditioners for elliptic problems by substructuring I. *Math. Comp.*, 47:103–134, 1986.

[4] S.C. Brenner. A new look at FETI. In *Domain Decomposition Methods in Sciences and Engineering*, Proceedings of the 13th Int. Conference on Domain Decomposition, October 9-12, 2000, Lyon, France. ddm.org, 2001.

[5] S.C. Brenner. An additive Schwarz preconditioner for the FETI method. *Numer. Math.*, 2002. (to appear).

[6] J. Breuer, O. Steinbach, and W. L. Wendland. Wavelet approximation of the symmetric boundary integral formulation. *In preparation*, 2003.

[7] C. Carstensen, M. Kuhn, and U. Langer. Fast parallel solvers for symmetric boundary element domain decomposition equations. *Numer. Math.*, 79:321–347, 1998.

[8] H. Cheng, L. Greengard, and V. Rokhlin. A fast adaptive multipole algorithm in three dimensions. *J. Comput. Phys.*, 155(2):468–498, 1999.

[9] M. Costabel. Boundary integral operators on Lipschitz domains: Elementary results. *SIAM J. Math. Anal. Appl.*, 19(3):613–626, 1988.

[10] M. Costabel and E. P. Stephan. Boundary integral equations for mixed boundary value problems in polygonal domains and galerkin approximation. In *Mathematical Models and Methods in Mechanics*, pages 175–251, Warsaw, 1985. Banach Center Publications 15.

[11] C. Farhat, M. Lesoinne, and K. Pierson. A scalable dual-primal domain decomposition method. *Numer. Linear Algebra Appl.*, 7(7-8):687–714, 2000.

[12] C. Farhat and F.-X. Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Int. J. Numer. Meth. Engrg.*, 32:1205–1227, 1991.

[13] C. Farhat and F.-X. Roux. Implicite parallel processing in structural mechanics. In J.T. Oden, editor, *Comput. Mech. Adv.*, pages 1–124, Amsterdam, 1994. North-Holland.

[14] L. Greengard and V. Rokhlin. A fast algorithm for particle simulations. *J. Comput. Physics*, 73:325–348, 1987.

[15] G. Haase, B. Heise, M. Kuhn, and U. Langer. Adaptive domain decompostion methods for finite and boundary element equations. In *Boundary Element Topics (ed. by W. Wendland)*, pages 121–147, Berlin, 1998. Springer-Verlag.

[16] G. Haase, U. Langer, and A. Meyer. The approximate Dirichlet decomposition method. part I and II. *Computing*, 47:137–167, 1991.

[17] W. Hackbusch, B. N. Khoromskij, and S. A. Sauter. On $\mathcal{H}^2$–matrices. In *Lectures on applied mathematics*, pages 9–29, Berlin, 2000. Springer-Verlag.

[18] W. Hackbusch and B.N. Khoromskij. A sparse $\mathcal{H}$-matrix arithmetic ii. application to multi-dimensional problems. *Computing*, 64(1):21–47, 2000.

[19] W. Hackbusch and Z.P. Nowak. On the fast matrix multiplication in the boundary element method by panel clustering. *Numer. Math.*, 54(4):463–491, 1989.

[20] G. C. Hsiao, E. Schnack, and W. L. Wendland. Hybrid coupled finite–boundary element methods for elliptic systems of second order. *Comput. Methods Appl. Mech. Engrg.*, 190:431–485, 2000.

[21] G. C. Hsiao, O. Steinbach, and W. L. Wendland. Domain decomposition methods via boundary integral equations. *J. Comput. Appl. Math.*, 125:521–537, 2000.

[22] G. C. Hsiao and W. L. Wendland. A finite element method for some integral equations of the first kind. *J. Math. Anal. Appl.*, 58:449–481, 1977.

[23] G.C. Hsiao and W.L. Wendland. Domain decomposition in boundary element methods. In *Proceedings of the Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations (ed. by R. Glowinski and Y.A. Kuznetsov and G. Meurant and J. Périaux and O. B. Widlund), Moscow, May 21-25, 1990*, pages 41–49, Philadelphia, 1991. SIAM.

[24] A. Klawonn and O.B. Widlund. A domain decomposition method with Lagrange multipliers and inexact solvers for linear elasticity. *SIAM J. Sci. Comput.*, 22(4):1199–1219, 2000.

[25] A. Klawonn and O.B. Widlund. FETI and Neumann–Neumann Iterative Substructuring Methods: Connections and New Results. *Comm. Pure Appl. Math.*, 54:57–90, January 2001.

[26] U. Langer. Parallel iterative solution of symmetric coupled FE/BE- equations via domain decomposition. *Contemp. Math.*, 157:335–344, 1994.

[27] J. Mandel. Balancing domain decomposition. *Comm. Numer.Methods Engrg*, 9:233 – 241, 1993.

[28] J. Mandel and M. Brezina. Balancing domain decomposition: theory and performance in two and three dimensions. Technical Report, Denver 1993.

[29] J. Mandel and R. Tezaur. Convergence of a substrucuring method with Lagrange multipliers. *Numer. Math.*, 73(4):473–487, 1996.

[30] W. McLean and O. Steinbach. Boundary element preconditioners for a hypersingular integral equation on an intervall. *Adv. Comput. Math.*, 11:437–459, 1999.

[31] J. C. Nédélec. Integral equations with non–integrable kernels. *Int. Eq. Operat. Th.*, 5:562–572, 1982.

[32] G. Of and O. Steinbach. A fast multipole boundary element method for a modified hypersingular boundary integral equation. In *Proceedings of the International Conference on Multifield Problems*. Springer Lecture Notes in Applied Mechanics, 2003.

[33] G. Of, O. Steinbach, and W. L. Wendland. The fast multipole method for the symmetric boundary integral formulation. *In preparation*, 2003.

[34] R. Schneider. *Multiskalen- und Wavelet-Matrixkompression*. Advances in Numerical Mathematics. B.G. Teubner-Verlag, Stuttgart, 1998.

[35] D. Stefanica. A numerical study of FETI algorithms for mortar finite element methods. *SIAM J. Sci. Comput.*, 23(4):1135–1160, 2001.

[36] O. Steinbach. Boundary elements in domain decomposition methods. *Contemp. Math.*, 180:343–348, 1994.

[37] O. Steinbach. Fast solution techniques for the symmetric boundary element method in linear elasticity. *Comput. Methods Appl. Mech. Engrg.*, 157:185–191, 1998.

[38] O. Steinbach. *Stability estimates for hybrid coupled domain domain decomposition methods*, volume 1809 of *Springer Lecture Notes in Mathematics*. Springer-Verlag, Berlin - Heidelberg - New York, 2003.

[39] O. Steinbach and W. L. Wendland. The construction of some efficient preconditioners in the boundary element method. *Adv. Comput. Math.*, 9:191–216, 1998.